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ABSTRACT

THE USE OF OPERATOR REGULARIZATION IN THE COMPUTATION OF EFFECTIVE FIELD THEORIES

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Current methods of computing low-energy effective field theories while being accurate are extremely cumbersome to implement. Operator regularization provides a way to calculate any desired effective field theory avoiding the tediousness of previous methods. This technique is shown to be an effective method in explicitly calculating the $1/m_{heavy}$ corrections that match an original full theory to its low-energy effective counterpart. This is demonstrated up to two loop order for the case of a charged two field ϕ_4^4 theory as well as up to one loop order for the Higgs sector of the Minimal Supersymmetric Model. Additionally, the renormalization group functions for the ϕ_4^4 model are calculated from the finite parts of the Greens functions.

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Chapter 1

Introduction

The physical world we observe is an incredibly rich and complex place. In order to focus on the important physics at a given scale, one usually divides the world into different regimes of important parameters (velocity, energy, distance, etc.). Typically there are quantities in a calculation which are small or large to a particular parameter of interest. The usual trick is to take these large quantities as infinite and the small ones as zero thus leaving an overall simpler theory. The physical effects of the small quantities are then taken as perturbations of the new simpler theory. This new perturbed simpler theory is called an effective theory [1].

For example, the classical properties of the motion of fluids can be completely described by using the rules of statistical quantum mechanics. That is, all the properties of the fluid can be calculated from the small distance behavior of the fluid. However, it is much easier to use classical fluid dynamics where the various properties of the fluid which arise from small scale effects (viscosity, pressure, etc.) are treated as macroscopic properties of the fluid. This is an example of an effective field theory where the relevant parameter is the distance scale.

The use of effective field theories in particle physics is particularly useful because all the particles that are too heavy to be produced at a given energy scale can be ignored. However, because of the need to regularize any quantum field theory, the process of constructing an effective field theory is a non-trivial matter.

Currently the physics community is beginning to experimentally probe physics above the energies of the Standard Model. For example, one interpretation of the recent data from the ZEUS experiment at HERA is Z boson exchange between leptoquarks, a particle predicted in various GUT models, or even between squarks the supersymmetric partners to quarks [2–4]. Models such as supersymmetry operate at smaller distance scales than is covered by the Standard Model.

There are many different theoretical models that have been developed to try to predict what we will see in this new energy regime beyond the Standard Model. As mentioned above, one class of these high energy models are the supersymmetric models [5]. In a supersymmetric model fermions and bosons are treated as different states of a larger supersymmetric particle doublet. This is analogous to the proton and neutron which are considered different isospin states of the nucleon isodoublet.

One of the more promising supersymmetric theories is the Minimal Supersymmetric Model [6]. As one can suppose by its name, the Minimal Supersymmetric Model adds supersymmetry to the Standard Model with a minimal number of new parameters. These parameters include the sundry supersymmetric partners to the familiar zoo of particles (sleptons, squarks, winos, zinos, photinos and their kin) and a new mixing angle [6]. Furthermore, adding supersymmetry to the Standard Model necessarily requires the addition of more scalar doublets to the Higgs sector of the model [5, 6]. The Minimal Supersymmetric Model achieves this by the addition of a single new scalar doublet.

When the Higgs mechanism is applied to the Minimal Supersymmetric Model, five physical Higgs fields as well as three Goldstone fields are generated [6]. It turns out that one of these physical Higgs fields can be associated with the Higgs field of the Standard Model [6]. As one would expect, the Standard Model Higgs is much lighter than the other four Higgs fields (all of whom have approximately the same mass) that arise in the Minimal Supersymmetric Model [6]. The Higgs sector of the Minimal Supersymmetric Model thus has two mass scales, one associated with the lighter Higgs and the other associated with the four other heavier Higgs. This "two scale" system is an ideal case for the application of an effective field theory. Experiments to probe for the existence and properties of the heavier Higgs can be inferred by experimental corrections to theoretical predictions of the Standard Model Higgs [7]. In fact, these corrections can be explicitly calculated by using an effective field theory generated from the Minimal Supersymmetric Model.

There are two methods for creating effective field theories. The first, called the "bottom up" method, starts with a known low energy theory and the high energy behavior is then extrapolated by symmetry arguments and phenomenological methods. Historically, this is mainly how models governing new interactions were discovered [1]. The second method, called the "top down" method, starts from a full theory that contains all the high and low energy behavior. The heavy particle degrees of freedom are then "integrated out" leaving the effective theory [1]. However, the main difference between the two methods is that in the "top down" approach all the constants that appear in the effective field theory are known explicitly in terms of the full theory parameters. In the "bottom up" approach the constants in the effective theory must be calculated through experimental means.

To be more specific, the first step in calculating an effective field theory in the "bottom up" approach is to write down all the terms constructed from the fields explicitly present in the desired effective theory

$$\mathcal{L}_{eff} = \sum_k c_k \hat{O}_k \tag{i}$$

where the set $\{\hat{O}_k\}$ are operators which are constructed from the fields present in the low energy theory and their covariant derivatives [8,9]. The constants c_k are calculated by phenomenological methods and have inverse mass dimension. A good example of a "bottom up" approach is the evolution of weak interactions from the Fermi $(\overline{\psi}\psi)^2$ theory to the current Standard Model of electroweak interactions [10]. This approach has the advantage that one can ask questions without having to know the physics at a higher energy scales.

While in the "top down" approach all quantities can be explicitly calculated, it is a tedious process. First, all the loop diagrams (with internal heavy fields and external light fields) for a particular n-point Green's function needs to be calculated. These loop diagrams then have to be regularized and renormalized so as to remove all the divergences. After all this is done the remaining finite parts are expanded out in a series in powers of the scaling parameter, i.e. $1/m_{heavy}$, to generate the effective Lagrangian

$$\mathcal{L}_{eff} = \sum_k \frac{g_k}{m^k_{heavy}} \hat{O}_k \tag{ii}$$

Note that the operators $\{\hat{O}_k\}$ necessarily contain the symmetries present in the initial Lagrangian [8,9].

The coefficients g_k in (ii) are generated from the various integrals over the Feynman parameters that enter into the loop diagrams with external light fields and internal heavy fields. However, these integrals are very sensitive to how the Feynman parameters were initially introduced into the loop diagrams and how the loop diagrams were calculated. Depending how one inputs the Feynman parameters, one can be left with an integral that is not calculatable. Further these integrals can, if one is not very careful, generate extra false infinities in the finite parts. This problem aside, the process of generating an effective Lagrangian can become quite involved especially when one goes beyond one loop processes and complications related to subdivergences start to manifest themselves.

Recently, a new technique by D.G.C. McKeon, T. Sherry and L. Culumovic called operator regularization has been developed to regularize and renormalize a given n-point function to any number of loops [11-19]. The remarkable thing about operator regularization is that at no time in the calculation do explicit infinities appear. Further, the regularization is done in such a way that all the manifest symmetries of the initial Lagrangian are preserved throughout the calculation. This greatly simplifies the tedious process of renormalization and eliminates the problems that arise when one is forced to artificially break the symmetries of the initial Lagrangian in order to regularize a theory.

In order to calculate renormalization group functions using operator regularization a low energy expansion called the DeWitt expansion is used [20]. This expansion is a series in the various fields that appear in the initial Lagrangian and their covariant derivatives. This type of series is exactly what is desired in order to perform a large mass expansion to generate an effective Lagrangian. It turns out by using terms in the DeWitt expansion of a higher order than those used to calculate renormalization group functions the effective Lagrangian can easily be generated.

This paper is organized as follows. Section 2 outlines the general technique and advantages of operator regularization for any order in $\hbar$. Section 3 applies operator regularization to a model with two charged scalar fields one of which is heavier than the other. Such a model is sufficient to illustrate the relevant points of the operator regularization technique without the complications introduced when particles with non-zero spin are considered. The two, three and four point renormalized Greens functions are calculated to two loop order and the renormalization group functions are generated. It is also established that the creation of an effective field theory can easily be done by using higher order DeWitt coefficients than those used to generate the renormalization group functions. Section 4 applies operator regularization to the Minimal Supersymmetric Model. Following a brief outline of the general Higgs two-doublet model, the effective Lagrangian for the Higgs sector of the Minimal Supersymmetric model, which includes up to the eight-point interaction terms, is generated up to one loop and to order $1/M^4$ (where M is the heavy Higgs mass). It is also confirmed that in the decoupling limit ($M \to \infty$) the Higgs Sector of the Minimal Supersymmetric Model reduces to the Higgs Sector of the Standard Model of electroweak interactions.

In this paper natural units ($\hbar = c = 1$) are assumed unless explicitly noted otherwise and all calculations are carried out in Euclidean space.

Chapter 2

Operator Regularization

2.1 Introduction to Operator Regularization

The process of regularizing divergences in quantum field theory without breaking the symmetries present in the initial Lagrangian is a considerable problem. The standard methods of regularization, i.e. dimensional regularization and Pauli-Villars regularization, introduce artificial regularizing parameters into the Lagrangian [9]. These parameters break the symmetries of the Lagrangian that one wishes to preserve throughout the calculation. For example, γ_5 is ill defined in dimensional regularization.

Operator regularization (OR) differs from the previous methods of regularizing in that it is the field operators that appear in the generating functional that are regularized by a redefinition of the logarithms and inverses of operators. Because no artificial parameter is introduced into the generating functional, any n-point function which is calculated using OR is manifestly finite to all orders in $\hbar$ [11–19]. Additionally, the risk of false anomalies arising by mathematical accident is removed: any anomalies that appear in the theory under consideration do so naturally from quantum effects [18].

The actual mathematics behind OR is not very far removed from the techniques one is already familiar with in quantum field theory. The fields under consideration are split into their quantum and classical parts, i.e. their interacting parts and their parts that can be experimentally detected. From this point the usual method of using a generating functional with an arbitrary source is utilized to produce the explicit form for any n-point Green's function. Regularization is applied only once the mathematical forms of the Green's functions are known, as well as before any integrations over loop momenta are attempted. This is very different from other methods of regularizing where the actual integrations over loop momenta are regularized via. some artificial limiting parameter. The regularization of ultraviolet divergences in OR is achieved by a clever redefinition of the logarithms and inverses of the matrix elements that appear in the Green's functions. This process removes all ultraviolet divergences thus leaving the Green's functions manifestly finite before any integrations over loop momenta are done. Applying a perturbative expansion to these regularized and finite matrix elements calculations can be carried out with out having to worry about ultraviolet divergences or the effects due to the introduction of artificial regularizing parameters. The final results achieved through OR are identical to those achieved through other methods of regularization up to an irrelevant additive constant.

2.2 The Technique

The background field formalism is the starting point for operator regularization [21, 22]. The fields $\phi_i(x)$ that appear in the Lagrangian, which can generally be fermionic or bosonic, are split into the sum

$$\phi_i(x) = f_i(x) + h_i(x) \tag{2.2.1}$$

where $f_i(x)$ is the classical part of the field and $h_i(x)$ is the quantum part.

Given a source $J_i(x)$, which corresponds to the field $\phi_i(x)$, the generating functional for the Euclidean space Green's functions is given by

$$Z[J] = \int \mathcal{D}\phi \, \exp\left[-\frac{1}{\hbar}(S[\phi] + \phi \cdot J)\right]. \tag{2.2.2}$$

where $\phi \cdot J = \int dx \, \phi_i(x) J_i(x)$.

Using the definition for the generating functional of the connected Green's functions, $W[J]$, [23]

$$Z[J] = \exp\left[-\frac{1}{\hbar}W[J]\right] \tag{2.2.3}$$

and the generating functional of one particle irreducible (1PI) Green's functions, $\Gamma[\phi]$,

$$\Gamma[\phi] = W[J] - \phi \cdot J \tag{2.2.4}$$

(2.2.3) takes the form

$$\exp\left(-\frac{1}{\hbar}\Gamma[\phi]\right) = \int \mathcal{D}h \, \exp\left[-\frac{1}{\hbar}(S[f+h] + h \cdot J)\right]. \tag{2.2.5}$$

In order to further evaluate $\Gamma[f+h]$ the Taylor series of the action

$$\begin{aligned} S[f+h] + h \cdot J = S[f] &+ \frac{1}{2}h_i S_{ij}(f) h_j \\ &+ \frac{1}{3!}S_{ijk}(f) h_i h_j h_k + \frac{1}{4!}S_{ijkl} h_i h_j h_k h_l \\ &+ h \cdot \left[J + \frac{\delta S}{\delta f}\right] \end{aligned} \tag{2.2.6}$$

is needed where $S_{ij} = \delta^2 S/\delta f_i \delta f_j$, etc. The Taylor series in (2.2.6) is truncated with S_{ijkl} because terms with higher powers of the fields do not exist in renormalizable theories in four dimensions.

Factoring the powers of h_i and using the source J it is found that.

$$\begin{aligned}\exp\left[-\frac{1}{\hbar}\Gamma[f]\right] = \exp\left[-\frac{1}{\hbar}S[f]\right] \operatorname{sdet}^{-1/2}[S_{ij}(f)/\mu^2]\sum_{n=0}^{\infty}\frac{1}{n!}\times \\ \times\left[\frac{\hbar^2\mu^{-3}}{3!}S_{ijk}(f)\frac{\delta^3}{\delta J_i\delta J_j\delta J_k} - \frac{\hbar^4\mu^{-4}}{4!}S_{ijkl}(f)\frac{\delta^4}{\delta J_i\delta J_j\delta J_k\delta J_l}\right]^n \times \\ \times\exp\left[\frac{1}{2\hbar}J_i\left[\frac{S_{ij}(f)}{\mu^2}\right]^{-1}J_j\right]\Bigg|_{\mu J=-\delta(\Gamma-S)/\delta f}\end{aligned} \tag{2.2.7}$$

where $S[f]$ is the classical action [18]. The scale parameter μ in (2.2.7) arises due to the arbitrariness in the normalization of the functional integral (2.2.2) [16, 24–26]. Note that the superdeterminant (defined below) is used vs. the ordinary determinant in (2.2.7). This is because in the general case the matrix S can contain both fermionic and bosonic elements.

Writing

$$\Gamma[f] = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots \tag{2.2.8}$$

equations (2.2.7) and (2.2.8) can now be used to generate the 1PI functions to any order in $\hbar$. Doing so to order $\hbar^2$,

$$\Gamma_1[f] = \frac{1}{2}\operatorname{str}[\ln(\hat{S}_{ij}[f])] \tag{2.2.9a}$$

$$\begin{aligned}\Gamma_2[f] = \frac{\mu^{-4}}{8}S_{ijkl}\left[\hat{S}_{ij}[f]\right]^{-1}\left[\hat{S}_{kl}[f]\right]^{-1} \\ - \frac{\mu^{-6}}{12}S_{ijk}S_{pqr}\left[\hat{S}_{ip}[f]\right]^{-1}\left[\hat{S}_{jq}[f]\right]^{-1}\left[\hat{S}_{kr}[f]\right]^{-1}.\end{aligned} \tag{2.2.9b}$$

where the relation

$$\operatorname{sdet} A = \exp[\operatorname{str}\ln A] \tag{2.2.10}$$

and the definition

$$\hat{S} \equiv S/\mu^2 \tag{2.2.11}$$

have been used.

The superdeterminant of the matrix S_{ij} is given by a ratio of the determinants of the fermionic and bosonic parts of S_{ij}. That is, if S_{ij} is written as

$$h_iS_{ij}h_j = bS_{bb}b + bS_{bf}f + fS_{fb}b + fS_{ff}f \tag{2.2.12}$$

(where b is a bosonic variable and f is a fermionic one) the superdeterminant is defined by either of the two following equations [27]

$$\operatorname{sdet} S \equiv \det(S_{bb} - S_{bf}S_{ff}^{-1}S_{fb})\det{}^{-1}S_{ff} \tag{2.2.13a}$$

or

$$\operatorname{sdet} S \equiv \det S_{bb} \det{}^{-1}(S_{ff} - S_{fb}S_{bb}^{-1}S_{bf}). \tag{2.2.13b}$$

Note that in the case where S is purely fermionic or purely bosonic, the superdeterminant reduces to the ordinary determinant.

The supertrace of the matrix S_{ij}, whose form is given by (2.2.12), is defined by [27]

$$\operatorname{str} S \equiv \operatorname{tr} S_{bb} - \operatorname{tr} S_{ff}. \tag{2.2.14}$$

Note that just like the superdeterminant, if S is purely bosonic or fermionic the supertrace reduces to the ordinary trace.

For the sake of convenience the following notation is introduced

$$M_{ij}(f) \equiv S_{ij}, \tag{2.2.15a}$$

$$a_{ijk}(f) \equiv S_{ijk}, \tag{2.2.15b}$$

$$b_{ijkl} \equiv S_{ijkl}. \tag{2.2.15c}$$

Equations (2.2.9a) and (2.2.9b) can thus be written as

$$\Gamma_1[f] = \frac{1}{2}\operatorname{str}[\ln(\hat{M}_{ij}[f])] \tag{2.2.16a}$$

$$\begin{aligned}\Gamma_2[f] = \int dx \Big[\frac{\mu^{-4}}{8} b_{ijkl} \langle x|\hat{M}_{ij}^{-1}(f)|x\rangle\langle x|\hat{M}_{kl}^{-1}(f)|x\rangle\Big] \\ - \frac{\mu^{-6}}{12}\int dx\,dy \Big[a_{ijk}(f(x))a_{pqr}(f(y))\langle x|\hat{M}_{ip}^{-1}(f)|y\rangle \\ \langle x|\hat{M}_{jq}^{-1}(f)|y\rangle\langle x|\hat{M}_{kr}^{-1}(f)|y\rangle\Big]\end{aligned} \tag{2.2.16b}$$

where $\hat{M} \equiv M/\mu^2$.

As we will see below, in operator regularization it is the matrix elements of $\hat{M}$ that are regularized. For the case of Γ_1, i.e. the one loop case, $\ln \hat{M}$ needs to be regularized. In the higher loop cases it is the inverses of $\hat{M}$ which requires the regularization.

The whole procedure of OR is based upon the fundamental regularizing equation [11, 18]

$$\begin{aligned}\ln(A) &= -\lim_{s\to 0}\frac{d^n}{ds^n}\frac{s^{n-1}}{n!}A^{-s} \\ &= -\mathcal{S}^{(n)}\left(s^{-1}A^{-s}\right), \quad (n = 1, 2, 3, \ldots),\end{aligned} \tag{2.2.17}$$

where

$$\mathcal{S}^{(n)} \equiv \lim_{s\to 0}\frac{d^n}{ds^n}\frac{s^n}{n!}. \tag{2.2.18}$$

Repeated differentiation of (2.2.17) gives the regulated form of the inverse power of an operator

$$A^{-n} = \lim_{s\to 0}\frac{d^n}{ds^n}\left[\frac{s^{n-1}}{n!}\frac{\Gamma(s+n)}{\Gamma(s)\Gamma(n)}A^{-s-n}\right]. \tag{2.2.19}$$

Using (2.2.19) the products of inverses of many operators may also be regularized

$$A_1^{-1}A_2^{-1}\cdots A_t^{-1} = \mathcal{S}^{(n)}\left(A_1^{-1-s}A_2^{-1-s}\cdots A_t^{-1-s}\right). \tag{2.2.20}$$

Additionally, using the integral definition of the gamma function, A^{-s} can be written as

$$A^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1}\exp(-At). \tag{2.2.21}$$

Substituting (2.2.17) and (2.2.21) into (2.2.16a) gives the regulated form of the one loop Green's function

$$\begin{aligned}\Gamma_1[f] &= \frac{1}{2}\,\mathrm{str}\left[-\lim_{s\to 0}\frac{d}{ds}\left(\hat{M}^{-s}\right)\right]\\ &= -\frac{1}{2}\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^{\infty} dt\; t^{s-1}\,\mathrm{str}\; e^{-\hat{M}t}\right].\end{aligned} \tag{2.2.22}$$

So far OR looks like the application of fancy mathematics. The critical question is that does this technique truly remove ultraviolet divergences from the Greens functions? The answer to this can been seen by denoting a general propagator by P. A general n-loop Green's function in d dimensions is thus

$$\Gamma^{(n)}_{Feynman} = \int \frac{d^dp_1\cdots d^dp_n}{(2\pi)^{dn}}\frac{1}{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_t^{\alpha_t}}. \tag{2.2.23}$$

When OR is used, the a n-loop Green's function remains basically the same except that the regulating parameter s is introduced into the exponent via. (2.2.20)

$$\Gamma^{(n)}_{OR} = \mathcal{S}^{(n)}\int \frac{d^4p_1\cdots d^4p_n}{(2\pi)^{4n}}\frac{1}{P_1^{\alpha_1+s}P_2^{\alpha_2+s}\cdots P_t^{\alpha_t+s}}. \tag{2.2.24}$$

Notice that if the limit $s\to 0$ is taken before $\mathcal{S}^{(n)}$ is applied $\Gamma^{(n)}_{OR}$ reduces back to the conventional $K_{Feynman}$ in four dimensions.

