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High-Energy Amplitudes in Gauge Theories in the Next-to-Leading-Order

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HIGH-ENERGY AMPLITUDES IN GAUGE THEORIES IN THE NEXT-TO-LEADING-ORDER

by

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ABSTRACT

HIGH-ENERGY AMPLITUDES IN GAUGE THEORIES IN THE NEXT-TO-LEADING-ORDER

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Scattering processes play a central role in physics, and high-energies experiments give us an insight into the fine structure of matter. The high-energy behavior of amplitudes in gauge theories can be reformulated in terms of the evolution of Wilson-line operators. In the leading order this evolution is governed by the non-linear Balitsky-Kovchegov (BK) equation. In order to see if this equation is relevant for existing or future deep inelastic scattering (DIS) accelerators (like Electron Ion Collider (EIC) or Large Hadron electron Collider (LHeC)) one needs to know how large are the next-to-leading order (NLO) corrections. In addition, the NLO corrections define the scale of the running-coupling constant in the BK equation and therefore determine the magnitude of the leading-order cross sections. The first main result of this thesis is the calculation of these NLO corrections. In Quantum Chromodynamics (QCD), the next-to-leading order BK equation has both conformal and non-conformal parts. To separate the conformally invariant effects from the running-coupling effects, we first restore the conformal NLO BFKL kernel out of the eigenvalues known from the forward NLO BFKL result using the requirement of Möbius invariance of $\mathcal{N}$=4 SYM amplitudes in the Regge limit, and then we calculate the NLO evolution of the color dipoles in the conformal $\mathcal{N}$=4 SYM theory. To this end we define the "composite dipole operator" with the rapidity cutoff preserving conformal invariance, and the resulting Möbius invariant kernel for this operator agrees with the forward NLO BFKL calculation of Ref. [47]. In QCD, the NLO kernel for the composite operators resolves in a sum of the conformal part and the running-coupling part.

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CHAPTER I

INTRODUCTION

The fundamental theory of strong interactions is the gauge theory of Quantum Chromodynamics (QCD). Confinement and asymptotic freedom are two important features of the theory. The first, confinement, expresses the impossibility of having unbound state for the degrees of freedom (gluons and quarks). The second, asymptotic freedom, allows one to consider the fundamental constituent of hadrons as free particles undergoing high energy or high momentum transfer in scattering processes.

Before quarks and gluons were accepted as fundamental degrees of freedom of the theory of strong interactions, many efforts were made to describe hadrons using Regge theory. This theory is based on three postulates. The scattering matrix operator S is Lorentz invariant, it is unitary ($SS^\dagger = S^\dagger S = 1$) and analytic with respect to the Mandelstam variables $s = (P_A + P_B)^2$, the energy square of the center of mass and $t = (P_A - P_B)^2$, the momentum transfer, where P_A and P_B are the momenta of the incoming particle A and B.

In Regge theory, the singularities of the scattering amplitude in the complex angular plane, called Regge poles and cuts, describe the exchange of particles in a scattering process. Regge poles and cuts are related to the masses and spins of hadrons which are referred to as Regge particles or Reggeons. The Pomeron is the pole with the largest real part and for this reason it is dominant in processes characterized by asymptotically high energy scattering. Besides, the Pomeron has the quantum number of the vacuum (i.e. isospin zero and even under charge conjugation).

Nowadays, the existence of the Pomeron is universally accepted. The reasons which motivate its existence are the following: Pomeranchuk theorem [1] says that the total cross sections of scattering processes with exchange of charged (or carrying any quantum number) particles vanish asymptotically for high energy. Experimentally it is observed that at high energies the total cross section rises slowly. This motivated Pomeranchuk to postulate the existence of an object, the pomeron, with the quantum number of the vacuum, responsible for the continuing rise of the cross section. It has been shown later on that the intercept of the pomeron $\alpha(0)$ (i.e. the pomeron itself) has to be greater than 1 [2].

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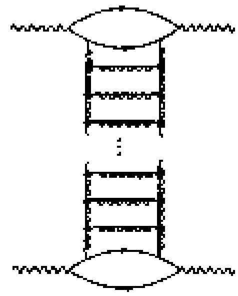

FIG. 1: Ladder diagram with Lipatov effective vertices.

When QCD was proposed as the fundamental theory of strong interactions, many efforts were devolved to reproduce results previously obtained in Regge theory. Therefore, objects like the Pomeron which had sense in Regge theory, were reproduced in the new "language" and new features of the strong interactions were discovered along the way.

In the framework of QCD, the Pomeron, having the quantum number of the vacuum, can be reproduced at the lowest order by two gluon in the t-channel [3]. In the Regge limit ($s \to \infty$), contribution proportional to $(\alpha_s \log(s))^n$ cannot be neglected. It turns out that these contributions coming from ladder diagrams consist of two vertical "reggeized" gluons coupled by "simple" gluons through the effective vertices also known as Lipatov vertices (see figure 1).

In general what is meant for reggized particles is the following. Consider a process in the t-channel where a particle with virtuality (or mass) m and angular momentum l is exchanged, then this particle is said to reggeize if the amplitude behaves asymptotically in s as

$$A(s,t) \propto s^{\alpha(t)}. \tag{1}$$

A gluon is said to be reggeized when in the LLA its propagator is proportional to

$$\frac{g^{\mu\nu}}{\mathbf{k}^2} s^{\alpha_{reg}(t)} \tag{2}$$

where $\alpha_{reg}(t)$ is the trajectory of the reggeized gluon in the complex angular momenta in the leading order in α_s.

Nowadays it is widely accepted that hadrons are objects which are a composition of asymptotically free particles whose properties cannot be measured directly because of their confinement. Deep inelastic scattering (DIS) has contributed significantly to our understanding of the structure of matter. At small value of the Bjorken scaling variable (x_{Bj}) it probes the parton distributions at high energy limit. In QCD one predicts that several novel phenomena will occur in this limit, such as a very strong increase in parton densities and parton saturation. Understanding the behavior of parton distributions in this new regime is one of the most interesting and challenging theoretical problem in QCD.

The small-x_{Bj} behavior of the structure functions of deep inelastic scattering is described by the evolution of color dipoles. In the leading order, it is given by the so-called Balitsky-Kovchegov (BK) equation [45] [20] which is now a starting point of discussion of the small-x_{Bj} evolution in the saturation regime. The region of applicability of the leading-order evolution equation can be determined by the next-to-leading order (NLO) corrections. Besides, NLO corrections are essential because unlike the Q^2 evolution, the argument of the coupling constant in the BK equation is left undetermined in the leading order, and usually is set by hand to be Q_s, the saturation scale.

The analysis of this argument is therefore very important. From a theoretical point of view, we need to know whether the coupling constant is determined by the size of the original dipole or by the size of the produced dipoles since one may get a very different behavior of the solution of the BK equation. On the experimental side instead, the cross section is proportional to some power of the coupling constant so the argument determines how big (or how small) the cross section is. The precise form of the argument of α_s should come from the solution of the BK equation with the running coupling constant, and the starting point of the analysis of the argument of α_s is the calculation of the NLO evolution equation. These are some of the motivations which drove my research so far.

In [40] we have performed the calculation of the small-x_{Bj} evolution of color dipole in the next-to-leading order. To obtain the evolution of the structure function in the NLO, one has to include also the probability for the creation of the dipole by the virtual photon (so called photon impact factor) in the next-to-leading order.

Another aspect which makes NLO corrections very important is the conformal properties of QCD at high energies. It is known QCD is not conformal invariant because of the running of the coupling constant. At leading order the BFKL [21] equation is invariant under the two-dimensional conformal Möbius transformation.

As expected, in the NLO the conformal invariance is lost, but it is not known whether the only source of non conformal invariance is due to the presence of the running coupling effects.

The $\mathcal{N}=4$ SYM theory is conformal invariant in 4 dimensions and this makes the theory a suitable playground for the studies of unknown conformal properties like the transition between the full conformal group in 4 dimension and the $SL(2,C)$ Möbius group at high energies. Motivated by these arguments, in [39] we have performed the calculation of the NLO evolution of color dipoles in the case of $\mathcal{N}=4$ SYM theory.

The NLO calculations may also help one to study the initial conditions of formation of quark-gluon plasma (studied in Brookhaven National Laboratory) by studying the scattering of two shock waves and the effective action at high energy in QCD.

The results presented in this thesis will be also relevant in future experiments like Electron Ion Collider (EIC) and Large Hadron electron Collider (LHeC) located in Geneva.

I.1 REGGE THEORY

As mentioned above, before the advent of Quantum Chromodynamics (QCD), Regge theory (based on three assumptions made on the S matrix) provided the tools necessary to study the scattering process of particles at high center-of-mass energies.

In this section we briefly present the implications arising as consequence of these postulates and we will show what one can learn about high-energy scattering processes. A complete treatment of this subject can be found in [4].

It can be shown that the S matrix, which transforms the initial state at time $t=-\infty$ to the state at time $t=+\infty$, can be written as

$$S_{ab} = \delta_{ab} + iT_{ab}, \tag{3}$$

where we denote by S_{ab} the element of the matrix operator and define $T_{ab} = (2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right) A_{ab}$ as the matrix element representing the interaction. The matrix element δ_{ab} represents the absence of interaction and A_{ab} is the amplitude of the process. The unitarity of the S operator leads to the Cutkosky rules:

$$2\mathrm{ImA_{ab}} = (2\pi)^4\delta^4\left(\sum_i \mathrm{p_i} - \sum_f \mathrm{p_f}\right)[\mathrm{AA^\dagger}]_{\mathrm{ab}}. \tag{4}$$

From (4) we can see that the imaginary part of forward amplitude A_{aa} is related to the total cross section. This result is known as the Optical theorem.

The third postulate allows one to consider the amplitude $A(s,t)$ as an analytic function of the Mandelstam variables, s and t. As a consequence of this property together with the unitariness, of S matrix, singularities and branch cuts in the complex s-plane take physical meaning.

We know that two particles can merge into one particle at discrete energies only. Besides, two particles can create two different particles only at energies higher then the sum of energies of the lightest pair of particles allowed by the conservation of quantum numbers. This means that there are intervals in the s-plane in which the cross section (and $\mathrm{Im}A_{aa}$) is zero. Under this condition we can apply Schwarz reflection principle[1] which in this case reads

$$A(s^*,t) = (A(s,t))^*, \tag{5}$$

in its domain of analyticity. The imaginary part of the amplitude is given by

$$\mathrm{Im}A(s,t) = \frac{1}{2i}\left(A(s,t) - A^*(s,t)\right), \tag{6}$$

which means that the amplitude $A(s,t)$ due to (5) would be zero on the entire real s-axis. Thus, the total cross section would assume always zero values. For this reason $A(s,t)$ has to have poles and cuts on the real s axis. The poles correspond to single particle production (resonance), and the branch cuts correspond to general particle production.

It is known that the amplitude $A(s,t)$ can be written in terms of spherical harmonics with zero angular momentum which are Legendre polynomials in the cosine of the scattering angle: This is the so called partial wave expansion. Considering a process of two particles in two particles we can write

$$A(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(1+2s/t) \tag{7}$$

where in the s-channel $\cos(\theta) = 1 + 2s/t$ and the coefficient a_l are the partial wave amplitudes. Using Sommerfeld-Watson transformation, based on the Cauchy's integral theorem, the right hand site of (7) can be written as a contour integral in the complex angular l plane over a function with singularities at positive integer values of l (zero included):

[1]Schwarz reflection principle says that when a function is analytic in a given domain and real on a straight line contained in that domain, then it takes values which are complex conjugate of each other at points which are mirror reflections with respect to that line.

$$A(s,t) = \oint dl \frac{1}{\sin \pi l}(2l+1)a(l,s)P(l,1+2s/t) \tag{8}$$

where the contour circles the positive real axis clockwise and it extends at infinity. Now using an asymptotic formula for the Legendre polynomials one obtains for each of the pole a contribution

$$A(s,t) \propto s^{\alpha(t)} \tag{9}$$

where $\alpha(t)$ is the position of the pole in the complex l plane. It can be deduced that at high energies the scattering amplitude is dominated by the Regge pole with the largest real part. Expression (9) is interpreted as the exchange of a Reggeon with angular momentum $\alpha(t)$ called Regge trajectory. It was found that in the t-channel process, mesons plotted in a diagram of angular momentum versus square mass lie in straight lines and it can be shown that this linearity holds also for negative value of t (s-channel):

$$\alpha(t) = \alpha(0) + \alpha' t. \tag{10}$$

$\alpha(0)$ is the Regge intercept of the Reggeon and α' is its Regge slope. The differential cross section for processes where a single Reggeon is exchanged is:

$$\frac{d\sigma}{dt} \propto s^{2\alpha_0 + 2\alpha' t - 2}. \tag{11}$$

The asymptotic behavior of the total cross section can also be obtained:

$$\sigma_{tot} \propto s^{(\alpha(0)-1)}. \tag{12}$$

I.2 THE GAUGE THEORY OF QUANTUM CHROMODYNAMICS

In this section we describe the gauge theory of Quantum Chormodynamics (QCD). The origin of the theory relies on the discovery of the renormalizability of field theories due to 't Hooft and Veltman [5], [6], [7].

QCD is a gauge theory based on the $SU(3)$ symmetry group, called also the color group. On the contrary, if the spontaneously broken symmetry of the electroweak sector of the standard model based on the gauge group $SU(2) \times U(1)_Y$ [8], [9], [10], where $SU(2)$ is the gauge group of the weak interactions and $U(1)_Y$ denote the hypercharge, the $SU(3)$ gauge symmetry of QCD is an exact symmetry.

In QCD quarks may exist in three different quantum numbers called the color number (red, green, blue):

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix} \tag{13}$$

where q_r, q_g, q_b are Dirac spinor. The theory is a gauge theory based on the invariance of the lagrangian for local transformation applied to q in the color quantum space

$$\psi'(x) = U(x)\psi(x) \quad \text{and} \quad \overline{\psi}'(x) = \overline{\psi}(x)U^{\dagger}(x) \tag{14}$$

where U, is the unitary matrix of the color $SU(3)$. The lagrangian density for n non-interacting quarks with mass m_i is [11]

$$\mathcal{L}_{quark} = \sum_i^n \overline{q}_i^a (i\not{\partial} - m_i)_{ab} q_i^b, \tag{15}$$

where $\not{\partial} = \gamma^\mu \partial_\mu$ and $(i\not{\partial} - m_i)_{ab}$ is proportional to the identity matrix in the color space. This lagrangian density is invariant for global $SU(N)$ transformations with N the number of colors.

$$q_a \longrightarrow q'_a = \exp(it \cdot \theta)_{ab} q_b. \tag{16}$$

In order to have a local gauge invariance in space-time one introduces the covariant derivative

$$D_{\mu,ab} = \partial_\mu \mathbf{1}_{ab} - ig_s(t \cdot A_\mu)_{ab}, \tag{17}$$

where A_μ^a are color vector fields such that

$$D'_{\mu,ab} q'_b(x) = \exp(it \cdot \theta(x))_{ab} D_{\mu,bc} q_c(x). \tag{18}$$

The kinetic term for the field A^μ is,

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} \tag{19}$$

where $F_{\mu\nu}^A$ is the non-abelian field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c. \tag{20}$$

where f^{abc} are the structure constants and t^a are the matrix with dimension of the representation of the color gauge group $SU(3)$. The term $gf^{abc}A^b_\mu A^c_\nu$ has important theoretical consequences: it implies that the gluon fields can interact among themselves through vertices involving three or four gluons.

Since the Lagrangian is invariant for gauge transformations, the propagator of the theory cannot be calculated unless we introduce a "gauge fixing" term, and this choice is in principle arbitrary.

Furthermore, in the non-abellian gauge theory it is necessary to introduce the so called ghost field term. This term describe the propagation of non-physical modes in the theory, and it is introduced to preserve the unitarity of the theory. However such a term does not appear in some choices of the gauge such as the light-cone gauge.

The QCD lagrangin density is therefore

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A + \sum_i^n \bar{q}^a_i(i\not{\partial} - m_i)_{ab}q^b_i - \frac{1}{2\lambda}(\partial^\mu A_\mu)^2 + \mathcal{L}_{ghost}. \tag{21}$$

The only parameters of the theory are the coupling constant g and the mass of quarks. In what follow we will set the mass of the light quarks to be zero and the masses of the heavy quarks to be infinitely large. In this way the only parameter of the theory is the coupling constant g.

A peculiar feature of QCD is the asymptotic freedom [12]: the coupling constant g decreases with increasing values of the interacting energy. At energies of the order of 1 GeV the coupling constant is such that the theory can be studied perturbatively.

In general the lagrangian of a theory has some parameters, like the coupling constant and the mass of the particles. These parameters are called the bare parameters and they are not directly measured in experiments. If for example one is interested to measure the electric charge of the electron in experiments where electrons collide into each other, we observe that the result we obtain depends on the momenta of the electrons. One way to understand the origin of these phenomena is to consider the polarization of the vacuum induced by the electric charge of the electron. The screening effect due to the polarization of virtual positive and negative particles which can pop up and annihilate from the vacuum, reduces the effective charge of the electron. In natural units ($\hbar = c = 1$) the momentum is inversely proportional to the distances one is probing, so the larger is the momenta of the colliding electrons the smaller is the distance at which the electric charge is measured: the screening effect of the vacuum is consequently reduced. One can deduce that physical coupling constants

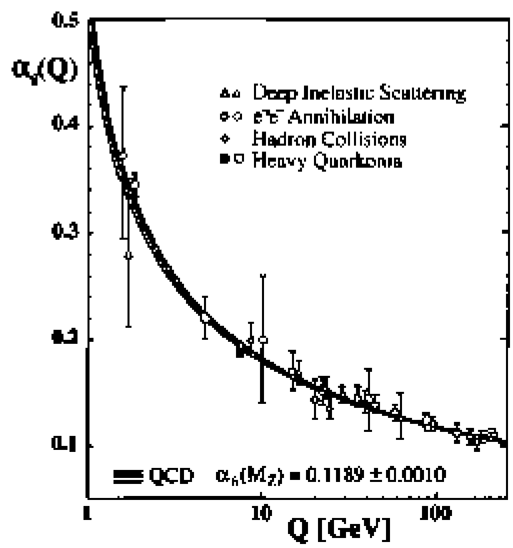

FIG. 2: Running-coupling: experiments data and QCD predictions.

depend on the distance scale, and so the smaller is the distance scale probed the larger is the charge measured.

In calculation these types of phenomena appears as infinities of the theory, and the renormalization group is designed to cure this problem.

As we have mentioned above, in QCD we observe the opposite phenomena [12]: in scattering processes the coupling constant decreases with increasing energies. At one loop level it is possible to solve exactly the renormalization group equation and derive $\alpha_s \equiv \frac{g_s}{4\pi}$ at some scale Q, as a function of its value at the renormalization scale μ:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{1}{4\pi}\alpha_s(\mu^2)\beta_0 \ln \frac{Q^2}{\mu^2}}, \quad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f. \tag{22}$$

Here N_f is the number of quark flavors and the number of quark colors N_c is the eigenvalue of the quadratic Casimir operator in $SU(N)$.

In Fig. 2 a summary of measurements of $\alpha_s(Q)$ as a function of the respective energy scale Q is presented [13]. Note that open symbols indicate (resummed) NLO, and filled symbols NNLO QCD calculations used in the respective analysis. The curves are the QCD predictions for the combined world average value of $\alpha_s(M_{Z^0})$, in 4-loop approximation and using 3-loop threshold matching at the heavy quark pole masses $M_c = 1.5$ GeV and $M_b = 4.7$ GeV.

I.3 DEEP INELASTIC SCATTERING

It is well established that hadrons are made of asymptotically free particles which are quasi on-shell excitations. Because of their confining properties it is not possible to measure them directly, and therefore Deep Inelastic Scattering (DIS) experiments are used to study the structure functions of hadrons which to lowest order in QCD are related to the number density of partons inside hadrons. These are processes in which a lepton with momentum k scatter off a parton inside a hadron with momentum P through an exchange of a virtual photon with virtuality $Q^2 = -q^2$ which probes the structure of the targer. From the measurements of the final momentum k' of the scattered lepton one obtains the momentum $q = k' - k$ transferred by the virtual photon to the hadron. The process is illustrated in Fig. 3.

A suitable frame in which the parton picture is revealed is the infinite-momentum frame of the hadron. In this frame the characteristic time of the γ^*-partons interactions is much shorter than any interactions between the partons. In such a frame, Lorentz contraction in the longitudinal direction reduces the proton to a pancake and time dilation enhances the lifetime of the fluctuating partons so that the hadron constituents are effectively frozen. In the limit in which the effective mass of the partons is negligible and therefore have small rest mass (partons are asymptotically free) the interactions can be considered incoherent. If we now consider the Breit frame in which the longitudinal momentum of the virtual photon is zero ($q^\mu = (q_0, q_\perp, 0)$), we have that $Q^2 = q_\perp^2$. In this frame the virtual photon act as a microscope resolving partons in the transverse plane of dimension of the order of $\frac{1}{q_\perp}$.

It has been observed in experiments like H1 or Zeus at Hera (see Fig. 4), that the structure functions of gluons and sea quarks grow with decreasing values of the Lorentz invariant variable x_B [14]. The diagram in Fig. 4 (taken from Ref. [16]) shows the gluon, valence, and sea quark momentum distributions in the nucleon (note that the gluon and sea quark distributions are scaled down by a factor of 20).

The small-x region is obtained in the so called Regge limit. When the center of the mass energy square s and the momentum transfer Q^2 are much greater then the mass of the hadron which is being probed, the Bjorken variable is approximatively given by $x_B \equiv \frac{Q^2}{2P \cdot q} \sim \frac{Q^2}{s}$. The Regge limit is then achived when the parameter s is taken very large while keeping the momentum transfer Q^2 fixed, consequently x_B decreases.

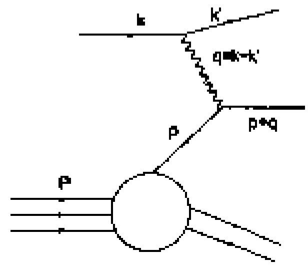

FIG. 3: Deep inelastic scattering kinematics

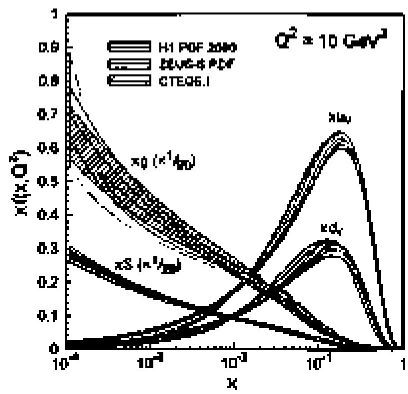

FIG. 4: Gluon, valence, and sea quark momentum distributions in the nucleon.

A better understood limit is the Bjorken limit. This limit is obtained when both the energy and the momentum transfer goes to infinity while their ratio is kept fixed. In the Bjorken limit the structure functions obey the DGLAP (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi) equation [4] which is an evolution equation in the momentum Q^2: the power resolution of the virtual photon probing the target is inversely proportional to the momentum transfer and so, the higher is the momentum transfer the smaller is the size of the parton resolved by the virtual photon in the transverse direction. DGLAP equation which is a resummation of radiative corrections proportional to $(\alpha_s \ln Q^2)^n$ for all $n \geq 1$, predicts the increase of the parton distributions, but the system evolves towards a dilute regime. Besides, although DGLAP has been successful in explaining many features of HERA data, in the region of low momentum transfer its predictive power is lost. In this case DGLAP needs higher twist corrections ($1/Q^2$ corrections) which are not negligible anymore at low Q^2.

It is clear then that, at high energies a different kind of resummation is needed. Indeed, at high energies contributions proportional to $(\alpha_s \ln s)^n$ for all $n \geq 1$ cannot be neglected anymore, and therefore they have to be resummed. The equation governing the evolution of the gluon distribution, and of the scattering amplitude at high energies is the BFKL (Balitsky, Fadin, Kuraev, Lipatov) equation.

I.4 BFKL EQUATION AND THE POMERON

In this section we will illustrate the main idea of BFKL [21] and we will also introduce the definition of the so called photon impact factor (for a review [17]). To this end we consider the scattering process of two virtual photon ($\gamma^*\gamma^*$ scattering). The lowest order diagram consists of one loop diagram that can be calculated by the known Feynman rules for loops (see figure 5).

However, as originally was done in [21] it is more convenient to use the dispersion relation techniques.

Writing the amplitude $A(s,t)$ using Cauchy's integral in the complex s-plane and taking into account the cuts on the real axis of the s-plane, we can relate the amplitude of the process to its discontinuity. On the other hand we know that

$$\mathrm{Disc}\, A(s,t) = 2i \mathrm{Im}\, A(s,t). \tag{23}$$

Therefore, $A(s,t)$ is given by an integration on the real s-axis of $\frac{1}{\pi}\mathrm{Im}\, A(s,t)$. The imaginary part of the amplitude is obtained using Cutkosky rules. At high energy it is convenient to introduce the Sudakov decomposition

$$p^\mu = \alpha_p p_1^\mu + \beta_p p_2^\mu + p_\perp^\mu. \tag{24}$$

The imaginary part of the amplitude to the LLA is then

$$\mathrm{Im}\, A(s,t) \propto s \int \frac{d^2k}{4\pi^2} I^A(k_\perp, r_\perp) \frac{1}{\vec{k}_\perp^2 (\vec{r}-\vec{k})_\perp^2} \left(\frac{s}{m^2}\right)^{\frac{g^2 N_c}{2\pi^2}\hat{K}_r} I^B(-k_\perp, -r_\perp) \tag{25}$$

where r is the momentum transfer and $\hat{K}_r$ is the operator defined by its kernel (the BFKL kernel)

$$(\hat{K}_r f)(\vec{k}_\perp) = \int \frac{d^2k'}{4\pi^2} K(k_\perp, k'_\perp, r) f(\vec{k}'_\perp). \tag{26}$$

I^A and I^B are the so called photon impact factors which account for the coupling of the pomeron with the hadrons. Taking the Mellin transform of (25) in the complex angular momentum l one obtains

$$\mathrm{Im}A(l,t) \propto \int \frac{d^2k}{4\pi^2} \frac{I^A(k_\perp, r_\perp)}{\vec{k}_\perp^2 (\vec{r}-\vec{k})_\perp^2} \frac{1}{\omega - \frac{g^2}{2\pi} N_c \hat{K}_r} I^B(-k_\perp, -r_\perp), \tag{27}$$

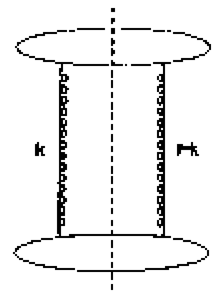

FIG. 5: Lowest-order diagrams for the high-energy scattering of virtual photons.

We note now that the rightmost singularity of the integrand in (27) will determine the behavior of the amplitude at $s \to \infty$. The BFKL equation can be written as

$$2\alpha_s N_c (\hat{K}_r f)(\vec{k}_\perp) = \omega f(\vec{k}_\perp). \tag{28}$$

whose maximal eigenvalue gives the position of the rightmost singularity. It represents the evolution of the ladder gluons.

The BFKL equation can be solved in the non-forward direction ($t \neq 0$) thanks to its invariance under global conformal transformations in transverse space. It is then possible to see the BFKL pomeron as a state of two reggized gluons evolving according to conformally invariant two-dimensional quantum mechanics. The time-like variable is the rapidity because the rung gluons are strongly ordered in rapidity which at high energies is given by $\ln \frac{1}{x_B}$

According to the behaviour of the solution of the BFKL equation the scattering amplitude of the process grows as power of s i.e. s^{ω_P} where $\omega_P = \frac{12 \ln 2}{\pi} \alpha_s$ is the Pomeron intercept. At very high energies such a solution would violate the unitarity bound (the amplitude grows infinitely). Furthermore, the behaviour of the BFKL solution violates the Froissart bound: Based on general principles the Froissart-Martin theorem [15] states that the total cross section cannot rise faster then $\ln^2 s$.

1.4.1 The need for a nonlinear equation

As was previously mentioned, the power resolution of the virtual photon is inversaly proportioual to the momentum transfer. We have also seen that the DGLAP equation is an evolution equation in momentum transfer, and therefore it describes an evolution towards a dilute regime. Besides, we observed that the DGALAP equations need higher twist corrections for low values of Q^2. This type of corrections have some theoretical difficulties.

On the other hand, if we fix the momentum transfer and we let the hadron evolve with energy, the parton density increases. Fixing the momentum Q^2 means that the virtual photon resolves in the hadron partons with fixed size in the transverse direction while their number increases. The BFKL equation provides a nice description of such an increase in density, but it fails to describe the evolution of parton density at very high energies. The BFKL equation only considers the emission process, but it does not take into account the possibility for partons to interact each other: BFKL is a linear equation.

The recombination phenomena of partons occurring at very high energies are governed by non-linear effects (coherency effects) which can be taken into account only by a non-linear equation. Furthermore, in order to recover unitarization of the theory, the system eventually evolves towards a saturation region [48].

Since coherency effects at high energies are important, the Breit frame introduced above is not anymore a suitable frame to describe DIS processes. Instead, we consider the so called *dipole frame*: We perform a Lorentz boost in a frame in which the hadron still carries most of the energy, but the virtual photon has enough energy to split into a quark anti-quark ($q\bar{q}$) in a color singlet state pair long before scattering. In this frame the lifetime of this $q\bar{q}$ fluctuation is much larger then the interaction time between this pair and the hadron. This frame is a natural frame work for the description of multiple scattering which as we have seen becomes important at high energy. In the next chapter we will describe the propagation of a $q\bar{q}$ pair at high energy through a medium generated by color field.

CHAPTER II

THE BALITSKY-KOVCHEGOV EQUATION

As pointed out in the previous section, in order to describe non linear effects, a non linear equation is in order. In this section we describe how a semiclassical approach to the description of the DIS at small-x_B leads to a non-linear equation: the Balitsky-Kovchegov (BK) equation.

II.1 LEADING ORDER EVOLUTION OF COLOR DIPOLES

A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path. In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line - the infinite gauge link ordered along the straight line collinear to particle's velocity n^μ:

$$U^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty} du\ n_\mu\ A^\mu(un + x_\perp)\Big\}, \tag{29}$$

Here A_μ is the gluon field of the target, $x_\perp$ is the transverse position of the particle which remains unchanged throughout the collision, and the index η labels the rapidity of the particle. Repeating the above argument for the target (moving fast in the spectator's frame) we see that particles with very different rapidities perceive each other as Wilson lines and therefore these Wilson-line operators form the convenient effective degrees of freedom in high-energy QCD (for a review, see ref. [17]).