To explicitly see that operator regularization does indeed remove ultraviolet divergences we expand out the operator regulated n-point Green's function in s in a general Laurent series

$$\Gamma^{(n)}_{OR} = \mathcal{S}^{(n)}\left[\frac{C_{-n}}{s^n}+\frac{C_{-n+1}}{s^{n-1}}+\cdots+C_0+sC_1+\cdots\right] \tag{2.2.25}$$

Applying the explicit form of $\mathcal{S}^{(n)}$ given in (2.2.18) to (2.2.25)

$$\begin{aligned}\Gamma^{(n)}_{OR} &= \lim_{s\to 0}\frac{d^n}{ds^n}\frac{s^n}{n!}\left[\frac{C_{-n}}{s^n}+\frac{C_{-n+1}}{s^{n-1}}+\cdots+C_0+sC_1+\cdots\right]\\ &= \lim_{s\to 0}\frac{d^n}{ds^n}\frac{1}{n!}\left[C_{-n}+sC_{-n+1}+\cdots+s^nC_0+s^{n+1}C_1+\cdots\right]\\ &= C_0.\end{aligned} \tag{2.2.26}$$

Thus OR does remove all the poles, i.e. ultraviolet divergences, from the Green's functions. Further, because the application of OR comes down to a creative rewriting of logarithms and inverses the end result is the same (up to an additive constant) as the one would get in more conventional regularizing and renormalizing schemes as long as limit procedure is valid [18]. Do note that even though we have only considered a scalar Green's function, the above argument follows identically for those Green's functions that contain Lorentz indices, i.e. those with powers of momentum in their numerator's.

The above approach of showing the validity of the OR technique, while confirming that OR regulates and removes ultraviolet divergences, is incomplete in one very important aspect. In order to regularize the multi-loop Green's functions there are some important subtleties to be considered. In particular, at the multi-loop level there is the problem of subdivergent diagrams, i.e. divergent lower loop contributions embedded into the higher loop diagram. The contributions of these subdivergences must be systematically subtracted away from the higher loop diagram in order to preserve unitarity of the Green's function [18]. There are two standard methods for doing this, the counterterm method [23, 28, 29] and the BPHZ recursion scheme [23, 30–32]. Because the infinities in OR are, in a sense, automatically subtracted, the application of the counterterm method is not very useful. Therefore, in OR the BPHZ scheme is utilized.

Define the general renormalized n-point Green's function by

$$\bar{\Gamma}^{(n)} \equiv \mathcal{F}^{(n)}\Gamma^{(n)} \tag{2.2.27}$$

where the operator $\mathcal{F}^{(n)}$ acts upon the unrenormalized Green's function $\Gamma^{(n)}$ in a way that preserves unitarity. Culumovic, Leblanc, Mann, McKeon and Sherry have shown by solely demanding unitarity of the multi-loop Green's function the form of $\mathcal{F}^{(n)}$ can be explicitly generated using the topological language of the BPHZ recursion scheme. [18] Their result follows,

$$\begin{aligned} \bar{\Gamma}_G^{(n)}(\mu^2) &= \mathcal{F}^{(n)}\Gamma_G^{(n)}(s,\mu^2) \\ &\equiv \mathcal{S}^{(n)}\left[\Gamma_G^{(n)}(s,\mu^2) + \sum_{\substack{\gamma_1\cdots\gamma_r \\ \gamma_a\cap\gamma_b=\emptyset}} \Gamma_{G/\{\gamma_1\cdots\gamma_r\}}\left(\prod_{a=1}^{r}\left[-\hat{K}\Gamma_{\gamma_a}\right]\right)\right]. \end{aligned} \tag{2.2.28}$$

where each γ_a is a subgraph of the order of $O(\hbar^m)$ $(m < n)$ and $\Gamma_{G/\gamma_a}(X)$ is the Green's function in which the subgraph γ_a is replaced by X. The function $\hat{K}\Gamma_{\gamma_a}$ is defined by

$$\hat{K}\Gamma_{\gamma_a}(s,\mu^2) = \sum_{k=0}^{m-1}\mathcal{S}'^{(k)}\left(\frac{s'}{s}\right)^{m-k}\Gamma_{\gamma_a}(s',\mu^2) - \chi\Gamma_{\gamma_a}. \tag{2.2.29}$$

The action of χ upon Γ_{γ_a} corresponds to the addition of finite renormalization terms. The addition of the χ term is necessary because in OR the one loop Green's functions are treated in a different way than the way for multi-loop Green's functions (regularization of $\ln M$ vs. M^{-1}). Because of this difference, one must be sure that the scale independent parts of the sub-divergent diagrams are the same as their lower loop counterparts in order to preserve unitarity.

Substituting (2.2.19), (2.2.20) and (2.2.21) into (2.2.16b) and then applying to it (2.2.28) with $n = 2$ gives the regulated form of the two point Green's function

$$\begin{aligned}
\Gamma_2[f] = \mathcal{S}^{(2)} \int d^4x\, d^4y \Bigg\{ & \frac{b_{ijkl}}{8}\delta^4(x-y)\Bigg[\mu^{4s}\langle x|\hat{M}_{ij}^{-1-s}|y\rangle\langle x|\hat{M}_{kl}^{-1-s}|y\rangle \\
& - \left[\mathcal{S}'^{(0)}\left(\frac{s'}{s}\right)\mu^{2s+2s'} - \chi\right]\Big(\langle x|\hat{M}_{ij}^{-1-s}|y\rangle\langle x|\hat{M}_{kl}^{-1-s'}|y\rangle \\
& +\langle x|\hat{M}_{ij}^{-1-s'}|y\rangle\langle x|\hat{M}_{kl}^{-1-s}|y\rangle\Big)\Bigg] \\
& - \frac{a_{ijk}(x)a_{pqr}(y)}{12}\Bigg[\mu^{6s}\langle x|\hat{M}_{ip}^{-1-s}|y\rangle\langle x|\hat{M}_{jq}^{-1-s}|y\rangle\langle x|\hat{M}_{kr}^{-1-s}|y\rangle \\
& - \left[\mathcal{S}'^{(0)}\left(\frac{s'}{s}\right)\mu^{2s+4s'} - \chi\right] \times \\
& \times \Big(\langle x|\hat{M}_{ip}^{-1-s}|y\rangle\langle x|\hat{M}_{jq}^{-1-s'}|y\rangle\langle x|\hat{M}_{kr}^{-1-s'}|y\rangle \\
& + \langle x|\hat{M}_{ip}^{-1-s'}|y\rangle\langle x|\hat{M}_{jq}^{-1-s}|y\rangle\langle x|\hat{M}_{kr}^{-1-s'}|y\rangle \\
& +\langle x|\hat{M}_{ip}^{-1-s'}|y\rangle\langle x|\hat{M}_{jq}^{-1-s'}|y\rangle\langle x|\hat{M}_{kr}^{-1-s}|y\rangle\Big)\Bigg]\Bigg\} \\
\equiv \mathcal{S}^{(2)}[B(s) - A(s)] & \qquad (2.2.30)
\end{aligned}$$

where the terms containing by b_{ijkl} are collected into $B(s)$ while those containing by $a_{ijk}(x)a_{pqr}(y)$ are denoted by $A(s)$.

Substituting (2.2.21) into (2.2.30) $A(s)$ is obtained

$$\begin{aligned}
A(s) = \int d^4x\, d^4y\, & \frac{a_{ijk}(x)a_{pqr}(y)}{12}\int_0^\infty dt_1\, dt_2\, dt_3 \\
& \Bigg[\frac{\mu^{6s}t_1^s\, t_2^s\, t_3^s}{\Gamma[s+1]^3}\langle x|e^{-\hat{M}t_1}|y\rangle_{ip}\langle x|e^{-\hat{M}t_2}|y\rangle_{jq}\langle x|e^{-\hat{M}t_3}|y\rangle_{kr} \\
& - \left[\mathcal{S}'^{(0)}\left(\frac{s'}{s}\right)\mu^{2s+4s'} - \chi\right]\frac{t_1^s\, t_2^{s'}\, t_3^{s'}}{\Gamma[s+1]\Gamma[s'+1]^2} \times \qquad (2.2.31) \\
& \times \Big(\langle x|e^{-\hat{M}t_1}|y\rangle_{ip}\langle x|e^{-\hat{M}t_2}|y\rangle_{jq}\langle x|e^{-\hat{M}t_3}|y\rangle_{kr} \\
& + \langle x|e^{-\hat{M}t_2}|y\rangle_{ip}\langle x|e^{-\hat{M}t_1}|y\rangle_{jq}\langle x|e^{-\hat{M}t_3}|y\rangle_{kr} \\
& + \langle x|e^{-\hat{M}t_2}|y\rangle_{ip}\langle x|e^{-\hat{M}t_3}|y\rangle_{jq}\langle x|e^{-\hat{M}t_1}|y\rangle_{kr}\Big)\Bigg]
\end{aligned}$$

In a similar manner. $B(s)$ is obtained as well

$$
\begin{aligned}
B(s) = \int d^4x \, \frac{b_{ijkl}}{8} \int_0^\infty dt_1 dt_2 \Bigg[& \frac{\mu^{4s} t_1^s t_2^s}{\Gamma[s+1]^2} \langle x | e^{-\dot{M}t_1} | x \rangle_{ij} \langle x | e^{-\dot{M}t_2} | x \rangle_{kl} \\
& - \left[S'^{(0)} \left(\frac{s'}{s} \right) \mu^{2s+2s'} - \chi \right] \frac{t_1^s t_2^{s'}}{\Gamma[s+1]\Gamma[s'+1]} \times \\
& \times \Big(\langle x | e^{-\dot{M}t_1} | x \rangle_{ij} \langle x | e^{-\dot{M}t_2} | x \rangle_{kl} \\
& + \langle x | e^{-\dot{M}t_2} | x \rangle_{ij} \langle x | e^{-\dot{M}t_1} | x \rangle_{kl} \Big) \Bigg] .
\end{aligned}
\qquad (2.2.32)
$$

To visualize these involved expressions for both one loop and two loop Green's functions the matrix element $\langle x | e^{-\dot{M}t} | y \rangle$ is graphically represented by a thick line and a thin line is used to represent the ordinary fields. e.g. the fields that appear in the $a_{ijk}(f)$ permutation symbols (see Figs. 1-2).

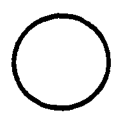

Figure 1: Graphical representation of Γ_1

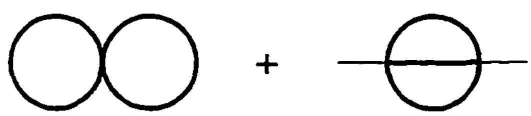

Figure 2: Graphical representation of Γ_2

The first diagram in Fig. 2. usually called the "double bubble" or "double scoop" diagram. arises from (2.2.32). The second diagram in Fig. 2. termed the "setting-sun" or "London" diagram. comes from (2.2.31).

In order to do any practical calculations. the exponential of $\dot{M}$ must be expanded out in a perturbative series. Depending on what the overall desired results are. this can be handled in one of two ways in OR. If the momentum structure of the Green's functions is desired (for cross-sections and the like) the two perturbative expansions proposed by Schwinger need to be used [33]:

$$\begin{aligned}\operatorname{str} e^{-(M_0+M_I)t} = \operatorname{str} \Big[& e^{-M_0 t} + \frac{(-t)}{1!} e^{-M_0 t} M_I \\ & + \frac{(-t)^2}{2!} \int_0^1 du\, e^{-(1-u)M_0 t} M_I e^{uM_0 t} M_I + \cdots \Big]\end{aligned} \tag{2.2.33a}$$

and

$$\begin{aligned}e^{-(M_0+M_I)t} = & e^{-M_0 t} + (-t) \int_0^1 du\; e^{-(1-u)M_0 t} M_I e^{-uM_0 t} \\ & + (-t)^2 \int_0^1 du\; u \int_0^1 dv\; {}^{-(1-u)M_0 t} M_I e^{-u(1-v)M_0 t} M_I e^{-uvM_0 t} + \cdots\end{aligned} \tag{2.2.33b}$$

where M has been split into the sum of a part M_0, which is diagonalizable on some space of states, and a perturbatively interacting part, M_I. Using the graphical notation where the factors of e^{M_0} are drawn thin lines and factors of M_I are represented by dots, a graphical view of the two Schwinger expansions can be visualized by Fig. 3 and Fig. 4.

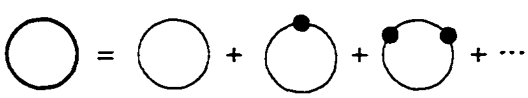

Figure 3: Graphical representation of $\operatorname{str} e^{-(M_0+M_I)t}$.

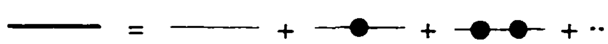

Figure 4: Graphical representation of $e^{-(M_0+M_I)t}$.

If, on the other hand, the various renormalization group functions or the low energy behavior of the theory is desired the DeWitt expansion

$$\langle x | e^{-\left\{(p+\hat{A})^2+\hat{B}\right\}t} | y \rangle = \frac{e^{-(x-y)^2/4t}}{(4\pi t)^2} \sum_{k=0}^{\infty} \hat{a}_k(x, y; \phi_i) t^k \tag{2.2.34}$$

is more useful [34]. Here the new matrices $\hat{A}$ and $\hat{B}$ have been introduced such that the matrix M has the general form

$$M \equiv \left(p + \hat{A}\right)^2 + \hat{B} + \hat{\Xi}\,(m_i^2) \tag{2.2.35}$$

The elements of the matrix $\hat{\Xi}$ consist of the masses that appear in the initial Lagrangian. The DeWitt coefficients that appear in (2.2.34) consist of the various fields, gauge strength tensors and their numerous covariant derivatives that appear in the model of interest. The method for calculating the DeWitt coefficients, $a_k(x, y; \phi_i)$, is outlined in Appendix A.

Chapter 3

Two Field Quantum Scalar Dynamics

3.1 Preliminaries

The mathematical rigor of OR developed in the last section is valid for any model one wishes to consider. However, in order to illustrate the actual use of OR and how to calculate things like the renormalization group functions and terms that contribute to an effective theory the model of scalar QED with two complex scalar fields, one of the scalar fields being heavier than the other, with ϕ_4^4 type couplings will be explicitly considered. For the sake of convenience, symmetry under interchange of heavy and light fields is imposed in our model.

The most general Lagrangian in Euclidean space that satisfies the stated symmetry requirements is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\phi_L^* D_\mu^\dagger D^\mu \phi_L - \frac{1}{2}m_L^2\phi_L^*\phi_L - \frac{1}{2}\phi_H^* D_\mu^\dagger D^\mu \phi_H - \frac{1}{2}m_H^2\phi_H^*\phi_H \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\lambda_0\left(\phi_L^{*2}\phi_L^2 + \phi_H^{*2}\phi_H^2\right) - \frac{1}{4}\lambda_c\left(\phi_L^{*2}\phi_H^2 + \phi_H^{*2}\phi_L^2\right) \\ &- \overline{\lambda}\,\phi_L^*\phi_L\phi_H^*\phi_H\end{aligned} \tag{3.1.1}$$

where the subscripts L/H denote light/heavy fields ($m_L \ll m_H$) and D_μ is the covariant derivative $D_\mu = (p - ieA)_\mu$.

To apply OR, the Lagrangian first needs to be brought into the form [11]

$$\mathcal{L} = -\frac{1}{2}\Phi^\dagger M\Phi + \frac{1}{3!}a_{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{4!}b_{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l. \tag{3.1.2}$$

Comparing (3.1.1) to (3.1.2), Φ is expressed as the column matrix,

$$\Phi = \begin{pmatrix} A_\mu \\ \phi_L^* \\ \varphi_L \\ \phi_H^* \\ \varphi_H \end{pmatrix}. \tag{3.1.3}$$

After applying the background field formalism to (3.1.1) and (3.1.3)

$$A_\mu = Q_\mu + V_\mu \tag{3.1.4a}$$

$$\varphi_{L,H} = f_{L,H} + h_{L,H} \tag{3.1.4b}$$

$$\phi_{L,H}^* = f_{L,H}^* + h_{L,H}^* \tag{3.1.4c}$$

where,

$$\{Q, f\} = \text{classical fields}$$

$$\{V, h\} = \text{quantum fields},$$

we can determine the forms of the matrix M and the permutation symbols a_{ijk} and b_{ijkl} in (3.1.2).

In particular, the matrix M is generated from those terms in the Lagrangian that are bilinear in the quantum fields. The elements of the matrix M

$$M = \begin{bmatrix} M_{11} & \cdots\cdots & M_{15} \\ \vdots & \ddots & \vdots \\ M_{51} & \cdots\cdots & M_{55} \end{bmatrix}, \tag{3.1.5}$$

are thus given by

$$M_{11} = 2e^2\left(f_L^* f_L + f_H^* f_H\right) + g_{\mu\nu}k^2 \tag{3.1.6a}$$

$$M_{12} = M_{31} = -ieT_\mu f_L + 2e^2 f_L Q_\mu \tag{3.1.6b}$$

$$M_{13} = M_{21} = -ieT_\mu f_L^* + 2e^2 f_L^* Q_\mu \tag{3.1.6c}$$

$$M_{14} = M_{51} = -ieT_\mu f_H + 2e^2 f_H Q_\mu \tag{3.1.6d}$$

$$M_{15} = M_{41} = -ieT_\mu f_H^* + 2e^2 f_H^* Q_\mu \tag{3.1.6e}$$

$$\begin{aligned} M_{22} = M_{33} = p^2 - ieT_\mu Q_\mu + e^2 Q^2 + \lambda_0 f_L^* f_L \\ + \overline{\lambda} f_H^* f_H + m_L^2 f_L^* f_L \end{aligned} \tag{3.1.6f}$$

$$M_{23} = \frac{1}{2}\left(\lambda_0 f_L^{*2} + \lambda_c f_H^{*2}\right) \tag{3.1.6g}$$

$$M_{24} = M_{53} = \lambda_c f_L f_H^* + \overline{\lambda} f_L^* f_H \tag{3.1.6h}$$

$$M_{25} = M_{43} = \overline{\lambda} f_L^* f_H^* \tag{3.1.6i}$$

$$M_{32} = \frac{1}{2}\left(\lambda_0 {f_L}^2 + \lambda_c {f_H}^2\right) \tag{3.1.6j}$$

$$M_{34} = M_{52} = \overline{\lambda} f_L f_H \tag{3.1.6k}$$

$$M_{35} = M_{42} = \lambda_c f_L^* f_H + \overline{\lambda} f_L f_H^* \tag{3.1.6l}$$

$$\begin{aligned} M_{44} = M_{55} &= p^2 - ieT_\mu Q_\mu + e^2 Q^2 + \lambda_0 f_H^* f_H \\ &\quad + \overline{\lambda} f_L^* f_L + m_H^2 f_H^* f_H \end{aligned} \tag{3.1.6m}$$

$$M_{45} = \frac{1}{2}\left(\lambda_0 {f_H^*}^2 + \lambda_c {f_L^*}^2\right) \tag{3.1.6n}$$

$$M_{54} = \frac{1}{2}\left(\lambda_0 {f_H}^2 + \lambda_c {f_L}^2\right) \tag{3.1.6o}$$

where, $T_\mu = (p + p')_\mu$ is the momentum structure that comes from the current terms in the Lagrangian, p_μ is the momentum of the scalar field and k_μ is the momentum of the electromagnetic field.

Additionally, the permutation symbols have the values

$$a_{[123]} = a_{[145]} = 2e^2 Q_\mu - ieT_\mu \tag{3.1.7a}$$

$$a_{[113]} = 4e^2 f_L^* \tag{3.1.7b}$$

$$a_{[112]} = 4e^2 f_L \tag{3.1.7c}$$

$$a_{[115]} = 4e^2 f_H^* \tag{3.1.7d}$$

$$a_{[114]} = 4e^2 f_H \tag{3.1.7e}$$

$$a_{[233]} = -\lambda_0 f_L^* \tag{3.1.7f}$$

$$a_{[223]} = -\lambda_0 f_L \tag{3.1.7g}$$

$$a_{[455]} = -\lambda_0 f_H^* \tag{3.1.7h}$$

$$a_{[445]} = -\lambda_0 f_H \tag{3.1.7i}$$

$$a_{[255]} = -\lambda_c f_L^* \tag{3.1.7j}$$

$$a_{[225]} = -\lambda_c f_L \tag{3.1.7k}$$

$$a_{[344]} = -\lambda_c f_H^* \tag{3.1.7l}$$

$$a_{[334]} = -\lambda_c f_H \tag{3.1.7m}$$

$$a_{[345]} = -\overline{\lambda} f_L^* \tag{3.1.7n}$$

$$a_{[245]} = -\overline{\lambda} f_L \tag{3.1.7o}$$

$$a_{[235]} = -\overline{\lambda} f_H^* \tag{3.1.7p}$$

$$a_{[234]} = -\overline{\lambda} f_H \tag{3.1.7q}$$

and

$$b_{[1123]} = b_{[1145]} = 2e^2 g_{\mu\nu} \tag{3.1.8a}$$

$$b_{[2233]} = b_{[4455]} = -\lambda_0 \tag{3.1.8b}$$

$$b_{[2255]} = b_{[3344]} = -\lambda_c \tag{3.1.8c}$$

$$b_{[2345]} = -\overline{\lambda} \tag{3.1.8d}$$

where $[a_1 a_2 \cdots a_i]$ represents all possible permutations of the set $\{a_1 a_2 \cdots a_i\}$.

Before the formal calculation can proceed, an appropriate expansion of the exponential of M must be chosen. Because the goal of our calculation is the renormalization group functions and the generation of an effective field theory, the DeWitt expansion will be used as our perturbative expansion.