Let us consider the deep inelastic scattering from a hadron at small $x_B = Q^2/(2p\cdot q)$. The virtual photon decomposes into a pair of fast quarks moving along straight lines separated by some transverse distance. The propagation of this quark-antiquark pair reduces to the "propagator of the color dipole" $U(x_\perp)U^\dagger(y_\perp)$ - two Wilson lines ordered along the direction collinear to quarks' velocity. The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{\mathcal{U}}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\} \tag{30}$$

switched between the target's states ($N_c = 3$ for QCD). The gluon parton density is approximately

$$x_B G(x_B, \mu^2 = Q^2) \ \simeq \ \langle p|\ \hat{\mathcal{U}}^\eta(x_\perp, 0)|p\rangle\Big|_{x_\perp^2 = Q^{-2}} \tag{31}$$

where $\eta = \ln \frac{1}{x_B}$. (As usual, we denote operators by "hat"). The energy dependence of the structure function is translated then into the dependence of the color dipole on the slope of the Wilson lines determined by the rapidity η.

Thus, the small-x behavior of the structure functions is governed by the rapidity evolution of color dipoles [18, 19]. At relatively high energies and for sufficiently small dipoles we can use the leading logarithmic approximation (LLA) where $\alpha_s \ll 1$, $\alpha_s \ln x_B \sim 1$ and get the non-linear BK evolution equation for the color dipoles [45, 20]:

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z\ \frac{(x-y)^2}{(x-z)^2(z-y)^2}[\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(y,z) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)] \quad (32)$$

The first three terms correspond to the linear BFKL evolution [21] and describe the parton emission while the last term is responsible for the parton annihilation. For sufficiently high x_B the parton emission balances the parton annihilation so the partons reach the state of saturation [48] with the characteristic transverse momentum Q_s growing with energy $1/x_B$ (for a review, see [23])

II.2 DERIVATION OF THE BK EQUATION

Before discussing the small-x evolution of color dipole in the next-to-leading approximation we will derive the leading-order (BK) evolution equation. The dependence of the structure functions on x_B comes from the dependence of Wilson-line operators

$$\hat{U}^\eta(x_\perp) = \mathrm{Pexp}\Big\{ig\int_{-\infty}^{\infty}du\ n_\mu\ \hat{A}^\mu(un+x_\perp)\Big\}, \quad n \equiv p_1 + e^{-2\eta}p_2 \quad (33)$$

on the slope of the supporting line. The momenta p_1 and p_2 are the light-like vectors such that $q = p_1 - x_B p_2$ and $p = p_2 + \frac{m^2}{s}p_1$ where p is the momentum of the target and m is the mass. We use the Sudakov variables $p = \alpha p_1 + \beta p_2 + p_\perp$ and the notations $x_\bullet \equiv x_\mu p_1^\mu$ and $x_* \equiv x_\mu p_2^\mu$ related to the light-cone coordinates: $x_* = x^+\sqrt{s/2}$, $x_\bullet = x^-\sqrt{s/2}$.

To find the evolution of the color dipole (30) with respect to the slope of the Wilson lines in the leading log approximation we consider the matrix element of the color dipole between (arbitrary) target states and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2 = \eta_1 - \Delta\eta$ leaving the gluons with $\eta < \eta_2$ as a background field (to be integrated over later). In the frame of gluons with $\eta \sim \eta_1$ the fields with $\eta < \eta_2$ shrink to a pancake and we obtain the four diagrams shown in Fig. 6.

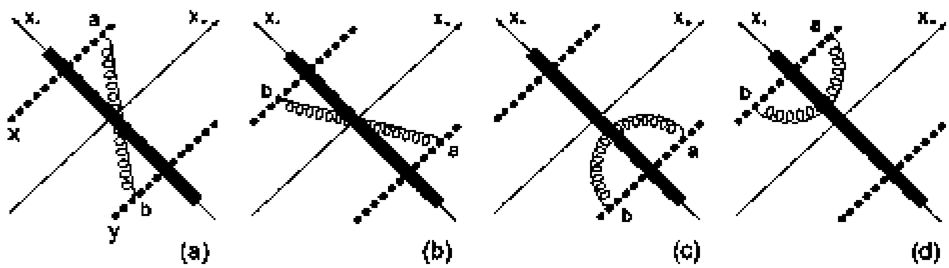

FIG. 6: Leading-order diagrams for the small-x evolution of color dipole. Gauge links are denoted by dotted lines.

Technically, to find the kernel in the leading-ordrer approximation we write down the general form of the operator equation for the evolution of the color dipole

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_x \hat U^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x \hat U^\dagger_y\} + ... \tag{34}$$

(where dots stand for the higher orders of the expansion) and calculate the l.h.s. of Eq. (34) in the shock-wave background

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{shockwave}} \tag{35}$$

In what follows we replace $\langle ...\rangle_{\mathrm{shockwave}}$ by $\langle ...\rangle$ for brevity. With future NLO computation in view, we will perform the leading-order calculation in the lightcone gauge $p_2^\mu A_\mu = 0$. The gluon propagator in a shock-wave external field has the form [30, 26]

$$\begin{aligned}
\langle \hat A^a_\mu(x)\hat A^b_\nu(y)\rangle \ =\ & \theta(x_* y_*)\delta^{ab}\frac{s}{2}\int d\alpha d\beta\,(x_\perp|\frac{d_{\mu\nu}}{i(\alpha\beta s - p_\perp^2 + i\epsilon)}|y_\perp) \\
- \theta(x_*)\theta(-y_*)\int_0^\infty & d\alpha\ \frac{e^{-i\alpha(x-y)_*}}{2\alpha}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\Big[g^\perp_{\mu\xi} - \frac{2}{\alpha s}(p^\perp_\mu p_{2\xi} + p_{2\mu}p^\perp_\xi)\Big]U^{ab} \\
& \times\Big[g^{\perp\xi}_\nu - \frac{2}{\alpha s}(p^\xi_2 p^\perp_\nu + p_{2\nu}p^\xi_\perp)\Big]e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp) \\
- \theta(-x_*)\theta(y_*)\int_0^\infty & d\alpha\ \frac{e^{i\alpha(x-y)_*}}{2\alpha}(x_\perp|e^{i\frac{p_\perp^2}{\alpha s}x_*}\Big[g^\perp_{\mu\xi} - \frac{2}{\alpha s}(p^\perp_\mu p_{2\xi} + p_{2\mu}p^\perp_\xi)\Big]U^{\dagger ab} \\
& \times\Big[g^{\perp\xi}_\nu - \frac{2}{\alpha s}(p^\xi_2 p^\perp_\nu + p_{2\nu}p^\xi_\perp)\Big]e^{-i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)
\end{aligned} \tag{36}$$

where

$$d_{\mu\nu}(k) \equiv g^\perp_{\mu\nu} - \frac{2}{s\alpha}(k^\perp_\mu p_{2\nu} + k^\perp_\nu p_{2\mu}) - \frac{4\beta}{s\alpha}p_{2\mu}p_{2\nu} \tag{37}$$

Hereafter we use Schwinger's notations $(x_\perp|F(p_\perp)|y_\perp) \equiv \int d\!\!{}^-p\ e^{i(p,x-y)_\perp}F(p_\perp)$ (the scalar product of the four-dimensional vectors in our notations is $x\cdot y = \frac{2}{s}(x_* y_\bullet +$

$x_* y_*) - (x,y)_\perp)$. Note that the interaction with the shock wave does not change the α-component of the gluon momentum.

We obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\ \langle \hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle_{\mathrm{Fig.6a}}$$
$$= -4\alpha_s\int_0^\infty\frac{d\alpha}{\alpha}(x_\perp|\frac{p_i}{p_\perp^2+\alpha^2e^{-2\eta_1}s}U^{ab}\frac{p_i}{p_\perp^2+\alpha^2e^{-2\eta_1}s}|y_\perp) \tag{38}$$

(with power accuracy $\sim \frac{m^2}{s}$ one can replace $n_\mu A^\mu$ by $A_\bullet$). Formally, the integral over α diverges at the lower limit, but since we integrate over the rapidities $\eta > \eta_2$ we get (in the LLA)

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\ \langle \hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle_{\mathrm{Fig.6a}} = -4\alpha_s\Delta\eta(x_\perp|\frac{p_i}{p_\perp^2}U^{ab}\frac{p_i}{p_\perp^2}|y_\perp) \tag{39}$$

and therefore

$$\langle \hat{U}_x\otimes\hat{U}^\dagger_y\rangle^{\eta_1}_{\mathrm{Fig.6a}} = -\frac{\alpha_s}{\pi^2}\Delta\eta\ (t^aU_x\otimes t^bU^\dagger_y)\int d^2z_\perp\frac{(x-z,y-z)_\perp}{(x-z)^2_\perp(y-z)^2_\perp}U^{ab}_z \tag{40}$$

The contribution of the diagram in Fig. 6b is obtained from Eq. (40) by the replacement $t^aU_x\otimes t^bU^\dagger_y\to U_xt^b\otimes U^\dagger_yt^a$, $x\leftrightarrow y$ and the two remaining diagrams are obtained from Eq. (39) by taking $y=x$ (Fig. 6c) and $x=y$ (Fig. 6d). Finally, one obtains

$$\langle \hat{U}_x\otimes\hat{U}^\dagger_y\rangle^{\eta_1}_{\mathrm{Fig.6}} = -\frac{\alpha_s\Delta\eta}{\pi^2}(t^aU_x\otimes t^bU^\dagger_y + U_xt^b\otimes U^\dagger_yt^a)\int d^2z_\perp\frac{(x-z,y-z)_\perp}{(x-z)^2_\perp(y-z)^2_\perp}U^{ab}_z$$
$$+\frac{\alpha_s\Delta\eta}{\pi^2}(t^aU_xt^b\otimes U^\dagger_y)\int\frac{d^2z_\perp}{(x-z)^2_\perp}U^{ab}_z + \frac{\alpha_s\Delta\eta}{\pi^2}(U_x\otimes t^bU^\dagger_yt^a)\int\frac{d^2z_\perp}{(y-z)^2_\perp}U^{ab}_z \tag{41}$$

so

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle^{\eta_1}_{\mathrm{Fig.6}} = \frac{\alpha_s\Delta\eta}{2\pi^2}\int d^2z_\perp\frac{(x-y)^2_\perp}{(x-z)^2_\perp(y-z)^2_\perp}[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}] \tag{42}$$

There are also contributions coming from the diagrams shown in Fig. 7 (plus graphs obtained by reflection with respect to the shock wave). These diagrams are proportional to the original dipole $\mathrm{Tr}\{U_xU^\dagger_y\}$ and therefore the corresponding term can be derived from the contribution of Fig.6 graphs using the requirement that the r.h.s. of the evolution equation should vanish for $x=y$ since $\lim_{x\to y}\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\}=0$). It is easy to see that this requirement leads to

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle^{\eta_1} = \frac{\alpha_s\Delta\eta}{2\pi^2}\int d^2z_\perp\frac{(x-y)^2_\perp}{(x-z)^2_\perp(y-z)^2_\perp}[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}] \tag{43}$$

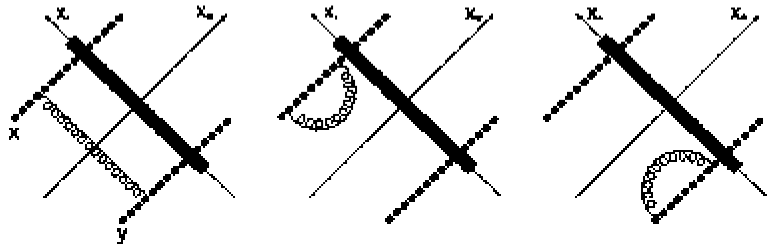

FIG. 7: Leading-order diagrams proportional to the original dipole.

which is equivalent to the BK equation for the evolution of the color dipole (32).

CHAPTER III

NLO EVOLUTION OF COLOR DIPOLES

In order to get the region of application of the leading-order evolution equation one needs to find the next-to-leading order (NLO) corrections. In the case of the small-x evolution equation (32) there is another reason why NLO corrections are important. Unlike the DGLAP evolution, the argument of the coupling constant in Eq. (32) is left undetermined in the LLA, and usually it is set by hand to be Q_s. Careful analysis of this argument is very important from both theoretical and experimental points of view. From the theoretical viewpoint, we need to know whether the coupling constant is determined by the size of the original dipole $|x-y|$ or of the size of the produced dipoles $|x-z|$ and/or $|z-y|$ since we may get a very different behavior of the solutions of the equation (32). On the experimental side, the cross section is proportional to some power of the coupling constant so the argument determines how big (or how small) is the cross section. The typical argument of α_s is the characteristic transverse momenta of the process. For high enough energies, they are of order of the saturation scale Q_s which is $\sim 2 \div 3$ GeV for the LHC collider, so even the difference between $\alpha(Q_s)$ and $\alpha(2Q_s)$ can make a substantial impact on the cross section. The precise form of the argument of α_s should come from the solution of the BK equation with the running coupling constant, and the starting point of the analysis of the argument of α_s in Eq. (32) is the calculation of the NLO evolution.

Let us present our result for the NLO evolution of the color dipole (hereafter we use notations $X \equiv x-z$, $X' \equiv x-z'$, $Y \equiv y-z$, and $Y' \equiv y-z'$)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\ \frac{(x-y)^2}{X^2Y^2} \\
&\times\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + \left(\frac{67}{9}-\frac{\pi^2}{3}\right)N_c - \frac{10}{9}n_f \\
&\qquad - 2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}\ [\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_y\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}] \\
&+\frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\Big[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} \\
&+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
&\qquad\times\ [\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_{z'}\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}^\dagger_y\} - \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\hat{U}_{z'}U^\dagger_y\hat{U}_z\hat{U}^\dagger_{z'}\} - (z'\to z)]
\end{aligned}
\tag{44}
$$

$$+ \Big\{\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] - \frac{(x-y)^4}{X^2Y'^2X'^2Y^2}\Big\}$$
$$\times \ln\frac{X^2Y'^2}{X'^2Y^2}\,\mathrm{Tr}\{\hat U_x\hat U^\dagger_z\}\mathrm{Tr}\{\hat U_z\hat U^\dagger_{z'}\}\mathrm{Tr}\{\hat U_{z'}\hat U^\dagger_y\}$$
$$+ 4n_f\Big\{\frac{4}{(z-z')^4} - 2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}$$
$$\times\mathrm{Tr}\{t^a\hat U_x t^b\hat U^\dagger_y\}[\mathrm{Tr}\{t^a\hat U_z t^b\hat U^\dagger_{z'}\} - (z'\to z)]\Big]$$

Here μ is the normalization point in the $\overline{MS}$ scheme and $b = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the first coefficient of the β-function. The first result we have obtained is the gluon part of the evolution, the quark part of Eq. (44) proportional to n_f was found earlier [24, 52]. Also, the terms with cubic nonlinearities were previously found in the large-N_c approximation in Ref. [26]. The NLO kernel is a sum of the running-coupling part (proportional to b), the non-conformal double-log term $\sim \ln\frac{(x-y)^2}{(x-z)^2}\ln\frac{(x-y)^2}{(y-z)^2}$ and the three conformal terms which depend on the two four-point conformal ratios $\frac{X^2Y'^2}{X'^2Y^2}$ and $\frac{(x-y)^2(z-z')^2}{X^2Y'^2}$. Note that the logarithm of the second conformal ratio $\ln\frac{(x-y)^2(z-z')^2}{X^2Y'^2}$ is absent. It is interesting to observe that the term $\left(\frac{67}{9}-\frac{\pi^2}{3}\right)N_c - \frac{10}{9}n_f$ is exactly the term appearing in the cusp anomalous dimension which is a universal function controlling the IR-induced properties of perturbative QCD [27].

It should be emphasized that the NLO result itself does not lead automatically to the argument of coupling constant α_s in Eq. 32. In order to get this argument one can use the renormalon-based approach [28]: first get the quark part of the running coupling constant coming from the bubble chain of quark loops and then make a conjecture that the gluon part of the β-function will follow that pattern. The Eq. (44) proves this conjecture in the first nontrivial order: the quark part of the β-function $\frac{2}{3}n_f$ calculated earlier gets promoted to full b. The analysis of the argument of the coupling constant was performed in Refs. [24, 52] and we briefly review it in Sect. 7 for completeness. Roughly speaking, the argument of α_s is determined by the size of the smallest dipole: $\min(|x-y|, |x-z|, |y-z|)$.

III.1 DIAGRAMS WITH TWO GLUON-SHOCKWAVE INTERSECTIONS

III.1.1 "Cut self-energy" diagrams

In the next-to-leading order there are three types of diagrams. Diagrams of the first type have two intersections of the emitted gluons with the shock wave, diagrams of

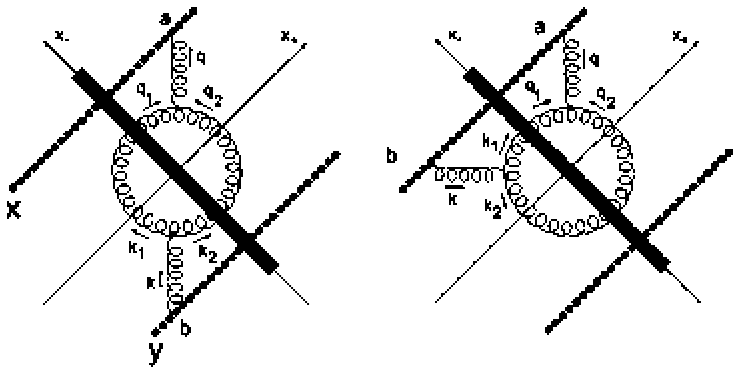

FIG. 8: "Cut self-energy" diagram.

the second type have one intersection, and finally there are diagrams of the third type without intersections. In principle, there could have been contributions coming from the gluon loop which lies entirely in the shock wave, but we will demonstrate below that such terms are absent (see the discussion at the end of Sect. V.4).

For the NLO calculation we use the light-cone gauge $p_2^\mu A_\mu = 0$. Also, we find it convenient to change the prescription for the cutoff in the longitudinal direction. We consider the light-like dipoles (in the p_1 direction) and impose the cutoff on the maximal α emitted by any gluon from the Wilson lines so

$$
\begin{aligned}
U_x^\eta &= \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big] \\
A_\mu^\eta(x) &= \int d^4k\, \theta(e^\eta - |\alpha_k|) e^{-ik\cdot x} A_\mu(k)
\end{aligned}
\tag{45}
$$

As we will see below, the (almost) conformal result (44) comes from the regularization (45). In Appendix A.2 we will present the NLO kernel for the cutoff with the slope (33). We start with the calculation of the Fig. 8a diagram. Multiplying two propagators (36), two 3-gluon vertices and two bare propagators we obtain

$$
\begin{aligned}
& g^2 \int_0^\infty du \int_{-\infty}^0 dv (\hat{A}_\bullet^a(up_1 + x_\perp) \hat{A}_\bullet^b(vp_1 + y_\perp)) \qquad (46) \\
&= \frac{1}{2} g^4 \frac{s^2}{4} f^{anl} f^{bn'l'} \int d\alpha d\alpha_1 d\beta d\beta' d\beta_1 d\beta_1' d\beta_2 d\beta_2' \int d^2z d^2z' \int d^2q_1 d^2q_2 d^2k_1 d^2k_2 \\
& e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp} \frac{4\alpha_1(\alpha-\alpha_1) U_z^{nn'} U_{z'}^{ll'} e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp}}{(\beta - \beta_1 - \beta_2 + i\epsilon)(\beta' - \beta_1' - \beta_2' + i\epsilon)(\beta - i\epsilon)(\beta' - i\epsilon)} \\
& \frac{d_{\bullet\lambda}(\alpha p_1 + \beta p_2 + q_{1\perp} + k_{1\perp})}{\alpha\beta s - (q_1+q_2)_\perp^2 + i\epsilon} \frac{d_{\lambda'\bullet}(\alpha p_1 + \beta' p_2 + q_{2\perp} + k_{2\perp})}{\alpha\beta' s - (k_1+k_2)_\perp^2 + i\epsilon}
\end{aligned}
$$

$$\frac{d_{\mu\xi}(\alpha_1 p_1+\beta_1 p_2+q_{1\perp})}{\alpha_1\beta_1 s-q_{1\perp}^2+i\epsilon}\frac{d^{\xi}{}_{\mu'}(\alpha_1 p_1+\beta'_1 p_2+k_{1\perp})}{\alpha_1\beta'_1 s-k_{1\perp}^2+i\epsilon}\frac{d_{\nu\eta}((\alpha-\alpha_1)p_1+\beta_2 p_2+q_{2\perp})}{(\alpha-\alpha_1)\beta_2 s-q_{2\perp}^2+i\epsilon}$$
$$\frac{d^{\eta}{}_{\nu'}((\alpha-\alpha_1)p_1+\beta'_2 p_2+k_{2\perp})}{(\alpha-\alpha_1)\beta'_2 s-k_{2\perp}^2+i\epsilon}\Gamma^{\mu\nu\lambda}(\alpha p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\Gamma^{\mu'\nu'\lambda'}(\alpha p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})$$

where

$$\Gamma_{\mu\nu\lambda}(p,k,-p-k)=(p-k)_\lambda g_{\mu\nu}+(2k+p)_\mu g_{\nu\lambda}+(-2p-k)_\nu g_{\lambda\mu} \qquad (47)$$

In this formula $\frac{1}{\beta-i\epsilon}$ comes from the integration over u parameter in the l.h.s. and $\frac{1}{\beta-\beta_1-\beta_2+i\epsilon}$ from the integration of the left three-gluon vertex over the half-space $x_* > 0$. Similarly, we get $\frac{1}{\beta'-i\epsilon}$ from the integration over v parameter and $\frac{1}{\beta'-\beta'_1-\beta'_2+i\epsilon}$ from the integration of the right three-gluon vertex over the half-space $x_* < 0$. The factor $\frac{1}{2}$ in the r.h.s. is combinatorial. Note that in the light-cone gauge one can always neglect the $\beta p_{2\xi}$ components of the momenta in the three-gluon vertex since they are always multiplied by some $d_{\xi\eta}$.

Taking residues at $\beta=\beta'=0$ and $\beta_2=-\beta_1$, $\beta'_2=-\beta'_1$ we obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(up_1+x_\perp)\hat{A}^b_\bullet(vp_1+y_\perp)\rangle \qquad (48)$$
$$=\frac{1}{2}g^4\frac{s^2}{4}f^{anl}f^{bn'l'}\int d\alpha d\alpha_1 d\beta_1 d\beta'_1\int d^2z d^2z'\int d^2q_1 d^2q_2 d^2k_1 d^2k_2\; e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp}$$
$$4\frac{\alpha_1(\alpha-\alpha_1)}{\alpha^2}U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1)z-i(q_2-k_2)z'}\;\frac{(q_{1\perp}+q_{2\perp})_\lambda}{(q_1+q_2)_\perp^2}\frac{(k_{1\perp}+k_{2\perp})_{\lambda'}}{(k_1+k_2)_\perp^2}$$
$$\frac{d_\mu{}^\xi(\alpha_1 p_1+q_{1\perp})}{\alpha_1\beta_1 s-q_{1\perp}^2+i\epsilon}\frac{d_{\xi\mu'}(\alpha_1 p_1+k_{1\perp})}{\alpha_1\beta'_1 s-k_{1\perp}^2+i\epsilon}\frac{d_\nu{}^\eta((\alpha-\alpha_1)p_1+q_{2\perp})}{-(\alpha-\alpha_1)\beta_1 s-q_{2\perp}^2+i\epsilon}\frac{d_{\eta\nu'}((\alpha-\alpha_1)p_1+k_{2\perp})}{-(\alpha-\alpha_1)\beta'_1 s-k_{2\perp}^2+i\epsilon}$$
$$\Gamma^{\mu\nu\lambda}(\alpha_1 p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\Gamma^{\mu'\nu'\lambda'}(\alpha_1 p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})$$

We have omitted terms $\sim\beta p_2$ in the arguments of $d_{\xi\eta}$ since they do not contribute to $d_{\mu\xi}d^{\xi\mu'}$, see Eq. (37). Introducing the variable $u=\alpha_1/\alpha$ and taking residues at $\beta_1=\frac{q_\perp^2}{\alpha_1 s}$ and $\beta'_1=\frac{k_\perp^2}{\alpha_1 s}$ we obtain

$$-\frac{g^4}{8\pi^2}f^{anl}f^{bn'l'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\;\bar{u}u\int d^2z d^2z'\int d^2q_1 d^2q_2 d^2k_1 d^2k_2$$
$$e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp-i(q_1-k_1,z)-i(q_2-k_2,z')}U_z^{nn'}U_{z'}^{ll'}\frac{(q_{1\perp}+q_{2\perp})_\lambda(k_{1\perp}+k_{2\perp})_{\lambda'}}{(q_1+q_2)_\perp^2(k_1+k_2)_\perp^2}$$

$$\frac{d_{\mu\xi}(u\alpha p_1+q_{1\perp})d^{\xi}{}_{\mu'}(u\alpha p_1+k_{1\perp})}{q_{1\perp}^2\bar u+q_{2\perp}^2 u}\frac{d_{\nu\eta}(\bar u\alpha p_1+q_{2\perp})d^{\eta}{}_{\nu'}(\bar u\alpha p_1+k_{2\perp})}{k_{1\perp}^2\bar u+k_{2\perp}^2 u}$$
$$\Gamma^{\mu\nu\lambda}(u\alpha p_1+q_{1\perp},\bar u\alpha p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\Gamma^{\mu'\nu'\lambda'}(u\alpha p_1+k_{1\perp},\bar u\alpha p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp}) \tag{49}$$

where we have imposed a cutoff $\alpha<\sigma$ in accordance with Eq. (45).

Using formulas

$$\begin{aligned}
&d_{\mu\xi}(u\alpha p_1+q_\perp)d^{\xi\mu'}(u\alpha p_1+k_\perp) \;=\; \Big(g^\perp_{\mu\xi}-\frac{2}{s\alpha u}p_{2\mu}q^\perp_\xi\Big)\Big(g_\perp^{\xi\mu'}-\frac{2}{s\alpha u}k_\perp^\xi\Big),\\
&-(q_1+q_2)^\lambda_\perp\Gamma_{\mu\nu\lambda}(\alpha u p_1+q_{1\perp},\alpha\bar u p_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp)\\
&\;\times(g_\perp^{\mu i}-\frac{2}{s\alpha u}p_2^\mu q_1^i)(g_\perp^{\nu j}-\frac{2}{s\alpha\bar u}p_2^\nu q_2^j)\\
&=\;(q_{1\perp}^2-q_{2\perp}^2)g_\perp^{ij}+\frac{2}{u}q_1^i(q_1+q_2)^j-\frac{2}{\bar u}(q_1+q_2)^i q_2^j
\end{aligned} \tag{50}$$

we can represent the contribution of diagram in Fig. 8a in the form

$$\begin{aligned}
\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.8a}} \;=\; &-\frac{g^4}{8\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\\
\times\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\;\bar u u&\int d^2q_1d^2q_2d^2k_1d^2k_2\;\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,y-z)_\perp-i(k_2,y-z')_\perp}}{(q_1+q_2)^2(k_1+k_2)^2(q_1^2\bar u+q_2^2u)(k_1^2\bar u+k_2^2u)}\\
\times\;\Big[(q_1^2-q_2^2)\delta_{ij}&-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar u}(q_1+q_2)_iq_{2j}\Big]\\
\times\;\Big[(k_1^2-k_2^2)\delta_{ij}&-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar u}(k_1+k_2)_ik_{2j}\Big]
\end{aligned} \tag{51}$$

Throughout this thesis we use Greek letters for indices $\mu=0,1,2,3$ (with $g^{\mu\nu}=(1,-1,-1,-1)$) and Latin letters for transverse indices $i=1,2$.

The diagram shown in Fig. 8b is obtained by the substitution $e^{-i(k_1+k_2,y_\perp)}\to -e^{-i(k_1+k_2,x)_\perp}$ (the different sign comes from the replacement $[-\infty p_1,0]_y$ by $[0,-\infty p_1]_x$). We get

$$\begin{aligned}
&\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.8a+b}}\\
&=\frac{g^4}{8\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\;\bar uu\int d^2q_1d^2q_2d^2k_1d^2k_2\\
&\quad\times\frac{e^{i(q_1+q_2,x)_\perp-i(q_1-k_1,z)-i(q_2-k_2,z')}\big[e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp}\big]}{(q_1+q_2)^2(k_1+k_2)^2(q_1^2\bar u+q_2^2u)(k_1^2\bar u+k_2^2u)}\\
&\quad\times\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar u}(q_1+q_2)_iq_{2j}\Big]\\
&\quad\times\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar u}(k_1+k_2)_ik_{2j}\Big]
\end{aligned} \tag{52}$$

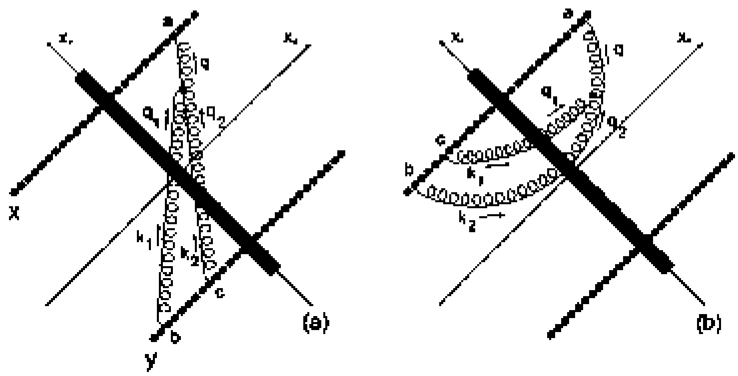

FIG. 9: "Cut vertex" diagrams.