Recall that in order to use the DeWitt expansion the matrix M should be written in terms of new matrices $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$

$$M \equiv \left(p + \hat{\mathcal{A}}\right)^2 + \hat{\mathcal{B}} + \hat{\Xi}\left(m_i^2\right). \tag{2.2.35}$$

For the Lagrangian given in (3.1.1) the non-zero elements of the matrix $\hat{\mathcal{A}}$ are given by

$$\hat{\mathcal{A}}_{22} = \hat{\mathcal{A}}_{33} = \hat{\mathcal{A}}_{44} = \hat{\mathcal{A}}_{55} = eQ_\mu \tag{3.1.9a}$$

$$\hat{\mathcal{A}}_{12} = \hat{\mathcal{A}}_{31} = ef_L \tag{3.1.9b}$$

$$\hat{\mathcal{A}}_{13} = \hat{\mathcal{A}}_{21} = ef_L^* \tag{3.1.9c}$$

$$\hat{\mathcal{A}}_{14} = \hat{\mathcal{A}}_{51} = ef_H \tag{3.1.9d}$$

$$\hat{\mathcal{A}}_{15} = \hat{\mathcal{A}}_{41} = ef_H^*. \tag{3.1.9e}$$

Similarly, the elements of the matrix $\hat{\mathcal{B}}$ are

$$\hat{\mathcal{B}}_{11} = 0 \tag{3.1.10a}$$

$$\hat{\mathcal{B}}_{12} = \hat{\mathcal{B}}_{31} = e^2 f_L T_\mu Q_\mu \tag{3.1.10b}$$

$$\hat{\mathcal{B}}_{13} = \hat{\mathcal{B}}_{21} = e^2 f_L^* T_\mu Q_\mu \tag{3.1.10c}$$

$$\hat{\mathcal{B}}_{14} = \hat{\mathcal{B}}_{41} = e^2 f_H T_\mu Q_\mu \tag{3.1.10d}$$

$$\hat{\mathcal{B}}_{15} = \hat{\mathcal{B}}_{41} = e^2 f_H^* T_\mu Q_\mu \tag{3.1.10e}$$

$$\hat{\mathcal{B}}_{22} = \hat{\mathcal{B}}_{33} = -e^2 f_L^* f_L + \lambda_0 f_L^* f_L + \overline{\lambda} f_H^* f_H \tag{3.1.10f}$$

$$\hat{\mathcal{B}}_{23} = -e^2 f_L^{*2} + \frac{1}{2}\lambda_0 f_L^{*2} + \frac{1}{2}\lambda_c f_H^{*2} \tag{3.1.10g}$$

$$\hat{\mathcal{B}}_{24} = \hat{\mathcal{B}}_{53} = -e^2 f_L^* f_H + \lambda_c f_L f_H^* + \overline{\lambda} f_L^* f_H \tag{3.1.10h}$$

$$\hat{\mathcal{B}}_{25} = \hat{\mathcal{B}}_{43} = -e^2 f_L^* f_H^* + \overline{\lambda} f_L^* f_H^* \tag{3.1.10i}$$

$$\hat{\mathcal{B}}_{32} = -e^2 f_L{}^2 + \frac{1}{2}\lambda_0 f_L{}^2 + \frac{1}{2}\lambda_c f_H{}^2 \tag{3.1.10j}$$

$$\hat{\mathcal{B}}_{34} = \hat{\mathcal{B}}_{52} = -e^2 f_L f_H + \overline{\lambda} f_L f_H \tag{3.1.10k}$$

$$\hat{\mathcal{B}}_{35} = \hat{\mathcal{B}}_{42} = -e^2 f_L f_H^* + \lambda_c f_L^* f_H + \overline{\lambda} f_L f_H^* \tag{3.1.10l}$$

$$\hat{\mathcal{B}}_{44} = \hat{\mathcal{B}}_{55} = -e^2 f_H^* f_H + \lambda_0 f_H^* f_H + \overline{\lambda} f_L^* f_L \tag{3.1.10m}$$

$$\hat{\mathcal{B}}_{45} = -e^2 {f_H^*}^2 + \frac{1}{2}\lambda_0 {f_H^*}^2 + \frac{1}{2}\lambda_c {f_L^*}^2 \tag{3.1.10n}$$

$$\hat{\mathcal{B}}_{54} = -e^2 {f_H}^2 + \frac{1}{2}\lambda_0 {f_H}^2 + \frac{1}{2}\lambda_c {f_L}^2. \tag{3.1.10o}$$

Following the sample calculation of the DeWitt coefficients given in Appendix A. we find the relevant coefficients in terms of the matrix $\hat{\mathcal{B}}$, a new general "gauge-field" matrix

$$\hat{\mathcal{F}}_{\mu\nu} \equiv \hat{\mathcal{A}}_{\nu,\mu} - \hat{\mathcal{A}}_{\mu,\nu} + i\left[\hat{\mathcal{A}}_\mu, \hat{\mathcal{A}}_\nu\right] \tag{3.1.11}$$

and the difference coordinate $\Delta = x - y$:

$$\hat{a}_0 = 1 \tag{3.1.12a}$$

$$\begin{aligned}\hat{a}_1 = & -\hat{\mathcal{B}} - \frac{i}{6}\Delta_\alpha \hat{\mathcal{F}}_{\alpha\mu;\mu} - \frac{1}{24}\Delta_\alpha\Delta_\beta \hat{\mathcal{B}}_{;\alpha\beta} - \frac{1}{12}\Delta_\alpha\Delta_\beta \hat{\mathcal{F}}_{\mu\alpha}\hat{\mathcal{F}}_{\mu\beta} \\ & - \frac{1}{720}\Delta_\alpha\Delta_\beta\Delta_\gamma\Delta_\delta \hat{\mathcal{F}}_{\mu\alpha;\beta}\hat{\mathcal{F}}_{\mu\gamma;\delta} + \cdots\end{aligned} \tag{3.1.12b}$$

$$\begin{aligned}\hat{a}_2 = & \frac{1}{2}\hat{\mathcal{B}}^2 - \frac{1}{6}\hat{\mathcal{B}}_{;\mu\mu} - \frac{1}{12}\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}_{\mu\nu} \\ & - \Delta_\alpha\Delta_\beta\left(\frac{1}{180}\hat{\mathcal{F}}_{\mu\nu;\nu}\hat{\mathcal{F}}_{\mu\alpha;\beta} - \frac{1}{108}\hat{\mathcal{F}}_{\alpha\mu;\nu}\hat{\mathcal{F}}_{\beta\mu;\nu} - \frac{1}{72}\hat{\mathcal{F}}_{\alpha\mu;\mu}\hat{\mathcal{F}}_{\beta\nu;\nu} \right. \\ & \left. + \frac{2}{135}\hat{\mathcal{F}}_{\alpha\mu;\nu}\hat{\mathcal{F}}_{\mu\nu;\beta} + \frac{1}{135}\hat{\mathcal{F}}_{\mu\nu;\alpha}\hat{\mathcal{F}}_{\mu\nu;\beta}\right) + \cdots\end{aligned} \tag{3.1.12c}$$

$$\hat{a}_3 = \frac{1}{12}\hat{\mathcal{B}}_{;\mu}\hat{\mathcal{B}}_{;\mu} - \frac{2}{135}\hat{\mathcal{F}}_{\mu\nu;\lambda}\hat{\mathcal{F}}_{\mu\lambda;\nu} - \frac{2}{135}\hat{\mathcal{F}}_{\mu\nu;\lambda}\hat{\mathcal{F}}_{\mu\nu;\lambda} + \frac{1}{180}\hat{\mathcal{F}}_{\mu\nu;\nu}\hat{\mathcal{F}}_{\mu\lambda;\lambda} + \cdots. \tag{3.1.12d}$$

The covariant derivative in (3.1.12) is given by the notation

$$\hat{\mathcal{X}}_{;\mu} \equiv D_\mu \hat{\mathcal{X}} = \hat{\mathcal{X}}_{,\mu} + i\left[\hat{\mathcal{A}}_\mu, \hat{\mathcal{X}}\right]. \tag{3.1.13}$$

The sundry DeWitt coefficients given in (3.1.12) are completely general. i.e. they are valid for any Lagrangian one wished to consider. However, do note in our specific case of two field scalar QED the "gauge-field" matrix $\hat{\mathcal{F}}_{\mu\nu}$ reduces to

$$\hat{\mathcal{F}}_{\mu\nu} = \hat{\mathcal{A}}_{\nu,\mu} - \hat{\mathcal{A}}_{\mu,\nu}. \tag{3.1.14}$$

Furthermore, one needs to be careful with the covariant derivative when taking multiple derivatives. Unlike normal differentiation, the order of covariant differentiation does matter. Specifically, the commutator of a covariant derivative acting upon a general matrix $\hat{\mathcal{X}}$ is given by

$$[D_\mu, D_\nu]\hat{\mathcal{X}} = \left[\hat{\mathcal{F}}_{\mu\nu}, \hat{\mathcal{X}}\right]. \tag{3.1.15}$$

3.2 One Loop Green's Function

Starting from (2.2.22) and substituting in the DeWitt expansion (2.2.34) the one loop Green's function is obtained

$$\Gamma_1 = -\frac{1}{2}\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{tr}\, e^{-\hat{M}t}\right] \tag{2.2.22}$$

$$= -\frac{1}{2}\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\sum_n\int_0^\infty \frac{dt}{(4\pi)^2}t^{s+n-3}\,\mathrm{tr}\left(\hat{a}_n(x,x)e^{-\hat{\Xi}t}\right)\right]. \tag{3.2.1}$$

To proceed further the form of $\hat{\Xi}$ needs to be specified. From the explicit form of the Lagrangian (3.1.1) it is obvious that $\hat{\Xi}$ will just be a matrix that is diagonal in the masses. However, because the DeWitt expansion is a low energy expansion an artificial mass for the electromagnetic field, m_γ, must be introduced to handled the infra-red divergences. Despite the breaking of the symmetries in the theory, this does not violate any of the principles of QR. In fact, the introduction of parameters to regulate infra-red divergences is common to all regularizing schemes who's focus is the regularizing of ultra-violet divergences in the theory. The full form of $\hat{\Xi}$ is thus

$$\hat{\Xi} = \mu^{-2}\begin{bmatrix} m_\gamma^2 & 0 & 0 & 0 & 0 \\ 0 & m_L^2 & 0 & 0 & 0 \\ 0 & 0 & m_L^2 & 0 & 0 \\ 0 & 0 & 0 & m_H^2 & 0 \\ 0 & 0 & 0 & 0 & m_H^2 \end{bmatrix}. \tag{3.2.2}$$

In what follows we have defined

$$\hat{\Lambda} \equiv \frac{\Lambda}{(4\pi)^2}, \qquad \Lambda \in \{\lambda_0, \lambda_c, \overline{\lambda}\} \tag{3.2.3a}$$

$$\hat{\alpha} \equiv \frac{\alpha}{4\pi} \tag{3.2.3b}$$

$$\hat{m}^2 \equiv m^2/\mu^2. \tag{3.2.3c}$$

The method for the calculation of various n-point Green's functions at one-loop order is a straight forward process. Each DeWitt coefficient given in (3.1.12) must be calculated using the matrices $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ given in (3.1.9) and (3.1.10) respectfully. At one loop order the difference coefficient, Δ_μ, is zero, therefore, the actual number of terms in each DeWitt coefficient decreases significantly. Once the full forms of the DeWitt coefficients are calculated, the trace of the product of each coefficient with exponential of the mass matrix can be taken and the integrals over t and the regularizing with respect to s can be done.

As an example, consider the simplest non-trivial case, $n = 1$. The $n = 1$ DeWitt coefficient is simply

$$\hat{a}_1 = -\hat{\mathcal{B}}.$$

Thus,

$$\begin{aligned}\mathrm{tr}\left(\hat{a}_1 e^{-\Xi t}\right) = &-f_L^* f_L\left[2(\lambda_0 - e^2)e^{-m_L^2 t/\mu^2} + 2\overline{\lambda} e^{-m_H^2 t/\mu^2}\right] \\ &- f_H^* f_H\left[2(\lambda_0 - e^2)e^{-m_H^2 t/\mu^2} + 2\overline{\lambda} e^{-m_L^2 t/\mu^2}\right].\end{aligned} \tag{3.2.4}$$

Carrying out the integration over t and the regularization gives the result

$$\begin{aligned}\Gamma_1^{(1)} &= -\frac{1}{2}\lim_{s\to 0}\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{(4\pi)^2}t^{s-2}\,\mathrm{tr}\left(\hat{a}_1(x,x)e^{-\hat{\Xi} t}\right)\right] \\ &= -\frac{1}{(4\pi)^2}\lim_{s\to 0}\frac{d}{ds}\frac{\Gamma(-1+s)}{\Gamma(s)}\times \\ &\quad\times\left[m_L^2(\hat{m}_L^2)^{-s}\left\{(\lambda_0 - e^2)f_L^* f_L + \overline{\lambda} f_H^* f_H\right\}\right. \\ &\quad\left.+m_H^2(\hat{m}_H^2)^{-s}\left\{(\lambda_0 - e^2)f_H^* f_H + \overline{\lambda} f_L^* f_L\right\}\right] \\ &= f_L^* f_L\left[m_L^2(1-\ln\hat{m}_L^2)(\hat{\lambda}_0 - \hat{\alpha}) + m_H^2(1-\ln\hat{m}_H^2)\hat{\overline{\lambda}}\right] \\ &\quad + f_H^* f_H\left[m_H^2(1-\ln\hat{m}_H^2)(\hat{\lambda}_0 - \hat{\alpha}) + m_L^2(1-\ln\hat{m}_L^2)\hat{\overline{\lambda}}\right].\end{aligned} \tag{3.2.5}$$

Calculation of the rest of the two, three and four point functions proceed in a similar manner using the DeWitt coefficient $\hat{a}_2$. The results for all the one loop Green's functions are summarized below

$$\Gamma_1^{f_L^* f_L} = f_L^* f_L\left[m_L^2\left(1-\ln\hat{m}_L^2\right)\left[\hat{\lambda}_0 - \hat{\alpha}\right] + m_H^2\left(1-\ln\hat{m}_H^2\right)\hat{\overline{\lambda}}\right] \tag{3.2.6a}$$

$$\Gamma_1^{f_H^* f_H} = f_H^* f_H\left[m_H^2\left(1-\ln\hat{m}_H^2\right)\left[\hat{\lambda}_0 - \hat{\alpha}\right] + m_L^2\left(1-\ln\hat{m}_L^2\right)\hat{\overline{\lambda}}\right] \tag{3.2.6b}$$

$$\Gamma_1^{|F_{\mu\nu}|^2} = F_{\mu\nu}F^{\mu\nu}\left[\frac{1}{12}\hat{\alpha}\left(\ln\hat{m}_L^2 + \ln\hat{m}_H^2\right)\right] \tag{3.2.6c}$$

$$\Gamma_1^{Q_\mu f_L^* f_L} = i\,T_\mu Q^\mu f_L^* f_L\left[-\frac{1}{12}e\,\hat{\alpha}\ln\hat{m}_L^2\right] \tag{3.2.6d}$$

$$\Gamma_1^{Q_\mu f_H^* f_H} = i\,T_\mu Q^\mu f_H^* f_H\left[-\frac{1}{12}e\,\hat{\alpha}\ln\hat{m}_H^2\right] \tag{3.2.6e}$$

$$\Gamma_1^{f_L^{*2} f_L^2} = f_L^{*2} f_L^2\left[-\lambda_0\left(\hat{\lambda}_0 - \hat{\alpha}\right)\ln\hat{m}_L^2 - \frac{1}{2}\left(\overline{\lambda}\,\hat{\overline{\lambda}} + \frac{1}{4}\lambda_c\hat{\lambda}_c\right)\ln\hat{m}_H^2 + \frac{1}{2}\lambda_0\,\hat{\alpha}\ln\hat{m}_\gamma^2\right] \tag{3.2.6f}$$

$$\Gamma_1^{f_H^{*2} f_H^2} = f_H^{*2} f_H^2\left[-\lambda_0\left(\hat{\lambda}_0 - \hat{\alpha}\right)\ln\hat{m}_H^2 - \frac{1}{2}\left(\overline{\lambda}\,\hat{\overline{\lambda}} + \frac{1}{4}\lambda_c\hat{\lambda}_c\right)\ln\hat{m}_L^2 + \frac{1}{2}\lambda_0\,\hat{\alpha}\ln\hat{m}_\gamma^2\right] \tag{3.2.6g}$$

$$\Gamma_1^{f_L^{*2} f_H^2} = f_L^{*2} f_H^2\left[-\lambda_c\left(\hat{\overline{\lambda}} + \frac{1}{4}\hat{\lambda}_0 - \frac{1}{2}\hat{\alpha}\right)\left(\ln\hat{m}_L^2 + \ln\hat{m}_H^2\right) + \frac{1}{2}\lambda_c\,\hat{\alpha}\ln\hat{m}_\gamma^2\right] \tag{3.2.6h}$$

$$\Gamma_1^{f_H^{*2} f_L^2} = f_H^{*2} f_L^2\left[-\lambda_c\left(\hat{\overline{\lambda}} + \frac{1}{4}\hat{\lambda}_0 - \frac{1}{2}\hat{\alpha}\right)\left(\ln\hat{m}_L^2 + \ln\hat{m}_H^2\right) + \frac{1}{2}\lambda_c\,\hat{\alpha}\ln\hat{m}_\gamma^2\right] \tag{3.2.6i}$$

$$\begin{aligned}\Gamma_1^{f_L^* f_L f_H^* f_H} = f_L^* f_L f_H^* f_H&\left[-\left(\overline{\lambda}\left[\hat{\overline{\lambda}} + \hat{\lambda}_0\right] + \lambda_c\hat{\lambda}_c - 2\overline{\lambda}\hat{\alpha}\right)\left(\ln\hat{m}_L^2 + \ln\hat{m}_H^2\right)\right. \\ &\left. + 2\overline{\lambda}\,\hat{\alpha}\ln\hat{m}_\gamma^2\right]\end{aligned} \tag{3.2.6j}$$

$$\Gamma_1^{Q^2 f_L^* f_L} = Q^2 f_L^* f_L \left[-\frac{4}{3} e^2 \hat{\alpha} \ln \hat{m}_L^2 - \frac{5}{3} e^2 \hat{\alpha} \ln \hat{m}_\gamma^2 \right] \tag{3.2.6k}$$

$$\Gamma_1^{Q^2 f_H^* f_H} = Q^2 f_H^* f_H \left[-\frac{4}{3} e^2 \hat{\alpha} \ln \hat{m}_H^2 - \frac{5}{3} e^2 \hat{\alpha} \ln \hat{m}_\gamma^2 \right] . \tag{3.2.6l}$$

The terms that generate an effective field theory from the full theory, i.e. terms proportional to powers of $1/m_H$, occur naturally in the DeWitt expansion by going to coefficients of higher order than $\hat{a}_2$. In the one loop case, the $O(1/m_H^2)$ terms come from the $\hat{a}_3$ coefficient. Again, the calculation proceeds exactly like before. The results of the calculation are

$$\begin{aligned}
\Gamma_1^{LM} &= \frac{1}{m_H^2} \left[\frac{1}{6} e^2 \dot{\bar{\lambda}} F_{\mu\nu} F^{\mu\nu} \phi_L^* \phi_L + \frac{1}{6} \left(\bar{\lambda} \dot{\bar{\lambda}} + \frac{1}{4} \lambda_c \dot{\lambda}_c \right) [\phi_L^{*2} \phi_L^{2}]_{,\mu\mu} \right. \\
&\quad \left. - \frac{1}{6} \bar{\lambda} \dot{\bar{\lambda}} (\phi_L^* \phi_L)_{,\mu} (\phi_L^* \phi_L)_{,\mu} - \frac{1}{12} \lambda_c \dot{\lambda}_c (\phi_L^{*2})_{,\mu} (\phi_L^{2})_{,\mu} \right] \\
&\quad + O\left(1/m_H^4\right) \\
&= \frac{1}{m_H^2} \left[\frac{1}{6} e^2 \dot{\bar{\lambda}} F_{\mu\nu} F^{\mu\nu} \phi_L^* \phi_L + \frac{2}{3} \bar{\lambda} \dot{\bar{\lambda}} (\phi_L^* \phi_L)(\phi_{L,\mu}^* \phi_{L,\mu}) \right. \\
&\qquad + \frac{1}{6} [\phi_L^{*2} (\phi_{L,\mu} \phi_{L,\mu}) + h.c.] \left(\bar{\lambda} \dot{\bar{\lambda}} + \frac{1}{2} \lambda_c \dot{\lambda}_c \right) \\
&\qquad \left. + \frac{1}{3} [\phi_L^{*2} \phi_L \phi_{L,\mu\mu} + h.c.] \left(\bar{\lambda} \dot{\bar{\lambda}} + \frac{1}{4} \lambda_c \dot{\lambda}_c \right) \right] \\
&\qquad + O\left(1/m_H^4\right)
\end{aligned} \tag{3.2.7}$$

3.3 Two Loop Green's Function

The calculation of the two loop Green's function follows the same basic formalism as the one loop Green's function. The only major difference is the more complex nature of the renormalization scheme needed to handle sub-divergences.

Starting with the $B(s)$ term in (2.2.32)

$$\begin{aligned} B(s) = \int d^4x \, \frac{b_{ijkl}}{8} \int_0^\infty dt_1 dt_2 \Bigg[& \frac{\mu^{4s} t_1^s t_2^s}{\Gamma[s+1]^2} \langle x|e^{-\hat{M}t_1}|x\rangle_{ij} \langle x|e^{-\hat{M}t_2}|x\rangle_{kl} \\ & - \left[S'^{(0)}\left(\frac{s'}{s}\right) \mu^{2s+2s'} - \chi \right] \frac{t_1^s t_2^{s'}}{\Gamma[s+1]\Gamma[s'+1]} \times \\ & \times \Big(\langle x|e^{-\hat{M}t_1}|x\rangle_{ij} \langle x|e^{-\hat{M}t_2}|x\rangle_{kl} \\ & + \langle x|e^{-\hat{M}t_2}|x\rangle_{ij} \langle x|e^{-\hat{M}t_1}|x\rangle_{kl} \Big) \Bigg] \end{aligned} \tag{2.2.32}$$

and substituting the DeWitt expansion (2.2.34) into (2.2.32)

$$\begin{aligned} B(s) = \int d^4x \, \frac{b_{ijkl}}{8\,(4\pi)^2} \int_0^\infty dt_1 dt_2 \sum_{n,n'} \Bigg[& \frac{\mu^{4s} t_1^{s+n} t_2^{s+n'}}{\Gamma[s+1]^2} \langle x|\hat{a}_n e^{\hat{\Xi}t_1}|x\rangle_{ij} \langle x|\hat{a}_{n'} e^{\hat{\Xi}t_2}|x\rangle_{kl} \\ & - \left[S'^{(0)}\left(\frac{s'}{s}\right) \mu^{2s+2s'} - \chi \right] \frac{t_1^{s+n} t_2^{s'+n'}}{\Gamma[s+1]\Gamma[s'+1]} \times \\ & \times \Big(\langle x|\hat{a}_n e^{\hat{\Xi}t_1}|x\rangle_{ij} \langle x|\hat{a}_{n'} e^{\hat{\Xi}t_2}|x\rangle_{kl} \\ & + \langle x|\hat{a}_{n'} e^{\hat{\Xi}t_2}|x\rangle_{ij} \langle x|\hat{a}_n e^{\hat{\Xi}t_1}|x\rangle_{kl} \Big) \Bigg]. \end{aligned} \tag{3.3.1}$$

$$\equiv \sum_{n,n'} B^{(n,n')}(s) \tag{3.3.2}$$

While looking complicated, all the above integrals are straight forward to calculate. To be explicit, all the $B(s)$ integrals reduce to the general form

$$\begin{aligned} I_B^{(n,n')}(s) &= \int_0^\infty dt_1 dt_2 \frac{1}{\Gamma[1+s]\Gamma[1+s']} t_1^{s+n} t_2^{s'+n'} e^{-m_1^2 t_1} e^{-m_2^2 t_2} \\ &= \frac{\Gamma[1+s+n]\Gamma[1+s'+n']}{\Gamma[1+s]\Gamma[1+s']} m_1^{-1-n-s} m_2^{-1-n'-s'}. \end{aligned} \tag{3.3.3}$$

For example, the calculation of $B^{(0,1)}$ (only keeping terms proportional to $f_L^* f_L$) is

$$
\begin{aligned}
B^{(0,1)}(s) = B^{(1,0)}(s) &= \int d^4x \, \frac{b_{ijkl}}{8\,(4\pi)^2} \int_0^\infty dt_1 dt_2 \times \\
&\times \mathcal{F}^{(2)} \frac{\mu^{4s} t_1^s t_2^{s+1}}{\Gamma[s+1]^2} \langle x| e^{\bar{\Xi} t_1} |x\rangle_{ij} \langle x| \hat{a}_1 e^{\bar{\Xi} t_2} |x\rangle_{kl} \\
&= \int d^4x \, \frac{1}{2\,(4\pi)^2} \int_0^\infty dt_1 dt_2 \times \\
&\times \mathcal{F}^{(2)} \frac{\mu^{4s} t_1^{s-2} t_2^{s-1}}{\Gamma[s+1]^2} \Big[e^{m_L^2 t_1/\mu^2} e^{m_L^2 t_2/\mu^2} \left(\lambda_0{}^2 f_L^* f_L\right) \\
&+ e^{m_L^2 t_1/\mu^2} e^{m_H^2 t_2/\mu^2} \left(\overline{\lambda}^2 f_L^* f_L\right) + e^{m_H^2 t_1/\mu^2} e^{m_L^2 t_2/\mu^2} \left(\lambda_0 \overline{\lambda} f_L^* f_L\right) \\
&+ e^{m_H^2 t_1/\mu^2} e^{m_H^2 t_2/\mu^2} \left(\overline{\lambda}\lambda_0 f_L^* f_L\right) \Big] \\
&= \frac{1}{2} \int d^4x \, f_L^* f_L \Big[\hat{\lambda}_0{}^2 m_L^2 \left(\ln \hat{m}_L^2 - \frac{1}{4} \ln^2 \hat{m}_L^2 \right) \\
&+ \hat{\overline{\lambda}}^2 m_L^2 \left(\ln \hat{m}_H^2 - \frac{1}{4} \ln^2 \hat{m}_H^2 \right) \\
&+ \lambda_0 \overline{\lambda} m_H^2 \left(\ln \hat{m}_L^2 + \ln \hat{m}_H^2 - \frac{1}{4} \ln^2 \hat{m}_L^2 - \frac{1}{4} \ln^2 \hat{m}_H^2 \right) \Big]
\end{aligned}
\tag{3.3.4}
$$

The integrals involved with the $A(s)$ part of the two point Green's function are much more complicated. In both the one loop case and in the calculation of the $B(s)$ integrals the difference coordinate $\Delta = x - y$ is zero. For the $A(s)$ functions this is obviously not the case. Additionally, the permutation symbols a_{ijk} depend on the fields and thus indirectly upon the coordinates which further complicates the integrals. The first step in calculating the $A(s)$ integrals is to simplify them by expanding out the permutation symbols in a Taylor series in the coordinates.

Let

$$U = \frac{1}{2}(x + y) \tag{3.3.5a}$$

$$\Delta = x - y. \tag{3.3.5b}$$

Thus for an arbitrary function f

$$f(x) = f(U) + \frac{1}{2} f_{,\mu}(U)\Delta_\mu + \frac{1}{8} f_{,\mu\nu}(U)\Delta_\mu\Delta_\nu + \cdots \tag{3.3.6a}$$

$$f(y) = f(U) - \frac{1}{2} f_{,\mu}(U)\Delta_\mu + \frac{1}{8} f_{,\mu\nu}(U)\Delta_\mu\Delta_\nu + \cdots . \tag{3.3.6b}$$

Using (3.3.5a) and (3.3.5b) as well as (3.3.6a) and (3.3.6b), the permutation symbols can be expanded out

$$\begin{aligned}
a_{ijk}(x)a_{pqr}(y) &= a_{ijk}(U)a_{pqr}(U) \\
&\quad + \frac{1}{2}\left[(a_{ijk})_{,\mu}(U)a_{pqr}(U) - a_{ijk}(U)(a_{pqr})_{,\mu}(U)\right]\Delta_\mu \\
&\quad + \frac{1}{8}\left[(a_{ijk})_{,\mu\nu}(U)a_{pqr}(U) - 2(a_{ijk})_{,\mu}(U)(a_{pqr})_{,\nu}(U)\right. \\
&\qquad \left. + a_{ijk}(U)(a_{pqr})_{,\mu\nu}(U)\right]\Delta_\mu\Delta_\nu + \cdots \\
&\equiv a^{(0)}_{ijk;pqr}(U) + \frac{1}{2}\left(a^{(1)}_{ijk;pqr}(U)\right)_{,\mu}\Delta_\mu \\
&\quad + \frac{1}{8}\left(a^{(2)}_{ijk;pqr}(U)\right)_{,\mu\nu}\Delta_\mu\Delta_\nu + \cdots
\end{aligned} \tag{3.3.7}$$

Further defining

$$A(s) \equiv \mathcal{F}^{(2)}\tilde{A}(s) \tag{3.3.8}$$

gives

$$\begin{aligned}
\tilde{A}(s) &= \frac{1}{12}\int d^4U d^4\Delta \int_0^\infty dt_1 dt_2 dt_3 \, \times \\
&\times e^{-\Delta^2[1/t_1+1/t_2+1/t_3]/4} \sum_{k,k',k''} t_1^{s+k-2} t_2^{s-k'-2} t_3^{s+k''-2} \, \times \\
&\times \left[a^{(0)}_{ijk;pqr}(U) + \frac{1}{2}\left[a^{(1)}_{ijk;pqr}(U)\right]_{,\mu}\Delta_\mu + \cdots\right] \times \\
&\times \left[\hat{a}_k(\Delta)e^{-\Xi t_1}\right]_{ip}\left[\hat{a}_{k'}(\Delta)e^{-\Xi t_2}\right]_{jq}\left[\hat{a}_{k''}(\Delta)e^{-\Xi t_3}\right]_{kr}
\end{aligned} \tag{3.3.9}$$

which can be written as

$$\tilde{A}(s) = \int d^4U d^4\Delta\left[\tilde{A}^{(0)}(s) + \Delta_{\alpha_1}\tilde{A}^{(1)}_{\alpha_1}(s) + \Delta_{\alpha_1}\Delta_{\alpha_2}\tilde{A}^{(2)}_{\alpha_1\alpha_2} + \cdots\right]. \tag{3.3.10}$$

Because the Δ integrals are simply Gaussians, those integrals with an odd number of Δs vanish. After integrating over Δ the following general expression is found.

$$\begin{aligned}
\tilde{A}(m_1, m_2, m_3; s) &= \int d^4U \int_0^\infty dt_1 dt_2 dt_3 \, e^{-m_1^2 t_1} e^{-m_2^2 t_2} e^{-m_3^2 t_3} \, \times \\
&\times \sum_{k,k',k''}\left[\frac{1}{12} t_1^{s+k} t_2^{s+k'} t_3^{s+k''} \tau^2 \tilde{A}^{(0)}_{k,k',k''}\right. \\
&\quad \left. + \sum_{n=1}^\infty \frac{\Gamma[2n]}{6\,\Gamma[n]} t_1^{s+k+n} t_2^{s+k'+n} t_3^{s+k''+n} \tau^{n+2} \tilde{A}^{(2n)}_{k,k',k''}\right]
\end{aligned} \tag{3.3.11}$$

where

$$\tau = \frac{t_1 t_2 + t_1 t_3 + t_2 t_3}{t_1 t_2 t_3}. \tag{3.3.12}$$

The remaining $\tilde{A}^{(n)}_{k,k',k''}$ terms include the contractions over the sundry indices that arise from the integration over the Δ_μ coordinates as well as the contractions over the permutation indices and their derivatives given in (3.3.7). The only remaining integrals to be calculated are those over t_i. These are quite involved and are discussed in detail in Appendix B.

As an example of a setting-sun type two loop diagram, consider $k = k' = k'' = 0$. Limiting attention to terms proportional to $f_L^* f_L$ we obtain

$$\begin{aligned}
\tilde{A}_{0,0,0} = \frac{1}{(4\pi)^2}\mathcal{F}^{(2)} \int d^4U \Bigg[& \frac{f_L^* f_L}{12}\Big(3\lambda_0{}^2 I(s,s,s;1,1,1;2;m_L) \\
& + \left(3\lambda_c{}^2 + 6\bar{\lambda}^2\right) I(s,s,s;m_L/m_H,1,1;2;m_H)\Big) \\
& - p^2\frac{f_L^* f_L}{6}\Big(3\lambda_0{}^2 I(s+1,s+1,s+1;1,1,1;2;m_L) \\
& + \left(3\lambda_c{}^2 + 6\bar{\lambda}^2\right) I(s+1,s+1,s+1;m_L/m_H,1,1;2;m_H)\Big)\Bigg] \\
= \int d^4U f_L^* f_L \Bigg[& p^2\left(\frac{3}{2}\hat{\lambda}_0{}^2 \ln\hat{m}_L^2 + \frac{1}{2}\left(\hat{\lambda}_c{}^2 + 2\hat{\bar{\lambda}}^2\right)\right) \\
& + m_L^2\Big\{\frac{9}{8}\hat{\lambda}_0{}^2\left(2\ln\hat{m}_L^2 - \ln^2\hat{m}_L^2\right) \\
& + \frac{3}{4}\left(\hat{\lambda}_c{}^2 + 2\hat{\bar{\lambda}}^2\right)\left(2\ln\hat{m}_H^2 + \ln^2\hat{m}_H^2\right)\Big\} \\
& + m_H^2\frac{3}{4}\left(\hat{\lambda}_c{}^2 + 2\hat{\bar{\lambda}}^2\right)\left(2\ln\hat{m}_H^2 + \ln^2\hat{m}_H^2\right)\Bigg]
\end{aligned} \tag{3.3.13}$$

where the functions $I(s,s',s'';\alpha,\beta,\gamma;\sigma;m)$ are given in Appendix B.

Following (2.2.30) by subtracting each setting-sun integral for a particular n-point function from its double scoop integral gives the overall two loop n-point functions. Thus,

$$\begin{aligned}
\Gamma_2^{f_L^* f_L} = f_L^* f_L \Bigg[& m_L^2\Bigg(\left[-\frac{1}{4}\ln\hat{m}_L^2 + \frac{5}{8}\ln^2\hat{m}_L^2\right]\hat{\lambda}_0{}^2 + \left[-\frac{3}{2}\ln\hat{m}_H^2 - \frac{3}{4}\ln^2\hat{m}_H^2\right]\hat{\lambda}_c{}^2 \\
& + \left[-\ln\hat{m}_H^2 - 2\ln^2\hat{m}_H^2\right]\hat{\bar{\lambda}}^2\Bigg) \\
& + m_H^2\left(-\left[\hat{\lambda}_c{}^2 + 2\hat{\bar{\lambda}}^2\right]\left(\frac{3}{2}\ln\hat{m}_H^2 + \frac{3}{4}\ln^2\hat{m}_H^2\right) \right. \\
& \left. + \hat{\bar{\lambda}}\hat{\lambda}_0\left(\ln\hat{m}_L^2 + \ln\hat{m}_H^2 - \frac{1}{4}\ln^2\hat{m}_L^2 - \frac{1}{4}\ln^2\hat{m}_H^2\right)\right)\Bigg) \\
& - p^2\left(\frac{3}{2}\hat{\lambda}_0{}^2\ln\hat{m}_L^2 + \frac{1}{2}\left(\hat{\lambda}_c{}^2 + 2\hat{\bar{\lambda}}^2\right)\ln\hat{m}_H^2\right)\Bigg]
\end{aligned} \tag{3.3.14a}$$

$$
\begin{aligned}
\Gamma_2^{f_H^* f_H} = f_H^* f_H \Bigg[& m_H^2 \left(\left[-\frac{1}{4}\ln \hat{m}_H^2 + \frac{5}{8}\ln^2 \hat{m}_H^2\right] \hat{\lambda}_0^{\,2} + \left[-\frac{3}{2}\ln \hat{m}_L^2 - \frac{3}{4}\ln^2 \hat{m}_L^2\right] \hat{\lambda}_c^{\,2} \right. \\
& \left. + \left[-\ln \hat{m}_L^2 - 2\ln^2 \hat{m}_L^2\right] \hat{\bar{\lambda}}^2 \right) \\
& + m_L^2 \left(- \left[\hat{\lambda}_c^{\,2} + 2\hat{\bar{\lambda}}^2\right] \left(\frac{3}{2}\ln \hat{m}_L^2 + \frac{3}{4}\ln^2 \hat{m}_L^2\right) \right. \\
& \left. + \hat{\bar{\lambda}}\hat{\lambda}_0 \left(\ln \hat{m}_H^2 + \ln \hat{m}_L^2 - \frac{1}{4}\ln^2 \hat{m}_H^2 - \frac{1}{4}\ln^2 \hat{m}_L^2\right)\right) \\
& - p^2 \left(\frac{3}{2}\hat{\lambda}_0^{\,2}\ln \hat{m}_H^2 + \frac{1}{2}\left(\hat{\lambda}_c^{\,2} + 2\hat{\bar{\lambda}}^2\right)\ln \hat{m}_L^2\right)\Bigg]
\end{aligned} \tag{3.3.14b}
$$

$$
\Gamma_2^{Q_\mu f_L^* f_L} = i T_\mu Q_\mu f_L^* f_L \left[-\frac{9}{2}e\,\hat{\lambda}_0^{\,2}\ln \hat{m}_L^2 - \frac{3}{2}e\left(\hat{\lambda}_c^{\,2} + 2\hat{\bar{\lambda}}^2\right)\ln \hat{m}_H^2\right] \tag{3.3.14c}
$$

$$
\Gamma_2^{Q_\mu f_H^* f_H} = i T_\mu Q_\mu f_H^* f_H \left[-\frac{9}{2}e\,\hat{\lambda}_0^{\,2}\ln \hat{m}_H^2 - \frac{3}{2}e\left(\hat{\lambda}_c^{\,2} + 2\hat{\bar{\lambda}}^2\right)\ln \hat{m}_L^2\right] \tag{3.3.14d}
$$

$$
\begin{aligned}
\Gamma_2^{f_L^{*2} f_L^2} = f_L^{*2} f_L^2 \Bigg[& \lambda_0\hat{\lambda}_0^{\,2}\left(\frac{9}{4}\ln \hat{m}_L^2 + \frac{3}{8}\ln \hat{m}_H^2\right) \\
& + \lambda_0\hat{\lambda}_c^{\,2}\left(\frac{1}{2}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 + \frac{35}{8}\ln^2 \hat{m}_H^2\right) \\
& + \lambda_0\hat{\bar{\lambda}}^2\left(4\ln \hat{m}_H^2 + 2\ln \hat{m}_L^2 + 2\ln \hat{m}_L^2 \ln \hat{m}_H^2 + \frac{25}{8}\ln^2 \hat{m}_H^2\right) \\
& - \bar{\lambda}\left[\hat{\bar{\lambda}}^2 + \hat{\lambda}_c^{\,2}\right]\left(3\ln \hat{m}_H^2 + 9\ln^2 \hat{m}_H^2\right)\Bigg]
\end{aligned} \tag{3.3.14e}
$$

$$
\begin{aligned}
\Gamma_2^{f_H^{*2} f_H^2} = f_H^{*2} f_H^2 \Bigg[& \lambda_0\hat{\lambda}_0^{\,2}\left(\frac{9}{4}\ln \hat{m}_H^2 + \frac{3}{8}\ln \hat{m}_L^2\right) \\
& + \lambda_0\hat{\lambda}_c^{\,2}\left(\frac{1}{2}\ln \hat{m}_L^2 + \frac{1}{4}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 + \frac{35}{8}\ln^2 \hat{m}_L^2\right) \\
& + \lambda_0\hat{\bar{\lambda}}^2\left(4\ln \hat{m}_L^2 + 2\ln \hat{m}_H^2 + 2\ln \hat{m}_L^2 \ln \hat{m}_H^2 + \frac{25}{8}\ln^2 \hat{m}_L^2\right) \\
& - \bar{\lambda}\left[\hat{\bar{\lambda}}^2 + \hat{\lambda}_c^{\,2}\right]\left(3\ln \hat{m}_L^2 + 9\ln^2 \hat{m}_L^2\right)\Bigg]
\end{aligned} \tag{3.3.14f}
$$

$$\begin{aligned}
\Gamma_2^{f_L^{*2} f_H{}^2} = f_L^{*2} f_H{}^2 \lambda_c \Big[\bar{\bar{\lambda}}^2 & \Big(-\frac{1}{2}\ln \hat{m}_L^2 - \frac{1}{2}\ln \hat{m}_H^2 + \frac{1}{8}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& -\frac{11}{48}\ln^2 \hat{m}_L^2 - \frac{11}{48}\ln^2 \hat{m}_H^2 \Big) \\
+ \hat{\lambda}_0{}^2 & \Big(\frac{1}{2}\ln \hat{m}_L^2 + \frac{1}{2}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& + \frac{1}{4}\ln^2 \hat{m}_L^2 + \frac{1}{4}\ln^2 \hat{m}_H^2 \Big) \\
+ \hat{\lambda}_c{}^2 & \Big(\frac{1}{4}\ln \hat{m}_L^2 + \frac{1}{4}\ln \hat{m}_H^2 + \frac{1}{8}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& + \frac{1}{8}\ln^2 \hat{m}_L^2 + \frac{1}{8}\ln^2 \hat{m}_H^2 \Big) \\
- \frac{3}{4}\hat{\lambda}_0 \hat{\bar{\lambda}} & \left(\ln \hat{m}_L^2 + \ln \hat{m}_H^2 \right) \\
+ \hat{\lambda}_c \hat{\bar{\lambda}} & \Big(-\frac{1}{4}\ln \hat{m}_L^2 - \frac{1}{4}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& -\frac{1}{16}\ln^2 \hat{m}_L^2 - \frac{1}{16}\ln^2 \hat{m}_H^2 \Big) \Big]
\end{aligned} \tag{3.3-14g}$$

$$\begin{aligned}
\Gamma_2^{f_H^{*2} f_L{}^2} = f_H^{*2} f_L{}^2 \lambda_c \Big[\bar{\bar{\lambda}}^2 & \Big(-\frac{1}{2}\ln \hat{m}_L^2 - \frac{1}{2}\ln \hat{m}_H^2 + \frac{1}{8}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& -\frac{11}{48}\ln^2 \hat{m}_L^2 - \frac{11}{48}\ln^2 \hat{m}_H^2 \Big) \\
+ \hat{\lambda}_0{}^2 & \Big(\frac{1}{2}\ln \hat{m}_L^2 + \frac{1}{2}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& + \frac{1}{4}\ln^2 \hat{m}_L^2 + \frac{1}{4}\ln^2 \hat{m}_H^2 \Big) \\
+ \hat{\lambda}_c{}^2 & \Big(\frac{1}{4}\ln \hat{m}_L^2 + \frac{1}{4}\ln \hat{m}_H^2 + \frac{1}{8}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& + \frac{1}{8}\ln^2 \hat{m}_L^2 + \frac{1}{8}\ln^2 \hat{m}_H^2 \Big) \\
- \frac{3}{4}\hat{\lambda}_0 \hat{\bar{\lambda}} & \left(\ln \hat{m}_L^2 + \ln \hat{m}_H^2 \right) \\
+ \hat{\lambda}_c \hat{\bar{\lambda}} & \Big(-\frac{1}{4}\ln \hat{m}_L^2 - \frac{1}{4}\ln \hat{m}_H^2 + \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \\
& -\frac{1}{16}\ln^2 \hat{m}_L^2 - \frac{1}{16}\ln^2 \hat{m}_H^2 \Big) \Big]
\end{aligned} \tag{3.3-14h}$$