III.1.2 "Cut vertex" diagrams

Next, consider the "cut vertex" diagram in Fig. 9a. The analog of Eq. (48) has the form:

$$
\begin{aligned}
& g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv(\hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+y_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)) \\
&= 2g^4 s f^{mna}\int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2\int d^2q_1 d^2q_2 d^2k_1 d^2k_2 \\
&\times\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_{1\perp},\alpha_2p_1+q_{2\perp},-(\alpha_1+\alpha_2)p_1-(q_1+q_2)_\perp) \\
&\times\frac{\alpha_1\theta(\alpha_1)\theta(\alpha_2)e^{i(q_1+q_2,x)_\perp}\Big((q_1+q_2)_\perp+2(\beta_1+\beta_2)p_2\Big)_\lambda}{(\alpha_1+\alpha_2)(\beta_1+\beta_2-i\epsilon)} \\
&\times\frac{\Big(k_1+\frac{2(k_1,q_1)_\perp}{\alpha_1 s}p_2\Big)_\mu\Big(k_2+\frac{2(k_2,q_2)_\perp}{\alpha_2 s}p_2\Big)_\nu}{[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s-(q_1+q_2)_\perp^2+i\epsilon]} \\
&\times\int d^2z d^2z'\frac{e^{-i(k_1+k_2,y)_\perp}U^{mb}_z U^{nc}_{z'}e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}}{k_1^2(k_1^2\alpha_2+k_2^2\alpha_1)(\alpha_1\beta_1 s-q_{1\perp}^2+i\epsilon)(\alpha_2\beta_2 s-q_{2\perp}^2+i\epsilon)}
\end{aligned}
\tag{53}
$$

Going to variables $\alpha=\alpha_1+\alpha_2$, $u=\alpha_1/\alpha$ and taking residues at $\beta_1+\beta_2=0$ and $\beta_1=\frac{q_1^2}{\alpha_1}$ we get

$$
\begin{aligned}
& g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv(\hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+y_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)) \\
&= \frac{g^4}{2\pi^2}f^{mna}\int_0^\sigma\frac{d\alpha}{\alpha^2}\int_0^1 du\, u\int d^2q_1 d^2q_2 d^2k_1 d^2k_2\int d^2z d^2z' U^{mb}_z U^{nc}_{z'} \\
&\frac{e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}}{(q_1+q_2)_\perp^2(q_1^2\bar{u}+q_2^2u)k_1^2(k_1^2\bar{u}+k_2^2u)}\Big(k_1+\frac{2(k_1,q_1)_\perp}{\alpha u s}p_2\Big)_\mu\Big(k_2+\frac{2(k_2,q_2)_\perp}{\alpha\bar{u}s}p_2\Big)_\nu \\
&\times(q_1+q_2)_{\perp\lambda}\Gamma^{\mu\nu\lambda}(\alpha up_1+q_{1\perp},\alpha\bar{u}p_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp)
\end{aligned}
\tag{54}
$$

and therefore

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.9a}} = -i\frac{g^4}{2\pi^2}f^{mna}\ \mathrm{Tr}\{t^aU_xt^bt^cU_y^\dagger\}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\ \bar{u}u\int d^2k_1d^2k_2d^2q_1d^2q_2$$
$$\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}\frac{(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)} \tag{55}$$
$$\frac{k_{1i}k_{2j}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}$$

where we have used the formula

$$\begin{aligned}&\Big(k_1+\frac{2(k_1,q_1)_\perp}{\alpha us}p_2\Big)_\mu\Big(k_2+\frac{2(k_2,q_2)_\perp}{\alpha\bar{u}s}p_2\Big)_\nu(q_1+q_2)_{\perp\lambda}\\&\times\Gamma^{\mu\nu\lambda}(\alpha up_1+q_{1\perp},\alpha\bar{u}p_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp)\\&=k_{1i}k_{2j}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big]\end{aligned} \tag{56}$$

following from Eq. (50).

The contribution of the diagram shown in Fig. 9b differs from Eq. (55) by the substituion $e^{-i(k_1+k_2,y)_\perp}\ \to\ e^{-i(k_1+k_2,x)_\perp}$ and changing the order of t^b, t^c matrices. (Similarly to the case of the Fig. (8)b diagram, this prescription follows from the replacement $[-\infty p_1,0]_y$ by $[0,\infty p_1]_x$ but now we consider the second term of the expansion in the gauge field). We get

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.9a+b}} = -i\frac{g^4}{2\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\ \bar{u}u\int d^2k_1d^2k_2d^2q_1d^2q_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}$$
$$\frac{(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}e^{i(q_1+q_2,x)_\perp-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}$$
$$\times\frac{k_{1i}k_{2j}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}\ f^{mna}\mathrm{Tr}\{t^aU_x[t^ct^be^{-i(k_1+k_2,x)}+t^bt^ce^{-i(k_1+k_2,y)}]U_y^\dagger\} \tag{57}$$

There is another type of diagrams with two gluon-shockwave intersections shown in Fig. 10

$$g^3\int_0^\infty dt\int_{-\infty}^0 du\int_{-\infty}^0 dv\ \langle\hat{A}_\bullet^a(tp_1+x_\perp)\hat{A}_\bullet^b(up_1+x_\perp)\hat{A}_\bullet^c(vp_1+y_\perp)\rangle \tag{58}$$
$$\begin{aligned}&=\ 2g^4s\int d\alpha_1d\alpha_2d\beta_1d\beta_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}\int\frac{d^2k_1d^2k_2}{k_1^2k_2^2}\ \int d^2q_1d^2q_2\ \theta(\alpha_1)\theta(\alpha_2)\\&\times f^{mna}\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_1,\alpha_2p_1+q_2,-(\alpha_1+\alpha_2)p_1-(q_1+q_2)_\perp)\\&\times\frac{[(q_1+q_2)_\perp+2(\beta_1+\beta_2)p_2]_\lambda}{(\alpha_1+\alpha_2)(\beta_1+\beta_2-i\epsilon)[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s-(q_1+q_2)_\perp^2+i\epsilon]}\\&\times\frac{k_{1\mu}+\frac{2(k_1,q_1)_\perp}{\alpha_1s}p_{2\mu}}{\alpha_1\beta_1s-q_{1\perp}^2+i\epsilon}\ \frac{k_{2\nu}+\frac{2(k_2,q_2)_\perp}{\alpha_2s}p_{2\nu}}{\alpha_2\beta_2s-q_{2\perp}^2+i\epsilon}e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}\end{aligned}$$

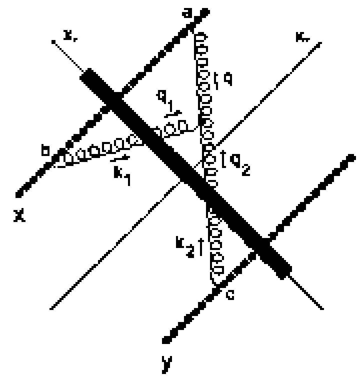

FIG. 10: Another type of diagrams with two gluon-shockwave intersections .

Taking residues at $\beta_1+\beta_2=0$ and at $\beta_1=\frac{q_1^2}{\alpha_1 s}$ and going to variables $\alpha=\alpha_1+\alpha_2$ and $u=\alpha_1/\alpha$ we get

$$g^3\int_0^\infty dt\int_{-\infty}^0 du\int_{-\infty}^0 dv\ \langle\hat{A}^a_*(tp_1+x_\perp)\hat{A}^b_*(up_1+x_\perp)\hat{A}^c_*(vp_1+y_\perp)\rangle \tag{59}$$
$$=\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha^2}\int_0^1 du\int d^2q_1d^2q_2d^2k_1d^2k_2\ U_z^{mb}U_{z'}^{nc}$$
$$\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}}{k_1^2k_2^2(q_1+q_2)^2(\bar{u}q_1^2+uq_2^2)}\Big(k_1+\frac{2(q_1,k_1)_\perp p_2}{s\alpha u}\Big)^\mu\Big(k_2+\frac{2(q_2,k_2)_\perp p_2}{s\alpha\bar{u}}\Big)^\nu$$
$$\times(q_1+q_2)^\lambda\Gamma_{\mu\nu\lambda}(\alpha up_1+q_1,\alpha\bar{u}p_1+q_2,-\alpha p_1-(q_1+q_2)_\perp)$$
$$=\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^2q_1d^2q_2d^2k_1d^2k_2\ U_z^{mb}U_{z'}^{nc}$$
$$\times\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}}{k_1^2k_2^2(q_1+q_2)^2(\bar{u}q_1^2+uq_2^2)}$$
$$\times\Big[(q_{1\perp}^2-q_{2\perp}^2)\delta^{ij}-\frac{2}{u}q_1^i(q_1+q_2)^j+\frac{2}{\bar{u}}(q_1+q_2)^iq_2^j\Big]k_{1i}k_{2j}$$

and therefore

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.10}}=i\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^2zd^2z'\int d^2q_1d^2q_2\int d^2k_1d^2k_2 \tag{60}$$
$$e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}U_z^{mm'}U_{z'}^{nn'}\frac{e^{i(q_1+q_2,x)_\perp-i(k_1,x)_\perp-i(k_2,y)_\perp}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}$$
$$\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big]\frac{k_{1i}k_{2j}}{k_1^2k_2^2}\mathrm{Tr}\{t^aU_xt^{m'}t^{n'}U_y^\dagger\}$$

where again we have used formula (56).

The sum of the contributions (52), (57) and (60) can be represented as follows

$$\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.8+Fig.9+Fig.10}} \equiv \langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.11\ I+III+V+VII+IX}} \tag{61}$$
$$= \frac{g^2}{8\pi^2}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^1 du\ \bar u u \int d^2z d^2z' \int d^2q_1 d^2q_2 \int d^2k_1 d^2k_2\ e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp} U_z^{mm'} U_{z'}^{nn'}$$
$$\times f^{amn}\mathrm{Tr}\Big\{t^a \frac{e^{i(q_1+q_2,x)_\perp}}{(q_1+q_2)^2(q_1^2\bar u + q_2^2 u)}\Big[(q_1^2-q_2^2)\delta_{ij} - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar u}(q_1+q_2)_i q_{2j}\Big]U_x$$
$$\times\Big[t^b f^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar u + k_2^2 u)}\Big[(k_1^2-k_2^2)\delta_{ij} - \frac{2}{u}k_{1i}(k_1+k_2)_j + \frac{2}{\bar u}(k_1+k_2)_i k_{2j}\Big]$$
$$-4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)} - \frac{e^{-i(k_1,x)_\perp - i(k_2,y)_\perp}}{\bar u u k_1^2 k_2^2}t^{m'}t^{n'} + \frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)}\Big)\Big]U_y^\dagger\Big\}$$

If we add contribution of the diagrams with the gluon on the right side of the shock wave attached to the Wilson line at the point y instead of x (which differs from Eq. (61) by the substitution $e^{i(q_1+q_2,x)_\perp} \to -e^{i(q_1+q_2,y)_\perp}$) we obtain

$$\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.11\ I+II+...+X}} \tag{62}$$
$$= \frac{g^2}{8\pi^2}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^1 du\ \bar u u \int d^2z d^2z' \int d^2q_1 d^2q_2 \int d^2k_1 d^2k_2\ e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp} U_z^{mm'} U_{z'}^{nn'}$$
$$\times f^{amn}\mathrm{Tr}\Big\{t^a \frac{(e^{i(q_1+q_2,x)_\perp} - e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar u + q_2^2 u)}\Big[(q_1^2-q_2^2)\delta_{ij} - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar u}(q_1+q_2)_i q_{2j}\Big]U_x$$
$$\times\Big[t^b f^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar u + k_2^2 u)}\Big[(k_1^2-k_2^2)\delta_{ij} - \frac{2}{u}k_{1i}(k_1+k_2)_j + \frac{2}{\bar u}(k_1+k_2)_i k_{2j}\Big]$$
$$-4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)} - \frac{e^{-i(k_1,x)_\perp - i(k_2,y)_\perp}}{\bar u u k_1^2 k_2^2}t^{m'}t^{n'} + \frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)}\Big)\Big]U_y^\dagger\Big\}$$

The result (62) can be obtained from the self-energy contribution (52) by the replacement of the term corresponding to the emission of the two gluons via the three-gluon vertex

$$t^b f^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar u + k_2^2 u)}\Big[(k_1^2-k_2^2)\delta_{ij} - \frac{2}{u}k_{1i}(k_1+k_2)_j + \frac{2}{\bar u}(k_1+k_2)_i k_{2j}\Big]$$

with similar contribution containing the "effective vertex"

$$S^{m'n'}(k_1,k_2;x,y) \equiv$$
$$t^b f^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar u + k_2^2 u)}\Big[(k_1^2-k_2^2)\delta_{ij} - \frac{2}{u}k_{1i}(k_1+k_2)_j + \frac{2}{\bar u}(k_1+k_2)_i k_{2j}\Big]$$
$$-\ 4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)} - \frac{e^{-i(k_1,x)_\perp - i(k_2,y)_\perp}}{\bar u u k_1^2 k_2^2}t^{m'}t^{n'} + \frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar u k_1^2(k_1^2\bar u + k_2^2 u)}\Big) \tag{63}$$

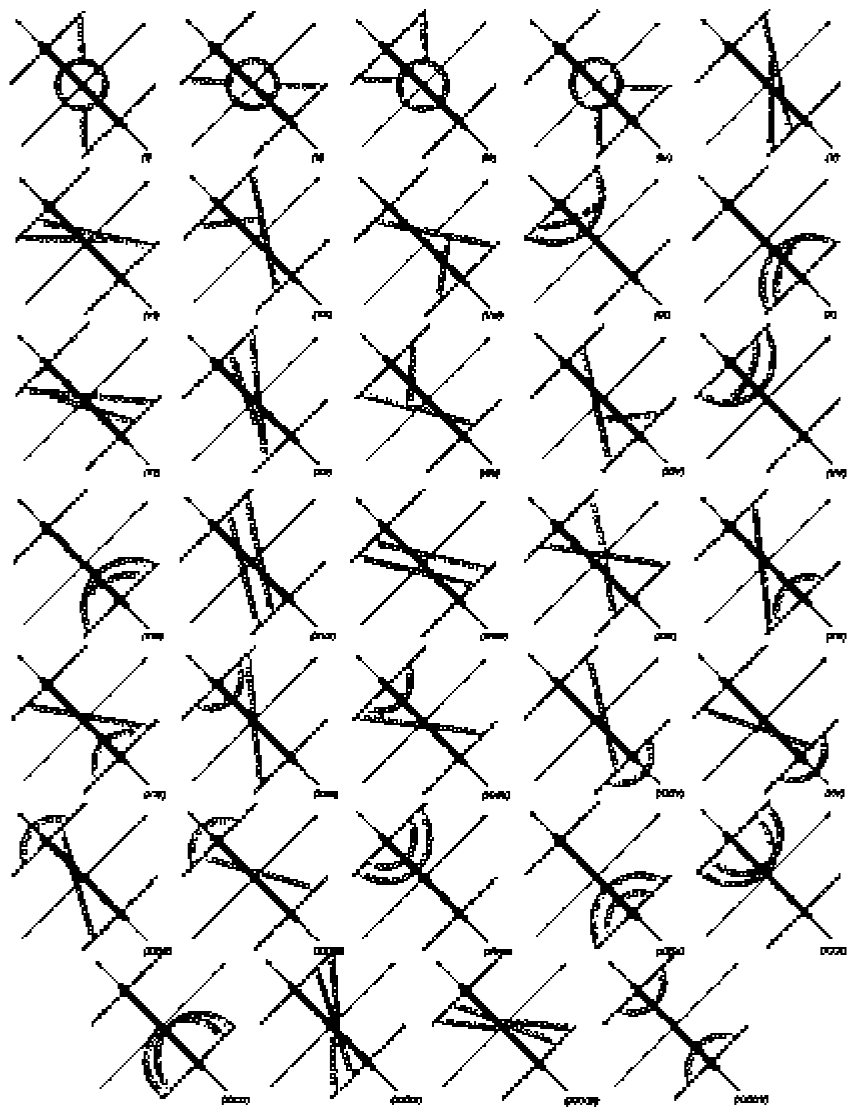

FIG. 11: Diagrams with two cuts.

It can be demonstrated that the sum of the contributions of the diagrams shown in Fig. 11 I,..., IV, XI,..., XVI can be obtained from the self-energy contribution (52) by replacing the gluon vertex

$$t^a f^{amn}\frac{(e^{i(q_1+q_2,x)_\perp}-e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_i q_{2j}\Big] \tag{64}$$

with similar "effective vertex"

$$t^a f^{amn}\frac{(e^{i(q_1+q_2,x)_\perp}-e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_i q_{2j}\Big]$$
$$+\,4iq_{1i}q_{2j}\left(\frac{t^mt^ne^{i(q_1+q_2,x)_\perp}}{\bar{u}q_1^2(q_1^2\bar{u}+q_2^2u)}-\frac{e^{i(q_1,x)_\perp+i(q_2,y)_\perp}}{\bar{u}uq_1^2q_2^2}t^nt^m+\frac{t^nt^me^{i(q_1+q_2,y)_\perp}}{\bar{u}q_1^2(q_1^2\bar{u}+q_2^2u)}\right) \tag{65}$$

Note that (65) is equal to $S^{lmn}(q_1,q_2;x,y)$. Let us consider now the box diagrams topology shown in fig. 11 XVII-XXXIV. The calculation of these diagrams is similar to the above calculation of "cut self-energy" and "cut vertex" diagrams so we present here only the final result

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.11\ XVII+...+XXXIV}} \tag{66}$$
$$=\frac{g^2}{2\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\ \bar{u}u\int d^2zd^2z'\int d^2q_1d^2q_2\int d^2k_1d^2k_2\ e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}$$
$$U_z^{mm'}U_{z'}^{nn'}\mathrm{Tr}\Big\{\Big[q_{1i}q_{2j}\Big(\frac{t^mt^ne^{i(q_1+q_2,x)_\perp}}{\bar{u}q_1^2(q_1^2\bar{u}+q_2^2u)}+\frac{t^nt^me^{i(q_1+q_2,x)_\perp}}{uq_2^2(q_1^2\bar{u}+q_2^2u)}-\frac{e^{i(q_1,x)_\perp+i(q_2,y)_\perp}}{\bar{u}uq_1^2q_2^2}t^nt^m$$
$$-\frac{e^{i(q_1,y)_\perp+i(q_2,x)_\perp}}{\bar{u}uq_1^2q_2^2}t^mt^n+\frac{t^nt^me^{i(q_1+q_2,y)_\perp}}{\bar{u}q_1^2(q_1^2\bar{u}+q_2^2u)}+\frac{t^mt^ne^{i(q_1+q_2,y)_\perp}}{uq_2^2(q_1^2\bar{u}+q_2^2u)}\Big)\Big]U_x$$
$$\times\Big[k_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,x)_\perp}}{uk_2^2(k_1^2\bar{u}+k_2^2u)}-\frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{m'}t^{n'}$$
$$-\frac{e^{-i(k_2,x)_\perp-i(k_1,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{n'}t^{m'}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}+\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,y)_\perp}}{uk_2^2(k_1^2\bar{u}+k_2^2u)}\Big)\Big]U_y^\dagger\Big\}$$

This expression agrees with the sum of "box topology" diagrams in Ref. [26].

Now we observe that each three-gluon vertex diagram is equal to its own cross diagram (the same cannot be said for about diagrams). Thus we may redefine the "effective vertex" (63) in the following way

$$S^{m'n'}(k_1,k_2;x,y)=$$
$$t^bf^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar{u}+k_2^2u)}\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\Big]$$

$$- i2k_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)} + \frac{t^{m'}t^{n'}e^{-i(k_1+k_2,x)_\perp}}{uk_2^2(k_1^2\bar{u}+k_2^2u)} - \frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{m'}t^{n'}$$
$$-\frac{e^{-i(k_2,x)_\perp-i(k_1,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{n'}t^{m'} + \frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)} + \frac{t^{n'}t^{m'}e^{-i(k_1+k_2,y)_\perp}}{uk_2^2(k_1^2\bar{u}+k_2^2u)}\Big) \quad (67)$$

which corresponds to writing each contribution of the three-gluon vertex diagrams as a sum of two equal terms.

A similar expression can be written for the "effective vertex" (65) and therefore the sum of all diagrams with two gluon-shockwave intersections can be written as

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}_{\mathrm{Fig.11}} = \frac{g^2}{8\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\ \bar{u}u\int d^2zd^2z'\int d^2q_1d^2q_2\int d^2k_1d^2k_2$$
$$e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}U_z^{mm'}U_{z'}^{nn'}\mathrm{Tr}\Big\{[S^{lmn}(q_1,q_2;x,y)U_x][S^{m'n'}(k_1,k_2;x,y)U_y^\dagger]\Big\} \quad (68)$$

Separating the contributions of different color structures one obtains

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}_{\mathrm{Fig.11}} = \frac{g^2}{8\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1\frac{du}{\bar{u}u}\ \int d^2zd^2z'\int d^2q_1d^2q_2\int d^2k_1d^2k_2\ U_z^{mm'}U_{z'}^{nn'}$$
$$\times\mathrm{Tr}\Big\{t^a f^{amn}\frac{(e^{i(q_1,X)+i(q_2,X')} - x\leftrightarrow y)}{(q_1^2\bar{u}+q_2^2u)} \quad (69)$$
$$\times\Big[\frac{(q_1^2-q_2^2)\bar{u}u\delta_{ij} - 2\bar{u}q_{1i}(q_1+q_2)_j + 2u(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2} - u\frac{q_{1i}q_{2j}}{q_1^2} + \bar{u}\frac{q_{1i}q_{2j}}{q_2^2}\Big]$$
$$-t^a f^{amn}\frac{q_{1i}q_{2j}}{q_1^2q_2^2}(e^{i(q_1,X)+i(q_2,Y')} - x\leftrightarrow y) \quad (70)$$
$$+i\{t^m,t^n\}\frac{q_{1i}q_{2j}}{q_1^2q_2^2}(e^{i(q_1,X)} - e^{i(q_1,Y)})(e^{i(q_2,X')} - e^{i(q_2,Y')})\Big\}U_x$$
$$\times\Big\{t^b f^{bm'n'}\frac{(e^{-i(k_1,X)-i(k_2,X')} - x\leftrightarrow y)}{(k_1^2\bar{u}+k_2^2u)} \quad (71)$$
$$\Big[\frac{(k_1^2-k_2^2)\bar{u}u\delta_{ij} - 2\bar{u}k_{1i}(k_1+k_2)_j + 2u(k_1+k_2)_ik_{2j}}{(k_1+k_2)^2} - u\frac{k_{1i}k_{2j}}{k_1^2} + \bar{u}\frac{k_{1i}k_{2j}}{k_2^2}\Big]$$
$$- t^b f^{bm'n'}\frac{k_{1i}k_{2j}}{k_1^2k_2^2}(e^{-i(k_1,X)-i(k_2,Y')} - x\leftrightarrow y) \quad (72)$$
$$-i\{t^{m'},t^{n'}\}\frac{k_{1i}k_{2j}}{k_1^2k_2^2}(e^{-i(k_1,X)} - e^{-i(k_1,Y)})(e^{-i(k_2,X')} - e^{-i(k_2,Y')})\Big\}U_y$$

This result agrees with Ref. [26].

Performing the Fourier transformation

$$\int d^2q_1d^2q_2\ e^{i(q_1,x_1)+i(q_2,x_2)}\ \frac{q_{1i}q_{2j}}{q_1^2(q_1^2\bar{u}+q_2^2u)} \ = \ -\frac{x_{1i}x_{2j}}{4\pi^2x_2^2(ux_1^2+\bar{u}x_2^2)}$$
$$\int d^2q_1d^2q_2\ e^{i(q_1,x_1)+i(q_2,x_2)}\ \frac{\delta_{ij}(q_1^2-q_2^2) - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}$$

$$= \frac{-(x_1^2 - x_2^2)\delta_{ij} + \frac{2}{u}(x_1 - x_2)_i x_{2j} + \frac{2}{\bar{u}} x_{1i}(x_1 - x_2)_j}{4\pi^2 (x_1 - x_2)^2 (u x_1^2 + \bar{u} x_2^2)} \tag{73}$$

we get

$$\frac{d}{d\ln\sigma}\langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.11}} = \frac{\alpha_s^2}{8\pi^4}\int_0^1 du\ u\bar{u}\int d^2z d^2z'\ U_z^{bb'} U_{z'}^{cc'}$$
$$\times\ \mathrm{Tr}\Big\{t^a f^{abc}\Big[\frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_i X_j'}{uX^2} - \frac{X_i X_j'}{\bar{u}X'^2}}{uX^2 + \bar{u}X'^2} - \frac{X_i Y_j'}{\bar{u}uX^2 Y'^2} - (x \leftrightarrow y)\Big]$$
$$+ i\frac{\{t^b, t^c\}}{\bar{u}u}\Big(\frac{X_i}{X^2} - \frac{Y_i}{Y^2}\Big)\Big(\frac{X_j'}{X'^2} - \frac{Y_j'}{Y'^2}\Big)\Big\} U_x$$
$$\times\ \Big\{t^{a'} f^{a'b'c'}\Big[\frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_i X_j'}{uX^2} - \frac{X_i X_j'}{\bar{u}X'^2}}{uX^2 + \bar{u}X'^2} - \frac{X_i Y_j'}{\bar{u}uX^2 Y'^2} - (x \leftrightarrow y)\Big]$$
$$- i\frac{\{t^{b'}, t^{c'}\}}{\bar{u}u}\Big(\frac{X_i}{X^2} - \frac{Y_i}{Y^2}\Big)\Big(\frac{X_j'}{X'^2} - \frac{Y_j'}{Y'^2}\Big)\Big\} U_y^\dagger \tag{74}$$

where we introduced the notations

$$X_{ij} \equiv (X^2 - X'^2)\delta_{ij} + \frac{2}{u}(z - z')_i X_j' + \frac{2}{\bar{u}} X_i (z - z')_j$$
$$Y_{ij} \equiv (Y^2 - Y'^2)\delta_{ij} + \frac{2}{u}(z - z')_i Y_j' + \frac{2}{\bar{u}} Y_i (z - z')_j \tag{75}$$

III.1.3 Subtraction of the $(LO)^2$ contribution

It is easy to see that result the (74) for the sum of diagrams in Fig. 11 diverges as $u \to 0$ and $u \to 1$. If we put a lower cutoff $\alpha > \sigma'$ on the α integrals we would get a contribution $\sim \ln^2 \frac{\sigma}{\sigma'}$ coming from the region $\alpha_2 \gg \alpha_1 > \sigma'$ (or $\alpha_1 \gg \alpha_2 > \sigma'$) which corresponds to the the square of the leading-order BK kernel rather than to the NLO kernel. To get the NLO kernel we need to subtract this $(LO)^2$ contribution. Indeed, the operator form of the evolution equation for the color dipole up to the next-to-leading order looks like

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \tag{76}$$

where $\eta = \ln\sigma$. Our goal is to find K_{NLO} by considering the l.h.s. of this equation in the external shock-wave background so

$$\langle K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} - \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} \tag{77}$$

The subtraction (77) leads to the $\left[\frac{1}{u}\right]_+$ prescription for the terms divergent as $\frac{1}{u}$ (and similarly $\frac{1}{\bar{u}} \to \left[\frac{1}{\bar{u}}\right]_+$ for the contribution divergent as $u \to 1$). Here we define $\left[\frac{1}{u}\right]_+$ in the usual way

$$\int_0^1 du\ f(u)\Big[\frac{1}{u}\Big]_+ \ \equiv\ \int_0^1 du\ \frac{f(u)-f(0)}{u}, \qquad \int_0^1 du\ f(u)\Big[\frac{1}{\bar u}\Big]_+ \ \equiv\ \int_0^1 du\ \frac{f(u)-f(1)}{\bar u} \tag{78}$$

To illustrate this prescription, consider the divergent terms in Eq. (74) proportional to $(X,Y)(Y',z-z')$ or $(X',Y')(Y,z-z')$

$$\begin{aligned}
\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle \ =\ & \frac{\alpha_s^2}{4\pi^4}\int_0^1\frac{du}{\bar u}\int d^2z d^2z'\ U^{bb'}_z U^{cc'}_{z'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\Big[\frac{1}{uX^2+\bar uX'^2}\\
&\Big[f^{abc}\mathrm{Tr}\{t^aU_x\Big(\frac{uf^{a'b'c'}t^{a'}}{uY^2+\bar uY'^2}-\frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\}\\
&\qquad +f^{a'b'c'}\mathrm{Tr}\{\Big(t^a\frac{uf^{abc}}{uY^2+\bar uY'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\}\Big]\\
&+\frac{1}{X^2(uY^2+\bar uY'^2)}\mathrm{Tr}\{(f^{abc}t^a+i\{t^b,t^c\})U_xf^{a'b'c'}t^{a'}U^\dagger_y+t^af^{abc}U_x(f^{a'b'c'}t^{a'}-i\{t^{b'},t^{c'}\})U^\dagger_y\}\Big]\\
&+\frac{\alpha_s^2}{4\pi^4}\int_0^1\frac{du}{u}\int d^2zd^2z'\ U^{bb'}_zU^{cc'}_{z'}\frac{(X',Y')(Y,z-z')}{(z-z')^2Y^2}\Big[\frac{1}{uX^2+\bar uX'^2}\\
&\Big[f^{abc}\mathrm{Tr}\{-t^aU_x\Big(\frac{\bar uf^{a'b'c'}t^{a'}}{(uY^2+\bar uY'^2)}+\frac{i}{Y'^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\}\\
&\qquad +f^{a'b'c'}\mathrm{Tr}\{\Big(-t^a\frac{\bar uY_iY'_jf^{abc}}{uY^2+\bar uY'^2}+\frac{i}{Y'^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\}\Big]\\
&+\frac{1}{X'^2(uY^2+\bar uY'^2)}\mathrm{Tr}\{(-f^{abc}t^a+i\{t^b,t^c\})U_xf^{a'b'c'}t^{a'}U^\dagger_y-t^af^{abc}U_x(f^{a'b'c'}t^{a'}+i\{t^{b'},t^{c'}\})U^\dagger_y\}\Big]
\end{aligned} \tag{79}$$

Note that the second term is equal to the first one after the replacement $u\leftrightarrow\bar u$, $z\leftrightarrow z'$ and $b\leftrightarrow c, b'\leftrightarrow c'$.