$$
\begin{aligned}
\Gamma_2^{f_L^* f_L f_H^* f_H} = f_L^* f_L f_H^* f_H \Bigg[& \bar{\lambda}\bar{\hat{\lambda}}^2 \left(\frac{3}{2}\ln \hat{m}_L^2 + \frac{3}{2}\ln \hat{m}_H^2 - \frac{5}{2}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \right. \\
& \qquad \left. - \frac{13}{8}\ln^2 \hat{m}_L^2 - \frac{13}{8}\ln^2 \hat{m}_H^2 \right) \\
& + \bar{\lambda}\hat{\lambda}_c{}^2 \left(-\frac{1}{4}\ln \hat{m}_L^2 - \frac{1}{4}\ln \hat{m}_H^2 - \frac{1}{4}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \right. \\
& \qquad \left. + \frac{1}{16}\ln^2 \hat{m}_L^2 + \frac{1}{16}\ln^2 \hat{m}_H^2 \right) \\
& + \bar{\lambda}\hat{\lambda}_0{}^2 \left(6\ln \hat{m}_L^2 + 6\ln \hat{m}_H^2 - 2\ln \hat{m}_L^2 \ln \hat{m}_H^2 \right. \\
& \qquad \left. - \frac{7}{8}\ln^2 \hat{m}_L^2 - \frac{7}{8}\ln^2 \hat{m}_H^2 \right) \\
& + \bar{\hat{\lambda}}\hat{\lambda}_c \left(\frac{1}{4}\ln \hat{m}_L^2 + \frac{1}{4}\ln \hat{m}_H^2 - \frac{1}{2}\ln \hat{m}_L^2 \ln \hat{m}_H^2 \right. \\
& \qquad \left. - \frac{7}{16}\ln^2 \hat{m}_L^2 - \frac{7}{16}\ln^2 \hat{m}_H^2 \right) \\
& + \bar{\lambda}\lambda_0 \left(-3\ln \hat{m}_L^2 - 3\ln \hat{m}_H^2 + \frac{3}{2}\ln^2 \hat{m}_L^2 + \frac{3}{2}\ln^2 \hat{m}_H^2 \right) \\
& + \frac{3}{4}\bar{\lambda}\hat{\lambda}_0\hat{\lambda}_c \left(\ln^2 \hat{m}_L^2 + \ln^2 \hat{m}_H^2 \right) \\
& + \lambda_0\hat{\lambda}_c{}^2 \left(-\frac{3}{2}\ln \hat{m}_L^2 - \frac{3}{2}\ln \hat{m}_H^2 - \frac{3}{8}\ln^2 \hat{m}_L^2 - \frac{3}{8}\ln^2 \hat{m}_H^2 \right) \Bigg]
\end{aligned} \tag{3.3.14i}
$$

$$
\Gamma_2^{Q^2 f_L^* f_L} = Q^2 f_L^* f_L \left[-\frac{9}{2}e^2 \hat{\lambda}_0{}^2 \ln \hat{m}_L^2 - \frac{3}{2}e^2 \left(\hat{\lambda}_c{}^2 + 2\bar{\hat{\lambda}}^2 \right) \ln \hat{m}_H^2 \right] \tag{3.3.14j}
$$

$$
\Gamma_2^{Q^2 f_H^* f_H} = Q^2 f_H^* f_H \left[-\frac{9}{2}e^2 \hat{\lambda}_0{}^2 \ln \hat{m}_H^2 - \frac{3}{2}e^2 \left(\hat{\lambda}_c{}^2 + 2\bar{\hat{\lambda}}^2 \right) \ln \hat{m}_L^2 \right]. \tag{3.3.14k}
$$

In the one loop case terms that are part of the effective theory were simply generated by higher order DeWitt coefficients. However, this is not the complete story in the two-loop case. The "double scoop" integrals have the structure of two one loop integrals multiplied together. Therefore, any effective operators that arise from "double scoop" integrals are not unique, i.e. they are just products of one-loop effective operators. Thus, all the new effective operators at two-loop order come only from the "setting-sun" type terms. The forms of these effective operators arise both from the DeWitt coefficients as well as higher derivative terms from the expansion of the a_{ijk} permutation symbols (3.3.12). For example, some of the $O(1/m_H^2)$ terms arise from the terms with two Δ's in the DeWitt coefficient $\hat{a}_l$ multiplied by the zero derivative terms in the expansion of the a_{ijk} symbols. Further, the numerical coefficients of the new terms strictly arise from the Case II setting-sun integrals given in Appendix B, i.e. "setting-sun" diagrams where the internal loop structure consists of two heavy fields and one light field. The results of this involved calculation follow

$$
\begin{aligned}
\Gamma_2^{LM} = \frac{1}{m_H^2} \Bigg[& -\frac{1}{3} e^2 \left(\dot{\lambda}_c{}^2 + 2\dot{\bar{\lambda}}^2 \right) F_{\mu\nu} F^{\mu\nu} \phi_L^* \phi_L \\
& - \left[\left(\phi_L^{*2}\right)_{,\mu\mu} \phi_L{}^2 + \phi_L^{*2} \left(\phi_L{}^2\right)_{,\mu\mu} \right] \left(\frac{1}{6} \bar{\lambda} \dot{\lambda}_c{}^2 + \frac{49}{432} \lambda_0 \dot{\bar{\lambda}}^2 \right) \\
& - \left(\phi_L^* \phi_L\right)_{,\mu\mu} \phi_L^* \phi_L \left(\frac{2}{3} \bar{\lambda} \dot{\bar{\lambda}}^2 + \frac{1}{3} \bar{\lambda} \dot{\lambda}_c{}^2 + \frac{43}{432} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \dot{\lambda}_c{}^2 \right) \right) \Bigg] \\
& + O\left(1/m_H^4\right) \\
= -\frac{1}{m_H^2} \Bigg[& \frac{1}{3} e^2 \left(\dot{\lambda}_c{}^2 + 2\dot{\bar{\lambda}}^2 \right) F_{\mu\nu} F^{\mu\nu} \phi_L^* \phi_L \\
& + (\phi_L^* \phi_L)(\phi_{L,\mu}^* \phi_{L,\mu}) \left[\frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \frac{5}{2} \dot{\lambda}_c{}^2 \right) + \frac{145}{144} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \frac{43}{435} \dot{\lambda}_c{}^2 \right) \right] \\
& + \left[\phi_{L,\mu\mu}^* \phi_L^* \phi_L{}^2 + h.c. \right] \left[\frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \dot{\lambda}_c{}^2 \right) + \frac{47}{144} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \frac{43}{144} \dot{\lambda}_c{}^2 \right) \right] \\
& + \left[\phi_L^{*2} (\phi_{L,\mu} \phi_{L,\mu}) + h.c. \right] \left[\frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \frac{1}{2} \dot{\lambda}_c{}^2 \right) + \frac{43}{432} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \dot{\lambda}_c{}^2 \right) \right] \Bigg] \\
& + O\left(1/m_H^4\right)
\end{aligned}
\tag{3.3.15}
$$

Adding together the one loop effective Lagrangian (3.2.7) and the above two loop effective Lagrangian (3.3.15) gives the total effective Lagrangian up to two loops for two field QSD

$$
\begin{aligned}
\mathcal{L}_{eff}^{QSD} = \frac{1}{m_H^2} \Bigg[& \frac{1}{6} e^2 \left(\dot{\bar{\lambda}} + 2(\dot{\lambda}_c{}^2 + 2\dot{\bar{\lambda}}^2) \right) F_{\mu\nu} F^{\mu\nu} \phi_L^* \phi_L \\
& + \left\{ \frac{2}{3} \bar{\lambda} \dot{\bar{\lambda}} - \frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \frac{5}{2} \dot{\lambda}_c{}^2 \right) - \frac{145}{144} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \frac{43}{435} \dot{\lambda}_c{}^2 \right) \right\} \times \\
& \qquad \times (\phi_L^* \phi_L)(\phi_{L,\mu}^* \phi_{L,\mu}) \\
& + \left\{ \frac{1}{3} \left(\bar{\lambda} \dot{\bar{\lambda}} + \frac{1}{4} \lambda_c \dot{\lambda}_c \right) - \frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \dot{\lambda}_c{}^2 \right) - \frac{47}{144} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \frac{43}{144} \dot{\lambda}_c{}^2 \right) \right\} \times \\
& \qquad \times \left[\phi_{L,\mu\mu}^* \phi_L^* \phi_L{}^2 + \phi_{L,\mu\mu} \phi_L \phi_L^{*2} \right] \\
& + \left\{ \frac{1}{6} \left(\bar{\lambda} \dot{\bar{\lambda}} + \frac{1}{2} \lambda_c \dot{\lambda}_c \right) - \frac{2}{3} \bar{\lambda} \left(\dot{\bar{\lambda}}^2 + \frac{1}{2} \dot{\lambda}_c{}^2 \right) - \frac{43}{432} \lambda_0 \left(\dot{\bar{\lambda}}^2 + \dot{\lambda}_c{}^2 \right) \right\} \times \\
& \qquad \times \left[\phi_L^{*2} (\phi_{L,\mu} \phi_{L,\mu}) + \phi_L{}^2 (\phi_{L,\mu}^* \phi_{L,\mu}^*) \right] \Bigg] \\
& + O(1/m_H^4).
\end{aligned}
\tag{3.3.16}
$$

3.4 The Renormalization Group

In (2.2.7) the scale μ was introduced due to the arbitrariness of the functional integral in (2.2.2). Because of this arbitrariness, changes in μ's value must be compensated by changes in the values of the parameters in the theory. In other words, the n-point Green's function cannot depend on μ, i.e. [9]

$$\mu\frac{d}{d\mu}\Gamma^{(n)} = 0. \tag{3.4.1}$$

Applying (3.4.1) to the two field QSD model generates the renormalization group equation

$$\begin{aligned}
&\left(\mu\frac{\partial}{\partial\mu} + \beta_0\frac{\partial}{\partial\lambda_0} + \beta_c\frac{\partial}{\partial\lambda_c} + \overline{\beta}\frac{\partial}{\partial\overline{\lambda}} + \beta_e\frac{\partial}{\partial e}\right.\\
&- \begin{bmatrix} m_L^2 & m_H^2 \end{bmatrix} \begin{bmatrix} \gamma_{LL} & \gamma_{LH} \\ \gamma_{HL} & \gamma_{HH} \end{bmatrix} \begin{bmatrix} \partial/\partial m_L^2 \\ \partial/\partial m_H^2 \end{bmatrix}\\
&- \gamma_{\phi_L}\left[f_L^*\frac{\partial}{\partial f_L^*} + f_L\frac{\partial}{\partial f_L}\right] - \gamma_{\phi_H}\left[f_H^*\frac{\partial}{\partial f_H^*} + f_H\frac{\partial}{\partial f_H}\right]\\
&\left.-\gamma_A A_\mu\frac{\partial}{\partial A_\mu}\right)\Gamma^{(n)} = 0
\end{aligned} \tag{3.4.2}$$

where the various renormalization group functions are defined by

$$\beta_0 \equiv \mu\frac{\partial\lambda_0}{\partial\mu} \tag{3.4.3a}$$

$$\beta_c \equiv \mu\frac{\partial\lambda_c}{\partial\mu} \tag{3.4.3b}$$

$$\overline{\beta} \equiv \mu\frac{\partial\overline{\lambda}}{\partial\mu} \tag{3.4.3c}$$

$$\beta_e \equiv \mu\frac{\partial e}{\partial\mu} \tag{3.4.3d}$$

$$\gamma_{\phi_L} \equiv -\mu\frac{\partial\ln\phi_L^*}{\partial\mu} = -\mu\frac{\partial\ln\phi_L}{\partial\mu} \tag{3.4.3e}$$

$$\gamma_{\phi_H} \equiv -\mu\frac{\partial\ln\phi_H^*}{\partial\mu} = -\mu\frac{\partial\ln\phi_H}{\partial\mu} \tag{3.4.3f}$$

$$(\gamma_{LL}m_L^2 + \gamma_{LH}m_H^2) \equiv -\mu\frac{\partial m_L^2}{\partial\mu} \tag{3.4.3g}$$

$$(\gamma_{HL}m_L^2 + \gamma_{HH}m_H^2) \equiv -\mu\frac{\partial m_H^2}{\partial\mu}. \tag{3.4.3h}$$

The method for deriving the renormalization group functions is very similar to the one used in conventional MS renormalization [9, 23, 29]. However, in the case of operator regularization the logarithms of the masses play the role that the poles play in the conventional regularization schemes [18, 19].

To start, consider the general finite expansion of the Greens function

$$
\begin{aligned}
\Gamma = \sum_i \Gamma_i = & -\frac{1}{2} f_L^* \partial^2 f_L \left[1 + \hat{a}_1 + \hat{a}_2 \cdots\right] \\
& - \frac{1}{2} f_H^* \partial^2 f_H \left[1 + \hat{b}_1 + \hat{b}_2 \cdots\right] \\
& + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \left[1 + \hat{c}_1 + \hat{c}_2 + \cdots\right] \\
& + \frac{1}{2} m_L^2 f_L^* f_L \left[1 + \hat{d}_1 + \hat{d}_2 + \cdots\right] \\
& + \frac{1}{2} m_H^2 f_H^* f_H \left[1 + \hat{e}_1 + \hat{e}_2 + \cdots\right] \\
& + \frac{1}{4} \lambda_0 {f_L^*}^2 {f_L}^2 \left[1 + \hat{f}_1 + \hat{f}_2 + \cdots\right] \\
& + \frac{1}{4} \lambda_0 {f_H^*}^2 {f_H}^2 \left[1 + \hat{g}_1 + \hat{g}_2 + \cdots\right] \\
& + \frac{1}{4} \lambda_c {f_L^*}^2 {f_H}^2 \left[1 + \hat{h}_1 + \hat{h}_2 + \cdots\right] \\
& + \frac{1}{4} \lambda_c {f_H^*}^2 {f_L}^2 \left[1 + \hat{j}_1 + \hat{j}_2 + \cdots\right] \\
& + f_L^* f_L f_H^* f_H \left[1 + \hat{k}_1 + \hat{k}_2 + \cdots\right] \\
& - ieA_\mu f_L^* \overleftrightarrow{\partial} f_L \left[1 + \hat{l}_1 + \hat{l}_2 + \cdots\right] \\
& - ieA_\mu f_H^* \overleftrightarrow{\partial} f_H \left[1 + \hat{n}_1 + \hat{n}_2 + \cdots\right] \\
& + e^2 A^2 f_L^* f_L \left[1 + \hat{o}_1 + \hat{o}_2 + \cdots\right] \\
& + e^2 A^2 f_H^* f_H \left[1 + \hat{q}_1 + \hat{q}_2 + \cdots\right]
\end{aligned}
\tag{3.4.4}
$$

where,

$$
\begin{aligned}
\hat{x}_1 = & \sum_{\xi_i = \{\lambda_0, \lambda_c, \overline{\lambda}, e\}} \xi_i \left[x_{10}^i + \left(x_{11}^{L,\xi_i} \ln \hat{m}_L^2 + x_{11}^{H,\xi_i} \ln \hat{m}_H^2 + x_{11}^{\gamma,\xi_i} \ln \hat{m}_\gamma^2\right)\right] \\
\hat{x}_2 = & \sum_{\xi_i, \xi_j = \{\lambda_0, \lambda_c, \overline{\lambda}, e\}} \xi_i \xi_j \left[x_{20}^{\xi_i,\xi_j}\right. \\
& + \left(x_{21}^{L,\xi_i,\xi_j} \ln \hat{m}_L^2 + x_{21}^{H,\xi_i,\xi_j} \ln \hat{m}_H^2 + x_{21}^{\gamma,\xi_i,\xi_j} \ln \hat{m}_\gamma^2\right) \\
& + \left(x_{22}^{LL,\xi_i,\xi_j} \ln^2 \hat{m}_L^2 + x_{22}^{HH,\xi_i,\xi_j} \ln^2 \hat{m}_H^2 + x_{22}^{\gamma\gamma,\xi_i,\xi_j} \ln^2 \hat{m}_\gamma^2\right. \\
& \quad + x_{22}^{LH,\xi_i,\xi_j} \ln \hat{m}_L^2 \ln \hat{m}_H^2 + x_{22}^{L\gamma,\xi_i,\xi_j} \ln \hat{m}_L^2 \ln \hat{m}_\gamma^2 \\
& \quad \left.\left. + x_{22}^{H\gamma,\xi_i,\xi_j} \ln \hat{m}_H^2 \ln \hat{m}_\gamma^2\right)\right].
\end{aligned}
$$

Further, define the renormalization group functions in terms of a series

$$
\beta_0 = \sum_i \dot{A}_i \tag{3.4.5a}
$$

$$\beta_c = \sum_i \hat{B}_i \tag{3.4.5b}$$

$$\overline{\beta} = \sum_i \hat{C}_i \tag{3.4.5c}$$

$$\beta_e = \sum_i \hat{D}_i \tag{3.4.5d}$$

$$(\gamma_{LL} m_L^2 + \gamma_{LH} m_H^2) = \sum_i \hat{E}_i \tag{3.4.5e}$$

$$(\gamma_{HL} m_L^2 + \gamma_{HH} m_H^2) = \sum_i \hat{F}_i \tag{3.4.5f}$$

$$2\gamma_{OL} = \sum_i \hat{H}_i \tag{3.4.5g}$$

$$2\gamma_{OH} = \sum_i \hat{J}_i \tag{3.4.5h}$$

$$2\gamma_A = \sum_i \hat{K}_i \tag{3.4.5i}$$

where

$$\hat{X}_1 = \sum_{\xi_i = \{\lambda_0, \lambda_c, \overline{\lambda}, e\}} \xi_i X_1^i$$

$$\hat{X}_2 = \sum_{\xi_i, \xi_j = \{\lambda_0, \lambda_c, \overline{\lambda}, e\}} \xi_i \xi_j X_2^{ij}$$

$$\hat{X}_3 = \sum_{\xi_i, \xi_j, \xi_k = \{\lambda_0, \lambda_c, \overline{\lambda}, e\}} \xi_i \xi_j \xi_k X_3^{ijk} \text{ .etc.}$$

Taking (3.4.4) and (3.4.5), substituting them into (3.4.2) and matching couplings and the various powers of the logarithms of the masses generates over ninety different recursion relations for the coefficients appearing in (3.4.5). For example, few of the recursion relations for the $\hat{A}_i$ coefficients are

$$\hat{A}_2^{\lambda_0\lambda_0} = 2\left(\hat{f}_{11}^{\lambda_0 L} + \hat{f}_{11}^{\lambda_0 H} + \hat{f}_{11}^{\lambda_0 \gamma}\right) + 2\hat{H}_1^{\lambda_0} \tag{3.4.6a}$$

$$\hat{A}_2^{\lambda_0\lambda_c} = 2\left(\hat{f}_{11}^{\lambda_c L} + \hat{f}_{11}^{\lambda_c H} + \hat{f}_{11}^{\lambda_c \gamma}\right) + +2\hat{H}_1^{\lambda_c} \tag{3.4.6b}$$

$$\hat{A}_2^{\lambda_0\overline{\lambda}} = 2\left(\hat{f}_{11}^{\overline{\lambda} L} + \hat{f}_{11}^{\overline{\lambda} H} + \hat{f}_{11}^{\overline{\lambda} \gamma}\right) + +2\hat{H}_1^{\overline{\lambda}} \tag{3.4.6c}$$

$$\hat{A}_2^{\lambda_0\alpha} = 2\left(\hat{f}_{11}^{\alpha L} + \hat{f}_{11}^{\alpha H} + \hat{f}_{11}^{\alpha \gamma}\right) + 2\hat{H}_1^{\alpha} \tag{3.4.6d}$$

$$\begin{aligned}\hat{A}_3^{\lambda_0\lambda_0\lambda_0} &= 2\left(\hat{f}_{21}^{\lambda_0\lambda_0 L} + \hat{f}_{21}^{\lambda_0\lambda_0 H} + \hat{f}_{21}^{\lambda_0\lambda_0\gamma}\right) - 2\hat{A}_2^{\lambda_0\lambda_0}\hat{f}_{10}^{\lambda_0} \\ &+ \left(\hat{E}_1^{\lambda_0}\hat{f}_{11}^{\lambda_0 L} + \hat{F}_1^{\lambda_0}\hat{f}_{11}^{\lambda_0 H} + \hat{G}_1^{\lambda_0}\hat{f}_{11}^{\lambda_0\gamma}\right) \\ &- \left(\hat{B}_2^{\lambda_0\lambda_0}\hat{f}_{10}^{\lambda_c} + \hat{C}_2^{\lambda_0\lambda_0}\hat{f}_{10}^{\overline{\lambda}} + \hat{D}_2^{\lambda_0\lambda_0}\hat{f}_{10}^{\alpha}\right) \\ &+ 2\hat{H}_1^{\lambda_0}\hat{f}_{10}^{\lambda_0} + 2\hat{H}_2^{\lambda_0\lambda_0}\end{aligned} \tag{3.4.6e}$$

Substituting in the values from the one and two loop Green's functions into these numerous recursion relations the renormalization group functions to two loop order are found to be

$$\begin{aligned}\beta_0 = \lambda_0 \Big[&-8\hat{\lambda}_0 - \frac{1}{2}\zeta_c\hat{\lambda}_c - 4\bar{\zeta}\,\hat{\bar{\lambda}} + 12\hat{\alpha} \\ &+10\hat{\lambda}_0^{\,2} + 5\hat{\lambda}_c^{\,2} + 44\hat{\bar{\lambda}}^{2} - 16\bar{\zeta}\,\hat{\bar{\lambda}}^{2} - 8\hat{\lambda}_0\hat{\bar{\lambda}} - 25\bar{\zeta}\hat{\lambda}_c^{\,2}\Big]\end{aligned} \tag{3.4.7a}$$

$$\beta_c = \lambda_c \left[-4\hat{\lambda}_0 - 16\hat{\bar{\lambda}} + 12\hat{\alpha} + 4\hat{\lambda}_c^{\,2} + 4\hat{\lambda}_0^{\,2} - 24\hat{\bar{\lambda}}^{2} - 32\hat{\lambda}_0\hat{\bar{\lambda}}\right] \tag{3.4.7b}$$

$$\begin{aligned}\bar{\beta} = \bar{\lambda}\Big[&-4\lambda_0 - 4\lambda_c - 4\zeta_c\bar{\zeta}^{-1}\hat{\lambda}_c + 12\hat{\alpha} \\ &+2\hat{\bar{\lambda}}^{2} + 20\hat{\lambda}_0^{\,2} - 5\hat{\lambda}_c^{\,2} - 20\hat{\lambda}_0\hat{\bar{\lambda}} + \hat{\lambda}_c\hat{\lambda}_c - 2\bar{\zeta}^{-1}\hat{\lambda}_c^{\,2}\Big]\end{aligned} \tag{3.4.7c}$$

$$\beta_e = e\left[\frac{1}{6}\hat{\alpha} + 9\hat{\lambda}_0^{\,2} + 3\hat{\lambda}_c^{\,2} + 6\hat{\bar{\lambda}}^{2}\right] \tag{3.4.7d}$$

$$\gamma_{LL} = \gamma_{HH} = \left[2\lambda_0 - 2\hat{\alpha} + 3\hat{\lambda}_0^{\,2} + 6\hat{\lambda}_c^{\,2} + 4\hat{\bar{\lambda}}^{2} + 2\hat{\lambda}_0\hat{\bar{\lambda}}\right] \tag{3.4.7e}$$

$$\gamma_{LH} = \gamma_{HL} = \left[2\hat{\bar{\lambda}} + 6\hat{\lambda}_c^{\,2} + 10\hat{\bar{\lambda}}^{2} - 6\hat{\lambda}_0\hat{\bar{\lambda}}\right] \tag{3.4.7f}$$

$$\gamma_{OL} = \gamma_{OH} = \left[3\hat{\lambda}_0^{\,2} + \hat{\lambda}_c^{\,2} + 2\hat{\bar{\lambda}}^{2}\right] \tag{3.4.7g}$$

$$\gamma_A = -\frac{2}{3}\hat{\alpha} \tag{3.4.7h}$$

where,

$$\bar{\zeta} \equiv \bar{\lambda}/\lambda_0$$
$$\zeta_c \equiv \lambda_c/\lambda_0.$$

Chapter 4

Higgs Sector of the Minimal Supersymmetric Model

4.1 Physics Beyond the Standard Model

There are a number of reasons rooted within the Higgs sector to expect that the Standard Model is a low energy theory of a larger structured theory of interactions. These include the problems of naturalness and hierarchy [35, 36]. For example. in the Standard Model the first order correction to the Higgs boson mass yields a quadratically divergent expression [8]. This implies that it is not natural to have Higgs bosons that are relatively light unless this divergence is controlled by the structure of the theory. Unfortunately the Standard Model does not supply a mechanism to control this.

In theories with supersymmetry the quadratic divergence is canceled by the one loop corrections due to the supersymmetric partners of the particles in the loop. Thus. the corrections to the tree-level mass squared of the Higgs boson is limited by the extent of supersymmetry breaking [35]. Further. in order for naturalness and other problems to be resolved the scale of supersymmetry breaking cannot exceed $O(1\text{ TeV})$ [35]. Supersymmetric theories are also interesting in that they are the only theories which do provide a structure to cure the problems of hierarchy and naturalness while retaining Higgs bosons and elementary spin-0 particles [35].

The Higgs sector of the Standard Model consists of one Higgs doublet which gives rise to only one neutral physical Higgs who's mass is a free parameter in the theory [36]. However. this choice of one scalar doublet is completely arbitrary. In fact, there are only two major constraints on the formulation of the Higgs sector.

The first is the experimental fact that $\rho = m_W^2/(m_Z^2 \cos^2\theta_W)$ is very close to one [37]. In the Standard Model this ρ parameter determined by the Higgs structure of the model. It is also known that any model with any number of Higgs doublets (and/or singlets) produces $\rho = 1$ at tree level [38]. Therefore, any version of the Standard Model with any number of Higgs doubles will satisfy this experimental requirement.

The second major constraint is from the limits on flavor-changing neutral currents (FCNC). In the Standard Model these currents are absent because the operations that diagonalize the mass matrix automatically diagonalize the Higgs-fermion couplings [36]. However, in models with more than one Higgs doublet this ceases to be the case. Fortunately a theorem due to Glashow and Weinberg [39] states that tree level FCNCs mediated by Higgs bosons will be absent if all fermions of a given charge couple to no more than one Higgs doublet. Requiring this theorem to be satisfied constrains the Higgs-fermion couplings but is not an unique constraint [6].

The Higgs sector of the Minimal Supersymmetric Model (MSSM) consists of two scalar doublets. MSSM is an attractive extension of the Standard Model because it not only satisfies the above constraints on the Higgs sector, but adds new phenomena to the theory with the addition of a minimal number of free parameters [35].

In what follows a brief overview of the structure of a general Higgs Two Doublet model is given. Of all the Higgs Two Doublet models, MSSM is of interest because it introduces new and rich physics beyond the description of the Standard Model with a minimal number of new parameters. In the decoupling limit, the Higgs sector of MSSM reduces to a system with two masses one of these being much heavier than the other. Operator Regularization is applied in order to calculate, to one loop order, the effective Lagrangian for this system in the decoupling limit. This, in turn, gives the corrections to the Standard Model one would expect if MSSM is the correct theory for physics beyond the Standard Model.

4.2 Formulation of the Higgs Two Doublet Model

The Higgs Sector of the Minimal Supersymmetric Model is a sub-case of the general Higgs Two Doublet Model [40–45]. In this model, two complex Y=1, SU(2) scalar doublet fields, denoted by $\phi_{1,2}$, are used

$$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix} \tag{4.2.1a}$$

$$\phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}. \tag{4.2.1b}$$

The vacuum expectation values of the doublets are

$$\langle\phi_1\rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \tag{4.2.2a}$$

$$\langle\phi_2\rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} e^{i\xi} \tag{4.2.2b}$$

where ξ has been introduced such that if $\sin\xi \neq 0$ there is CP violation in the Higgs sector. Note that the vacuum expectation values, v_1 and v_2, are constrained by the W mass, $m_W^2 = g^2(v_1^2 + v_2^2)/2$ [8].

The Higgs potential which spontaneously breaks $SU(2)_L \times U(1)_Y$ down to $U(1)_{EM}$ and contains the discrete symmetry $\phi_1 \to -\phi_1$ (which is needed to insure that FCNCs are not large) is [46]

$$\begin{aligned} V_{\phi_{1,2}} = {} & \lambda_1(\phi_1^\dagger\phi_1 - v_1^2)^2 + \lambda_2(\phi_2^\dagger\phi_2 - v_2^2)^2 \\ & + \lambda_3\left[\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 - v_1^2 - v_2^2\right]^2 \\ & + \lambda_4\left[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)\right] \\ & + \lambda_5\left[\Re\left(\phi_1^\dagger\phi_2\right) - v_1 v_2\cos\xi\right]^2 \\ & + \lambda_6\left[\Im\left(\phi_1^\dagger\phi_2\right) - v_1 v_2\sin\xi\right]^2 \end{aligned} \tag{4.2.3}$$

where all the λ_i are real and non-negative to insure hermiticity and spontaneous symmetry breaking. From this point, only those theories with $\xi = 0$ will be considered.

As usual, it is desirable to write the ϕ fields in terms of the physical Higgs and Goldstone fields [6]

$$G^\pm = \phi_1^\pm\cos\beta + \phi_2^\pm\sin\beta \tag{4.2.4a}$$

$$G^0 = \sqrt{2}\left(\Im(\phi_1^0)\cos\beta + \Im(\phi_2^0)\sin\beta\right) \tag{4.2.4b}$$

$$h^0 = \sqrt{2}\left(-[\Re(\phi_1^0) - v_1]\sin\alpha + [\Re(\phi_2^0) - v_2]\cos\alpha\right) \tag{4.2.4c}$$

$$H^0 = \sqrt{2}\left(\left[\Re(\phi_1^0) - v_1\right]\cos\alpha + \left[\Re(\phi_2^0) - v_2\right]\sin\alpha\right) \tag{4.2.4d}$$

$$A^0 = \sqrt{2}\left(-\Im(\phi_1^0)\sin\beta + \Im(\phi_2^0)\cos\beta\right) \tag{4.2.4e}$$

$$H^\pm = -\phi_1^\pm \sin\beta + \phi_2^\pm \cos\beta \tag{4.2.4f}$$

where the angle β is defined by

$$\tan\beta \equiv v_2/v_1 \tag{4.2.5}$$

and the angle α arises from the mixing of the two physical Higgs scalars via the mass squared matrix

$$\mathcal{M} = \begin{pmatrix} 4v_1^2(\lambda_1 + \lambda_3) + v_2^2\lambda_5 & (4\lambda_3 + \lambda_5)v_1 v_2 \\ (4\lambda_3 + \lambda_5)v_1 v_2 & 4v_2^2(\lambda_2 + \lambda_3) + v_1^2\lambda_5 \end{pmatrix} \tag{4.2.6}$$

and is obtained via. [6]

$$\sin 2\alpha = \frac{2\mathcal{M}_{12}}{\sqrt{(\mathcal{M}_{11} - \mathcal{M}_{22})^2 + 4\mathcal{M}_{12}^2}} \tag{4.2.7a}$$

$$\cos 2\alpha = \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{\sqrt{(\mathcal{M}_{11} - \mathcal{M}_{22})^2 + 4\mathcal{M}_{12}^2}}. \tag{4.2.7b}$$

The masses for the physical fields in terms of the λ_i's and the vacuum expectation values are [6]

$$m_{H^0}^2 = \frac{1}{2}\left\{\mathcal{M}_{11} + \mathcal{M}_{22} + \sqrt{(\mathcal{M}_{11} - \mathcal{M}_{22})^2 + 4\mathcal{M}_{12}^2}\right\} \tag{4.2.8a}$$

$$m_{h^0}^2 = \frac{1}{2}\left\{\mathcal{M}_{11} + \mathcal{M}_{22} - \sqrt{(\mathcal{M}_{11} - \mathcal{M}_{22})^2 + 4\mathcal{M}_{12}^2}\right\} \tag{4.2.8b}$$

$$m_{H^\pm}^2 = \lambda_4(v_1^2 + v_2^2) \tag{4.2.8c}$$

$$m_{A^0}^2 = \lambda_6(v_1^2 + v_2^2). \tag{4.2.8d}$$

Thus in the Higgs Two Doublet Model the Higgs sector particle spectrum consists of five Higgs fields. These are two neutral CP-even (H^0, h^0), two charged ($H^\pm$) and a neutral CP-odd (psudoscalar) (A^0) Higgs fields.

4.3 The Minimal Supersymmetric Model

The potential for the Higgs sector of the Minimal Supersymmetric Model (MSSM) takes the form [6]

$$\begin{aligned} V_{MSSM} &= m_1^2 H_1^{i*} H_1^i + m_2^2 H_2^{i*} H_2^i - m_{12}^2 \epsilon_{ij} \left(H_1^i H_2^j + H_1^{i*} H_2^{j*} \right) \\ &+ \frac{1}{8} \left(g^2 + g'^2 \right) \left[H_1^{i*} H_1^i - H_2^{j*} H_2^j \right]^2 + \frac{1}{2} g^2 \left| H_1^{i*} H_2^i \right| \end{aligned} \tag{4.3.1}$$

where g and g' are the standard electroweak couplings and the two supersymmetric Higgs doublets have been introduced

$$H_1 = \begin{pmatrix} H_1^1 \\ H_1^2 \end{pmatrix} \tag{4.3.2a}$$

$$H_2 = \begin{pmatrix} H_2^1 \\ H_2^2 \end{pmatrix} \tag{4.3.2b}$$

with vacuum expectation values

$$\langle H_1 \rangle = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \tag{4.3.3a}$$

$$\langle H_2 \rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} . \tag{4.3.3b}$$

Writing the doublet elements in terms of the physical states [6]

$$H_1^1 = v_1 + \frac{1}{\sqrt{2}} \left[H^0 \cos\alpha - h^0 \sin\alpha + i \left(A^0 \sin\beta - G^0 \cos\beta \right) \right] \tag{4.3.4a}$$

$$H_1^2 = H^- \sin\beta - G^- \cos\beta \tag{4.3.4b}$$

$$H_2^1 = H^+ \cos\beta + G^+ \sin\beta \tag{4.3.4c}$$

$$H_2^2 = v_2 + \frac{1}{\sqrt{2}} \left[H^0 \sin\alpha + h^0 \cos\alpha + i \left(A^0 \cos\beta + G^0 \sin\beta \right) \right] \tag{4.3.4d}$$

the potential (4.3.1) becomes

$$\begin{aligned} V_{MSSM} &= m_{H^\pm}^2 H^+ H^- + \frac{1}{2} m_A^2 A^{0^2} + \frac{1}{2} m_{H^0}^2 H^{0^2} + \frac{1}{2} m_{h^0}^2 h^{0^2} \\ &+ g V_{MSSM}^{(3)} + g^2 V_{MSSM}^{(4)} \end{aligned} \tag{4.3.5}$$

where

$$\begin{aligned} V_{MSSM}^{(3)} &= \lambda_{31} h^0 H^+ H^- + \lambda'_{31} H^0 H^+ H^- + \lambda_{32} h^{0^3} + \lambda'_{32} H^{0^3} \\ &- \lambda_{33} h^0 H^{0^2} + \lambda'_{33} H^0 h^{0^2} + \lambda_{34} H^0 A^{0^2} - \lambda'_{34} h^0 A^{0^2} \end{aligned} \tag{4.3.6a}$$

$$\begin{aligned} V_{MSSM}^{(4)} = \lambda_{41} & \left(H^{+2}H^{-2} + \frac{1}{2}A^{02}H^{+}H^{-} + \frac{3}{2}A^{04}\right) \\ & + \lambda_{42}H^{02}H^{+}H^{-} + \lambda_{43}h^{02}H^{+}H^{-} + \lambda_{44}H^{0}h^{0}H^{+}H^{-} \\ & + \lambda_{45}\left(H^{04} + h^{04}\right) + \lambda_{46}\left(H^{0}h^{03} - h^{0}H^{03}\right) \\ & + \lambda_{47}H^{02}h^{02} + \lambda_{48}\left(h^{02}A^{02} - H^{02}A^{02}\right) + \lambda_{49}H^{0}h^{0}A^{02}. \end{aligned} \tag{4.3.6b}$$

The various λ_i coefficients in (4.3.6) are complicated functions of the two mixing angles α and β, the masses of the W and Z gauge fields and the Weinberg angle. The complete listing of these couplings are tabled in Appendix C. The couplings involving the Goldstone fields are not considered because the Goldstone fields are absorbed as the longitudinal degrees of freedom of the W and Z gauge fields thus giving the gauge fields their masses.

Now that the full form of the potential is known, it can be minimized giving constraints between the parameters in (4.3.5), the vacuum expectation values and the physical Higgs fields masses. As a result the tree-level mass relations are obtained

$$m_{H^0}^2 = \frac{1}{2}\left\{m_{A^0}^2 + m_W^2 + \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2 \cos^2 2\beta}\right\} \tag{4.3.7a}$$

$$m_{h^0}^2 = \frac{1}{2}\left\{m_{A^0}^2 + m_W^2 - \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2 \cos^2 2\beta}\right\} \tag{4.3.7b}$$

$$m_{H^\pm}^2 = m_{A^0}^2 + m_W^2 \tag{4.3.7c}$$

as well as the relations between the mixing angles

$$\sin 2\alpha = -\sin 2\beta\left(\frac{m_{H^0}^2 + m_{h^0}^2}{m_{H^0}^2 - m_{h^0}^2}\right), \tag{4.3.8a}$$

$$\cos 2\alpha = -\cos 2\beta\left(\frac{m_{A^0}^2 + m_Z^2}{m_{H^0}^2 - m_{h^0}^2}\right), \tag{4.3.8b}$$

where in both (4.3.7) and (4.3.8) the angle β and $m_{A^0}^2$ have been chosen as free parameters.

Notice that in the limit $m_{A^0} \to \infty$ at fixed β the heavy Higgs fields $A^0, H^\pm, H^0$ all decouple ($m_{A^0} \approx m_{H^0} \approx m_{H^\pm}$) and all that is left is a Higgs sector consisting of one CP-even scalar, h^0. It will be shown in the next section that this is identical to the Higgs sector of the Standard Model. Further, the interactions of h^0 with both the gauge fields and fermions are the same as those of the Higgs field in the Standard Model [8]. Thus the Higgs sector of the Standard Model can be viewed as a large mass expansion limit of the Higgs sector of the Minimal Supersymmetric Model.

4.4 Calculation of the Effective Lagrangian at One Loop

The computation of the large mass expansion to one loop order using OR follows the same method as in Section 2. The general form for one loop Green's function after substituting in the DeWitt expansion is given by (3.2.1). Thus to proceed further the matrix $\hat{M}$, the mass matrix Ξ and the relevant DeWitt coefficients need to be specified.

The matrix $\hat{M}$ is generated exactly the same way as in Section 3. Defining the column matrix of fields

$$\Phi = \begin{bmatrix} H^+ \\ H^- \\ H^0 \\ h^0 \\ A^0 \end{bmatrix}. \tag{4.4.1}$$

and expanding in the background field formalism, the elements of $\hat{M}$ are found to be

$$\begin{aligned} M_{11} = M_{22} = p_{H^\pm}^2 + \frac{1}{2}m_{H^\pm}^2 H^+H^- + \lambda_{31}h^0 + \lambda'_{31}H^0 \\ + \lambda_{41}\left(\frac{1}{2}A^{0^2} + 4H^+H^-\right) + \lambda_{42}H^{0^2} + \lambda_{43}h^{0^2} + \lambda_{44}h^0H^0 \end{aligned} \tag{4.4.2a}$$

$$M_{12} = \lambda_{41}H^{-^2} \tag{4.4.2b}$$

$$M_{13} = M_{32} = \lambda_{31}H^- + 2\lambda_{42}H^0H^- + \lambda_{44}h^0H^- \tag{4.4.2c}$$

$$M_{14} = M_{42} = \lambda'_{31}H^- + 2\lambda_{43}H^0H^- + \lambda_{44}H^0H^- \tag{4.4.2d}$$

$$M_{15} = M_{51} = \lambda_{41}A^0H^- \tag{4.4.2e}$$

$$M_{21} = \lambda_{41}H^{+^2} \tag{4.4.2f}$$

$$M_{23} = M_{31} = \lambda_{31}H^+ + 2\lambda_{42}H^0H^+ + \lambda_{44}h^0H^+ \tag{4.4.2g}$$

$$M_{24} = M_{41} = \lambda'_{31}H^+ + 2\lambda_{43}h^0H^+ + \lambda_{44}H^0H^+ \tag{4.4.2h}$$

$$M_{25} = M_{52} = \lambda_{41}A^0H^+ \tag{4.4.2i}$$

$$\begin{aligned} M_{33} = p_{H^0}^2 + \frac{1}{2}m_{H^0}^2 H^{0^2} + \frac{1}{2}3\lambda'_{32}H^0 - \lambda_{33}h^0 \\ + \lambda_{42}H^+H^- + 6\lambda_{45}H^{0^2} - 3\lambda_{46}h^0H^0 + \lambda_{47}h^{0^2} - \lambda_{48}A^{0^2} \end{aligned} \tag{4.4.2j}$$

$$\begin{aligned} M_{34} = M_{43} = 2\lambda'_{33}h^0 - 2\lambda_{33}H^0 + \lambda_{44}H^+H^- \\ + 3\lambda_{46}\left(h^{0^2} - H^{0^2}\right) + 4\lambda_{47}h^0H^0 + \lambda_{49}A^{0^2} \end{aligned} \tag{4.4.2k}$$

$$M_{35} = M_{53} = -2\lambda'_{34}A^0 + 2\lambda_{49}h^0A^0 + -4\lambda_{48}H^0A^0 \tag{4.4.2l}$$

$$\begin{aligned} M_{44} = p_{h^0}^2 + \frac{1}{2}m_{h^0}^2 h^{0^2} + 3\lambda_{32}h^0 + \lambda'_{33}H^0 + \lambda_{43}H^+H^- + 3\lambda_{46}h^0H^0 \\ + \lambda_{47}H^{0^2} + \lambda_{48}A^{0^2} \end{aligned} \tag{4.4.2m}$$

$$M_{45} = M_{54} = 2\lambda_{34}A^0 + 2\lambda_{49}A^0h^0 + 4\lambda_{48}A^0H^0 \tag{4.4.2n}$$

$$\begin{aligned} M_{55} = {} & p_{A^0}^2 + \frac{1}{2} m_{A^0}^2 A^{0^2} + \lambda_{34} h^0 - \lambda'_{34} H^0 + \lambda_{41} \left(9A^{0^2} + \frac{1}{2} H^+ H^- \right) \\ & + \lambda_{49} h^0 H^0 + \lambda_{48} \left(h^{0^2} - H^{0^2} \right) \end{aligned} \tag{4.4.2o}$$

where the various p_k^2 are the momenta for the various "k" fields.