It is convenient to return back to the notation α_1 and $\alpha_2=\sigma-\alpha_1$ (after application of $\frac{d}{d\ln\sigma}$ the value of α is set equal to σ).

$$\begin{aligned}
\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle = & \frac{\alpha_s^2}{2\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2zd^2z'\ U^{bb'}_zU^{cc'}_{z'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\
&\Big[\frac{\sigma}{\alpha_1X^2+\alpha_2X'^2}\Big[f^{abc}\mathrm{Tr}\{t^aU_x\Big(\frac{\alpha_1f^{a'b'c'}t^{a'}}{\alpha_1Y^2+\alpha_2Y'^2}-\frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\}\\
&\qquad +f^{a'b'c'}\mathrm{Tr}\{\Big(\frac{\alpha_1f^{abc}t^a}{\alpha_1Y^2+\alpha_2Y'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\}\Big]\\
&+\frac{2i\sigma}{X^2(\alpha_1Y^2+\alpha_2Y'^2)}\mathrm{Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U^\dagger_y-t^af^{abc}U_xt^{b'}t^{c'}U^\dagger_y\}\Big]
\end{aligned} \tag{80}$$

The corresponding term in $K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}$ is (see Eq. (32))

$$K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\} \;=\; -\frac{4\alpha_s}{\pi^2}\int d^2z\;\frac{(x-z,y-z)}{(x-z)^2(y-z)^2}{\rm Tr}\{t^a\hat U_x t^b\hat U^\dagger_y\}{\rm Tr}\{t^a\hat U_z t^b\hat U^\dagger_z\} \qquad (81)$$

The relevant term in the "matrix element" $\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle$ in the external shock-wave background comes from $\hat U_x,\hat U^\dagger_z$ taken in the leading order in α_s (so that $\hat U_x\to U_x$, $\hat U^\dagger_z\to U^\dagger_z$) and $\hat U_z\otimes\hat U^\dagger_y$ taken in the first order in α_s

$$\langle\hat U_z\otimes\hat U^\dagger_y\rangle\sim-\frac{\alpha_s}{\pi^2}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2z'\;\frac{(z-z',y-z')}{(z-z')^2(y-z')^2}(t^cU_z\otimes t^{c'}U^\dagger_y+U_zt^{c'}\otimes U^\dagger_yt^c)U^{cc'}_{z'}, \qquad (82)$$

or vice versa: $\hat U_x\to U_x$, $\hat U_z\to U_z$ and

$$\langle\hat U^\dagger_z\otimes\hat U^\dagger_y\rangle\sim\frac{\alpha_s}{\pi^2}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2z'\;\frac{(z-z',y-z')}{(z-z')^2(y-z')^2}(t^{c'}U^\dagger_z\otimes U^\dagger_yt^c+U^\dagger_zt^c\otimes t^{c'}U^\dagger_y)U^{cc'}_{z'} \qquad (83)$$

Here we have used the leading-order equations for Wilson lines with arbitrary color indices [45, 31]. Substituting eqs. (82) and (83) in Eq. (81) we obtain

$$\begin{aligned}\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle=&\frac{2\alpha_s^2}{\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2zd^2z'\frac{(X,Y)(Y',z-z')}{X^2Y^2Y'^2(z-z')^2}\\ &\times{\rm Tr}\{(if^{a'b'c'}t^ct^bU_xt^{b'}U^\dagger_y-if^{abc}t^aU_xt^{b'}t^{c'}U^\dagger_y)U^{bb'}_zU^{cc'}_{z'}\}\end{aligned} \qquad (84)$$

From Eq. (77) we get

$$\begin{aligned}&\langle K_{\rm NLO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle=\frac{\alpha_s^2}{2\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2zd^2z'\;U^{bb'}_zU^{cc'}_{z'} \qquad (85)\\ &\times\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\Big[\frac{\sigma}{\alpha_1X^2+\alpha_2X'^2}\Big[f^{abc}{\rm Tr}\{t^aU_x\Big(\frac{\alpha_1f^{a'b'c'}t^{a'}}{\alpha_1Y^2+\alpha_2Y'^2}-\frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\}\\ &\qquad+f^{a'b'c'}{\rm Tr}\{\Big(\frac{\alpha_1f^{abc}t^a}{\alpha_1Y^2+\alpha_2Y'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\}\Big]\\ &+\frac{2i\sigma}{X^2(\alpha_1Y^2+\alpha_2Y'^2)}{\rm Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U^\dagger_y-t^af^{abc}U_xt^{b'}t^{c'}U^\dagger_y\}\Big]\\ &-\frac{2\alpha_s^2}{\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2z\;\frac{(X,Y)(Y',z-z')}{X^2Y^2Y'^2(z-z')^2}{\rm Tr}\{if^{a'b'c'}t^ct^bU_xt^{b'}U^\dagger_y-if^{abc}t^aU_xt^{b'}t^{c'}U^\dagger_y\}U^{bb'}_zU^{cc'}_{z'}\\ &=\frac{\alpha_s^2}{2\pi^4}\int_0^1\frac{du}{\bar u}\int d^2zd^2z'\;U^{bb'}_zU^{cc'}_{z'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\ &\Big[f^{abc}f^{a'b'c'}{\rm Tr}\{t^aU_xt^{a'}U^\dagger_y\}\Big[\frac{2u}{(uX^2+\bar uX'^2)(uY^2+\bar uY'^2)}-\frac{2}{X^2Y^2}\Big]\end{aligned}$$

$$-\frac{i}{Y^2}\Big[\frac{1}{uX^2+\bar uX'^2}-\frac{1}{X^2}\Big]f^{abc}\mathrm{Tr}\{t^aU_x\{t^{b'},t^{c'}\}U_y^\dagger\}$$
$$+\frac{i}{Y^2}\Big[\frac{1}{uX^2+\bar uX'^2}-\frac{1}{X^2}\Big]f^{a'b'c'}\mathrm{Tr}\{\{t^b,t^c\}U_xt^{a'}U_y^\dagger\}$$
$$+\frac{2i}{X^2}\Big[\frac{1}{uY^2+\bar uY'^2}-\frac{1}{Y^2}\Big]\mathrm{Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U_y^\dagger-t^af^{abc}U_xt^{b'}t^{c'}U_y^\dagger\}\Big]$$

which corresponds to the $\left[\frac{1}{\bar u}\right]_+$ prescription (78) (the same prescription was used in Ref. [26]). Note that the "plus" prescription (78) is a consequence of the "rigid" cutoff $|\alpha|<\sigma$ (45); with the "smooth" cutoff (33) we would get different results - see Appendix A.2.

III.1.4 Assembling the result for 1→3 dipoles transition

There are four color structures in the r.h.s. of Eq. (74). Three of them reduce to

$$
\begin{aligned}
f^{abc}f^{a'b'c'}U_z^{bb'}U_{z'}^{cc'}\mathrm{Tr}\{t^aU_xt^{a'}U_y^\dagger\}&=\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\\
&-\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}+(z\leftrightarrow z')\\
if^{abc}U_z^{bb'}U_{z'}^{cc'}\mathrm{Tr}(t^aU_x\{t^{b'},t^{c'}\}U_y^\dagger)&=-\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\\
&+\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}-(z\leftrightarrow z')\\
if^{a'b'c'}U_z^{bb'}U_{z'}^{cc'}\mathrm{Tr}(\{t^b,t^c\}U_xt^{a'}U_y^\dagger)&=\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\\
&+\frac{1}{4}\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}-(z\leftrightarrow z')
\end{aligned}
\tag{86}
$$

Indeed, we will not need the explicit form of the fourth color structure $U_z^{aa'}U_{z'}^{bb'}\mathrm{Tr}(\{t^a,t^b\}U_x\{t^{a'},t^{b'}\}U_y^\dagger)$ since it is multiplied by pure LO[2] integral $\int\frac{du}{\bar uu}=\int\frac{du}{\bar u}+\int\frac{du}{u}$ and does not contribute to the NLO kernel.

Performing integration over u using the prescription (78) after some algebra we get

$$
\begin{aligned}
&\int_0^1du\;u\bar u\Big[\frac{1}{uX^2+\bar uX'^2}\Big(\frac{X_{ij}}{(z-z')^2}+\frac{X_iX'_j}{uX^2}-\frac{X_iX'_j}{\bar uX'^2}\Big)-\frac{X_iY'_j}{\bar uuX^2Y'^2}-(x\leftrightarrow y)\Big]^2\\
&=\frac{1}{(z-z')^4}\Big[-4+2\frac{X^2Y'^2+X'^2Y^2-4\Delta^2(z-z')^2}{X^2Y'^2-X'^2Y^2}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big]\\
&+\Big(\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big)\ln\frac{X^2Y'^2}{X'^2Y^2}
\end{aligned}
\tag{87}
$$

and

$$\int_0^1du\Big[\frac{1}{uX^2+\bar uX'^2}\Big(\frac{X_{ij}}{(z-z')^2}+\frac{X_iX'_j}{uX^2}-\frac{X_iX'_j}{\bar uX'^2}\Big)-\frac{X_iY'_j}{\bar uuX^2Y'^2}-(x\leftrightarrow y)\Big]$$

$$\times\Big(\frac{X_i}{X^2}-\frac{Y_i}{Y^2}\Big)\Big(\frac{X'_j}{X'^2}-\frac{Y'_j}{Y'^2}\Big)$$
$$= -\frac{(x-y)^4}{2X^2Y^2X'^2Y'^2}\ln\frac{X^2Y'^2}{X'^2Y^2}+\frac{(x-y)^2}{2(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{X'^2Y^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2} \tag{88}$$

so the two-cut contribution (74) reduces to

$$\begin{aligned}\frac{d}{d\eta}(\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\})_{\mathrm{Fig.11}} &= \frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\Big[\Big\{-\frac{4}{(z-z')^4}\\ &+\Big(2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}\\ &+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big)\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\\ &\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}]\\ &+\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}\\ &\times\ln\frac{X^2Y'^2}{X'^2Y^2}\quad\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big]\end{aligned} \tag{89}$$

This result agrees with the 1→3 dipoles kernel calculated in Ref. [26].

III.1.5 Subtraction of the UV part

The integral in the r.h.s. of Eq. (89) diverges as $z\to z'$. It is convenient to separate the divergent term by subtracting and adding the contribution at $z=z'$:

$$\begin{aligned}&\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}]\\ &=[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}-(z'\to z)]\\ &\quad+[N_c\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\mathrm{Tr}\{U_xU_y^\dagger\}]\end{aligned} \tag{90}$$

For the last line in the r.h.s. of Eq. (89) the subtraction is redundant since

$$\begin{aligned}&\int\frac{d^2z'}{\pi}\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\\ &=4\zeta(3)[\delta(Y)-\delta(X)]\end{aligned} \tag{91}$$

The easiest way to prove this at $z\neq x,y$ is to set $y=0$ and make an inversion $x\to 1/\tilde x$ so the integral (330) reduces to

$$\int d^2\tilde z'\,\frac{(\tilde x-\tilde z,\tilde x-\tilde z')}{(\tilde x-\tilde z')^2(\tilde z-\tilde z')^2}\ln\frac{(\tilde x-\tilde z)^2}{(\tilde x-\tilde z')^2}=0$$

The δ-function terms in the r.h.s. of Eq. (330) can be restored from the formula

$$\int\frac{d^2zd^2z'}{\pi^2}\Big\{\frac{-(x-y)^2(x-y,Y)}{X^2Y'^2X'^2Y^2} + \frac{(x-y,Y)}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2} = 4\zeta(3) \tag{92}$$

Thus, separating the UV-divergent terms in Eq. (89) according to Eq. (90) we obtain

$$\begin{aligned}
\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.11}} &= \frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4}+ \\
&\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] \\
&+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
&\times[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_zU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}-(z'\to z)] \\
&+\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\} \\
&\times\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}\Big] \\
&+\frac{\alpha_s^2}{16\pi^4}\int d^2z[N_c\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_y\}]\int d^2z'\Big[-\frac{4}{(z-z')^4}+ \\
&\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} \\
&+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big]
\end{aligned} \tag{93}$$

The first term is now finite while the second term contains the UV divergent contribution which reflects the usual UV divergency of the one-loop diagrams. To find the second term we use the dimensional regularization in the transverse space and set $d_\perp = 2-\epsilon$. Because the Fourier transforms (73) are more complicated at $d_\perp \neq 2$ it is convenient to return back to Eq. (69) and calculate the subtracted term in the the momentum representation. The calculation is performed in Appendix A.1 and here we only quote the final result (285)

$$\begin{aligned}
\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.11}} &= \frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4} \\
&+\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}
\end{aligned} \tag{94}$$

$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$
$$\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}-(z'\to z)]$$
$$+\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}$$
$$\times\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big]$$
$$-\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{(x-y)^2}\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]$$
$$\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$

where μ is the normalization scale in the $\overline{MS}$ scheme. It is worth noting that the $d_\perp=2-\epsilon$ regularization in the transverse space is independent of the ("rigid" or "smooth") cutoff in the longitudinal direction.

III.2 DIAGRAMS WITH ONE GLUON-SHOCKWAVE INTERSECTION

III.2.1 "Running coupling" diagrams

The relevant diagrams are shown in Fig. 12 (plus permutations). Let us start from the sum of diagrams Fig. 12 a and b. It has the form:

$$\int_0^\infty du\int_{-\infty}^0 dv\ \langle\hat A^a_\bullet(up_1+x_\perp)\hat A^b_\bullet(vp_1+y_\perp)\rangle_{\mathrm{Fig.12a+b}} \tag{95}$$
$$=g^2N_c\frac{s}{2}\int d^2k_\perp d^2k'_\perp\frac{d^2q_\perp}{q_\perp^2}\int d^2z\ U_z^{ab}\ e^{i(q,x-z)_\perp-i(k,y-z)_\perp}\int_0^\infty\frac{d\alpha}{\alpha}\int d\alpha'\int\frac{d\beta d\beta'}{\beta-i\epsilon}$$
$$\Big[\frac{(q+\frac{2(k,q)_\perp}{\alpha s}p_2)_\lambda}{(k^2+i\epsilon)^2}\frac{d_{\mu\mu'}(k-k')}{(k-k')^2+i\epsilon}$$
$$\times\Gamma^{\mu\nu\lambda}((\alpha-\alpha')p_1+(k-k')_\perp,\alpha'p_1+k'_\perp,-\alpha p_1-k_\perp)\frac{d_{\nu\nu'}(k')}{k'^2+i\epsilon}(k+2\beta p_2)_{\lambda'}$$
$$\times\Gamma^{\mu'\nu'\lambda'}((\alpha-\alpha')p_1+(k-k')_\perp,\alpha'p_1+k'_\perp,-\alpha p_1-k_\perp)$$
$$-2\frac{(k+2\beta p_2)_\nu(q_\mu+\frac{2(k,q)_\perp}{\alpha s}p_{2\mu})}{(k^2+i\epsilon)^2}\frac{g^{\mu\nu}d^\xi_\xi(k')-d^{\mu\nu}(k')}{k'^2+i\epsilon}\Big]$$

where the first term in the square brackets comes from Fig. 12a and the second from Fig. 12b. We use the principal-value prescription for the $1/\alpha'$ terms in $d_{\mu\nu}(k')$ in loop integrals.

To regularize the UV divergence we change the dimension of the transverse space to 2-ϵ. After some algebra one obtains

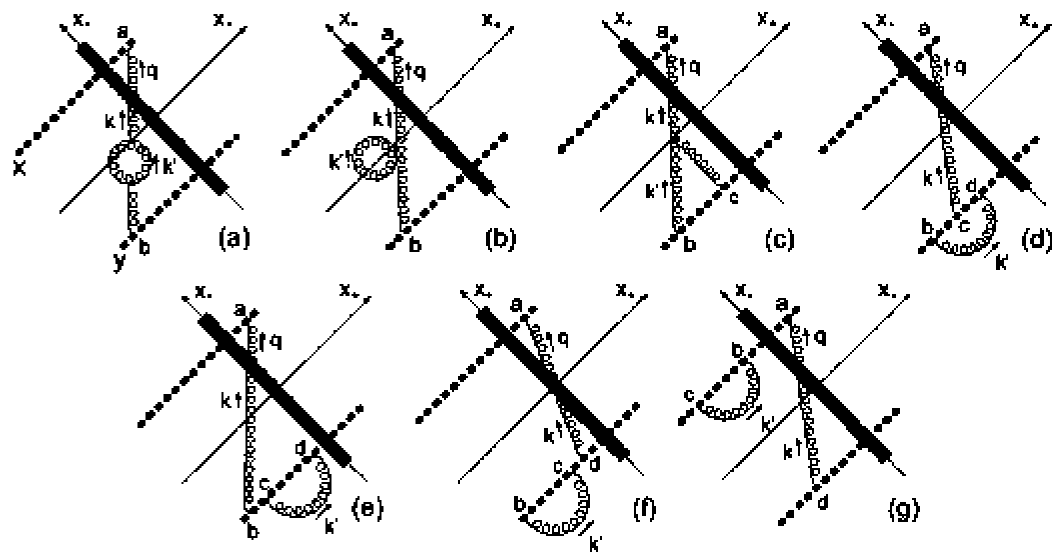

FIG. 12: "Running coupling" diagrams.

$$
\begin{aligned}
&\int_0^\infty du \int_{-\infty}^0 dv\ \langle \hat{A}^a_\bullet(up_1+x_\perp)\hat{A}^b_\bullet(vp_1+y_\perp)\rangle_{\rm Fig.12a+b} = g^2N_c\mu^{2\varepsilon}\int d^{2-\varepsilon}k\, d^{2-\varepsilon}k'\, d^{2-\varepsilon}q \qquad (96)\\
&\int d^{2-\varepsilon}z\, U^{ab}_z\int_0^\infty d\alpha\int d\beta d\beta' \frac{e^{i(q,X)_\perp - i(k,Y)_\perp}}{\alpha(\beta-i\epsilon)(\alpha\beta s-k_\perp^2+i\epsilon)^2q^2}\\
&\times\frac{s}{2}\int d\alpha' d\beta' \frac{1}{(\alpha'\beta' s-k'^2_\perp+i\epsilon)[(\alpha-\alpha')(\beta-\beta')s-(k-k')^2_\perp+i\epsilon]}\\
&\times\Big\{-\varepsilon[(\alpha-2\alpha')\beta s+k'^2_\perp-(k-k')^2_\perp][\frac{\alpha-2\alpha'}{\alpha}(k,q)_\perp+(2k'-k,q)_\perp]\\
&+2\frac{(\alpha-2\alpha')^2}{\alpha}(k,q)_\perp\beta s+\frac{(\alpha-2\alpha')}{\alpha}(k,q)_\perp(2k'-k,k)_\perp+(\alpha-2\alpha')\beta s(2k'-k,q)_\perp\\
&+(q,k)_\perp(k-2k',k-k')_\perp+(q,k')_\perp(k,2k'-k)_\perp+\frac{\alpha+\alpha'}{\alpha-\alpha'}\Big[\frac{\alpha-2\alpha'}{\alpha}(q,k)_\perp(k,k-k')_\perp+\\
&(q,2k'-k)_\perp(k,k-k')_\perp+(q,k)_\perp(k'^2_\perp-k^2_\perp)+(q,k-k')_\perp(k,2k-k')_\perp\\
&+(\alpha-2\alpha')(q,k-k')_\perp\beta s+(q,k-k')_\perp(k'^2_\perp-(k-k')^2_\perp)\\
&-(q,k')_\perp(k',k-k')_\perp+(q,k-k')_\perp(k-k')^2_\perp\Big]\\
&+\frac{\alpha'-2\alpha}{\alpha'}\Big[\frac{\alpha-2\alpha'}{\alpha}(q,k)_\perp(k,k')_\perp+(q,2k'-k)_\perp(k,k')_\perp-(q,k')_\perp(k,k+k')_\perp\\
&+(q,k)_\perp(k',2k-k')_\perp+(\alpha-2\alpha')(q,k')_\perp\beta s+(q,k')_\perp(k,2k'-k)_\perp
\end{aligned}
$$

$$-(q,k')_\perp k'^2_\perp + (q,k-k')_\perp(k',k-k')_\perp\Big]$$
$$+\frac{\alpha+\alpha'}{\alpha-\alpha'}\frac{\alpha'-2\alpha}{\alpha'}\Big[(k,k')_\perp(q,k-k')_\perp + (q,k')_\perp(k,k-k')_\perp\Big] + [(k-k')^2_\perp + k'^2](q,k)_\perp +$$
$$+4\alpha(q,k)_\perp\Big[\frac{\alpha-\alpha'}{\alpha'}+\frac{\alpha}{\alpha-\alpha'}\Big]\beta s - 4\alpha(q,k)_\perp\Big[\frac{k'^2_\perp}{\alpha-\alpha'}+\frac{(k-k')^2_\perp}{\alpha'}\Big]\Big\}$$

where we have omitted the contribution

$$\int d\alpha' d\beta' d^2k' \frac{1}{\alpha'(\alpha'\beta' s - k'^2_\perp + i\epsilon)} = \int d^4k' \frac{1}{k'^2+i\epsilon}\, V.p.\frac{1}{(k',p_2)} = 0 \tag{97}$$

Taking residues at $\beta = 0$ and $\beta' = \frac{k'^2_\perp}{\alpha' s}$ and changing to variable $u = \frac{\alpha'}{\alpha}$ we obtain

$$\int_0^\infty du \int_{-\infty}^0 dv\ \langle \hat{A}^a_\bullet(up_1 + x_\perp)\hat{A}^b_\bullet(vp_1 + y_\perp)\rangle_{\mathrm{Fig.12a+b}} \tag{98}$$
$$= -\frac{g^2N_c}{8\pi^2}\mu^{2\epsilon}\int d^{2-\epsilon}k d^{2-\epsilon}k' d^{2-\epsilon}q \int d^{2-\epsilon}z\ U^{ab}_z \int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1 du\ \frac{e^{i(q,x-z)_\perp - i(k,y-z)_\perp}}{k^4q^2[k'^2\bar{u} + (k-k')^2u]}$$
$$\times\Big\{-2\epsilon(2k'-k,k)(q,k'-ku) + (1-2u)(k,q)(2k'-k,k) + (q,k)(k-2k',k-k')$$
$$+(q,k')(k,2k'-k) + \frac{1+u}{\bar{u}}\Big[(1-2u)(q,k)(k,k-k') + 2(q,k)k'^2 - 2(k,k')_\perp(q,k')\Big]$$
$$-\frac{2-u}{u}\Big[(1-2u)(q,k)(k,k') + 2(q,k)(k-k',k') - 2(q,k')(k-k',k)\Big]$$
$$-\frac{(1+u)(2-u)}{u\bar{u}}[(k,k')(q,k-k') + (q,k')(k,k-k')]$$
$$+(q,k)\Big[(k-k')^2 + k'^2 - 4\frac{k'^2}{\bar{u}} - 4\frac{(k-k')^2}{u}\Big]\Big\}$$

Using the "plus" - prescription (78) to subtract the $(\mathrm{LO})^2$ contribution we get

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\ 12a+b}} \tag{99}$$
$$= -2\alpha_s^2N_c\mu^{2\epsilon}\int d^{2-\epsilon}z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}]$$
$$\times\int d^{2-\epsilon}k d^{2-\epsilon}k' d^{2-\epsilon}q \frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}\Bigg[\Big(\frac{(q,2k-k')}{k'^2} - \frac{(q,k+k')}{(k-k')^2}\Big)\ln\frac{(k-k')^2}{k'^2}$$
$$+\int_0^1 du \frac{(2-\epsilon)(q,ku-k')(k,k-2k') + 2(q,k)k^2}{k^2[k'^2\bar{u} + (k-k')^2u]}\Bigg]$$

Next we calculate diagram shown in Fig. 12c.

$$g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv\ \langle \hat{A}^a_\bullet(tp_1+x)\hat{A}^b_\bullet(up_1+y)\hat{A}^c_\bullet(vp_1+y)\rangle_{\mathrm{Fig.12c}} \tag{100}$$

$$= -2i g^4 f^{lbc}\int d\!\!{}^-\alpha\, d\!\!{}^-\beta\, d\!\!{}^-\alpha'\, d\!\!{}^-\beta'\, d\!\!{}^-\beta''\, d\!\!{}^{-2-\epsilon}q\, d\!\!{}^{-2-\epsilon}k\, d\!\!{}^{-2-\epsilon}k'\, e^{i(q,x-z)_\perp - i(k,y-z)_\perp}$$
$$\frac{\theta(\alpha)U_z^{al}}{q_\perp^2(\alpha\beta s - k^2 + i\epsilon)}\frac{(k'_\perp + 2\beta' p_2)_\mu}{\alpha'\beta' s - k'^2 + i\epsilon}\frac{(q_\perp + \frac{2}{\alpha s}(q,k)_\perp p_2)_\lambda}{\alpha'(\alpha - \alpha')}$$
$$\frac{((k-k')_\perp + 2\beta'' p_2)_\nu}{(\alpha-\alpha')\beta'' s - (k-k')_\perp^2 + i\epsilon}\frac{\Gamma^{\mu\nu\lambda}(\alpha' p_1 + k'_\perp, (\alpha-\alpha')p_1 + (k-k')_\perp, -\alpha p_1 - k_\perp)}{(\beta' - i\epsilon)(\beta'' + \beta' - i\epsilon)(\beta - \beta' - \beta'' - i\epsilon)}$$

There are 2 regions of integration over α's: $\alpha > |\alpha'|$ and $\alpha < |\alpha'|$. Taking relevant residues, we obtain

$$-\frac{g^4}{2\pi^2} f^{abl}\mu^{2\epsilon}\int_0^\infty \frac{d\alpha}{\alpha}\int d\!\!{}^{-2-\epsilon}k\, d\!\!{}^{-2-\epsilon}k'\, d\!\!{}^{-2-\epsilon}q\int d^2 z_\perp\, U_z^{cl}\ \frac{e^{i(q,X)-i(k,Y)}}{k'^2 q^2 k^2}\int_0^1 du$$
$$\Big\{\frac{1}{k'^2\bar u + (k-k')^2 u}\Big[\frac{(q,k')}{\bar u}[k'^2 + k^2] + \ \frac{k'^2}{u}(q, 2k - k') - 2(q,k)(k', k-k')\Big]$$
$$+\frac{1}{(k-k')^2\bar u + k^2 u}\Big[\frac{1}{u}[(q,k)k'^2 + (q,k')k^2] - (k', 2k-k')(q,k)\Big]\Big\} \qquad (101)$$

where we have introduced the variable $u = |\alpha'|/\alpha$ as usual. After integration over u with help of Eq. (78) this reduces to

$$-\frac{g^4}{2\pi^2} f^{abl}\mu^{2\epsilon}\int_0^\infty \frac{d\alpha}{\alpha}\int d\!\!{}^{-2-\epsilon}k\, d\!\!{}^{-2-\epsilon}k'\, d\!\!{}^{-2-\epsilon}q\int d^{2-\epsilon}z_\perp\ U_z^{cl}$$
$$\Big\{\Big[\frac{(q,k')}{(k-k')^2}[k'^2 + k^2] + (q, k' - 2k)\Big]\ln\frac{(k-k')^2}{k'^2}$$
$$+\Big[\frac{1}{(k-k')^2}[(q,k)k'^2 + (q,k')k^2] + (q,k)\Big]\ln\frac{(k-k')^2}{k^2}$$
$$-\int_0^1 du\ \frac{2(q,k)(k', k-k')}{k'^2\bar u + (k-k')^2 u}\Big\}\ \frac{e^{i(q,X)-i(k,Y)}}{k'^2 q^2 k^2} \qquad (102)$$

and therefore

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\mathrm{Fig.\ 12c}} = -\frac{g^4 N_c}{8\pi^2}\mu^{2\epsilon}\int d^{2-\epsilon}z[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\times\int d\!\!{}^{-2-\epsilon}k\, d\!\!{}^{-2-\epsilon}k'\, d\!\!{}^{-2-\epsilon}q\ \frac{e^{i(q,X)-i(k,Y)}}{k^2 q^2}\Big\{\Big[\frac{(q,k')}{(k-k')^2} + \frac{(q,k')k^2}{k'^2(k-k')^2} + \frac{(q,k'-2k)}{k'^2}\Big]\ln\frac{(k-k')^2}{k^2}$$
$$-\Big[\frac{(q,k)}{k'^2} + \frac{(q,k)}{(k-k')^2} + \frac{(q,k-k')k^2}{k'^2(k-k')^2}\Big]\ln\frac{k^2}{k'^2} - 2\int_0^1 du\ \frac{(q,k)(k',k-k')}{k'^2[k'^2\bar u + (k-k')^2 u]}\Big\} \qquad (103)$$

Next we calculate the sum of diagrams in Fig. 12 d,e, and f. The contribution of the diagram shown in Fig. 12d is

$$\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\mathrm{Fig.\ 12d}} = \int d\!\!{}^-\alpha\, d\!\!{}^-\alpha'\, d\!\!{}^-\beta\, d\!\!{}^-\beta'\int d\!\!{}^{-2-\epsilon}k\, d\!\!{}^{-2-\epsilon}k'\, d\!\!{}^{-2-\epsilon}q\int d^{2-\epsilon}z\ U_z^{ab} \qquad (104)$$

$$\times \frac{4g^4\mu^{2\epsilon}\theta(\alpha)(q,k)_\perp e^{i(q,x-z)_\perp - i(k,y-z)_\perp}\mathrm{Tr}\{t^aU_xt^ct^bt^cU_y^\dagger\}}{\alpha\alpha'(\beta+\beta'-i\epsilon)(\beta-i\epsilon)(\alpha'\beta's-k'^2_\perp+i\epsilon)(\alpha\beta s-k_\perp^2+i\epsilon)q_\perp^2}$$

$$= -\frac{g^4}{\pi^2}\mu^{2\epsilon}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^{2-\epsilon}q d^{2-\epsilon}k d^{2-\epsilon}k' \int d^{2-\epsilon}z\ U_z^{ab} \quad (105)$$

$$\times \frac{(q,k)_\perp e^{i(q,x-z)_\perp - i(k,y-z)_\perp}}{uk^2(uk^2+\bar{u}k'^2)q^2}\mathrm{Tr}\{t^aU_xt^ct^bt^cU_y^\dagger\}$$

where we took residues at $\beta = \frac{k_\perp^2}{\alpha s}$, $\beta' = \frac{k'^2_\perp}{\alpha' s}$ and introduced the variable $u = \frac{\alpha}{\alpha+\alpha'}$. It should be noted that the cutoff $\alpha < \sigma$ in the r.h.s. of this equation translates into $\int_0^\infty d\alpha d\alpha'\ \theta(\sigma-\alpha-\alpha')$ while our cutoff (45) corresponds to $\int_0^\infty d\alpha d\alpha'\ \theta(\sigma-\alpha)\theta(\sigma-\alpha')$. Fortunately, the difference

$$\int_0^\infty \frac{d\alpha d\alpha'}{\alpha'(\alpha k'^2+\alpha' k^2)}\left[\theta(\sigma-\alpha)\theta(\sigma-\alpha')-\theta(\sigma-\alpha-\alpha')\right] = \frac{1}{k'^2}\int_0^1\frac{dv}{v}\ln\frac{k'^2+k^2v}{k'^2\bar{v}+k^2v} \quad (106)$$

does not contain $\ln\sigma$ and hence does not contribute to the NLO kernel. Similarly, one can impose the cutoff $\alpha_1+\alpha_2<\sigma$ instead of the cutoff $\alpha_1,\alpha_2<\sigma$ in other diagrams whenever convenient.

Before calculating the diagrams in Fig. 12e and Fig. 12f it is convenient to make the replacement

$$\int_{-\infty}^0 du\int_u^0 dv\int_v^0 dt\ \hat{A}^a(u)\hat{A}^b(v)\hat{A}^c(t) \ \rightarrow\ \frac{1}{2}\int_{-\infty}^0 du\int_u^0 dvdt\ \hat{A}^a(u)\hat{A}^b(v)\hat{A}^c(t) \quad (107)$$

which can be performed since the color indices b and c in $...t^bt^c...$ are contracted. For the diagram in Fig. 12e we get

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.12e}} = 2g^4\mu^{2\epsilon}c_F\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d\alpha d\alpha' d\beta d\beta'\int d^{2-\epsilon}kd^{2-\epsilon}k'd^{2-\epsilon}q \quad (108)$$

$$\int d^{2-\epsilon}z\ U_z^{ab}\frac{e^{i(q,X)_\perp-i(k,Y)_\perp}}{\alpha'(\beta-i\epsilon)^2(k'^2+i\epsilon)}\Big(\frac{\beta'}{\beta+\beta'-i\epsilon}+\frac{\beta'}{\beta-\beta'-i\epsilon}\Big)\frac{\theta(\alpha)(k,q)_\perp}{\alpha(k^2+i\epsilon)q_\perp^2}$$

$$= 4g^4\mu^{2\epsilon}c_F\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d\alpha d\alpha' d\beta d\beta'\int d^{2-\epsilon}kd^{2-\epsilon}k'd^{2-\epsilon}q\int d^{2-\epsilon}z\ U_z^{ab}e^{i(q,X)_\perp-i(k,Y)_\perp}$$

$$\times\frac{\beta'}{\alpha\alpha'(\beta-i\epsilon)(\alpha'\beta's-k'^2_\perp+i\epsilon)(\beta+\beta'-i\epsilon)(\beta-\beta'-i\epsilon)}\frac{\theta(\alpha)(k,q)_\perp}{(\alpha\beta s-k_\perp^2+i\epsilon)q_\perp^2}$$

where $c_F = \frac{N_c^2-1}{2N_c}$. Taking residues at $\beta'=-\beta$, $\beta=\frac{k_\perp^2}{\alpha s}$ at $\alpha'>0$ and $\beta'=\beta$, $\beta=\frac{k_\perp^2}{\alpha s}$ at $\alpha'<0$ we obtain

$$\langle\mathrm{Tr}\{U_xU_y^\dagger\}\rangle_{\mathrm{Fig.\ 12e}} = \frac{g^4}{\pi^2}c_F\mu^{2\epsilon}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^{2-\epsilon}kd^{2-\epsilon}k'$$

$$\frac{e^{i(q,X)_\perp-i(k,Y)_\perp}}{uk^2(uk^2+\bar{u}k'^2)}\int d^{2-\epsilon}q\int d^{2-\epsilon}z\frac{(k,q)_\perp}{q_\perp^2}U_z^{db}\mathrm{Tr}\{t^dU_xt^bU_y^\dagger\} \quad (109)$$

The diagram in Fig. 12f yields

$$
\begin{aligned}
&\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 12f}} \qquad (110)\\
&= g^4\mu^{2\epsilon}\int d\alpha d\beta d^{2-\epsilon}k \int d\alpha' d\beta' d^{2-\epsilon}k' \frac{1}{\alpha'(\beta - i\epsilon)(\beta' - i\epsilon)(\alpha'\beta' s - k'^2_\perp + i\epsilon)}\\
&\times 2\theta(\alpha)\int d^{2-\epsilon}q\int d^{2-\epsilon}z e^{i(q,X)_\perp - i(k,Y)_\perp}\frac{(k,q)_\perp}{\alpha(\alpha\beta s - k_\perp^2 + i\epsilon)q_\perp^2}U_z^{ab}\mathrm{Tr}\{t^a U_x t^c t^c t^b U_y^\dagger\}\\
&= -\frac{g^4}{2\pi^2}\mu^{2\epsilon}\int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1\frac{du}{\bar u u}\int d^{2-\epsilon}d^{2-\epsilon}k'\int d^{2-\epsilon}q\int d^{2-\epsilon}z\; e^{i(q,x-z)-i(k,y-z)}\\
&\times\frac{(k,q)_\perp}{k^2k'^2q_\perp^2}U_z^{ab}\mathrm{Tr}\{t^a U_x t^c t^c t^b U_y^\dagger\}
\end{aligned}
$$

Adding Eqs. (105), (109), (110) and integrating over u using Eq. (78) we get

$$
\begin{aligned}
\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 12d+e+f}} &= -\frac{g^4N_c}{4\pi^2}\mu^{2\epsilon}\int d^{2-\epsilon}q d^{2-\epsilon}k d^{2-\epsilon}k'\int d^{2-\epsilon}z \qquad (111)\\
&\times\frac{(q,k)}{k^2k'^2q^2}\ln\frac{k^2}{k'^2}\; e^{i(q,X)-i(k,Y)}[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]
\end{aligned}
$$

Note that the diagram in Fig. 12f does not contribute to the NLO kernel.

The contribution of the last "running coupling" diagram shown in Fig. 12g has the form

$$
\begin{aligned}
&\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 12g}} = -\frac{g^4}{2}\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \qquad (112)\\
&\times\int_0^\infty du\int_{-\infty}^0 dv\langle \hat A_\bullet^a(up_1 + x_\perp)\hat A_\bullet^d(vp_1 + y_\perp)\rangle\int_{-\infty}^0 dtdw\langle \hat A^b(tp_1 + x_\perp)\hat A^c(wp_1 + x_\perp)\rangle
\end{aligned}
$$

where we have again replaced $\int_{-\infty}^0 dt\int_{-\infty}^t dw\; \hat A^b(t)\hat A^c(w)$ by $\frac{1}{2}\int_{-\infty}^0 dtdw\; \hat A^b(t)\hat A^c(w)$. Using the Eq. (38) we get

$$
\begin{aligned}
\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 12g}} &= \frac{ig^4}{\pi}\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\} \qquad (113)\\
&\times(x|\frac{p_i}{p_\perp^2}U^{ad}\frac{p_i}{p_\perp^2}|y)\int d\alpha' d\beta'\frac{\beta'}{\alpha'(\beta'^2+\epsilon^2)}(x|\frac{1}{\alpha'\beta' s - p_\perp^2 + i\epsilon}|x)\\
&= \frac{g^4}{2\pi^2}\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\}(x|\frac{p_i}{p_\perp^2}U^{ad}\frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)\int_0^\sigma\frac{d\alpha}{\alpha}\frac{d\alpha'}{\alpha'}
\end{aligned}
$$

which is obviously a $(\mathrm{LO})^2$ term which does not contribute to the NLO kernel.

Combining expressions (99), (103), and (111) we get

$$
\begin{aligned}
&\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 12}} = -2\alpha_s^2N_c\mu^{2\epsilon}\int d^{2-\epsilon}z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}] (114)\\
&\int d^{2-\epsilon}kd^{2-\epsilon}k'd^{2-\epsilon}q\;\frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}\Big\{\Big[\frac{k^2}{k'^2}(q,2k-k') - \frac{k^2}{(k-k')^2}(q,k+k')\Big]\ln\frac{(k-k')^2}{k'^2}
\end{aligned}
$$

$$+\int_0^1 du\ \frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k'^2\bar{u}+(k-k')^2u}$$
$$+\Big[\frac{(q,k')}{(k-k')^2}+\frac{(q,k')k^2}{k'^2(k-k')^2}+\frac{(q,k'-2k)}{k'^2}\Big]\ln\frac{(k-k')^2}{k'^2}$$
$$-\Big[\frac{(q,k)}{k'^2}+\frac{(q,k)}{(k-k')^2}+\frac{(q,k-k')k^2}{k'^2(k-k')^2}\Big]\ln\frac{k^2}{k'^2}$$
$$-2\int_0^1 du\ \frac{(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}+\frac{2(q,k)}{k'^2}\ln\frac{k^2}{k'^2}\Big\}$$
$$=-2\alpha_s^2N_c\mu^{2\varepsilon}\int d^{2-\varepsilon}z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\int d^{2-\varepsilon}k\,d^{2-\varepsilon}k'\,d^{2-\varepsilon}q\,\frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}\Big\{\ln\frac{k^2}{k'^2}\frac{3(q,k')k^2-4(q,k)(k,k')}{k'^2(k-k')^2}$$
$$+\int_0^1 du\ \frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k^2[k'^2\bar{u}+(k-k')^2u]}-\int_0^1 du\ \frac{2(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}\Big\}$$

Using the integral over k'

$$\int d^{2-\varepsilon}k'_\perp\ \Big\{\frac{3(q,k')k^2-4(q,k)(k,k')}{k'^2(k-k')^2}\ln\frac{k^2}{k'^2}$$
$$+\int_0^1 du\ \frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k^2[k'^2\bar{u}+(k-k')^2u]}$$
$$-\int_0^1 du\ \frac{2(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}\Big\}=\frac{(q,k)}{4\pi}\Big\{\frac{\Gamma(\varepsilon/2)}{(k^2)^{\varepsilon/2}}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Big[\frac{11}{3}-\varepsilon\frac{\pi^2}{6}\Big]+\frac{1}{9}\Big\}$$

one reduces the r.h.s. of Eq. (114) to

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\ 12}}=-\frac{\alpha_s^2N_c}{2\pi}\mu^{2\varepsilon}\int d^{2-\varepsilon}z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\int d^{2-\varepsilon}k\,d^{2-\varepsilon}q\ e^{i(q,X)-i(k,Y)}\ \frac{(q,k)}{k^2q^2}\Big\{\frac{\Gamma(\varepsilon/2)}{(k^2)^{\varepsilon/2}}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Big[\frac{11}{3}-\varepsilon\frac{\pi^2}{6}\Big]+\frac{1}{9}\Big\}\tag{115}$$

Next we subtract the counterterm

$$-\frac{22}{3}\frac{\alpha_s^2N_c}{\pi\varepsilon}\mu^\varepsilon\int d^{2-\varepsilon}z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}\tag{116}$$
$$-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]\int d^{2-\varepsilon}q\ d^{2-\varepsilon}k\ e^{i(q,X)-i(k,Y)}\frac{(q,k)}{q^2k^2}$$

corresponding to the poles $1/\varepsilon$ in the loop diagrams in Fig. 12 (we use the $\overline{MS}$ scheme). We obtain

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\ 12}}\ =\ -\frac{\alpha_s^2N_c}{2\pi}\int d^{2-\varepsilon}z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\int d^{2-\varepsilon}k\,d^{2-\varepsilon}k'\,d^{2-\varepsilon}q\ e^{i(q,X)-i(k,Y)}\ \frac{(q,k)}{k^2q^2}\Big\{\frac{11}{3}\ln\frac{\mu^2}{k^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big\}\tag{117}$$

The complete set of running-coupling diagrams is presented in Fig. 13.

The contribution of diagrams in Fig. 13 VII-XII differs from Eq. (117) by the replacement $e^{i(q,X)} \to e^{i(q,Y)}$ and sign change. There is also a symmetric set of diagrams XIII-XXIV obtained by the reflection of diagrams I-XII with respect to x_* axis. The result is obtained by the substitution $e^{-i(k,X)} \leftrightarrow e^{-i(k,Y)}$ and therefore the contribution of all diagrams in Fig. (13) takes the form

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 13 I+...+XXIV}} = \frac{\alpha_s^2 N_c}{2\pi}\int d^2z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}]$$
$$\times \int d^2kd^2k'd^2q\ [e^{i(q,X)} - e^{i(q,Y)}][e^{-i(k,X)} - e^{-i(k,Y)}]\frac{(q,k)}{k^2q^2}\Big\{\frac{11}{3}\ln\frac{\mu^2}{k^2} + \frac{67}{9} - \frac{\pi^2}{3}\Big\} \tag{118}$$

The remaining diagrams XXV-XXVIII contribute only to the $(\mathrm{LO})^2$. We have shown this for diagram XXVII (Fig. 12g), see Eq. (113). The diagram in Fig. 13 XXV is obtained from the equation (113) by the replacement $x \leftrightarrow y$, and the diagrams XXVI and XXVIII by the replacements $(x|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$ by $(x|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|x)(y|\frac{1}{p_\perp^2}|y)$ and $(y|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$, respectively. Thus, the diagrams XXV-XXVII do not contribute to the NLO kernel.

There is another set of diagrams obtained by the reflection of diagrams shown in Fig. (13) with respect to the shock-wave line. Their contribution is obtained from Eq. (118) by the replacement $q \leftrightarrow k$ in the logarithm so the final result for the sum of all "running coupling" diagrams of Fig. 13 type has the form

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 13 total}} = \frac{\alpha_s^2 N_c}{2\pi}\int d^2z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}] \tag{119}$$
$$\times \int d^2kd^2k'd^2q[e^{i(q,X)} - e^{i(q,Y)}][e^{-i(k,X)} - e^{i(k,Y)}]\frac{(q,k)}{k^2q^2}\Big\{\frac{11}{3}\ln\frac{\mu^4}{q^2k^2} + \frac{134}{9} + \frac{2\pi^2}{3}\Big\}$$
$$= \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}]$$
$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\mu^{-4}} + \frac{134}{9} - \frac{2\pi^2}{3}\Big] + \frac{11}{3}\Big[\frac{1}{X^2} - \frac{1}{Y^2}\Big]\ln\frac{X^2}{Y^2}\Big\}$$

III.2.2 Diagrams for 1→2 dipoles transition

There is one more class of diagrams with one gluon-shockwave intersection shown in Fig. 14. These diagrams are UV-convergent so we do not need to change the dimension of the transverse space to $2-\epsilon$. First we calculate diagrams shown in Fig. 14a,b.

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 14a+b}} = 4\mathrm{Tr}\{t^aU_xt^bt^ct^dU^\dagger_y\}\int d^{2-\epsilon}z\int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2$$

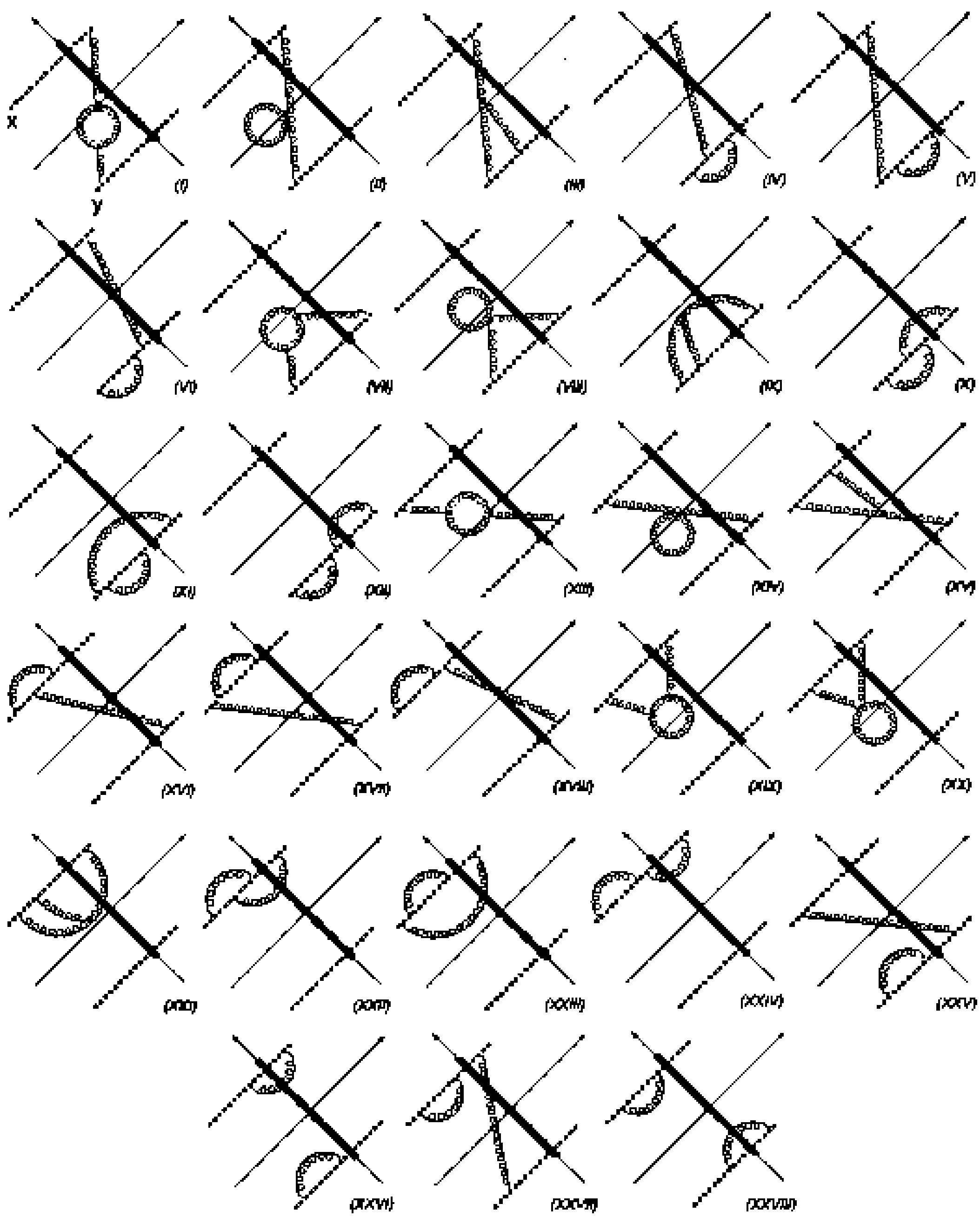

FIG. 13: The full set of "running coupling" diagrams.

$$\int d^{2-\epsilon}k_1 d^{2-\epsilon}k_2 e^{-i(k_1,y)_\perp - i(k_2,y)_\perp}\Big[\frac{\delta^{bd}U_z^{ac}}{(\beta_1 - i\epsilon)(\beta_1+\beta_2 - i\epsilon)} + \frac{\delta^{bc}U_z^{ad}}{(\beta_2 - i\epsilon)(\beta_1+\beta_2 - i\epsilon)}\Big]$$

$$\times\theta(\alpha_1)\int d^2q e^{i(q,x-z)_\perp + i(k_1,z)_\perp}\frac{(q_1,k_1)_\perp}{\alpha_1(\alpha_1\beta_1 s - k_{1\perp}^2 + i\epsilon)q_\perp^2}\frac{e^{i(k_2,x)_\perp}}{\alpha_2(\alpha_2\beta_2 s - k_{2\perp}^2 + i\epsilon)}$$

$$= -\frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bt^ct^dU_y^\dagger\}\int d^{2-\epsilon}z\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^{2-\epsilon}kd^{2-\epsilon}k'\ e^{-i(k,y)_\perp - i(k',y)_\perp}$$

$$\times\Big[\frac{\delta^{bd}U_z^{ac}}{k^2\bar u(k^2\bar u + k'^2u)} + \frac{\delta^{ab}U_z^{dc}}{k'^2u(k^2\bar u + k'^2 u)}\Big]\int d^2q e^{i(q,x-z)_\perp + i(k,z)_\perp + i(k',x)_\perp}\frac{(q,k)_\perp}{q_\perp^2}$$

$$= -\frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bt^ct^dU_y^\dagger\}\int d^2z\int_0^\infty\frac{d\alpha}{\alpha}\int d^2kd^2k'd^2q$$

$$\times\frac{(q,k)}{q^2}\ e^{i(q,X)-i(k,Y)_\perp + i(k',x-y)_\perp}\frac{\delta^{bc}U_z^{ad} - \delta^{bd}U_z^{ac}}{k^2k'^2}\ln\frac{k^2}{k'^2}$$

and therefore

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.\ 14a+b}} = -\frac{g^4N_c}{4\pi^2}\int d^2z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\int d^2kd^2k'd^2q\frac{(q,k)}{q^2k^2k'^2}\ e^{i(q,x-z)-i(k,y-z)_\perp + i(k',x-y)_\perp}\ln\frac{k^2}{k'^2} \tag{120}$$

The contribution of diagrams shown in Fig. 14c,d is obtained from Eq. (120) by the replacement $x \leftrightarrow y$ in the left part of the graph and the sign change so that $e^{-ik(y-z)+i(k',x-y)} \to -e^{-ik(x-z)-i(k',x-y)}$. The sum of the diagrams in Fig. 14a-d takes the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.\ 14a-d}} = -\frac{g^4N_c}{4\pi^2}\int d^2z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\int d^2kd^2k'd^2q\ \frac{(q,k)}{q^2k^2k'^2}\ e^{i(q,x-z)}\Big(e^{-i(k,y-z)_\perp - i(k',x-y)_\perp} - x\leftrightarrow y\Big)\ln\frac{k^2}{k'^2} \tag{121}$$

Next relevant diagram is shown in Fig. 14e

$$g^3\int_0^\infty du\int_{-\infty}^0 dv\int_{-\infty}^0 dt\ \langle\hat A_\bullet^a(up_1+x)\hat A_\bullet^b(vp_1+x)\hat A_\bullet^c(tp_1+y)\rangle_{\mathrm{Fig.14e}} \tag{122}$$

$$= 2g^4\int d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d^2k_1^\perp d^2k_2^\perp$$

$$\times\frac{\theta(\alpha_1+\alpha_2)f^{bcd}(k_{1\mu}^\perp + 2\beta_1p_{2\mu})(k_{2\nu}^\perp + 2\beta_2p_{2\nu})}{\alpha_1\alpha_2(\beta_1 - i\epsilon)(\alpha_1\beta_1 s - k_{1\perp}^2 + i\epsilon)(\beta_2 - i\epsilon)(\alpha_2\beta_2 s - k_{2\perp}^2 + i\epsilon)}$$

$$\times\int d^2z\ U_z^{ad}\int d^2q\ \frac{e^{i(q-k_1,x-z)_\perp - ik_2(y-z)_\perp}\Big[q + \frac{2(k_1+k_2,q)_\perp}{(\alpha_1+\alpha_2)s}p_2\Big]_\lambda}{q^2[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s - (k_1+k_2)_\perp^2 + i\epsilon]}\Gamma^{\mu\nu\lambda}(k_1,k_2,-k_1-k_2)$$

There are three regions of integration over α's: $\alpha_1,\alpha_2 > 0$, $\alpha_1 > -\alpha_2 > 0$ and $\alpha_2 > -\alpha_1 > 0$. Going to the variables $\alpha = \alpha_1 + \alpha_2$, $u = \alpha_2/\alpha$ in the first region, $\alpha = \alpha_1$, $u = -\alpha_2/\alpha$ in the second and $\alpha = \alpha_2$, $u = -\alpha_1/\alpha$ in the third, we obtain

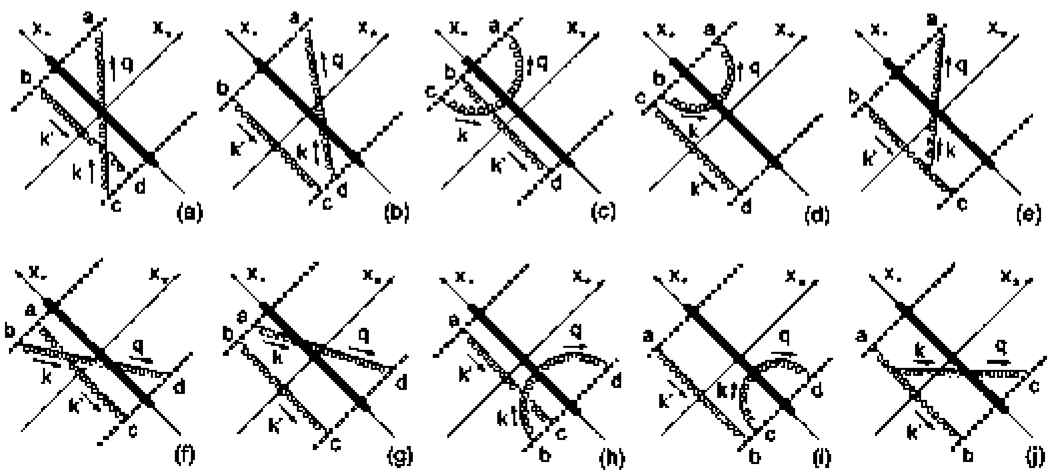

FIG. 14: 1→2 dipoles transition diagrams.

$$
\begin{aligned}
&g^3\int_0^\infty du\int_{-\infty}^0 dv\int_{-\infty}^0 dt\ \langle \hat{A}^a_\bullet(up_1+x)\hat{A}^b_\bullet(vp_1+x)\hat{A}^c_\bullet(tp_1+y)\rangle_{\rm Fig.14e} \qquad (123)\\
&=\frac{1}{2\pi^2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1 du\int d^2k_1^\perp d^2k_2^\perp\ f^{bad}U_z^{nd}\int\frac{d^2q}{q^2}\ e^{i(q-k_1,x-z)_\perp-i(k_2,y-z)_\perp}\\
&\times\Bigg[k^\perp_{1\mu}k^\perp_{2\nu}\Big(q_\lambda+\frac{2(k_1+k_2,q)_\perp}{\alpha s}p_{2\lambda}\Big)\frac{\Gamma^{\mu\nu\lambda}(\alpha\bar{u}p_1+k_1^\perp,\alpha up_1+k_2^\perp,-\alpha p_1-k_1^\perp-k_2^\perp)}{\bar{u}uk^2_{1\perp}k^2_{2\perp}(k_1+k_2)^2_\perp}\\
&-\bar{u}k^\perp_{1\mu}\Big(k^\perp_{2\nu}+2\frac{(k_1+k_2)^2_\perp}{\alpha\bar{u}s}p_{2\nu}\Big)\Big(q_\lambda+\frac{2(k_1+k_2,q)_\perp}{\alpha\bar{u}s}p_{2\lambda}\Big)\\
&\times\frac{\Gamma^{\mu\nu\lambda}(\alpha p_1+k_1^\perp,-u\alpha p_1+k_2^\perp,-\alpha\bar{u}-k_1^\perp-k_2^\perp)}{uk^2_{1\perp}(k_1+k_2)^2_\perp[u(k_1+k_2)^2_\perp+\bar{u}k^2_{2\perp}]}\\
&-\bar{u}\Big(k_1^\perp+2\frac{(k_1+k_2)^2_\perp}{\alpha\bar{u}s}p_2\Big)_\mu k^\perp_{2\nu}\Big(q+\frac{2(k_1+k_2,q)_\perp}{\alpha\bar{u}s}p_2\Big)_\lambda\\
&\times\frac{\Gamma^{\mu\nu\lambda}(-u\alpha p_1+k_1^\perp,\alpha p_1+k_2^\perp,-\alpha\bar{u}-k_1^\perp-k_2^\perp)}{uk^2_{2\perp}(k_1+k_2)^2_\perp[u(k_1+k_2)^2_\perp+\bar{u}k^2_{1\perp}]}\Bigg]
\end{aligned}
$$

Using the formula

$$
\begin{aligned}
&\Big(k_1^\perp+\frac{2A}{s}p_2\Big)_\mu\Big(k_2^\perp+\frac{2B}{s}p_2\Big)_\nu\Big(q+\frac{2C}{s}p_2\Big)_\lambda\\
&\times\Gamma^{\mu\nu\lambda}(\alpha_1p_1+k_{1\perp},\alpha_2p_1+k_{2\perp},-(\alpha_1+\alpha_2)p_1-(k_1+k_2)_\perp)\ =\\
&-C(\alpha_1-\alpha_2)(k_1,k_2)_\perp-A(\alpha_1+2\alpha_2)(q,k_2)_\perp+B(2\alpha_1+\alpha_2)(q,k_1)_\perp\\
&-[(q,k_1)_\perp(k_2,k_1+k_2)_\perp-(k_1\leftrightarrow k_2)]
\end{aligned}
$$

we get

$$\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.\ 14c}} = \frac{g^4 N_c}{8\pi^2}\int d^2k_1 d^2k_2 \frac{d^2q}{q^2}\int d^2z \int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1 du\ e^{i(q-k_1,x-z)-i(k_2,y-z)} \tag{124}$$
$$\times\ [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}]\Big\{\frac{(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)}{\bar u u k_1^2 k_2^2 (k_1+k_2)^2}$$
$$+\frac{(1+\bar u)(k_1+k_2)^2(q,k_1)-(1+u)(q,k_1+k_2)(k_1,k_2)}{u k_1^2 (k_1+k_2)^2 (k_2^2\bar u + (k_1+k_2)^2 u)}$$
$$+\frac{-\bar u[(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)]}{u k_1^2 (k_1+k_2)^2 (k_2^2\bar u + (k_1+k_2)^2 u)}$$
$$+\frac{-(1+\bar u)(k_1+k_2)^2(q,k_2)+(1+u)(q,k_1+k_2)(k_1,k_2)}{u k_2^2 (k_1+k_2)^2 (k_1^2\bar u + (k_1+k_2)^2 u)}$$
$$+\frac{-\bar u[(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)]}{u k_2^2 (k_1+k_2)^2 (k_1^2\bar u + (k_1+k_2)^2 u)}\Big\}$$

Performing the integration over u (with prescription (78)) we obtain

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.\ 14c}} = \frac{g^4 N_c}{8\pi^2}\int d^2z\ [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}] \tag{125}$$
$$\times\int \frac{d^2k_1 d^2k_2}{k_1^2 k_2^2 (k_1+k_2)^2}\frac{d^2q}{q^2}\ e^{i(q-k_1,X)-i(k_2,Y)} U^{cl}_z \Big\{(k_1+k_2)^2\Big[(q,k_2)\ln\frac{(k_1+k_2)^2}{k_1^2}$$
$$-(q,k_1)\ln\frac{(k_1+k_2)^2}{k_2^2}\Big] - (q,k_1+k_2)(k_1^2+k_2^2)\ln\frac{k_1^2}{k_2^2}\Big\}$$
$$= \frac{g^4 N_c}{8\pi^2}\int d^2z\ [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \mathrm{Tr}\{U_x U^\dagger_y\}]\int d^2k d^2k'\ e^{i(q-k',x-z)-i(k-k',y-z)}$$
$$\times\frac{d^2q}{q^2}U^{cl}_z\Big\{\frac{(q,k-k')}{k'^2(k-k')^2}\ln\frac{k^2}{k'^2} - \frac{(q,k')}{k'^2(k-k')^2}\ln\frac{k^2}{(k-k')^2}$$
$$+\frac{(q,k)}{k^2k'^2}\ln\frac{(k-k')^2}{k'^2} + \frac{(q,k)}{k^2(k-k')^2}\ln\frac{(k-k')^2}{k'^2}\Big\}$$

where we made the change of variables $k_1 \to k'$ and $k_2 \to k-k'$.