The relevant DeWitt coefficients for the large mass expansion to one loop are simply

$$\hat{a}_3 = -\frac{1}{6}\hat{B}^3 + \frac{1}{6}\hat{B}_{;\mu}\hat{B}_{;\mu} \tag{4.4.3a}$$

$$\hat{a}_4 = \frac{1}{24}\hat{B}^4 + \frac{1}{36}\hat{B}_{;\mu\mu}\hat{B}_{;\nu\nu} + \frac{1}{45}\hat{B}_{;\mu\nu}\hat{B}_{;\nu\mu}. \tag{4.4.3b}$$

Because interactions with the gauge fields are not considered, the matrix $\hat{M}$ (minus the factors of momentum and mass) is simply equal to $\hat{B}$. Further, in this case the covariant derivatives reduce to ordinary partial derivatives for the same reason.

The form of the mass matrix $\hat{\Xi}$ is straight forward

$$\hat{\Xi} = \mu^{-2} \begin{pmatrix} m_{H^\pm}^2 & 0 & 0 & 0 & 0 \\ 0 & m_{H^\pm}^2 & 0 & 0 & 0 \\ 0 & 0 & m_{H^0}^2 & 0 & 0 \\ 0 & 0 & 0 & m_{h^0}^2 & 0 \\ 0 & 0 & 0 & 0 & m_{A^0}^2 \end{pmatrix}. \tag{4.4.4}$$

In the large mass limit we have

$$\begin{gathered} m_{A^0}^2 \approx m_{H^0}^2 \approx m_{H^\pm}^2 \equiv M^2 \\ m_{h^0}^2 \equiv m^2 \\ m \ll M \end{gathered}$$

and $\hat{\Xi}$ reduces to a two parameter matrix

$$\hat{\Xi} = \mu^{-2} \begin{pmatrix} M^2 & 0 & 0 & 0 & 0 \\ 0 & M^2 & 0 & 0 & 0 \\ 0 & 0 & M^2 & 0 & 0 \\ 0 & 0 & 0 & m^2 & 0 \\ 0 & 0 & 0 & 0 & M^2 \end{pmatrix}. \tag{4.4.5}$$

In the previous section, it was shown that at one loop order the $\hat{a}_3$ DeWitt coefficient generates terms to $O(1/m_{heavy}^2)$. It is simple to show that the $\hat{a}_4$ coefficient will just add terms of $O(1/m_{heavy}^4)$ to the effective Lagrangian. From the explicit form of the potentials given in (4.3.6) it is obvious that the only vertices we have to create one loop processes are those involving three and four point interactions. Furthermore, from the form of the DeWitt coefficients in (4.4.3) dictates that in order to be complete we will need to calculate up to the eight point Green's function. If the light Higgs field h^0 is represented by a thin line and the set of heavy Higgs fields are represented by a thick line the various contributions to the three through eight point functions can be visualized (Fig. 5).

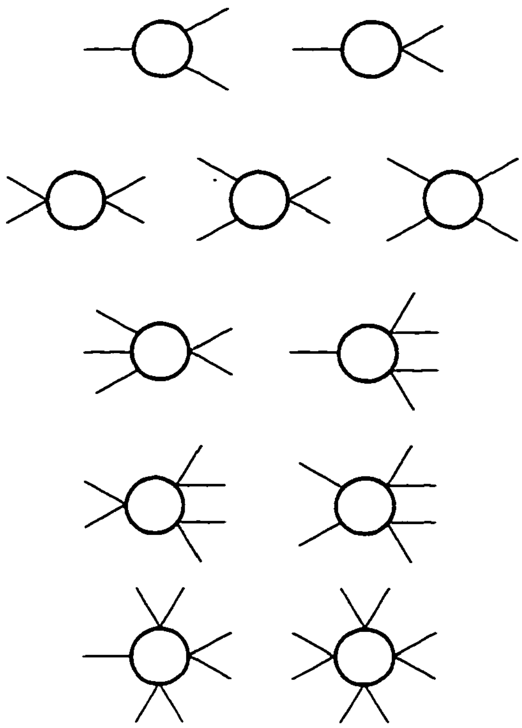

Figure 5: One Loop Contributions to the Three Through Eight Point Green's Functions

The new terms (up to the eight-point Green's function) due to the large mass expansion to $O(1/m_{heavy}^4)$ are calculated by substituting (4.4.2), (4.4.3) and either (4.4.4) or (4.4.5) into (3.2.1). The results of this calculation are presented below.

$$
\begin{aligned}
\Gamma^{(3)}_{LM} = & -\frac{g^3}{12\,(4\pi)^2} h^{0^3} \left[\frac{2}{m^2_{H^{\pm}}} \lambda^3_{31} + \frac{1}{m^2_{A^0}} \lambda^3_{34} + \frac{1}{m^2_{H^0}} \left(12 \lambda_{32} {\lambda_{33}}^2 - \lambda^3_{33} - 8 \lambda_{33} {\lambda'_{33}}^2 \right) \right] \\
& + \frac{g^3}{6\,(4\pi)^2} (h^{0^2})_{,\mu} h^0_{,\mu} \left[\frac{2}{m^2_{H^{\pm}}} \lambda_{31} \lambda_{43} + \frac{1}{m^2_{A^0}} \lambda_{34} \lambda_{48} \right. \\
& \qquad \left. + \frac{1}{m^2_{H^0}} \left(6 \lambda'_{33} \lambda_{46} - \lambda_{33} \lambda_{47} \right) \right] \qquad\qquad (4.4.6a) \\
& + \frac{g^3}{(4\pi)^2} \left\{ \frac{1}{36} (h^{0^2})_{,\mu\mu} h^0_{,\nu\nu} + \frac{1}{45} (h^{0^2})_{,\mu\nu} h^0_{,\mu\nu} \right\} \times \\
& \quad \times \left[\frac{2}{m^4_{H^{\pm}}} \lambda_{31} \lambda_{43} + \frac{1}{m^4_{A^0}} \lambda_{34} \lambda_{48} + \frac{1}{m^4_{H^0}} \left(6 \lambda'_{33} \lambda_{46} - \lambda_{33} \lambda_{47} \right) \right] \\
\Gamma^{(4)}_{LM} = & \frac{g^4}{(4\pi)^2} h^{0^4} \left[-\frac{1}{12} \left\{ \frac{6}{m^2_{H^{\pm}}} \lambda^2_{31} \lambda_{43} + \frac{3}{m^2_{A^0}} \lambda^2_{34} \lambda_{48} \right. \right. \\
& \qquad \left. + \frac{1}{m^2_{H^0}} \left(3 \lambda^2_{33} \lambda_{47} + 36 \lambda_{32} \lambda'_{33} \lambda_{46} + 8 {\lambda'_{33}}^2 \lambda_{43} - 24 \lambda_{33} \lambda'_{33} \lambda_{46} \right) \right\} \\
& + \frac{1}{48} \left\{ \frac{2}{m^4_{H^{\pm}}} \lambda^4_{31} + \frac{1}{m^4_{A^0}} \lambda^4_{34} \right. \\
& \quad + \frac{1}{m^4_{H^0}} \left(\lambda^4_{33} + 36 \lambda^2_{32} {\lambda'_{33}}^2 - 24 \lambda_{32} \lambda_{33} {\lambda'_{33}}^2 + 12 \lambda^2_{33} {\lambda'_{33}}^2 \right. \\
& \qquad \left. \left. \left. + 16 {\lambda'_{33}}^4 \right) \right\} \right] \qquad\qquad (4.4.6b) \\
& + \frac{g^4}{12\,(4\pi)^2} (h^{0^2})_{,\mu} (h^{0^2})_{,\mu} \times \\
& \quad \times \left[\frac{2}{m^2_{H^{\pm}}} \lambda^2_{43} + \frac{1}{m^2_{A^0}} \lambda^2_{48} + \frac{1}{m^2_{H^0}} \left(\lambda^2_{47} + 9 \lambda^2_{46} \right) \right] \\
& + \frac{g^4}{(4\pi)^2} \left[\frac{1}{36} (h^{0^2})_{,\mu\mu} (h^{0^2})_{,\nu\nu} + \frac{1}{45} (h^{0^2})_{,\mu\nu} (h^{0^2})_{,\mu\nu} \right] \times \\
& \quad \times \left[\frac{2}{m^4_{H^{\pm}}} \lambda^2_{43} + \frac{1}{m^4_{A^0}} \lambda^2_{48} + \frac{1}{m^2_{H^0}} \left(\lambda^2_{47} + 9 \lambda^2_{46} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{(5)}_{LM} = \frac{g^5}{(4\pi)^2} h^{05} \Bigg[& -\frac{1}{12}\Bigg\{ \frac{6}{m^2_{H^\pm}} \lambda_{31}\lambda_{43}^2 + \frac{3}{m^2_{A^0}} \lambda_{34}\lambda_{48}^2 \\
& + \frac{1}{m^2_{H^0}} \left(27\lambda_{32}\lambda_{46}^2 - 18\lambda_{33}\lambda_{46}^2 - 3\lambda_{33}\lambda_{47}^2 + 72\lambda_{45}\lambda_{46}\lambda'_{33}\right. \\
& \left. +24\lambda'_{33}\lambda_{46}\lambda_{47}\right)\Bigg\} \\
& + \frac{1}{48}\Bigg\{ \frac{1}{m^4_{H^0}} \left(-4\lambda_{33}^2\lambda_{47} + 108\lambda_{32}\lambda'_{33}\lambda_{46} - 72\lambda_{32}\lambda_{33}\lambda'_{33}\lambda_{46} + 36\lambda_{33}^2\lambda'_{33}\lambda_{46}\right. \\
& + 144\lambda_{32}{\lambda'_{33}}^2\lambda_{45} - 48\lambda_{33}{\lambda'_{33}}^2\lambda_{45} + 24\lambda_{32}{\lambda'_{33}}^2\lambda_{47} \\
& \left. -24\lambda_{33}{\lambda'_{33}}^2\lambda_{47} + 96{\lambda'_{33}}^3\lambda_{46}\right) \\
& + \frac{8}{m^4_{H^\pm}} \lambda_{31}^3\lambda_{43} \Bigg\}\Bigg]
\end{aligned}
\tag{4.4.6c}
$$

$$
\begin{aligned}
\Gamma^{(6)}_{LM} = \frac{g^6}{(4\pi)^2} h^{06} \Bigg[& -\frac{1}{12}\Bigg\{ \frac{2}{m^2_{H^\pm}} \lambda_{43}^3 + \frac{1}{m^2_{A^0}} \lambda_{48}^3 + \frac{1}{m^2_{H^0}} \left(18\lambda_{47}\lambda_{46}^2 + 54\lambda_{45} + \lambda_{47}^3\right)\Bigg\} \\
& + \frac{1}{48}\Bigg\{ \frac{6}{m^4_{A^0}} \lambda_{34}^2\lambda_{48}^2 + \frac{12}{m^4_{H^\pm}} \lambda_{31}^2\lambda_{43}^2 \\
& + \frac{1}{m^4_{H^0}} \left(81\lambda_{32}^2\lambda_{46}^2 - 54\lambda_{32}\lambda_{33}\lambda_{46}^2 + 27\lambda_{33}^2\lambda_{46}^2 + 6\lambda_{33}^2\lambda_{47}^2\right. \\
& + 432\lambda_{32}\lambda'_{33}\lambda_{45}\lambda_{46} - 144\lambda_{33}\lambda'_{33}\lambda_{45}\lambda_{46} \\
& + 72\lambda_{32}\lambda'_{33}\lambda_{46}\lambda_{47} - 72\lambda_{33}\lambda'_{33}\lambda_{46}\lambda_{47} + 144{\lambda'_{33}}^2\lambda_{45}^2 \\
& \left. +216{\lambda'_{33}}^2\lambda_{46}^2 + 48{\lambda'_{33}}^2\lambda_{45}\lambda_{47} + 12{\lambda'_{33}}^2\lambda_{47}^2\right)\Bigg\}\Bigg]
\end{aligned}
\tag{4.4.6d}
$$

$$
\begin{aligned}
\Gamma^{(7)}_{LM} = \frac{g^7}{48\,(4\pi)^2} h^{07} \Bigg[& \frac{4}{m^4_{A^0}} \lambda_{34}\lambda_{48}^3 + \frac{8}{m^4_{H^\pm}} \lambda_{31}\lambda_{43}^3 \\
& + \frac{1}{m^4_{H^0}} \left(324\lambda_{32}\lambda_{45}\lambda_{46}^2 - 108\lambda_{33}\lambda_{45}\lambda_{46}^2 + 54\lambda_{32}\lambda_{46}^2\lambda_{47}\right. \\
& - 54\lambda_{33}\lambda_{46}^2\lambda_{47} - 4\lambda_{33}\lambda_{47}^3 + 432\lambda'_{33}\lambda_{45}^2\lambda_{46} \\
& \left. +216\lambda'_{33}\lambda_{46}^3 + 144\lambda'_{33}\lambda_{45}\lambda_{46}\lambda_{47} + 36\lambda'_{33}\lambda_{46}\lambda_{47}^2\right)\Bigg]
\end{aligned}
\tag{4.4.6e}
$$

$$
\begin{aligned}
\Gamma^{(8)}_{LM} = \frac{g^8}{48\,(4\pi)^2} h^{08} \Bigg[& \frac{1}{m^4_{A^0}} \lambda_{48}^4 + \frac{2}{m^4_{H^\pm}} \lambda_{43}^4 \\
& + \frac{1}{m^4_{H^0}} \left(324\lambda_{45}^2\lambda_{46}^2 + 81\lambda_{46}^4 + 108\lambda_{45}\lambda_{46}^2\lambda_{47} + 27\lambda_{46}^2\lambda_{47}^2 + \lambda_{47}^4\right)\Bigg]
\end{aligned}
\tag{4.4.6f}
$$

In the large mass limit these become

$$\begin{aligned}
\Gamma^{(3)}_{LM} = &-\frac{g^3}{12M^2(4\pi)^2}h^{0^3}\left[2\lambda_{31}^3+\lambda_{34}^3+12\lambda_{32}\lambda_{33}^2-\lambda_{33}^3-8\lambda_{33}{\lambda'_{33}}^2\right] \\
&+\frac{g^3}{12M^2(4\pi)^2}(h^{0^2})_{,\mu}h^0_{,\mu}\left[2\lambda_{31}\lambda_{43}+\lambda_{34}\lambda_{48}+6\lambda'_{33}\lambda_{46}-\lambda_{33}\lambda_{47}\right] \\
&+\frac{g^3}{M^4(4\pi)^2}\left\{\frac{1}{36}(h^{0^2})_{,\mu\mu}h^0_{,\nu\nu}+\frac{1}{45}(h^{0^2})_{,\mu\nu}h^0_{,\mu\nu}\right\}\times \\
&\times\left[2\lambda_{31}\lambda_{43}+\lambda_{34}\lambda_{48}+6\lambda'_{33}\lambda_{46}-\lambda_{33}\lambda_{47}\right]
\end{aligned} \tag{4.4.7a}$$

$$\begin{aligned}
\Gamma^{(4)}_{LM} = &\frac{g^4}{(4\pi)^2}h^{0^4}\left[-\frac{1}{12M^2}\left\{6\lambda_{31}^2\lambda_{43}+3\lambda_{34}^2\lambda_{48}+3\lambda_{33}^2\lambda_{47}+36\lambda_{32}\lambda'_{33}\lambda_{46}\right.\right. \\
&\left.+8{\lambda'_{33}}^2\lambda_{43}-24\lambda_{33}\lambda'_{33}\lambda_{46}\right\} \\
&+\frac{1}{48M^4}\left\{2\lambda_{31}^4+\lambda_{34}^4+\lambda_{33}^4+36\lambda_{32}^2{\lambda'_{33}}^2-24\lambda_{32}\lambda_{33}{\lambda'_{33}}^2\right. \\
&\left.\left.+12\lambda_{33}^2{\lambda'_{33}}^2+16{\lambda'_{33}}^4\right\}\right] \\
&+\frac{g^4}{12M^2(4\pi)^2}(h^{0^2})_{,\mu}(h^{0^2})_{,\mu}\left[2\lambda_{43}^2+\lambda_{48}^2+\lambda_{47}^2+9\lambda_{46}^2\right] \\
&+\frac{g^4}{M^4(4\pi)^2}\left[\frac{1}{36}(h^{0^2})_{,\mu\mu}(h^{0^2})_{,\nu\nu}+\frac{1}{45}(h^{0^2})_{,\mu\nu}(h^{0^2})_{,\mu\nu}\right]\times \\
&\times\left[2\lambda_{43}^2+\lambda_{48}^2+\lambda_{47}^2+9\lambda_{46}^2\right]
\end{aligned} \tag{4.4.7b}$$

$$\begin{aligned}
\Gamma^{(5)}_{LM} = &\frac{g^5}{(4\pi)^2}h^{0^5}\left[-\frac{1}{12M^2}\left\{6\lambda_{31}\lambda_{43}^2+3\lambda_{34}\lambda_{48}^2+27\lambda_{32}\lambda_{46}^2-18\lambda_{33}\lambda_{46}^2-\right.\right. \\
&\left.3\lambda_{33}\lambda_{47}^2+72\lambda_{45}\lambda_{46}\lambda'_{33}+24\lambda'_{33}\lambda_{46}\lambda_{47}\right\} \\
&+\frac{1}{48M^4}\left\{-4\lambda_{33}^2\lambda_{47}+108\lambda_{32}\lambda'_{33}\lambda_{46}-72\lambda_{32}\lambda_{33}\lambda'_{33}\lambda_{46}+36\lambda_{33}^2\lambda'_{33}\lambda_{46}\right. \\
&+144\lambda_{32}{\lambda'_{33}}^2\lambda_{45}-48\lambda_{33}{\lambda'_{33}}^2\lambda_{45}+24\lambda_{32}{\lambda'_{33}}^2\lambda_{47} \\
&\left.\left.-24\lambda_{33}{\lambda'_{33}}^2\lambda_{47}+96{\lambda'_{33}}^3\lambda_{46}+8\lambda_{31}^3\lambda_{43}\right\}\right]
\end{aligned} \tag{4.4.7c}$$

$$\begin{aligned}
\Gamma^{(6)}_{LM} = \frac{g^6}{(4\pi)^2} h^{0^6} \Bigg[& -\frac{1}{12M^2}\left\{2\lambda_{43}^3 + \lambda_{48}^3 18\lambda_{47}\lambda_{46}^2 + 54\lambda_{45} + \lambda_{47}^3\right\} \\
& + \frac{1}{48M^4}\Big\{6\lambda_{34}^2\lambda_{48}^2 + 12\lambda_{31}^2\lambda_{43}^2 + 81\lambda_{32}^2\lambda_{46}^2 \\
& - 54\lambda_{32}\lambda_{33}\lambda_{46}^2 + 27\lambda_{33}^2\lambda_{46}^2 + 6\lambda_{33}^2\lambda_{47}^2 \\
& + 432\lambda_{32}\lambda_{33}'\lambda_{45}\lambda_{46} - 144\lambda_{33}\lambda_{33}'\lambda_{45}\lambda_{46} \\
& + 72\lambda_{32}\lambda_{33}'\lambda_{46}\lambda_{47} - 72\lambda_{33}\lambda_{33}'\lambda_{46}\lambda_{47} + 144\lambda_{33}'^2\lambda_{45}^2 \\
& + 216\lambda_{33}'^2\lambda_{46}^2 + 48\lambda_{33}'^2\lambda_{45}\lambda_{47} + 12\lambda_{33}'^2\lambda_{47}^2\Big\}\Bigg]
\end{aligned} \tag{4.4.7d}$$

$$\begin{aligned}
\Gamma^{(7)}_{LM} = \frac{g^7}{48M^4(4\pi)^2} h^{0^7} \Big[& 4\lambda_{34}\lambda_{48}^3 + 8\lambda_{31}\lambda_{43}^3 + 324\lambda_{32}\lambda_{45}\lambda_{46}^2 - 108\lambda_{33}\lambda_{45}\lambda_{46}^2 \\
& + 54\lambda_{32}\lambda_{46}^2\lambda_{47} - 54\lambda_{33}\lambda_{46}^2\lambda_{47} - 4\lambda_{33}\lambda_{47}^3 + 432\lambda_{33}'\lambda_{45}^2\lambda_{46} \\
& + 216\lambda_{33}'\lambda_{46}^3 + 144\lambda_{33}'\lambda_{45}\lambda_{46}\lambda_{47} + 36\lambda_{33}'\lambda_{46}\lambda_{47}^2\Big]
\end{aligned} \tag{4.4.7e}$$

$$\begin{aligned}
\Gamma^{(8)}_{LM} = \frac{g^8}{48M^4(4\pi)^2} h^{0^8} \Big[& \lambda_{48}^4 + 2\lambda_{43}^4 + 324\lambda_{45}^2\lambda_{46}^2 + 81\lambda_{46}^4 + 108\lambda_{45}\lambda_{46}^2\lambda_{47} \\
& + 27\lambda_{46}^2\lambda_{47}^2 + \lambda_{47}^4\Big]
\end{aligned} \tag{4.4.7f}$$

Adding the expressions (4.4.7) together the effective Lagrangian for the Higgs sector of MSSM to one loop order has the following structure

$$\begin{aligned}
\mathcal{L}^{MSSM}_{eff} = & -\frac{1}{M^2}\Lambda^{(3)}_{eff} h^{0^3} + \left(-\frac{1}{M^2}\Lambda^{(4.1)}_{eff} + \frac{1}{M^4}\Lambda^{(4.2)}_{eff}\right) h^{0^4} \\
& + \left(-\frac{1}{M^2}\Lambda^{(5.1)}_{eff} + \frac{1}{M^4}\Lambda^{(5.2)}_{eff}\right) h^{0^5} + \left(-\frac{1}{M^2}\Lambda^{(6.1)}_{eff} + \frac{1}{M^4}\Lambda^{(6.2)}_{eff}\right) h^{0^6} \\
& + \frac{1}{M^4}\Lambda^{(7)}_{eff} h^{0^7} + \frac{1}{M^4}\Lambda^{(8)}_{eff} h^{0^8} \\
& + \frac{1}{M^2}\left[\Omega^{(3.1)}_{eff} h^0 (h^0)_{,\mu}(h^0)_{,\mu} + \Omega^{(4.1)}_{eff} h^{0^2}(h^0)_{,\mu}(h^0)_{,\mu}\right] \\
& + \frac{1}{M^4}\Big[\Omega^{(3.2)}_{eff}(h^{0^2})_{,\mu\nu} h^0_{,\mu\nu} + \Omega^{(3.3)}_{eff}(h^{0^2})_{,\mu\mu} h^0_{,\nu\nu} \\
& + \Omega^{(4.2)}_{eff}(h^{0^2})_{,\mu\nu}(h^{0^2})_{,\mu\nu} + \Omega^{(4.3)}_{eff}(h^{0^2})_{,\mu\mu}(h^{0^2})_{,\nu\nu}\Big] \\
& + O(1/M^6).
\end{aligned} \tag{4.4.8}$$

The expression of the Λ_{eff} and Ω_{eff} coefficients in terms of the initial λ_i coefficients are given in equations (4.4.7a) – (4.4.7f).