The sum of diagrams shown in Fig. 14a-e can be represented as

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}\rangle_{\mathrm{Fig.\ 14a-e}} = 2\alpha_s^2 N_c\int d^2z [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}]$$
$$\times\int d^2k d^2k' d^2q e^{i(q,x-z)}\Big(e^{-i(k,y-z)_\perp - i(k',x-y)_\perp} - x \leftrightarrow y\Big)$$
$$\times\Big[\frac{(q,k-k')}{k'^2(k-k')^2} + \frac{(q,k)}{k^2(k-k')^2} - \frac{(q,k)}{k^2k'^2}\Big]\ln\frac{k^2}{k'^2} \tag{126}$$

Note that the expressions (121) and (125) are IR divergent as $k' \to 0$ but their sum (126) is IR stable. Once again, the contribution of the diagrams in Fig. 14f-k are obtained by replacement $e^{iq(x-z)} \to -e^{iq(y-z)}$ so the contribution of diagrams of Fig. 14 a-k has the form

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\ 14a-k}} = 2\alpha_s^2 N_c \int d^2z [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}]$$
$$\times \int d^2k d^2k' d^2q \times \ (e^{i(q,x-z)} - e^{i(q,y-z)})\Big(e^{-i(k,y-z)_\perp - i(k',x-y)_\perp} - x \leftrightarrow y\Big)$$
$$\times \Big[\frac{(q,k-k')}{k'^2(k-k')^2} + \frac{(q,k)}{k^2(k-k')^2} - \frac{(q,k)}{k^2k'^2}\Big] \ln\frac{k^2}{k'^2} \tag{127}$$

Performing the Fourier transformation with the help of the formulas

$$\int dk dk' \ \frac{e^{-i(k,y)-i(k',x-y)}}{k'^2}\Big(\frac{(k-k')_i}{(k-k')^2} - \frac{k_i}{k^2} + \frac{k_i k'^2}{k^2(k-k')^2}\Big) \ln\frac{k^2}{k'^2} \tag{128}$$
$$= \frac{i}{16\pi^2}\Big(\frac{x_i}{x^2} - \frac{y_i}{y^2}\Big) \ln\frac{(x-y)^2}{x^2} \ln\frac{(x-y)^2}{y^2}$$
$$+ \frac{i}{8\pi^2}\Big(\frac{(x,y)}{y^2}y_i - x_i\Big)\frac{1}{i\kappa}\Big\{ \int_0^1 du \ \Big[\frac{\ln u}{u - \frac{(x,y)-i\kappa}{x^2}} - \frac{\ln u}{u - \frac{(x,y)+i\kappa}{x^2}}\Big] + \frac{1}{2}\ln\frac{x^2}{y^2} \ln\frac{(x-y,y)+i\kappa}{(x-y,y)-i\kappa}\Big\}$$
$$+ \frac{i}{8\pi^2}\Big(\frac{(x,y)}{x^2}x_i - y_i\Big)\frac{1}{i\kappa}\Big\{ \int_0^1 du \ \Big[\frac{\ln u}{u - \frac{(x,x-y)-i\kappa}{(x-y)^2}} - \frac{\ln u}{u - \frac{(x,x-y)+i\kappa}{(x-y)^2}}\Big] - \frac{1}{2}\ln\frac{(x-y)^2}{x^2} \ln\frac{(x,y)+i\kappa}{(x,y)-i\kappa}\Big\}$$

(here $\kappa = \sqrt{x^2y^2 - (x,y)^2}$) one obtains

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\ 14a-k}} = -\frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}]\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2} \tag{129}$$

Note that the two last terms in the r.h.s. of Eq. (128) do not contribute.

The contribution of the diagram obtained by reflection of Fig. 14 with respect to the shock wave differs from Eq. (126) by replacement $q \leftrightarrow k$ which doubles result (129). The final expression for the contribution of all "dipole recombination diagrams" of Fig. 14 type has the form

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\ 14\ total}} = -\frac{\alpha_s^2 N_c}{4\pi^3}\int d^2z [\mathrm{Tr}\{U_x U^\dagger_z\}\mathrm{Tr}\{U_z U^\dagger_y\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U^\dagger_y\}]\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2} \tag{130}$$

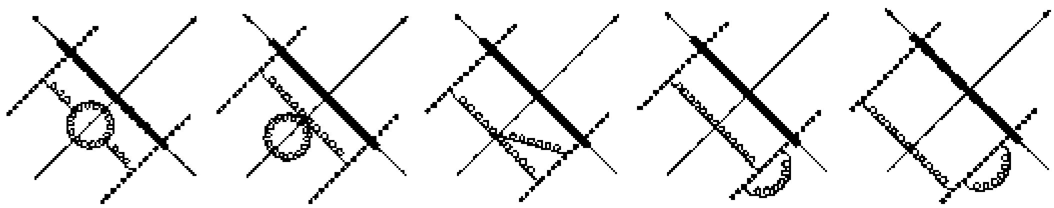

FIG. 15: Typical diagrams without the gluon-shockwave intersection.

III.3 ASSEMBLING THE NLO KERNEL

Adding results (89), (222) and (130) one obtains the contribution of the diagrams with one and two gluon intersections with the shock wave in the form:

$$\begin{aligned}
&\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig.11+Fig.13+Fig.14}} = \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z\,\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln(x-y)^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] \\
&+\frac{11}{3}\Big[\frac{1}{X^2}-\frac{1}{Y^2}\Big]\ln\frac{X^2}{Y^2} - 2\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
&\qquad\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}] \\
&+\frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\Big[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]} \\
&+\Big(\frac{(x-y)^4}{X^2Y'^2 - X'^2Y^2}\Big[\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2} - \frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
&\qquad\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} - (z'\to z)] \\
&+\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2} + \frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big)\Big\} \\
&\qquad\times\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big]
\end{aligned} \tag{131}$$

There are also diagrams without gluon-shockwave intersection like the graph shown in Fig. 15. They are proportional to the parent dipole $\mathrm{Tr}\{U_xU_y^\dagger\}$ and their contribution can be found from Eq. (131) using the requirement that the r.h.s. of the evolution equation must vanish at $x=y$ (since $U_xU_x^\dagger=1$). It is easy to see that the replacement $\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}$ by $\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}$ fulfills the above requirement so one obtains the final gluon contribution to the NLO kernel in the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z\,\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln(x-y)^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] \tag{132}$$

$$+\frac{11}{3}\Big[\frac{1}{X^2}-\frac{1}{Y^2}\Big]\ln\frac{X^2}{Y^2}-2\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\Big[\Big(-\frac{4}{(z-z')^4}+\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$
$$+\Big(\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$
$$\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}-(z'\to z)]$$
$$+\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}$$
$$\times\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big]$$

Promoting Wilson lines in the r.h.s of this equation to operators and adding the quark contribution from Ref. [24]

$$\frac{d}{d\eta}(\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\})_{\rm quark} = \frac{\alpha_s}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big[-\frac{\alpha_sn_f}{6\pi}\frac{(x-y)^2}{X^2Y^2}\Big(\ln(x-y)^2\mu^2+\frac{5}{3}\Big)+\frac{\alpha_sn_f}{6\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}\Big]$$
$$+\frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2zd^2z'\ \mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger-t^aU_zt^bU_z^\dagger\}\ \frac{1}{(z-z')^4}$$
$$\times\Big\{1-\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big\} \tag{133}$$

we obtain the full NLO kernel cited in Eq. (44).

III.3.1 Mean field Approximation

We now present the evolution equation for matrix elements of color dipoles for large nuclei in the leading-N_c approximation. Using the standard mean-field approximation [20]

$$\langle A|\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z\hat U_y^\dagger\}|A\rangle=\langle A|\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat U_z\hat U_y^\dagger\}|A\rangle,$$
$$\langle A|\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z\hat U_{z'}^\dagger\}\mathrm{Tr}\{\hat U_{z'}\hat U_y^\dagger\}|A\rangle$$
$$=\langle A|\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat U_z\hat U_{z'}^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat U_{z'}\hat U_y^\dagger\}|A\rangle$$

one easily obtains from Eq. (44) ($N(x,y)\equiv\langle A|\mathcal{U}(x,y)|A\rangle$)

$$\frac{d}{d\eta}N(x,y)=\frac{\alpha_sN_c}{2\pi^2}\int d^2z\ \frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_sN_c}{4\pi}\Big[\frac{11}{3}\ln(x-y)^2\mu^2 \tag{134}$$
$$-\frac{11}{3}\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+\frac{67}{9}-\frac{\pi^2}{3}-2\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}$$

$$[N(x,z)+N(z,y)-N(x,y)-N(x,z)N(z,y)]$$
$$+\frac{\alpha_s^2N_c^2}{8\pi^4}\int d^2zd^2z'\Big\{-\frac{2}{(z-z')^4}+\Big[\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}$$
$$+\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{X^2Y'^2(z-z')^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}$$
$$\times[N(z,z')-N(x,z)N(z,z')-N(z,z')N(z',y)-N(x,z)N(z',y)$$
$$+N(x,z)N(z,y)+N(x,z)N(z,z')N(z',y)]$$

In this closed form the NLO evolution equation (134) can be used for numerical simulations of the dipole evolution.

III.4 COMPARISON TO NLO BFKL KERNEL RESULTS

III.4.1 Linearized forward kernel

In this section we compare our kernel to the forward NLO BFKL results [29]. The linearized equation (44) has the form

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y)=\frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{(x-y)^2}{X^2Y^2}\Big\{1$$
$$+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2-b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+(\frac{67}{9}-\frac{\pi^2}{3})N_c-\frac{10}{9}n_f$$
$$-2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}[\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)]$$
$$+\frac{\alpha_s^2N_c^2}{16\pi^4}\int d^2zd^2z'\Big[-\frac{4}{(z-z')^4}+\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$
$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}$$
$$-\frac{n_f}{N_c^3}\Big\{\frac{4}{(z-z')^4}-2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big\}\Big]\hat{\mathcal{U}}(z,z')\quad(135)$$
$$+\frac{\alpha_s^2N_c^2}{16\pi^4}\int d^2zd^2z'\Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big)\Big\}$$
$$\times\ln\frac{X^2Y'^2}{X'^2Y^2}[\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,z')+\hat{\mathcal{U}}(z',y)]$$

For the case of forward scattering $\langle\hat{\mathcal{U}}(x,y)\rangle=\langle\hat{\mathcal{U}}(x-y)\rangle$, and the linearized equation (165) reduces to

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle=\frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{x^2}{(x-z)^2z^2}\Big\{1$$

$$+\frac{\alpha_s}{4\pi}\Big[b\ln x^2\mu^2 - b\frac{(x-z)^2-z^2}{x^2}\ln\frac{(x-z)^2}{z^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f$$
$$-2N_c\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big\}[\langle\hat{\mathcal{U}}(x-z)\rangle + \langle\hat{\mathcal{U}}(z)\rangle - \langle\hat{\mathcal{U}}(x)\rangle]$$
$$+\frac{\alpha_s^2N_c^2}{16\pi^4}\int d^2z d^2z'\Big[\Big(-\frac{4}{z^4} + \Big\{2\frac{(x-z-z')^2z'^2 + (x-z')^2(z+z')^2 - 4x^2z^2}{z^4[(x-z-z')^2z'^2 - (x-z')^2(z+z')^2]}$$
$$+\frac{x^4}{(x-z-z')^2z'^2 - (x-z')^2(z+z')^2}\Big[\frac{1}{(x-z-z')^2z'^2} + \frac{1}{(x-z')^2(z+z')^2}\Big]$$
$$+\frac{x^2}{z^2}\Big[\frac{1}{(x-z-z')^2z'^2} - \frac{1}{(x-z')^2(z+z')^2}\Big]\Big\}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2}$$
$$-\frac{n_f}{N_c^3}\Big\{\frac{4}{z^4} - 2\frac{(x-z-z')^2z'^2 + (x-z')^2(z+z')^2 - x^2z^2}{z^4[(x-z-z')^2z'^2 - (x-z')^2(z+z')^2]}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2}\Big\}\Big)\langle\hat{\mathcal{U}}(z)\rangle$$
$$+\langle\hat{\mathcal{U}}(x-z) - \hat{\mathcal{U}}(z)\rangle\Big[\frac{x^2}{(x-z)^2(z-z')^2z'^2} + \frac{x^2}{(x-z')^2(z-z')^2z^2}$$
$$-\frac{x^4}{(x-z)^2z^2(x-z')^2z'^2}\Big]\ln\frac{(x-z)^2z'^2}{(x-z')^2z^2}\Big] \qquad (136)$$

Using integral J_{13} from Ref. [32] we get

$$\frac{1}{\pi}\int d^2z'\ \Big[\frac{x^4}{(x-z-z')^2z'^2 - (x-z')^2(z+z')^2} + \frac{x^2}{z^2}\Big]$$
$$\times\frac{1}{z'^2(x-z-z')^2}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2}$$
$$= \frac{2x^2}{z^2}\Big\{\frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4} + 2Li_2\Big(-\frac{z^2}{x^2}\Big) - 2Li_2\Big(-\frac{x^2}{z^2}\Big)\Big]$$
$$-\Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\Big[\int_0^1 - \int_1^\infty\Big]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Big\} \qquad (137)$$

and therefore

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle = \frac{\alpha_sN_c}{2\pi^2}\int d^2z\ \frac{x^2}{(x-z)^2z^2}\Big\{1$$
$$+\frac{\alpha_s}{4\pi}\Big[b\ln x^2\mu^2 - b\frac{(x-z)^2-z^2}{x^2}\ln\frac{(x-z)^2}{z^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f$$
$$-2N_c\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big\}[\langle\hat{\mathcal{U}}(x-z)\rangle + \langle\hat{\mathcal{U}}(z)\rangle - \langle\hat{\mathcal{U}}(x)\rangle]$$
$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\int d^2z\ \frac{x^2}{z^2}\Big\{\Big(1+\frac{n_f}{N_c^3}\Big)\frac{3(x,z)^2-2x^2z^2}{16x^2z^2}\Big(\frac{2}{x^2}+\frac{2}{z^2}+\frac{x^2-z^2}{x^2z^2}\ln\frac{x^2}{z^2}\Big)$$
$$-\Big[3+\Big(1+\frac{n_f}{N_c^3}\Big)\Big(1-\frac{(x^2+z^2)^2}{8x^2z^2} + \frac{3x^4+3z^4-2x^2z^2}{16x^4z^4}(x,z)^2\Big)\Big]\int_0^\infty dt\frac{1}{x^2+t^2z^2}\ln\frac{1+t}{|1-t|}$$
$$+\frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4} + 2Li_2\Big(-\frac{z^2}{x^2}\Big) - 2Li_2\Big(-\frac{x^2}{z^2}\Big)\Big]$$

$$- \left(1 - \frac{(x^2 - z^2)^2}{(x-z)^2(x+z)^2}\right)\left[\int_0^1 - \int_1^\infty\right]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Bigg\} \mathcal{U}(z) + \frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)\mathcal{U}(x) \quad (138)$$

where we have used Eq. (330) to calculate the last term in the r.h.s.

III.4.2 Comparison of eigenvalues

To compare the eigenvalues of the Eq. (167) with NLO BFKL we expand $\mathcal{U}(x,0)$ in eigenfunctions

$$\langle\hat{\mathcal{U}}(x_\perp,0)\rangle = \sum_{n=-\infty}^{\infty}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{d\gamma}{2\pi i}\, e^{in\phi}(x_\perp^2\mu^2)^\gamma\, \langle\hat{\mathcal{U}}(n,\gamma)\rangle\,, \qquad (139)$$

compute the evolution of $\langle\hat{\mathcal{U}}(n,\gamma)\rangle$ from Eq. (167) and compare it to the calculation based on the NLO BFKL results from [29, 47]. (For the quark part of the NLO BK kernel the agreement with NLO BFKL was proved in Ref. [33]).

The relevant integrals have the form

$$\frac{1}{2\pi}\int d^2z\ [2(z^2/x^2)^\gamma e^{in\phi} - 1]\frac{x^2}{(x-z)^2z^2} = \chi(n,\gamma)$$

$$\frac{1}{\pi}\int d^2z\ [2(z^2/x^2)^\gamma e^{in\phi} - 1]\left(\frac{1}{(x-z)^2} - \frac{1}{z^2}\right)\ln\frac{(x-z)^2}{z^2} = \chi^2(n,\gamma) - \chi'(n,\gamma) - \frac{4\gamma\chi(\gamma)}{\gamma^2 - \frac{n^2}{4}}$$

$$\frac{1}{\pi}\int d^2z\ (z^2/x^2)^\gamma\frac{x^2}{(x-z)^2z^2}e^{in\phi}\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2} = \frac{1}{2}\chi''(n,\gamma) + \chi'(n,\gamma)\chi(n,\gamma) \qquad (140)$$

where $\chi(n,\gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})$, and

$$\frac{1}{\pi}\int d^2z\ (z^2/x^2)^{\gamma-1}e^{in\phi}\Bigg\{\left(1 + \frac{n_f}{N_c^3}\right)\frac{3(x,z)^2 - 2x^2z^2}{16x^2z^2}\left(\frac{2}{x^2} + \frac{2}{z^2} + \frac{x^2 - z^2}{x^2z^2}\right)\ln\frac{x^2}{z^2} \quad (141)$$

$$- \left[3 + \left(1 + \frac{n_f}{N_c^3}\right)\left(1 - \frac{(x^2+z^2)^2}{8x^2z^2} + \frac{3x^4 + 3x^4 - 2x^2z^2}{16x^4z^4}(x,z)^2\right)\right]$$

$$\times\int_0^\infty dt\frac{1}{x^2 + t^2z^2}\ln\frac{1+t}{|1-t|}\Bigg\}$$

$$= \Bigg\{ - \left[3 + \left(1 + \frac{n_f}{N_c^3}\right)\frac{2 + 3\gamma\bar{\gamma}}{(3-2\gamma)(1+2\gamma)}\right]\delta_{0n}$$

$$+ \left(1 + \frac{n_f}{N_c^3}\right)\frac{\gamma\bar{\gamma}}{2(3-2\gamma)(1+2\gamma)}\delta_{2n}\Bigg\}\frac{\pi^2\cos\pi\gamma}{(1-2\gamma)\sin^2\pi\gamma} \equiv F(n,\gamma)$$

$$\frac{1}{2\pi}\int d^2z\ (z^2/x^2)^{\gamma-1}e^{in\phi}\Bigg\{\frac{(x^2 - z^2)}{(x-z)^2(x+z)^2}\left[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4}\right.$$

$$\left. + 2Li_2\left(-\frac{z^2}{x^2}\right) - 2Li_2\left(-\frac{x^2}{z^2}\right)\right]$$

$$-\left(1 - \frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\right)\left[\int_0^1 - \int_1^\infty\right]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Bigg\} = -\Phi(n,\gamma) - \Phi(n,1-\gamma)$$

(142)

where [47]

$$\Phi(n,\gamma) = \int_0^1 \frac{dt}{1+t}\, t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big) - \mathrm{Li}_2(t) - \mathrm{Li}_2(-t)$$
$$-\Big(\psi(n+1) - \psi(1) - \ln(1+t) + \sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\} \quad (143)$$

The convenient way to calculate the integrals over angle ϕ is to represent $\cos n\phi$ as $T_n(\cos\phi)$ and use formulas for the integration of Chebyshev polynomials from Ref. [47].

Using integrals (140) - (142) one easily obtains the evolution equation for $\mathcal{U}(n,\gamma)$ in the form

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1 - \frac{b\alpha_s}{4\pi}\frac{d}{d\gamma} + \Big(\frac{67}{9} - \frac{\pi^2}{3}\Big)N_c - \frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma) - \frac{1}{2}\chi'(n,\gamma) - \frac{2\gamma\chi(n,\gamma)}{\gamma^2 - \frac{n^2}{4}}\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi''(n,\gamma) - 2\chi(n,\gamma)\chi'(n,\gamma) + 6\zeta(3) + F(n,\gamma)$$
$$-2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle \quad (144)$$

where $\chi'(n,\gamma) \equiv \frac{d}{d\gamma}\chi(n,\gamma)$ etc.

Next we calculate the same thing using NLO BFKL results [29, 47]. The impact factor $\Phi_A(q)$ for the color dipole $\mathcal{U}(x,y)$ is proportional to $\alpha_s(q)(e^{iqx} - e^{iqy})(e^{-iqx} - e^{-iqy})$ so one obtains the cross section of the scattering of color dipole in the form

$$\langle\hat{\mathcal{U}}(x,0)\rangle = \frac{1}{4\pi^2}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}\alpha_s(q)(e^{iqx}-1)(e^{-iqx}-1)\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^{\omega}G_\omega(q,q') \quad (145)$$

where $G_\omega(q,q')$ is the partial wave of the forward reggeized gluon scattering amplitude satisfying the equation

$$\omega G_\omega(q,q') = \delta^{(2)}(q-q') + \int d^2p K(q,p)G_\omega(p,q') \quad (146)$$

and $\Phi_B(q')$ is the target impact factor. The kernel $K(q,p)$ is symmetric with respect to $q \leftrightarrow p$ and the eigenvalues are

$$\int d^2p\Big(\frac{p^2}{q^2}\Big)^{\gamma-1}e^{in\phi}K(q,p) = \frac{\alpha_s(q)}{\pi}N_c\Big[\chi(n,\gamma) + \frac{\alpha_s N_c}{4\pi}\delta(n,\gamma)\Big], \quad (147)$$
$$\delta(n,\gamma) = -\frac{b}{2}[\chi'(n,\gamma) + \chi^2(n,\gamma)] + \Big(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N_c^3}\Big)\chi(n,\gamma) + 6\zeta(3)$$
$$-\chi''(n,\gamma) + F(n,\gamma) - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)\Big\}$$

The corresponding expression for $\langle\hat{\mathcal{U}}(n,\gamma)\rangle$ takes the form

$$\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{1}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}e^{-in\theta}$$
$$\times\alpha_s(q)\Big(\frac{q^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^{\omega}G_\omega(q,q') \qquad (148)$$

where θ is the angle between $\vec{q}$ and x axis. Using Eq. (168) we obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{1}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}e^{-in\theta} \qquad (149)$$
$$\times\alpha_s(q)\Big(\frac{q^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^{\omega}\int d^2pK(q,p)G_\omega(p,q')$$

The integration over q can be performed using

$$\int d^2q\alpha_s(q)\Big(\frac{q^2}{p^2}\Big)^{\gamma-1}e^{in\phi}K(q,p) = \frac{\alpha_s^2(p)}{\pi}N_c\Big[\chi(n,\gamma) - \frac{b\alpha_s}{4\pi}\chi'(n,\gamma) + \frac{\alpha_sN_c}{4\pi}\delta(n,\gamma)\Big] \qquad (150)$$

(recall that $K(q,p) = K(p,q)$ and $\alpha_s(p) = \alpha_s - \frac{b\alpha_s^2}{4\pi}\ln\frac{p^2}{\mu^2}$ with our accuracy). The result is

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_s}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-in\varphi}\Big(\frac{p^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q') \quad (151)$$
$$\times\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^{\omega}G_\omega(p,q')\frac{\alpha_s(p)}{\pi}N_c\Big[\chi(n,\gamma-\frac{\omega}{2})$$
$$-\frac{b\alpha_s}{4\pi}\chi'(n,\gamma-\frac{\omega}{2}) + \frac{\alpha_sN_c}{4\pi}\delta(n,\gamma-\frac{\omega}{2}))\Big]$$

where the angle φ corresponds to $\vec{p}$. Since $\omega\sim\alpha_s$ we can neglect terms $\sim\omega$ in the argument of δ and expand $\chi(n,\gamma-\frac{\omega}{2})\simeq\chi(n,\gamma)-\frac{\omega}{2}\chi'(n,\gamma)$. Using again Eq. (168) in the leading order we can replace extra ω by $\frac{\alpha_s}{\pi}N_c\chi(n,\gamma)$ and obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_s}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-in\varphi}\Big(\frac{p^2}{4\mu^2}\Big)^{\gamma} \qquad (152)$$
$$\times\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^{\omega}G_\omega(p,q')\frac{\alpha_s^2(p)}{\pi}N_c\Big[\chi(n,\gamma)$$
$$-\frac{b\alpha_s}{4\pi}\chi'(n,\gamma) + \frac{\alpha_sN_c}{4\pi}[\delta(n,\gamma) - 2\chi(n,\gamma)\chi'(n,\gamma)]\Big]$$

Finally, expanding $\alpha_s^2(p)\simeq\alpha_s(p)(\alpha_s - \frac{b\alpha_s^2}{4\pi}\ln\frac{p^2}{\mu^2})\alpha_s(\mu)$ we obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_sN_c}{2\pi^3}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\Big\{\chi(n,\gamma)\Big(1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}\Big) - \frac{b\alpha_s}{4\pi}\chi'(n,\gamma)$$

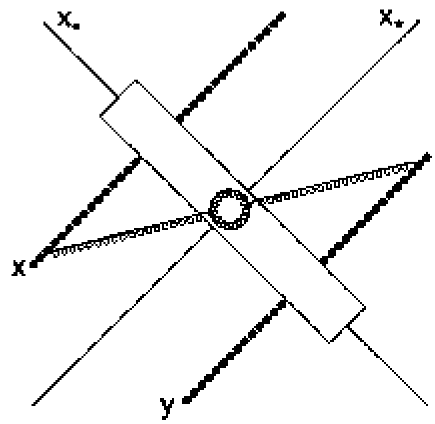

FIG. 16: Gluon loop inside the shock wave .

$$+\frac{\alpha_s N_c}{4\pi}[\delta(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)]\Big\}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-inp}\alpha_s(p)\Big(\frac{p^2}{4\mu^2}\Big)^\gamma\Phi_B(q')$$
$$\times\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^\omega G_\omega(p,q') \tag{153}$$

which can be rewritten as an evolution equation

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle=\frac{\alpha_s N_c}{\pi}\Big\{\Big(1+\frac{b\alpha_s}{4\pi}\Big[\chi(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}-\frac{d}{d\gamma}\Big]\Big)\chi(n,\gamma)$$
$$+\frac{\alpha_s N_c}{4\pi}[\delta(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)]\Big\}\langle\mathcal{U}(n,\gamma)\rangle$$
$$=\frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi''(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)$$
$$-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle \tag{154}$$

This eigenvalue coincides with Eq. (144) from Ref. [29]. It is worth noting that the eigenvalue Eq. (154) agrees with the $j\to 1$ asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators [34].

It should be emphasized that the coincidence of terms with the nontrivial γ dependence proves that there is no additional $O(\alpha_s)$ correction to the vertex of the gluon - shock wave interaction coming from the small loop inside the shock wave, see Fig. 16 (In other words, all the effects coming from the small loop in the shock wave are absorbed in the renormalization of coupling constant in the definition of the U operator (33)). In the case of quark loop, we proved that by the comparison of our results for $\mathrm{Tr}\{U_x U_y^\dagger\}$ in the shock-wave background with explicit light-cone calculation of the behavior of $\mathrm{Tr}\{U_x U_y^\dagger\}$ as $x \to y$ [24]. For the gluon loop, we can use the NLO BFKL results as an independent calculation. Let us repeat the arguments of Ref. [24] for this case. The characteristic transverse scale inside the shock wave is small (see the discussion in Ref. [24]) and therefore the contribution of the diagram in Fig. 16 reduces to the contribution of some operator *local* in the transverse space. This would bring the additional terms with the nontrivial z dependence to the kernel which translates into the nontrivial additional γ-dependent term in the eigenvalues. Such terms do not exist and therefore the gluon interaction with the shock wave does not get an extra $O(\alpha_s)$ correction.

III.5 ARGUMENT OF THE COUPLING CONSTANT IN THE BK EQUATION

In this section we briefly summarize the results of the renormalon-based analysis of the argument of the coupling constant carried in Refs. [24, 52]

To get an argument of coupling constant we can trace the quark part of the β-function (proportional to n_f). In the leading log approximation $\alpha_s \ln \frac{p^2}{\mu^2} \sim 1,\ \alpha_s \ll 1$ the quark part of the β-function comes from the bubble chain of quark loops in the shock-wave background. We can either have no intersection of quark loop with the shock wave (see Fig. 17a) or we may have one of the loops in the shock-wave background (see Fig. 17b).

The sum of these diagrams yields

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle = 2\alpha_s \mathrm{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2p\, d^2l\ [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}][e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}]$$
$$\times \frac{1}{p^2(1+\frac{\alpha_s}{6\pi}\ln\frac{\mu^2}{p^2})}\Big(1-\frac{\alpha_s n_f}{6\pi}\ln\frac{l^2}{\mu^2}\Big)\partial_\perp^2 U^{ab}(l)\frac{1}{(p-l)^2(1+\frac{\alpha_s}{6\pi}\ln\frac{\mu^2}{(p-l)^2})} \tag{155}$$

where we have left only the β-function part of the quark loop. Replacing the quark part of the β-function $-\frac{\alpha_s}{6\pi}n_f \ln\frac{p^2}{\mu^2}$ by the total contribution $\frac{\alpha_s}{4\pi}b\ln\frac{p^2}{\mu^2}$ we get

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle = 2\mathrm{Tr}\{t^a U_x t^b U_y^\dagger\} \int d^2p\, d^2q\ [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}]$$
$$\times[e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}]\frac{\alpha_s(p^2)}{p^2}\alpha_s^{-1}(l^2)\partial_\perp^2 U^{ab}(q)\frac{\alpha_s((p-l)^2)}{(p-l)^2} \tag{156}$$

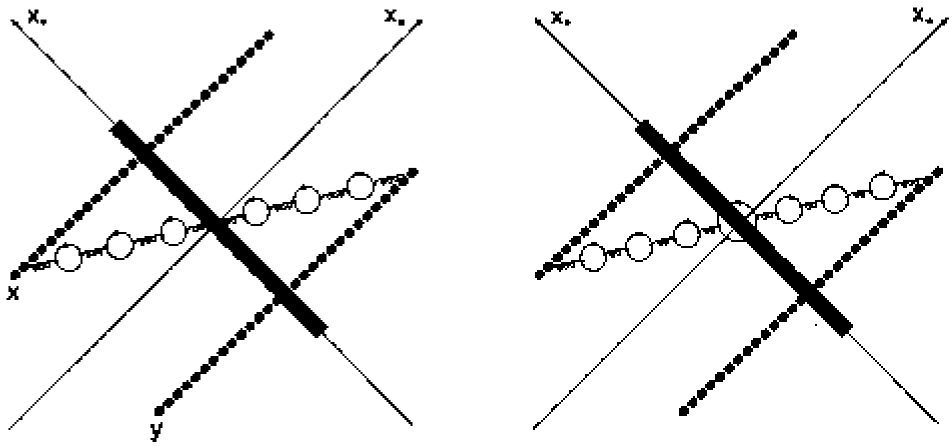

FIG. 17: Renormalon bubble chain of quark loops.

In principle, one should also include the "renormalon dressing" of the double-log and conformal terms in Eq. (44). We think, however, that they form a separate contribution which has nothing to do with the argument of the BK equation.

To go to the coordinate space, we expand the coupling constants in Eq. (156) in powers of $\alpha_s = \alpha_s(\mu^2)$, i.e. return back to Eq. (155) with $\frac{\alpha_s}{6\pi}n_f \to -b\frac{\alpha_s}{4\pi}$. Unfortunately, the Fourier transformation to the coordinate space can be performed explicitly only for a couple of first terms of the expansion $\alpha_s(p^2) \simeq \alpha_s - \frac{b\alpha_s}{4\pi}\ln p^2/\mu^2 + (\frac{b\alpha_s}{4\pi}\ln p^2/\mu^2)^2$. In the first order we get the running-coupling part of the NLO BK equation (44)

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle = \frac{\alpha_s}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}] \times\Big[\frac{(x-y)^2}{X^2Y^2}\Big(1+b\frac{\alpha_s}{4\pi}\ln(x-y)^2\mu^2\Big) - b\frac{\alpha_s}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}\Big] \tag{157}$$

The result of the Fourier transformation up to the second order has the form [24, 52]

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle = \frac{\alpha_s}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{b\alpha_s}{4\pi}\Big(\ln(x-y)^2\mu^2+\frac{5}{3}\Big) + \Big(\frac{b\alpha_s}{4\pi}\Big)^2\ln^2(x-y)^2\mu^2\Big] + \frac{b\alpha_s}{4\pi}\frac{1}{X^2}\ln\frac{X^2}{Y^2}\Big[1+\frac{b\alpha_s}{4\pi}\ln(x-y)^2\mu^2+\frac{b\alpha_s}{4\pi}\ln X^2\mu^2\Big] - \frac{b\alpha_s}{4\pi}\frac{1}{Y^2}\ln\frac{X^2}{Y^2}\Big[1+\frac{b\alpha_s}{4\pi}\ln(x-y)^2\mu^2+\frac{b\alpha_s}{4\pi}\ln Y^2\mu^2\Big]\Big\} + \dots \tag{158}$$

We extrapolate the $\ln + \ln^2$ terms in the above equation as follows:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_y\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}] \\ \times\Big[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\Big) + \frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\Big)\Big] + \ldots \tag{159}$$

where dots stand for the remaining conformal terms and $\ln^2$ term. (Here we promoted Wilson lines in the r.h.s. to operators).

When the sizes of the dipoles are very different the kernel of the above equation reduces to

$$\begin{aligned} &\tfrac{\alpha_s((x-y)^2)}{2\pi^2}\tfrac{(x-y)^2}{X^2Y^2} && |x-y| \ll |x-z|, |y-z| \\ &\tfrac{\alpha_s(X)^2)}{2\pi^2X^2} && |x-z| \ll |x-y|, |y-z| \\ &\tfrac{\alpha_s(Y)^2)}{2\pi^2Y^2} && |y-z| \ll |x-y|, |x-z| \end{aligned} \tag{160}$$

In Ref. [24], Eq. (159) was interpreted as an indication that the argument of the coupling constant is the size of the parent dipole $x-y$. We are grateful to G. Salam for pointing out that the proper interpretation is the size of the smallest dipole as follows from Eq. (160).

It is instructive to compare our result to the one in [52] where the NLO BK equation is rewritten in terms of three effective coupling constants. The authors of Ref. [52] extrapolate Eq. (158) in a different way

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{1}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_y\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}] \\ \times\Big[\frac{1}{X^2}\alpha_s(X^2) + \frac{1}{Y^2}\alpha_s(Y^2) - \frac{2(x-z,y-z)}{X^2Y^2}\frac{\alpha_s(X^2)\alpha_s(Y^2)}{\alpha_s(R^2)}\Big] \tag{161}$$

where R^2 is some scale interpolating between X^2 and Y^2 (the explicit form can be found in Ref. [52]). Theoretically, until the Fourier transformations in all orders in $\ln p^2/\mu^2$ are performed, both of these interpretations are models of the high-order behavior of running coupling constant. The convenience of these models can be checked by the numerical estimates of the size of the neglected term(s) in comparison to terms taken into account by the model, see the discussion in Ref. [37].

III.6 NLO EVOLUTION OF COLOR DIPOLE IN QCD

It is instructive also to present the evolution equation for composite operator (246) in QCD. The resulting equation will not be Möbius invariant because of the running coupling constant so composite operators (246) and (247) are not strictly speaking conformal. We will, however, keep the notation $[...]^{\rm conf}$ to indicate that these operators were conformal in $\mathcal{N}=4$ SYM.

To get the evolution equation for "conformal" composite operators (246) in QCD we subtract the scalar (242) and gluino (243) contributions from Eq. (162) and add the quark contribution calculated in Refs. [24, 52, 33]. We get

$$\begin{aligned}
&\frac{d}{d\eta}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\\
&+\frac{\alpha_s}{4\pi}\Big[b\ln z_{12}^2\mu^2 - b\frac{z_{13}^2-z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}+(\frac{64}{9}-\frac{\pi^2}{3})N_c-\frac{10}{9}n_f\Big]\\
&\times[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}\\
&+\frac{\alpha_s^2}{16\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Bigg[\Big\{\Big(-2+2\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}\\
&+\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2})+\frac{2z_{13}^2z_{24}^2-4z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big)+(z_3\leftrightarrow z_4)\Big\}\\
&\times[(\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\}-\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\})-(z_4\to z_3)]\\
&+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\qquad\times(\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\}-z_3\leftrightarrow z_4)\Bigg]\\
&+\frac{\alpha_s^2n_f}{2\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big\{2-\frac{z_{13}^2z_{24}^2+z_{23}^2z_{14}^2-z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\qquad\times\mathrm{tr}\{t^a\hat U^\eta_{z_1}t^b\hat U^{\dagger\eta}_{z_2}\}\mathrm{tr}\{t^a\hat U^\eta_{z_3}t^b(\hat U^{\dagger\eta}_{z_4}-\hat U^\eta_{z_3})\}
\end{aligned}\tag{162}$$

Let us demonstrate now that the kernel in formula 162reproduces the NLO BFKL eigenvalues [29]. In the two-gluon approximation the conformal dipoles (246) turn into

$$\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2)=\hat{\mathcal{U}}^\eta(z_1,z_2)-\frac{\alpha_sN_c}{4\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]\tag{163}$$

Also, using Eq. (324) it is easy to demonstrate that

$$[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}=-N_c[\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^\eta_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_1)]\tag{164}$$

and therefore the evolution equation (233) turns into

$$\frac{d}{d\eta}\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_s}{4\pi}\Big[b\ln z_{12}^2\mu^2 - b\frac{z_{13}^2-z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}$$
$$+(\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f\Big]\Big[\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^\eta_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^\eta_{\rm conf}(z_1,z_2)\Big]$$
$$+\frac{\alpha_s^2N_c^2}{8\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big\{2\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$-\frac{3z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$+(1+\frac{n_f}{N_c^3})\Big(\frac{z_{13}^2z_{24}^2+z_{14}^2z_{23}^2-z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}-2\Big)\Big\}\hat{\mathcal{U}}^\eta_{\rm conf}(z_3,z_4)$$
$$+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\hat{\mathcal{U}}^\eta(z_1,z_2) \qquad (165)$$

where we used formula

$$\int d^2z_4\ \Big\{\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big(2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big)-z_3\leftrightarrow z_4\Big\}$$
$$= 12\pi\zeta(3)[\delta(z_{23})-\delta(z_{13})] \qquad (166)$$

following from integrals (330) and (341) from Appendix B.5.

For the case of forward scattering $\langle\hat{\mathcal{U}}(x,y)\rangle = \mathcal{U}(x-y)$ and the linearized equation (165) can be reduced to an integral equation with respect to one variable $z\equiv z_{12}$. Using integrals (104)-(106) from Ref. [40] and integral

$$\int d\tilde{z}\frac{1}{\tilde{z}^2(z-z'-\tilde{z})^2}\ln\frac{z^2z'^2}{(z-\tilde{z}^2)(z'-\tilde{z}^2)} = -\frac{\pi}{(z-z')^2}\ln^2\frac{z^2}{z'^2}$$

we obtain

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z) = \frac{\alpha_sN_c}{2\pi^2}\int d^2z\ \frac{z^2}{(z-z')^2z'^2}\Big\{1$$
$$+\frac{\alpha_s}{4\pi}\Big[b\ln z^2\mu^2 - b\frac{(z-z')^2-z'^2}{z^2}\ln\frac{(z-z')^2}{z'^2}+(\frac{67}{9}-\frac{\pi^2}{3})N_c-\frac{10}{9}n_f\Big]$$
$$\times\ \big[\langle\hat{\mathcal{U}}(z-z')\rangle+\langle\hat{\mathcal{U}}(z')\rangle-\langle\hat{\mathcal{U}}(z)\rangle\big]$$
$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\int d^2z'\ \frac{z^2}{z'^2}\Big[-\frac{1}{(z-z')^2}\ln^2\frac{z^2}{z'^2}+F(z,z')+\Phi(z,z')\Big]\ \mathcal{U}(z')$$
$$+3\frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathcal{U}(z)$$

where

$$F(z,z') = \Big(1+\frac{n_f}{N_c^3}\Big)\frac{3(z,z')^2-2z^2z'^2}{16z^2z'^2}\Big(\frac{2}{z^2}+\frac{2}{z'^2}+\frac{z^2-z'^2}{z^2z'^2}\ln\frac{z^2}{z'^2}\Big)$$

$$-\Big[3+\Big(1+\frac{n_f}{N_c^3}\Big)\Big(1-\frac{(z^2+z'^2)^2}{8z^2z'^2}+\frac{3z^4+3z'^4-2z^2z'^2}{16z^4z'^4}(z,z')^2\Big)\Big]$$
$$\times\int_0^\infty dt\frac{1}{z^2+t^2z'^2}\ln\frac{1+t}{|1-t|}$$

and

$$\Phi(z,z')=\frac{(z^2-z'^2)}{(z-z')^2(z+z')^2}\Big[\ln\frac{z^2}{z'^2}\ln\frac{z^2z'^2(z-z')^4}{(z^2+z'^2)^4}+2\mathrm{Li}_2\Big(-\frac{z'^2}{z^2}\Big)-2\mathrm{Li}_2\Big(-\frac{z^2}{z'^2}\Big)\Big]$$
$$-\Big(1-\frac{(z^2-z'^2)^2}{(z-z')^2(z+z')^2}\Big)\Big[\int_0^1-\int_1^\infty\Big]\frac{du}{(z-z'u)^2}\ln\frac{u^2z'^2}{z^2}$$

The function $-\frac{1}{(q-q')^2}\ln^2\frac{q^2}{q'^2}+F(q,q')+\Phi(q,q')$ enters the NLO BFKL equation in the momentum space [29] and since the eigenfunctions of the forward BFKL equation are powers both in the coordinate and momentum space, it is clear that the corresponding eigenvalues coincide. As to the first term in the r.h.s. of Eq. (167), one can demonstrate using the analysis carried out in Ref. [40] that this term also agrees with the eigenvalues of Ref. [29]. This (somewhat lenghtly) analysis, however, would lead us away from the main topic of our result so we defer it to our future project. It should be also mentioned that the statement in our previous result [40] that our equation (245) disagrees with NLO BFKL was due to erroneous calculation of the integral (330) which was assumed to be zero. After taking into account the δ-function contributions in the r.h.s. of eq. (330) the disagreement disappears.

CHAPTER IV

CONFORMAL NLO BFKL KERNEL IN $\mathcal{N}$=4 SYM

The high-energy behavior of perturbative amplitudes is given by the BFKL pomeron [21]. In the leading order, the BFKL equation is conformally invariant under the Möbius SL(2,C) group of transformations of the transverse plane. In the next-to-leading order (NLO) the BFKL kernel in QCD is not invariant because of the running coupling, but the kernel in $\mathcal{N}=4$ SYM is expected to be invariant. The eigenvalues of this conformal kernel are known from the calculation of forward NLO BFKL in the momentum space [47]. In a conformal theory it is possible to recover the amplitude of the non-forward scattering of two reggeized gluons from the forward scattering amplitude. Using the NLO kernel for evolution of color dipoles in QCD [40] we guess the Möbius invariant kernel for $\mathcal{N}$=4 SYM and check that it reproduces known eigenvalues [47].

At high energies the typical forward scattering amplitude has the form

$$A(s,0) = s\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}F_A(q)F_B(q')\times\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\;f_+(\omega)\Big(\frac{s}{qq'}\Big)^{\omega}G_\omega(q,q') \tag{167}$$

where $F_A(q)$, $F_B(q')$ are the impact factors, $f_+(\omega)=\frac{1-e^{-i\pi\omega}}{\sin\pi\omega}$ is the signature factor, and $G_\omega(q,q')$ is the partial wave of the forward reggeized gluon scattering amplitude satisfying the BFKL equation

$$\omega G_\omega(q,q') = \delta^2(q-q') + \int d^2pK(q,p)G_\omega(p,q') \tag{168}$$

In $\mathcal{N}=4$ SYM the kernel $K(q,p)$ is known up to the next-to-leading order [47]

$$\begin{aligned}&\int d^2pK(q,p)f(p)\\&=\frac{\alpha_sN_c}{\pi^2}\int d^2p\,\Big\{\frac{1}{(q-p)^2}\Big(1-\frac{\alpha_sN_c\pi}{12}\Big)[f(p)-\frac{q^2}{2p^2}f(q)]\\&+\frac{\alpha_sN_c}{4\pi}[\Phi(q,p)-\frac{\ln^2q^2/p^2}{(q-p)^2}]\;f(p)\Big\}+\frac{3\alpha_s^2N_c^2}{2\pi^2}\zeta(3)f(p)\end{aligned} \tag{169}$$

where ζ is the Riemann zeta-function and

$$\Phi(q,p)=\frac{(q^2-p^2)}{(q-p)^2(q+p)^2}\Big[\ln\frac{q^2}{p^2}\ln\frac{q^2p^2(q-p)^4}{(q^2+p^2)^4}$$

$$+2\mathrm{Li}_2(-\frac{p^2}{q^2}) - 2\mathrm{Li}_2(-\frac{q^2}{p^2})\Big] - \Big[1 - \frac{(q^2-p^2)^2}{(q-p)^2(q+p)^2}\Big]$$
$$\times \Big[\int_0^1 - \int_1^\infty\Big]\frac{du}{(qu-p)^2}\ln\frac{u^2q^2}{p^2} \tag{170}$$

Here Li_2 is the dilogarithm.

The eigenvalues of the kernel (169) are [47]

$$\int d^2p\Big(\frac{p^2}{q^2}\Big)^{-\frac{1}{2}+i\nu} e^{in\phi}K(q,p) \; = \; \omega(n,\nu), \tag{171}$$
$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu) + \frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],$$
$$\delta(n,\gamma) = 6\zeta(3) - \frac{\pi^2}{3}\chi(n,\gamma) - \chi''(n,\gamma)$$
$$- 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)$$

where $\chi(n,\gamma) = 2\psi(1) - \psi(\gamma+\frac{n}{2}) - \psi(1-\gamma+\frac{n}{2})$ and

$$\Phi(n,\gamma) \; = \; \int_0^1 \frac{dt}{1+t}\, t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)$$
$$-\mathrm{Li}_2(t) - \mathrm{Li}_2(-t) - \Big(\psi(n+1) - \psi(1) + \ln(1+t)$$
$$+\sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\} \tag{172}$$

The Regge limit of the amplitude $A(x,y;x',y')$ in the coordinate space can be achieved as

$$x = \lambda x_* p_1 + x_\perp, \quad y = \lambda y_* p_2 + y_\perp,$$
$$x' = \rho x_\bullet p_2 + x'_\perp, \quad y' = \rho y'_\bullet p_2 + y'_\perp \tag{173}$$

with $\lambda,\rho \to \infty$ and $x_* > 0 > y_\bullet$, $x'_\bullet > 0 > y'_\bullet$. Hereafter we use the notations $x_\bullet = p_1^\mu x_\mu$, $x_* = p_2^\mu x_\mu$ where p_1 and p_2 are light-like vectors normalized by $2(p_1,p_2) = s$. These "Sudakov variables" are related to the usual light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$ by $x_* = x^+\sqrt{s/2}$, $x_\bullet = x^-\sqrt{s/2}$ so $x = \frac{2}{s}x_* p_1 + \frac{2}{s}x_\bullet p_2 + x_\perp$. We use the $(1,-1,-1,-1)$ metric so $x^2 = \frac{4}{s}x_\bullet x_* - \vec{x}_\perp^2$.

In the Regge limit (298) the full conformal group reduces to Möbius subgroup SL(2,C) leaving the transverse plane $(0,0,z_\perp)$ invariant. In a conformal theory the four-point amplitude $A(x,y;x',y')$ depends on two conformal ratios which can be chosen as

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2},$$
$$r \; = \; R\Big[1 - \frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2} + \frac{1}{R}\Big]^2 \tag{174}$$

The conformal ratio R scales as $\lambda^2\rho^2$ while r does not depend on λ or ρ. Following Ref. [41] (see also Ref. [42]) it is convenient to introduce two SL(2,C)-invariant vectors

$$\kappa = \frac{\sqrt{s}}{2x_*}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{\sqrt{s}}{2y_*}(p_1 - \frac{y^2}{s}p_2 + y_\perp) \tag{175}$$
$$\kappa' = \frac{\sqrt{s}}{2x'_*}(p_1 - \frac{x'^2}{s}p_2 + x'_\perp) - \frac{\sqrt{s}}{2y'_*}(p_1 - \frac{y'^2}{s}p_2 + y'_\perp)$$

such that

$$\kappa^2\kappa'^2 = \frac{1}{R} \qquad \text{and} \qquad 4(\kappa\cdot\kappa')^2 = \frac{r}{R} \tag{176}$$

(here $x^2 = -x_\perp^2$, $x'^2 = -x'^2_\perp$ and similarly for y). In the coordinate space the analog of Eq. (167) has the form:

$$A(x,y;x',y') = \int d^2z_1 d^2z_2 d^2z'_1 d^2z'_2 \; I_A(x,y;z_1,z_2)$$
$$\times \int \frac{d\omega}{2\pi i} \; R^{\frac{\omega}{2}} \tilde{f}_+(\omega) G_\omega(z_1,z_2;z'_1,z'_2) I_B(x',y';z'_1,z'_2) \tag{177}$$

where $\tilde{f}_+(\omega) = (e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor in the coordinate space. The partial wave of the scattering amplitude of two reggeized gluons satisfies the equation

$$\omega G_\omega(z_1,z_2;z'_1,z'_2) = -\frac{1}{2}\ln^2\frac{(z_1-z'_1)^2(z_2-z'_2)^2}{(z_2-z'_1)^2(z_1-z'_2)^2}$$
$$+\int d^2t_1 d^2t_2 \; K(z_1,z_2;t_1,t_2) G_\omega(t_1,t_2;z'_1,z'_2) \tag{178}$$

Here the first term in the r.h.s. is the leading-order contribution coming from two-gluon exchange.

The meaning of the Eq. (177) is that the amplitude is factorized into the product of three terms I_A, I_B, and G_ω corresponding to rapidities $\eta \sim \eta_A$, $\eta \sim \eta_B$, and $\eta_A > \eta > \eta_B$, respectively. With conformally invariant factorization of the amplitude into such a product the impact factors and G_ω should be separately Möbius invariant leading to invariant kernel $K(z_1,z_2;t_1,t_2)$. The eigenfunctions of a conformal kernel are [43]

$$E_{\nu,n}(z_{10},z_{20}) = \left[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\right]^{\frac{1}{2}+i\nu+\frac{n}{2}}\left[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\right]^{\frac{1}{2}+i\nu-\frac{n}{2}} \tag{179}$$

where $\tilde{z} = z_x + iz_y$, $\bar{z} = z_x - iz_y$ and $z_{10} \equiv z_1 - z_0$ etc. Denoting the eigenvalues of the kernel K by $\omega(n,\nu)$

$$\int d^2t_1 d^2t_2 \; K(z_1,z_2;t_1,t_2) E_{\nu,n}(t_1-z_0,t_2-z_0)$$
$$= \omega(n,\nu) E_{\nu,n}(z_{10},z_{20}) \tag{180}$$

and substituting the formal solution of Eq. (178) into Eq. (177) we obtain

$$
\begin{aligned}
A(x,y;x',y') = & \sum_{n=-\infty}^{\infty}\int\frac{d\nu}{\pi^2}\ \frac{-2(\nu^2+\frac{n^2}{4})R^{\frac{1}{2}\omega(n,\nu)}}{[\nu^2+\frac{(n-1)^2}{4}][\nu^2+\frac{(n+1)^2}{4}]}\\
\times & \tilde{f}_+(\omega(n,\nu))\int d^2z_0 d^2z_1 d^2z_2 I_A(x,y;z_1,z_2)E_{\nu,n}(z_{10},z_{20})\\
\times & \int d^2z'_1 d^2z'_2\ I_B(x',y';z'_1,z'_2)E^*_{\nu,n}(z'_1-z_0,z'_2-z_0)
\end{aligned}
$$

As demonstrated in Ref. [41], the impact factor depends on one conformal (Möbius invariant) ratio

$$
\begin{aligned}
I_A(x,y;z_1,z_2) &= \frac{1}{z_{12}^4}I_A\Big(\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big),\\
I_B(x',y';z'_1,z'_2) &= \frac{1}{z'^4_{12}}I_B\Big(\frac{\kappa'^2(\zeta'_1\cdot\zeta'_2)}{2(\kappa'\cdot\zeta'_1)(\kappa'\cdot\zeta'_2)}\Big)
\end{aligned}
$$

where $\zeta_1\equiv p_1+\frac{z_{1\perp}^2}{s}p_2+z_{1\perp}$ and similarly for other ζ's. This enables us to carry out integrations over z_i and z'_i. The formulas are especially simple when we consider the correlator of four scalar currents such as $\mathrm{Tr}\{Z^2\}$ ($Z=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$) so that only the term with $n=0$ contributes. From conformal (Möbius) invariance we get [41]

$$
\begin{aligned}
&\int\frac{d^2z_1d^2z_2}{z_{12}^4}\ I_A\Big(\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big)\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}+i\nu}\\
&= \frac{1+4\nu^2}{8\pi}\frac{\Gamma^2(\frac{1}{2}-i\nu)}{\Gamma(1-2i\nu)}\Big(\frac{\kappa^2}{4(\kappa\cdot\zeta_0)^2}\Big)^{\frac{1}{2}+i\nu}I_A(\nu)
\end{aligned}
\tag{181}
$$

(here $\zeta_0\equiv p_1+\frac{z_{0\perp}^2}{s}p_2+z_{0\perp}$) and therefore (cf. Ref. [41])

$$
\begin{aligned}
&(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle\\
&= \frac{i}{2}\int d\nu\ \tilde{f}_+(\nu)\frac{\tanh\pi\nu}{\nu}I_A(\nu)I_B(-\nu)\Omega(r,\nu)R^{\frac{1}{2}\omega(\nu)}
\end{aligned}
\tag{182}
$$

where $\mathcal{O}\equiv\frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{Z^2\}$, $\omega(\nu)\equiv\omega(0,\nu)$, $\tilde{f}_+(\nu)\equiv\tilde{f}_+(\omega(\nu))$, and

$$
\Omega(r,\nu)=\frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac{1}{2}-i\nu}
\tag{183}
$$

(Since the integral (183) does not scale with λ,ρ it can depend only on $\frac{(\kappa\cdot\kappa')^2}{\kappa^2\kappa'^2}=\frac{r}{4}$). Eq. (182), obtained in Ref. [44] from general consideration of the Regge limit in a conformal theory, proves the existence of the conformally invariant factorization (177). Note that all the dependence on large energy ($\equiv$ large λ,ρ) is contained in

$R^{\frac{1}{2}\omega(\nu)}$. For completeness, let us mention that in the leading order in perturbation theory $I(\nu)=\frac{2\pi^2 g_s}{\cosh\pi\nu}N_c/\sqrt{N_c^2-1}$.

To restore the NLO BFKL kernel in the coordinate representation (177) from the eigenvalues (171) in the momentum representation we must prove that Eq. (182) agrees with Eq. (167) with the same set of $\omega(\nu)$. (Strictly speaking, we need to demonstrate this property for arbitrary n but here we will do it only for $n=0$).

In order to perform Fourier transformation of the correlator (182) we need to relax the limit (298) by allowing small $x_* \sim y_* \sim 1/\lambda$ and $x'_* \sim y'_* \sim 1/\rho$. The conditions (176) for vectors (175) are now satisfied up to $\frac{1}{\lambda^2}$ and $\frac{1}{\rho^2}$ corrections. The correlator (182) takes the form

$$
\begin{aligned}
&(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle \\
&=\frac{i}{2\pi^3}\int d\nu\ \tilde{f}_+(\nu)\nu\tanh\pi\nu\Big[\frac{16x_*y_*x'_*y'_*}{s^2(x-y)^2(x'-y')^2}\Big]^{\frac{1}{2}\omega(\nu)} \\
&\times\int d^2z_0\left[\frac{\frac{(x-y)^2}{x_*y_*}}{(\frac{(x-z_0)^2_\perp}{x_*}-\frac{(y-z_0)^2_\perp}{y_*}-\frac{4}{s}(x-y)_*)^2}\right]^{\frac{1}{2}+i\nu}I_A(\nu) \\
&\times\left[\frac{\frac{(x'-y')^2}{x'_*y'_*}}{(\frac{(x'-z_0)^2_\perp}{x_*}-\frac{(y'-z_0)^2_\perp}{y'_*}-\frac{4}{s}(x'-y')_*)^2}\right]^{\frac{1}{2}-i\nu}I_B(-\nu)
\end{aligned}
\tag{184}
$$

The forward scattering amplitude can be defined as (cf. Ref. [17])

$$
\begin{aligned}
&A(s,0) \\
&=-i\int d^4z d^4x d^4y\langle\mathcal{O}(x_*,x_*+z_*,x_\perp+z_\perp)\mathcal{O}^\dagger(0,z_*,z_\perp) \\
&\times\mathcal{O}(y_*+z_*,y_*,y_\perp)\mathcal{O}^\dagger(z_*,0,0)\rangle\ e^{ip_A\cdot x+ip_B\cdot y}
\end{aligned}
\tag{185}
$$

where $p_A=p_1+\frac{p_A^2}{s}p_2$ and $p_B=p_2+\frac{p_B^2}{s}p_1$. Substituing Eq. (184) in Eq. (185) and performing the integrations over the coordinates we obtain

$$
\begin{aligned}
A(s,0)&=\frac{\pi^2 s}{(p_A^2p_B^2)^{3/2}}\int d\nu\ I_A(\nu)I_B(-\nu)\Big(\frac{s}{\sqrt{p_A^2p_B^2}}\Big)^{\omega(\nu)} \\
&\times f_+(\nu)\Big(\frac{p_A^2}{p_B^2}\Big)^{i\nu}\left|\frac{\Gamma^2(\frac{3}{2}+\frac{\omega(\nu)}{2}+i\nu)}{\Gamma(3+\omega(\nu)+2i\nu)}\frac{\Gamma^2(\frac{1}{2}-i\nu)}{\Gamma(1-2i\nu)}\right|^2
\end{aligned}
\tag{186}
$$

This should be compared with Eq. (167) which takes the form

$$
\begin{aligned}
A(s,0)\ &=\ s\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}F_A(q)F_B(q') \\
&\times\int\frac{d\nu}{2\pi^2}\ f_+(\nu)(q^2)^{-\frac{1}{2}+\frac{i\nu}{2}}(q'^2)^{-\frac{1}{2}-\frac{i\nu}{2}}\Big(\frac{s}{|q||q'|}\Big)^{\omega(\nu)}
\end{aligned}
\tag{187}
$$

$$= \int \frac{d\nu}{2\pi^2} \frac{s f_+(\nu)}{(p_A^2 p_B^2)^{\frac{3}{2}}} F_A(\nu) F_B(-\nu) \Big[\frac{s}{\sqrt{p_A^2 p_B^2}}\Big]^{\omega(\nu)} \Big(\frac{p_A^2}{p_B^2}\Big)^{i\nu}$$

It is clear that Eq. (186) and Eq. (187) coincide after the redefinition of the impact factor

$$F_A(\nu) = \sqrt{2}\pi^2 I_A(\nu) \frac{\Gamma^2(\frac{3}{2} - i\nu + \frac{\omega(\nu)}{2})\Gamma^2(\frac{1}{2} + i\nu)}{\Gamma(3 - 2i\nu + \omega(\nu))\Gamma(1 + 2i\nu)} \tag{188}$$

and similarly for F_B.

Now we are in a position to restore $K_{\rm NLO}(z_1, z_2; t_1, t_2)$ from the eigenvalues (171). Using the eigenvalues $\omega(n, \nu)$ and the requirement of conformal invariance it is possible to restore the conformal kernel for the BFKL equation [43]

$$\begin{aligned} K(z_1, z_2; z_3, z_4) &= \frac{1}{z_{34}^4} \sum_{n=-\infty}^{\infty} \int \frac{d\nu}{\pi^4} (\nu^2 + \frac{n^2}{4}) \omega(n, \nu) \\ &\times \int d^2 z_0 \; E_{\nu,n}(z_{10}, z_{20}) E^*_{\nu,n}(z_{30}, z_{40}) \end{aligned} \tag{189}$$

At the leading-order level K is given by the BFKL kernel in the dipole form (the linear part of the BK equation [45, 20])

$$\begin{aligned} K_{\rm LO}(z_1, z_2; z_3, z_4) &= \frac{\alpha_s N_c}{2\pi^2} \Big[\frac{z_{12}^2 \delta^2(z_{13})}{z_{14}^2 z_{24}^2} + \frac{z_{12}^2 \delta^2(z_{24})}{z_{13}^2 z_{23}^2} \\ &- \delta^2(z_{13}) \delta^2(z_{24}) \int d^2 z \; \frac{z_{12}^2}{(z_1 - z)^2 (z_2 - z)^2}\Big] \end{aligned} \tag{190}$$

At the NLO level, to perform explicitly the three summations and four integrations in Eq. (189) seems a formidable task. Instead, using the results of Ref. [40], we guess the NLO kernel in the form

$$\begin{aligned} K_{\rm NLO}(z_1, z_2; z_3, z_4) &= -\frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} K_{\rm LO}(z_1, z_2; z_3, z_4) \\ &+ \frac{\alpha_s^2 N_c^2}{8\pi^4 z_{34}^4} \Big[\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \Big\{\Big(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\ &+ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}\Big\} + 12\pi^2 \zeta(3) z_{34}^4 \delta(z_{13}) \delta(z_{24})\Big] \end{aligned} \tag{191}$$

and check that its eigenvalues coincide with $\omega_{\rm NLO}$ from Eq. (171). The explicit form of the NLO BFKL kernel (191) is the main result we will obtain. It is worth noting that the first term in braces in the r.h.s. corresponds to the analytic term in the conformal part of NLO BK kernel in QCD [40].