Notice that in the decoupling limit ($M \to \infty$) equation (4.4.8) vanishes. Therefore, in the decoupling limit the Higgs sector of the Minimal Supersymmetric Model reduces to

$$\mathcal{L} = \frac{1}{2}\left(p^2 - m_{h^0}^2\right) h^{0^2} - g\lambda_{32} h^{0^3} - g^2\lambda_{45} h^{0^4}. \tag{4.4.9}$$

By its structure one would expect that equation (4.4.9) is the Lagrangian of the Higgs sector of the Standard Model of electroweak interactions. This can be explicitly shown by making the correlations [10].

$$g\lambda_{32} = 3m_{h^0}^2 \left(G\sqrt{2}\right)^{1/2} \tag{4.4.10a}$$

$$g^2\lambda_{45} = 3m_{h^0}^2 G\sqrt{2} \tag{4.4.10b}$$

where G is the Fermi constant. Therefore, the effective Lagrangian in (4.4.8) in fact generates the explicit corrections to the Standard Model Higgs sector couplings.

$$\begin{aligned} V_{SM} &= h^{0^3}\left[3m_{h^0}^2\left(G\sqrt{2}\right)^{1/2} - \frac{1}{M^2}\Lambda_{eff}^{(3)} + O(1/M^6)\right] \\ &+ h^{0^4}\left[3m_{h^0}^2 G\sqrt{2} - \frac{1}{M^2}\Lambda_{eff}^{(4.1)} + \frac{1}{M^4}\Lambda_{eff}^{(4.2)} + O(1/M^6)\right]. \end{aligned} \tag{4.4.11}$$

As stated earlier, the only independent and unknown quantity in the Higgs couplings in MSSM is the mixing angle β (the other mixing angle α can be related to β via. equation (4.3.8)). Therefore, any measurements that show deviations from the Standard Model Higgs couplings give a way to assign a value to β. This, in turn, is important because the phenomenology of the Higgs sector is very sensitive to β [6].

Chapter 5

Conclusion

Operator Regularization has been shown to be an economical way to calculate n-point Greens functions and effective Lagrangians via. the examples of the two field QSD and the Minimal Supersymmetric Model. Due to the nature of Operator Regularization no divergent expressions are encountered at any stage in both applications. Nevertheless, an induced mass scale does arise. This scale and the DeWitt expansion are used to determine the renormalization group functions for the calculated renormalized Green's functions. This has been explicitly demonstrated in the example of two field QSD.

Using Operator Regularization to calculate effective field theories has been shown to be much easier than traditional methods of calculating effective field theories. By simply using "higher-order" DeWitt coefficients an effective field theory at one loop is easily generated. At higher loop order the calculations is more involved, but due to the nature regularizing and renormalizing in Operator Regularization the explicit calculation is much simpler than traditional approaches.

Applying this technique to the Higgs sector of the Minimal Supersymmetric model an effective field theory with only one Higgs field has been derived. This effective field theory has been confirmed to reduce to the Higgs sector of the Standard Model in the decoupling limit and generate supersymmetric corrections to the Standard Model couplings in this limit.

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Appendix A

The DeWitt Expansion

The DeWitt expansion of the matrix element

$$M = \langle x | e^{-\left[\frac{1}{2}(p-A)^2+V\right]t} | y \rangle \tag{A.1}$$

in powers of t has been calculated in a number of ways. In the case $x = y$ many approaches have been developed to calculate the expansion to high orders [20, 47–54]. For the off-diagonal case, $x \neq y$, there are a handful of methods that have been developed to calculate the expansion coefficients [54, 55]. The calculation which follows below is the general method developed by F.A. Dikes and D.G.C McKeon [55] using the quantum mechanical path integral and the Fock-Schwinger gauge.

In four dimensions the DeWitt expansion is

$$\begin{aligned} M &= \langle x | e^{-\left[\frac{1}{2}(p-A)^2+V\right]t} | y \rangle \\ &= \frac{e^{-\Delta^2/2t}}{(2\pi t)^2} \sum_{k=0}^{\infty} a_k\,(x_0, \Delta)\, t^n \end{aligned} \tag{A.2}$$

where

$$\Delta = x - y \tag{A.3}$$

$$x_0 = (x + y)/2. \tag{A.4}$$

The matrix element in (A.2) can be represented by a normalized quantum mechanical path integral

$$M = \int_x^y \mathcal{D}q(\tau)\mathcal{P} \exp \int_0^t d\tau \left[-\frac{\dot{q}^2(\tau)}{2} + i\,\dot{q}(\tau) \cdot A(q(\tau)) - V(q(\tau)) \right] \tag{A.5}$$

where the path ordered integration is implied over trajectories with endpoints x and y.

After defining a relative coordinate $\delta(\tau)$ about the midpoint x_0

$$q(\tau) = x_0 + \delta(\tau) \tag{A.6}$$

and imposing the Fock-Schwinger gauge condition [56–59]

$$\delta(\tau) \cdot A(x_0 + \delta(\tau)) = 0 \tag{A.7}$$

the gauge field and the potential can be expanded in powers of δ

$$A_\mu(x_0 + \delta(\tau)) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!} [\delta(\tau) \cdot D(x_0)]^n \, \delta_\sigma(\tau) F_{\sigma\mu}(x_0) \tag{A.8a}$$

$$V(x_0 + \delta(\tau)) = \sum_{n=0}^{\infty} \frac{1}{n!} [\delta(\tau) \cdot D(x_0)]^n \, V(x_0). \tag{A.8b}$$

where the covariant derivative at x_0 is denoted by $D(x_0)$.

Using (A.8a) and (A.8b), (A.5) is rewritten as

$$M = \int_{-\Delta/2}^{\Delta/2} \mathcal{D}\delta(\tau) \exp\left[-\int_0^t d\tau \frac{\dot\delta^2(\tau)}{2}\right] \sum_{k=0}^{\infty} \frac{1}{k!} \times$$
$$\times \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t d\tau \, [\delta(\tau) \cdot D(x_0)]^n \left[\frac{i}{n+2} \dot\delta_\mu(\tau) \delta_\nu(\tau) F_{\nu\mu}(x_0) - V(x_0)\right]\right)^k . \tag{A.9}$$

The functional integral in (A.9) can be evaluated by repeated functional differentiation of the standard integral [29, 60]

$$\int_{-\Delta/2}^{\Delta/2} \mathcal{D}\delta(\tau) \exp \int_0^t d\tau \left[-\frac{\dot\delta^2(\tau)}{2} + \omega(\tau) \cdot \delta(\tau)\right]$$
$$= \frac{e^{-\Delta^2/2t}}{(2\pi t)^2} \exp\left[\int_0^t d\tau \left(-\frac{1}{2} + \frac{\tau}{t}\right) - \frac{1}{2} \int_0^t d\tau d\tau' \, G(\tau, \tau') \omega(\tau) \cdot \omega(\tau')\right] \tag{A.10}$$

with respect to the source function $\omega(\tau)$ and then taking the limit $\omega(\tau) \to 0$. The Green's function in equation (A.10) and its derivatives with respect to τ are given by [55]

$$G(\tau, \tau') = \frac{1}{2} |\tau - \tau'| - \frac{1}{2}(\tau + \tau') + \frac{\tau\tau'}{t} \tag{A.11a}$$

$$G(\dot\tau, \tau') = \frac{1}{2}\mathrm{sgn}(\tau - \tau') - \frac{1}{2} + \frac{\tau'}{t} \tag{A.11b}$$

$$G(\dot\tau, \dot\tau') = \frac{1}{t} - \delta(\tau - \tau') - \delta(2t - \tau - \tau') - \delta(\tau + \tau'). \tag{A.11c}$$

Denoting

$$X(\tau) = -\frac{1}{2} + \frac{\tau}{t} \tag{A.12}$$

the following identities can be easily derived

$$G(\tau,\dot{\tau}')G(\tau',\dot{\tau}) = \frac{1}{t}G(\tau,\tau') \tag{A.13a}$$

$$G(\tau,\dot{\tau}) = X(\tau). \tag{A.13b}$$

As an simple example of this procedure consider $k = 1, n = 1$

$$\begin{aligned}
M_{xy}^{(1,1)} &= \int_{-\Delta/2}^{\Delta/2} \mathcal{D}\delta(\tau)\, exp\left[-\int_0^t d\tau \frac{\dot{\delta}^2(\tau)}{2}\right] \times \\
&\quad \times \int_0^t d\tau \left[\frac{i}{3}\delta_\alpha \delta_\beta \dot{\delta}_\gamma\, D_\alpha F_{\beta\gamma} - \delta_\alpha\, D_\alpha V\right] \\
&= \frac{e^{-\Delta^2/2t}}{(2\pi t)^2}\int_0^t d\tau \left[-\frac{1}{3}\Delta_\alpha\, D_\mu F_{\alpha\mu}\left\{X^2(\tau) - G(\tau,\tau)\frac{1}{t}\right\}\right. \\
&\quad \left. -\Delta_\alpha\, D_\alpha V\, X(\tau)\right] \\
&= -\frac{e^{-\Delta^2/2t}}{(2\pi t)^2}\frac{i}{12}t\,\Delta_\alpha F_{\alpha\mu;\mu}
\end{aligned} \tag{A.14}$$

where the semi-colon denotes gauge-covariant differentiation.

The DeWitt expansion used in the previous sections is of a slightly different form than above

$$\langle x|e^{\left\{-(p+A)^2+\dot{B}\right\}t}|y\rangle = \frac{e^{-\Delta^2/4t}}{(4\pi t)^2}\sum_{k=0}^{\infty} a_k(x,y;\phi_i)t^k. \tag{2.2.34}$$

However, (A.1) and the DeWitt coefficients calculated from it, can be transformed into the desired form by the simple transformation

$$t \to 2t,\ V \to \frac{1}{2}\dot{B}. \tag{A.15}$$

For example, under (A.15) the matrix element (A.14) transforms to

$$M_{xy}^{(1,1)} = -\frac{e^{-\Delta^2/4t}}{(4\pi t)^2}\frac{i}{6}t\,\Delta_\alpha F_{\alpha\mu;\mu}. \tag{A.16}$$

Appendix B

Calculation of Setting-Sun Integrals

Define the general setting sun type integral by

$$I(a,b,c;\alpha,\beta,\gamma;\sigma;m) = \int_0^\infty dt_1 dt_2 dt_3\, t_1^a t_2^b t_3^c\, T^{-\sigma} e^{-m^2(\alpha t_1+\beta t_2+\gamma t_3)} \tag{B.1}$$

where

$$T = t_1 t_2 + t_1 t_3 + t_2 t_3.$$

Inserting

$$1 = \int_0^\infty \frac{d\kappa}{\kappa}\delta\left(1 - \frac{t_1+t_2+t_3}{\kappa}\right) \tag{B.2}$$

into (B.1) and shifting the variables

$$t_i = \kappa\tau_i \tag{B.3}$$

gives

$$\begin{aligned} I(a,b,c;\alpha,\beta,\gamma;\sigma;m) = \int_0^\infty d\kappa\, \kappa^{2+a+b+c-2\sigma} \int_0^1 d\tau_1 d\tau_2 d\tau_3\, \delta(1-\tau_1-\tau_2-\tau_3) \\ \times\, \tau_1^a \tau_2^b \tau_3^c\,(\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3)^{-\sigma}\, e^{-\kappa^2 m^2(\alpha\tau_1+\beta\tau_2+\gamma\tau_3)}. \end{aligned} \tag{B.4}$$

Changing variables

$$\tau_1 = (1-x) \tag{B.5a}$$

$$\tau_2 = x(1-y) \tag{B.5b}$$

$$\tau_3 = xy \tag{B.5c}$$

and integrating over κ results in

$$\begin{aligned}
I = {} & \Gamma(3+a+b+c-2\sigma)\left(m^2\right)^{-3-a-b-c-2\sigma} \\
& \int_0^1 dx\, x \int_0^1 dy\, [\alpha(1-x)+\beta x(1-y)+\gamma xy]^{-3-a-b-c-2\sigma} \\
& \times (1-x)^a [x(1-y)]^b (xy)^c [x(1-x[1-y(1-y)])]^{-\sigma}.
\end{aligned} \tag{B.6}$$

There are two different cases to consider. Case I is where $\alpha = \beta = \gamma = 1$, that is the setting sun diagram will all three fields of equal mass. The second case, Case II, is given by $\alpha = \beta \wedge \alpha = \gamma \wedge \beta = \gamma$ and corresponds to the setting sun diagram where two of the fields are of the same mass but a different mass than the remaining field.

Case I has already been solved by Culumovic, Leblanc, Mann, McKeon and Sherry [18]. Their result is repeated below

$$\begin{aligned}
I(a,b,c;1,1,1;\sigma;m) = {} & \left(m^2\right)^{-3-a-b-c+2\sigma} \sum_{n=0}^{\infty} \frac{\Gamma[1+a-\sigma]\Gamma[\sigma-a]}{\Gamma[\sigma]}\frac{1}{n!} \\
& \times \left[\frac{\Gamma[\sigma+n]\Gamma[2+b+c-\sigma+n]}{\Gamma[\sigma-a+n]}\frac{\Gamma[1+c+n]\Gamma[1+b+n]}{\Gamma[2+b+c+2n]} \right. \\
& - \frac{\Gamma[3+a+b+c-2\sigma+n]\Gamma[1+a+n]}{\Gamma[2+a-\sigma+n]} \times \\
& \left. \times \frac{\Gamma[2+a+b-\sigma+n]\Gamma[2+a+c-\sigma+n]}{\Gamma[4+2a+b+c-2\sigma+2n]} \right].
\end{aligned} \tag{B.7}$$

At first glance, Case II looks like three sub-cases. However, because the choice of parameterization is arbitrary, Case II is actually one case. For simplicity choose

$$\begin{aligned}
\beta &= \gamma = 1 \\
\alpha &= \rho \equiv m_i^2/m^2 \\
m_i^2 &\neq m^2
\end{aligned}$$

Thus,

$$\begin{aligned}
I(a,b,c;\rho,1,1;\sigma;m) = {} & \Gamma[\xi]\left(m^2\right)^{-\xi} \int_0^1 dx\, x \int_0^1 dy \\
& [x+\rho(1-x)]^{-\xi}(1-x)^a(x(1-y))^b(xy)^c \\
& \times [x(1-xz)]^{-\sigma}
\end{aligned} \tag{B.8}$$

where

$$\xi \equiv 3+a+b+c-2\sigma \tag{B.9}$$

$$z \equiv 1 - y(1-y). \tag{B.10}$$

The integrals over x and y can be calculated by first applying the binomial expansion

$$[x + \rho(1-x)]^{-n} = \sum_{k=0}^{\infty} \frac{\Gamma[1-n]}{\Gamma[k+1]\Gamma[1-k-n]} x^{-n-k} \rho^k (1-x)^k \tag{B.11}$$

to (B.8). This gives

$$\begin{aligned} I = & \sum_{k=0}^{\infty} \left(m^2\right)^{-\xi} \frac{\Gamma[\xi]\Gamma[1-\xi]}{\Gamma[1+k]\Gamma[1-k+\xi]} \\ & \times \int_0^1 dx\, dy\; \rho^k x^{-2-a-k+\sigma} (1-x)^{a+k} y^c (1-y)^b (1-xz)^{-\sigma}. \end{aligned} \tag{B.12}$$

The integral over x is easily done using the hypergeometric function $F[a, b; c; z]$ [61]

$$\begin{aligned} I = & \sum_{k=0}^{\infty} \left(m^2\right)^{-\xi} \frac{\Gamma[\xi]\Gamma[1-\xi]}{\Gamma[1+k]\Gamma[1-k+\xi]} \frac{\Gamma[\sigma-1-a-k]\Gamma[1+a+k]}{\Gamma[\sigma]} \rho^k \\ & \times \int_0^1 dy\; y^c (1-y)^b F[\sigma, \sigma-1-a-k; \sigma; 1-z]. \end{aligned} \tag{B.13}$$

Transforming the hypergeometric function using

$$F[\beta, -n; \beta; -z] = (1+z)^n \tag{B.14}$$

and integrating over y gives the final result for Case II

$$\begin{aligned} I(a, b, c; \rho, 1, 1; \sigma; m) = & \sum_{k=0}^{\infty} \left(m^2\right)^{-\xi} \rho^k \frac{\Gamma[\xi]\Gamma[1-\xi]}{\Gamma[1+k]\Gamma[1-k+\xi]} \\ & \times \frac{\Gamma[\sigma-1-a-k]\Gamma[1+a+k]}{\Gamma[\sigma]} \\ & \times \frac{\Gamma[2+a+c-\sigma+k]\Gamma[2+a+b-\sigma+k]}{\Gamma[4+2a+b+c-2\sigma+2k]}. \end{aligned} \tag{B.15}$$

Appendix C

Higgs-Higgs Couplings in MSSM

All the following is a listing of the Higgs-Higgs couplings used in Section 4. For a complete listing of all the couplings in the Minimal Supersymmetric Model see Ref. [6].

The following definitions have been utilized

$$\delta \equiv \beta - \alpha$$
$$\delta' \equiv \beta + \alpha$$

where α and β are the mixing angles from Section 4.

Three Point Couplings

$$\begin{aligned}
\lambda_{31} &= m_W \cos\delta - \frac{m_Z}{2\cos\theta_W} \cos 2\beta \cos\delta' \\
\lambda'_{31} &= m_W \sin\delta + \frac{m_Z}{2\cos\theta_W} \cos 2\beta \sin\delta' \\
\lambda_{32} &= \frac{3}{2} \frac{m_Z}{\cos\theta_W} \cos 2\alpha \sin\delta' \\
\lambda'_{32} &= \frac{3}{2} \frac{m_Z}{\cos\theta_W} \cos 2\alpha \cos\delta' \\
\lambda_{33} &= \frac{1}{2} \frac{m_Z}{\cos\theta_W} (2\sin 2\alpha \cos\delta' + \cos 2\alpha \sin\delta') \\
\lambda'_{33} &= \frac{1}{2} \frac{m_Z}{\cos\theta_W} (2\sin 2\alpha \sin\delta' - \cos 2\alpha \cos\delta') \\
\lambda_{34} &= \frac{m_Z}{2\cos\theta_W} \cos 2\beta \sin\delta' \\
\lambda'_{34} &= \frac{m_Z}{2\cos\theta_W} \cos 2\beta \cos\delta'
\end{aligned}$$

Four Point Couplings

$$
\begin{aligned}
\lambda_{41} &= \frac{1}{2}\frac{\cos^2 2\beta}{\cos^2\theta_W} \\
\lambda_{42} &= \frac{1}{4}\left(1+\sin 2\beta\sin 2\alpha-\tan^2\theta_W\cos 2\alpha\cos 2\beta\right) \\
\lambda_{43} &= \frac{1}{4}\left(1-\sin 2\alpha\sin 2\beta+\tan^2\theta_W\cos 2\alpha\cos 2\beta\right) \\
\lambda_{44} &= \frac{1}{4}\left(\cos 2\alpha\sin 2\beta+\tan^2\theta_W\sin 2\alpha\cos 2\beta\right) \\
\lambda_{45} &= \frac{3}{4}\frac{\cos^2 2\alpha}{\cos^2\theta_W} \\
\lambda_{46} &= \frac{3}{4}\frac{\sin 2\alpha\cos 2\alpha}{\cos^2\theta_W} \\
\lambda_{47} &= \frac{1}{4\cos^2\theta_W}\left(3\sin^2 2\alpha-1\right) \\
\lambda_{48} &= \frac{1}{4}\frac{\cos 2\alpha\cos 2\beta}{\cos^2\theta_W} \\
\lambda_{49} &= \frac{1}{4}\frac{\sin 2\alpha\cos 2\beta}{\cos^2\theta_W}
\end{aligned}
$$

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