Equation (178) with the kernel (191) is obviously conformally invariant. Let us prove that its eigenvalues are given by Eq. (171). The integral

$$\int d^2z_3 d^2z_4 K_{\rm NLO}(z_1,z_2;z_3,z_4)E_{n,\nu}(z_{30},z_{40})$$
$$= [c(n,\nu)+\frac{\alpha_s^2N_c^2}{4\pi^2}(6\zeta(3)-\frac{\pi^2}{3}\chi(n,\nu))]E_{n,\nu}(z_{10},z_{20}) \tag{192}$$

can be reduced to

$$\frac{\alpha_s^2N_c^2}{8\pi^4}\int dz_3dz_4\frac{z_{12}^2}{z_{34}^2z_{13}^2z_{24}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}$$
$$+1\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\,(\frac{\tilde z_{34}}{\tilde z_{12}})^{\frac12+i\nu-\frac n2}(\frac{\bar z_{34}}{\bar z_{12}})^{\frac12+i\nu+\frac n2}=c(n,\nu)$$

by setting $z_0=0$ and making the inversion $x_i\to x_i/x^2$. Taking now $z_2=0$ we obtain

$$\frac{\alpha_s^2N_c^2}{8\pi^4}\int d^2z\,\frac{z_1^2}{z^2}(\frac{\tilde z}{\tilde z_1})^{\frac12+i\nu-\frac n2}(\frac{\bar z}{\bar z_1})^{\frac12+i\nu+\frac n2}\int d^2z'$$
$$\frac{1}{(z_1-z-z')^2z'^2}\,\Big\{2\ln\frac{z_1^2z^2}{(z_1-z')^2(z+z')^2}$$
$$+\Big[1+\frac{z_1^2z^2}{(z_1-z-z')^2z'^2-(z_1-z')^2(z+z')^2}\Big]$$
$$\times\ \ln\frac{(z_1-z-z')^2z'^2}{(z_1-z')^2(z+z')^2}\Big\}=c(n,\nu) \tag{193}$$

Using now the integral

$$\int\frac{d^2z'}{\pi}\,\frac{\ln(z_1-z')^2(z+z')^2/(z_1^2z^2)}{(z_1-z-z')^2z'^2}\ =\ \frac{1}{(z_1-z)^2}\ln^2\frac{z^2}{z_1^2} \tag{194}$$

and the integral J_{13} from Ref. [32]

$$\int\frac{d^2z'}{2\pi}\,\Big[1+\frac{z_1^2z^2}{(z_1-z-z')^2z'^2-(z_1-z')^2(z+z')^2}\Big]$$
$$\times\frac{z'^{-2}}{(z_1-z-z')^2}\ln\frac{(z_1-z-z')^2z'^2}{(z_1-z')^2(z+z')^2}\ =\Phi(z_1,z)$$

(see Eq. (170) for the definition of Φ) we obtain

$$\frac{\alpha_s^2N_c^2}{4\pi^3}\int d^2z\,(z^2/z_1^2)^{-\frac12+i\nu+\frac n2}e^{-in\phi}\Big[-\frac{1}{(z_1-z)^2}\ln^2\frac{z^2}{z_1^2}$$
$$+\ \Phi(z_1,z)\Big]\ =\ c(n,\nu) \tag{195}$$

where ϕ is the angle between $\vec z$ and $\vec z_1$. The final step is to use integrals [47]

$$\int\frac{d^2z}{\pi}\,\frac{1}{(z_1-z)^2}(z^2/z_1^2)^\gamma e^{in\phi}\ln^2\frac{z^2}{z_1^2}\ =\ \chi''(n,\gamma)$$
$$\int\frac{d^2z}{2\pi}\,(\frac{z^2}{z_1^2})^{\gamma-1}e^{in\phi}\Phi(z_1,z)=-\Phi(n,\gamma)-\Phi(n,1-\gamma)$$

Comparing to Eq. (169) we see that

$$c(n,\nu) = \frac{\alpha_s^2 N_c^2}{4\pi^2}\Big[-\chi''(n,\nu) - 2\Phi(n,\frac{1}{2}+i\nu) - 2\Phi(n,\frac{1}{2}-i\nu)\Big]$$

corresponds to $\omega_{\rm NLO}$ from Eq. (171).

Let us comment on the result in the literature that NLO BFKL in the coordinate space is not conformally invariant [46]. We think that the difference between our kernel and that of Ref. [46] is due to different cutoffs for longitudinal integrations. As we mentioned above, the conformal result for the NLO BFKL kernel (191) corresponds to the factorization in rapidity consistent with Möbius invariance. In other words, this kernel should describe the evolution of the color dipole with the conformally invariant rapidity cutoff. At present, there is no obvious way to impose such a cutoff although we believe that it can be done by constructing a "composite operator" for a color dipole, order by order in the perturbation theory. We also think that the Fourier transform of Eq. (184) in the non-forward case would give the precise cutoff for the longitudinal integrations in the momentum space and the change in the cutoff will lead to the transformation $K_{\rm NLO} \to K_{\rm NLO} - [O, K_{\rm NLO}]$ (Eq. (1) in Ref. [46]) which shall cure the discrepancy with the results of Ref. [46].

One can also restore the NLO QCD kernel with the same rapidity cutoff implicitly defined above to satisfy the requirement of the conformal invariance of the $\mathcal{N}=4$ kernel (191). Using the results of [24, 40] one obtains

$$\begin{aligned}
&K^{\rm QCD}_{\rm NLO}(z_1,z_2;z_3,z_4) = K_{\rm NLO}(z_1,z_2;z_3,z_4) \\
&+\frac{\alpha_s}{4\pi}(b\ln z_{12}^2\mu^2 + \frac{67}{9}N_c - \frac{10}{9}n_f)K_{\rm LO}(z_1,z_2;z_3,z_4) \\
&+\frac{\alpha_s^2 N_c}{8\pi^3}b\Big[\delta^2(z_{13})(\frac{1}{z_{14}^2}-\frac{1}{z_{24}^2})\ln\frac{z_{14}^2}{z_{24}^2} + \delta^2(z_{24})(\frac{1}{z_{13}^2} \\
&-\frac{1}{z_{23}^2})\ln\frac{z_{13}^2}{z_{23}^2} - \delta^2(z_{13})\delta^2(z_{24})\int d^2z_0(\frac{1}{z_{10}^2}-\frac{1}{z_{20}^2})\ln\frac{z_{10}^2}{z_{20}^2}\Big] \\
&+\frac{\alpha_s^2 N_c^2}{8\pi^4 z_{34}^4}\Big[-3\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
&+(1+\frac{n_f}{N_c^3})\Big(\frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} - 2\Big)\Big]
\end{aligned} \tag{196}$$

where $b = 11N_c/3 - 2n_f/3$ and μ is the normalization point in the $\overline{\rm MS}$ scheme. This kernel has the QCD eigenvalues $\omega(n,\nu)$ from Ref. [29]. Note that Eq. (196) is different from the NLO BK kernel for the evolution of color dipoles in Ref. [40] since the "rigid cutoff" $\alpha < \sigma$ we adopted in that paper is not conformally invariant.

CHAPTER V

NLO EVOLUTION OF COLOR DIPOLES IN $\mathcal{N}$=4 SYM

In this chapter we will derive the NLO kernel of the BK equation in the gauge theory of $\mathcal{N} = \Delta$ SYM theory. We then verify that the liner version of it coincides with the conformal BFKL NLO kernel obtained in previous chapter.

V.1 INTRODUCTION

Let us observe that the BK equation is conformally invariant in the two-dimensional space. This follows from the conformal invariance of the light-like Wilson lines. Indeed, it is easy to see that the Wilson line

$$U(x_\perp) = \text{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \; A_+(x^+, x_\perp)\right\} \tag{197}$$

is invariant under the inversion $x^\mu \to x^\mu/x^2$ (with respect to the point with zero (-) component). Indeed, $(x^+, x_\perp)^2 = -x_\perp^2$ so after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^- \to x^+/x_\perp^2$ and therefore

$$U(x_\perp) \;\to\; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \; A_+(\frac{x^+}{x_\perp^2}, x_\perp)\right\} = U(x_\perp/x_\perp^2) \tag{198}$$

It is easy to check that the Wilson line operators lie in the standard representation of the conformal Möbius group SL(2,C) with conformal spin 0 (see Appendix B.1).

The NLO evolution of color dipole in QCD [40] is not expected to be Möbius invariant due to the conformal anomaly leading to dimensional transmutation and running coupling constant. However, the NLO BK equation in QCD [40] has an additional terms violating Möbius invariance not related to the conformal anomaly. To understand the relation between the high-energy behavior of amplitudes and Möbius invariance of Wilson lines, it is instructive to consider the conformally invariant $\mathcal{N} = 4$ super Yang-Mills theory. This theory was intensively studied studied in recent years due to the fact that at large coupling constants it is dual to the IIB string theory in the AdS_5 background. In the light-cone limit, the contribution of scalar operators to Maldacena-Wilson line vanishes so one has the usual Wilson line constructed from gauge fields and therefore the LLA evolution equation for color dipoles in $\mathcal{N} = 4$ SYM has the same form as (32). At the NLO level, the contributions from gluino and scalar loops enter the game.

As we mentioned above, formally the light-like Wilson lines are Möbius invariant. Unfortunately, the light-like Wilson lines are divergent in the longitudinal direction and moreover, it is exactly the evolution equation with respect to the longitudinal cutoff which governs the high-energy behavior of amplitudes. It is impossible to find the conformally invariant cutoff in the longitudinal direction so when we use the non-invariant cutoff we can expect, as usual, the invariance to hold in the leading order but violated in higher orders in perturbation theory. We use the cutoff in longitudinal momentum of the gluons of Wilson line, and with this cutoff the NLO evolution equation in QCD has extra non-conformal parts not related to the running of coupling constant. Similarly, in $\mathcal{N}=4$ equation there will be non-conformal parts coming from the longitudinal cutoff of Wilson lines. We will demonstrate below that it is possible to construct the "composite conformal dipole operator" (order by order in perturbation theory) which mimics the conformal cutoff in the longitudinal direction so the corresponding evolution equation has no extra non-conformal parts. This is similar to the construction of the composite renormalized local operator in the case when the UV cutoff does not respect the symmetries of the bare operator - in this case, the symmetry of the UV-regularized operator is preserved order by order in perturbation theory by subtraction of the symmetry-restoring counterterms.

Let us present our result for the NLO evolution of the color dipole in the adjoint representation (hereafter we use notations $z_{13} \equiv x - z$, $z_{14} \equiv x - z_4$, $z_{23} \equiv y - z$, $z_{24} \equiv y - z_4$ and $(T^a)_{bc} = -if^{abc}$)

$$
\begin{aligned}
\frac{d}{d\eta}[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = {} & \frac{\alpha_s}{\pi^2}\int d^2 z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 - \frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big][\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{Tr}\{T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} \\
& \qquad - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \\
- \frac{\alpha_s^2}{8\pi^4}\int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} & \Bigg[\Big\{\Big(2\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big) \\
+ (z_3 \leftrightarrow z_4)\Big\} & \mathrm{Tr}\{[T^a, T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^\eta_{z_1}[T^{b'}, T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\} \\
& \times(\hat{U}^\eta_{z_3})^{aa'}[(\hat{U}^\eta_{z_4})^{bb'} - z_4 \to z_3] \\
+ \Big\{\Big(2\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\ln & \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big) - (z_3 \leftrightarrow z_4)\Big\} \\
& \times \mathrm{Tr}\{[T^a, T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^\eta_{z_1}[T^{b'}, T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4})^{bb'}\Bigg]
\end{aligned}
\tag{199}
$$

where

$$
[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} \tag{200}
$$

$$+\frac{\alpha_s}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}$$

is the "composite dipole" with the conformal longitudinal cutoff in the next-to-leading order and a is an arbitrary dimensional constant (see the discussion in the Appendix B.2). Similar expression for the conformal two-dipole operator in the r.h.s. of this equation is presented below, see Eq. (230). This kernel in the r.h.s. of Eq. (199) is obviously Möbius invariant since it depends on two four-point conformal ratios $\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$ and $\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}$. We will also demonstrate that Eq. (199) agrees with forward NLO BFKL calculation of Ref. [47].

V.2 CALCULATION OF THE NLO BK KERNEL

In the next-to-leading order the contributions to the kernel come from the one-loop diagrams for the color dipole in the shock-wave background. We will use the results for the gluon part of the NLO BK kernel from Ref. [40] and calculate the contribution of scalar and gluino loops. The $\mathcal{N}=4$ Lagrangian reads:

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(D^\mu\Phi^a_I)(D_\mu\Phi^a_I) - \frac{1}{4}g^2f^{abc}f^{lmc}\Phi^a_I\Phi^b_J\Phi^l_I\Phi^m_J \\ + \bar\lambda^a_{\dot\alpha A}\sigma^{\dot\alpha\beta}_\mu D^\mu\lambda^{aA}_\beta - i\,\lambda^{\alpha A}_a\bar\Sigma^a_{AB}A^a_b\lambda^B_{\alpha k}f^{abc} + i\,\bar\lambda^a_{\dot\alpha A}\Sigma^{aAB}A^a_b\bar\lambda^{\dot\alpha}_{Bc}f^{abc}\end{aligned} \quad (201)$$

so the bace propagators are

$$\begin{aligned}\langle\Phi^a_I(x)\Phi^b_J(y)\rangle &= i\delta^{ab}\delta_{IJ}\int d^4p\frac{e^{-ip\cdot(x-y)}}{p^2+i\epsilon}, \\ \langle\lambda^{aI}_\beta(x)\,\bar\lambda^{bJ}_{\dot\alpha}(y)\rangle &= \int d^4p e^{-ip\cdot(x-y)}\frac{ip_\mu\,\bar\sigma^\mu_{\beta\dot\alpha}}{p^2+i\epsilon},\end{aligned} \quad (202)$$

and the vertex of gluon emission in the momentum space is proportional to $(k_1-k_2)^\mu T^a\delta_{IJ}$ for the scalars and $\sigma^\mu T^a$ for gluinos. (We do not need Yukawa or four-scalar vertices at this level). The diagrams in the shock-way background are calculated similarly to the tree diagrams discussed in the previous Section.

V.2.1 Gluon contribution to NLO BK

Let us start with the gluon contribution to the NLO evolution kernel. There is no difference between the gluon part of the kernel in QCD and in $\mathcal{N}=4$ SYM so we will just copy it from Ref. [40]) replacing $\mathrm{tr}\{t^aU_{z_1}t^bU^\dagger_{z_2}\}$ in the fundamental representation by $\mathrm{Tr}\{T^aU_{z_1}T^bU^\dagger_{z_2}\}$ (we denote traces in the fundamental and adjoint representations of color group by tr{...} and Tr{...}, respectively)

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}_{\rm gluon}=\frac{\alpha_s}{\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1+\frac{\alpha_sN_c}{4\pi}\Big[\frac{11}{3}\ln z_{12}^2\mu^2-\frac{11}{3}\frac{z_{13}^2-z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}$$
$$+\frac{64}{9}-\frac{\pi^2}{3}-2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$+\frac{\alpha_s^2}{8\pi^4}\ln\frac{\alpha_{\max}}{\alpha_{\min}}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\,(\hat U^\eta_{z_3})^{aa'}\Big(2\Big[2-\frac{z_{13}^2z_{24}^2+z_{14}^2z_{23}^2-4z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big]$$
$$\times\mathrm{Tr}\{[T^a,T^b]U_{z_1}[T^{a'},T^{b'}]U^\dagger_{z_2}\}$$
$$-\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}+z_3\leftrightarrow z_4\Big]$$
$$\times\mathrm{Tr}\{[T^a,T^b]U_{z_1}T^{a'}T^{b'}U^\dagger_{z_2}+T^bT^aU_{z_1}[T^{b'},T^{a'}]U^\dagger_{z_2}\}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$
$$-\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}-z_3\leftrightarrow z_4\Big]$$
$$\times\mathrm{Tr}\{[T^a,T^b]U_{z_1}T^{a'}T^{b'}U^\dagger_{z_2}+T^bT^aU_{z_1}[T^{b'},T^{a'}]U^\dagger_{z_2}\}U^{bb'}_{z_4}\Big)\qquad(203)$$

where μ is the normalization point in the $\overline{MS}$ scheme. Note that the last term in r.h.s. is Möbius invariant. The coefficient $\frac{11}{3}$ stands in front of the non-conformal terms coming from the running of the coupling constant and, as we discussed in the Introduction, there is an additional non-conformal term ($\sim\ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2}$) coming from the non-invariance of the longitudinal cutoff (45).

It should be noted that there is, however, one small difference between QCD and $\mathcal{N}=4$ calculations of the gluon loop due to the fact that in supersymmetic theories it is more natural to use the dimensional reduction scheme instead of dimensional regularization. In dimensional reduction scheme the factor $g^\perp_{\mu\nu}g^{\perp\mu\nu}=d_\perp$ coming from the product of three-gluon vertices should be replaced by 2: $g^\perp_{\mu\nu}g^{\perp\mu\nu}\to 2$. Making proper replacement in formulas in Sect. IV of Ref. [40] one gets the factor $\frac{67}{9}$ from Eq. (5) in Ref. [40]) replaced by $\frac{64}{9}$ in the r.h.s. of the above equation.

V.2.2 Contribution of scalar particles

Diagrams with two scalar-shockwave intersections

First, we calculate the diagram with two scalar-shockwave intersections shown in Fig. 18.

The scalar propagator in the shock-wave background has the form [17]:

$$\langle\hat\Phi^a_I(x)\hat\Phi^b_J(y)\rangle=\theta(x_*y_*)\delta^{ab}\frac{s}{2}\int d\alpha d\beta(x_\perp|\frac{i\delta_{IJ}}{i(\alpha\beta s-p_\perp^2+i\epsilon)}|y_\perp)\qquad(204)$$

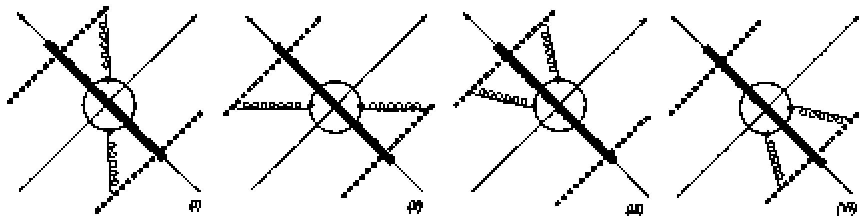

FIG. 18: Diagrams with the scalar loop bisected by the shock wave.

$$+\delta_{IJ}\int_0^\infty d\alpha\,\frac{e^{-i\alpha(x-y)_*}}{2\alpha}\Big[\theta(x_*)\theta(-y_*)(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}U^{ab}e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)$$
$$+\theta(-x_*)\theta(y_*)(x_\perp|e^{i\frac{p_\perp^2}{\alpha s}x_*}U^{\dagger ab}e^{-i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\Big]$$

We start with the calculation of the Fig. 18a diagram. Multiplying two propagators (225), two scalar-gluon vertices and two bare gluon propagators we obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(up_1+z_1)\hat{A}^b_\bullet(vp_1+z_2)\rangle = -6g^4\frac{s^2}{8}f^{anl}f^{bn'l'} \tag{205}$$

$$\int d\alpha d\alpha_1 d\beta d\beta' d\beta_1 d\beta_1' d\beta_2 d\beta_2'\int d^2z d^2z_4\int d^2q_1 d^2q_2 d^2k_1 d^2k_2\; e^{i(q_1+q_2,z_1)_\perp - i(k_1+k_2,z_2)_\perp}$$
$$\frac{4\alpha_1(\alpha-\alpha_1)U^{nn'}_{z_3}U^{ll'}_{z_4}e^{-i(q_1-k_1,z_3)_\perp - i(q_2-k_2,z_4)_\perp}}{(\beta-\beta_1-\beta_2+i\epsilon)(\beta'-\beta_1'-\beta_2'+i\epsilon)(\beta-i\epsilon)(\beta'-i\epsilon)} \tag{206}$$
$$\times\frac{d_{\bullet\lambda}(\alpha p_1+\beta p_2+q_{1\perp}+k_{1\perp})}{\alpha\beta s-(q_1+q_2)^2_\perp+i\epsilon}\frac{d_{\lambda'\bullet}(\alpha p_1+\beta' p_2+q_{2\perp}+k_{2\perp})}{\alpha\beta' s-(k_1+k_2)^2_\perp+i\epsilon}$$
$$\frac{[(2\alpha_1-\alpha)p_1+(q_1-q_2)_\perp]^\lambda[(2\alpha_1-\alpha)p_1+(k_1-k_2)_\perp]^{\lambda'}}{(\alpha_1\beta_1 s-q^2_{1\perp}+i\epsilon)(\alpha_1\beta_1' s-k^2_{1\perp}+i\epsilon)[(\alpha-\alpha_1)\beta_2 s-q^2_{2\perp}+i\epsilon][(\alpha-\alpha_1)\beta_2' s-k^2_{2\perp}+i\epsilon]}$$

Taking residues at $\beta=\beta'=0$ and $\beta_2=-\beta_1$, $\beta_2'=-\beta_1'$ we obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(up_1+z_1)\hat{A}^b_\bullet(vp_1+z_2)\rangle - 6g^4\frac{s^2}{8}f^{anl}f^{bn'l'} \tag{207}$$
$$=\int d\alpha d\alpha_1 d\beta_1 d\beta_1'\int d^2z d^2z_4\int d^2q_1 d^2q_2 d^2k_1 d^2k_2\; e^{i(q_1+q_2,z_1)_\perp - i(k_1+k_2,z_2)_\perp}$$
$$\times\,4\frac{\alpha_1(\alpha-\alpha_1)}{\alpha^2}U^{nn'}_{z_3}U^{ll'}_{z_4}e^{-i(q_1-k_1)z_3 - i(q_2-k_2)z_4}\;\frac{q^2_{1\perp}-q^2_{2\perp}}{(q_1+q_2)^2_\perp}\frac{k^2_{1\perp}-k^2_{2\perp}}{(k_1+k_2)^2_\perp}\frac{1}{(\alpha_1\beta_1 s-q^2_{1\perp}+i\epsilon)}$$
$$\times\frac{1}{(\alpha_1\beta_1' s-k^2_{1\perp}+i\epsilon)[-(\alpha-\alpha_1)\beta_1 s-q^2_{2\perp}+i\epsilon][-(\alpha-\alpha_1)\beta_1' s-k^2_{2\perp}+i\epsilon]}$$

Taking residues at $\beta_1=\frac{q_1^2}{\alpha_1 s}$ and $\beta_1'=\frac{k_1^2}{\alpha_1 s}$ we obtain

$$-6\frac{g^4}{8\pi^2}f^{anl}f^{bn'l'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\;\bar{u}u\int d^2z d^2z_4$$

$$\times \int d^2q_1 d^2q_2 d^2k_1 d^2k_2 \; e^{i(q_1+q_2,z_1)_\perp - i(k_1+k_2,z_2)_\perp - i(q_1-k_1,z_3) - i(q_2-k_2,z_4)}$$
$$\times U^{nn'}_{z_3} U^{ll'}_{z_4} \frac{(q^2_{1\perp} - q^2_{2\perp})(k^2_{1\perp} - k^2_{2\perp})}{(q_1+q_2)^2_\perp (k_1+k_2)^2_\perp} \; \frac{1}{(q^2_{1\perp}\bar{u} + q^2_{2\perp}u)(k^2_{1\perp}\bar{u} + k^2_{2\perp}u)} \tag{208}$$

where, as usually, $u = \alpha_1/\alpha$. The contribution of the diagram in Fig. 18b is obtained by replacing $e^{i(q_1+q_2,z_1)_\perp}$ by $-e^{i(q_1+q_2,z_2)_\perp}$ and the two remaining diagrams i n Fig. 18c and Fig. 18d are obtained by change $x \leftrightarrow y$. We get

$$\langle \mathrm{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}} = 6\frac{g^4}{8\pi^2} f^{anl} f^{bn'l'} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \; \bar{u}u \int d^2z d^2z_4$$
$$\int d^2q_1 d^2q_2 d^2k_1 d^2k_2 \; [e^{i(q_1+q_2,z_1)_\perp} - e^{i(q_1+q_2,z_2)_\perp}]$$
$$\times \; [e^{i(k_1+k_2,z_1)_\perp} - e^{-i(k_1+k_2,z_2)_\perp}] e^{-i(q_1-k_1,z_3) - i(q_2-k_2,z_4)} U^{nn'}_{z_3} U^{ll'}_{z_4}$$
$$\times \frac{(q^2_{1\perp} - q^2_{2\perp})(k^2_{1\perp} - k^2_{2\perp})}{(q_1+q_2)^2_\perp (k_1+k_2)^2_\perp} \; \frac{1}{(q^2_{1\perp}\bar{u} + q^2_{2\perp}u)(k^2_{1\perp}\bar{u} + k^2_{2\perp}u)} \tag{209}$$

Performing the Fourier transformation

$$\int d^2q_1 d^2q_2 \; e^{i(q_1,x_1)+i(q_2,x_2)} \; \frac{q_1^2 - q_2^2}{(q_1+q_2)^2(q_1^2\bar{u} + q_2^2 u)} = -\frac{x_1^2 - x_2^2}{4\pi^2(x_1-x_2)^2(ux_1^2 + \bar{u}x_2^2)} \tag{210}$$

we get

$$\langle \mathrm{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}} = 6\frac{g^4}{128\pi^6} f^{anl} f^{bn'l'} \int_0^\sigma \frac{d\alpha}{\alpha} \int_0^1 du \int d^2z d^2z_4 \; U^{nn'}(z) U^{ll'}(z_4) \; \frac{\bar{u}u}{z_{34}^4}$$
$$\times \Big[\frac{z_{13}^2 - z_{14}^2}{z_{13}^2 u + z_{14}^2\bar{u}} - \frac{z_{23}^2 - z_{24}^2}{z_{23}^2 u + z_{24}^2 \bar{u}}\Big]\Big[\frac{z_{13}^2 - z_{14}^2}{z_{13}^2 u + z_{14}^2\bar{u}} - \frac{z_{23}^2 - z_{24}^2}{z_{23}^2 u + z_{24}^2 \bar{u}}\Big] \mathrm{Tr}\{T^a U_{z_1} T^b U^\dagger_{z_2}\} \tag{211}$$

and therefore

$$\sigma\frac{d}{d\sigma}\langle \mathrm{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}} = -\frac{3\alpha^2}{4\pi^4}\int \frac{d^2z d^2z_4}{z_{34}^4}$$
$$\times \Big[-2 + \frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big] \; U^{aa'}_{z_3} U^{bb'}_{z_4} \mathrm{Tr}\{[T^a, T^b] U_{z_1} [T^{a'}, T^{b'}] U^\dagger_{z_2}\} \tag{212}$$

Following the method suggested in Refs. [24, 52, 40] we separate the UV-divergent part by adding and subtracting $z_4 \to z_3$ contribution: $U^{aa'}_{z_3} U^{bb'}_{z_4} = (U^{aa'}_{z_3} U^{bb'}_{z_4} - z_4 \to z_3 + U^{aa'}_{z_3} U^{bb'}_{z_3}$. We get

$$\sigma\frac{d}{d\sigma}\langle \mathrm{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}} = -\frac{3\alpha^2}{4\pi^4}\int \frac{d^2z_3 d^2z_4}{z_{34}^4} \; \Big[-2 + \frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big]$$
$$\times \; (U^{aa'}_{z_3} U^{bb'}_{z_4} - z_4 \to z_3) \mathrm{Tr}\{[T^a, T^b] U_{z_1} [T^{a'}, T^{b'}] U^\dagger_{z_2}\}$$
$$+ \frac{3\alpha^2 N_c}{4\pi^4} \int d^2z \mathrm{Tr}\{T^a U_{z_1} U^\dagger_{z_3} T^a U_{z_3} U^\dagger_{z_2}\} \int \frac{d^2z_4}{z_{34}^4} \; \Big[-2 + \frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big] \tag{213}$$

The second (UV-divergent) part should be calculated at $d_\perp \neq 2$. As in the case of gluon loop, the Fourier transform (210) at $d_\perp \neq 2$ is complicated so it is convenient to return to Eq. (209) in the momentum representation. After replacing $U^{mm'}_{z_3}U^{nn'}_{z_4}$ by $U^{mm'}_{z_3}U^{nn'}_{z_3}$, integrating over u and changing variables to $k_2 = q_2 = k'$, $p = q_1 + q_2$, $l = q_1 - k_1$ (so that $q_1 = p - k'$, $k_1 = p - l - k'$ and $k_1 + k_2 = p - l$) the Eq. (209) turns into

$$\langle \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}\ z_4\to z_3} = \frac{3g^4N_c}{4\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int d^2z_3\ \mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^aU_{z_3}U^\dagger_{z_2}\} \quad (214)$$
$$\times\int d^{2-\epsilon}p\,d^{2-\epsilon}l\ \ (e^{i(p,z_{13})} - e^{i(p,z_{23})})(e^{-i(p-l,z_{13})} - e^{-i(p-l,z_{23})})\Phi(p,l)$$

where

$$\Phi(p,l) = \mu^{2\epsilon}\int d^{2-\epsilon}k'\ \frac{1}{p^2(p-l)^2}\Big\{-2$$
$$-\frac{(p-k')^2+(p-k'-l)^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-k'-l)^2} + \frac{k'^2+(p-k')^2}{(p-k')^2-k'^2}\ln\frac{(p-k')^2}{k'^2}$$
$$+\frac{(p-k'-l)^2+k'^2}{(p-k'-l)^2-k'^2}\ln\frac{(p-k'-l)^2}{k'^2}\Big\}$$
$$= \frac{\Gamma^2(1-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{4\pi(3-\epsilon)\Gamma(2-\epsilon)}\frac{1}{p^2(p-l)^2}(p^{2-\epsilon}+|p-l|^{2-\epsilon}-l^{2-\epsilon})$$
$$= \frac{(p,p-l)}{2\pi p^2(p-l)^2}\Big(\frac{2}{\epsilon}-\ln\frac{l^2}{\mu^2}+\frac{8}{3}\Big) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2} + O(\epsilon) \quad (215)$$

Subtracting the pole in ϵ corresponding to counterterm (see the discussion in Refs [24, 40]) we get

$$\sigma\frac{d}{d\sigma}\langle \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}\ z_4\to z} = 3\frac{\alpha_s^2}{\pi}N_c\int d^2z\,\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^bU_{z_3}U^\dagger_{z_2}\}\int d^{2-\epsilon}p\ d^{2-\epsilon}l$$
$$\times(e^{i(p,z_{13})} - e^{i(p,z_{23})})(e^{-i(p-l,z_{13})} - e^{-i(p-l,z_{23})})$$
$$\times\Big[\frac{2(p,p-l)}{3p^2(p-l)^2}\Big(-\ln\frac{l^2}{\mu^2}+\frac{8}{3}\Big) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2}\Big] \quad (216)$$

Using the Fouriers integral from Appendix A.1 [40] we get

$$\int d^2p\,d^2l\ e^{i(p,z_{13})-i(p-l,z_{23})}\Big[\frac{2(p,p-l)}{3p^2(p-l)^2}\Big(-\ln\frac{l^2}{\mu^2}+\frac{8}{3}\Big) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2}\Big]$$
$$= \frac{(z_{13},z_{23})}{12\pi^2z_{13}^2z_{23}^2}\Big[2\ln\frac{z_{13}^2z_{23}^2\mu^2}{z_{12}^2}+\frac{16}{3}\Big] - \frac{1}{12\pi^2z_{23}^2}\ln\frac{z_{13}^2}{z_{12}^2} - \frac{1}{12\pi^2z_{13}^2}\ln\frac{z_{23}^2}{z_{12}^2}$$
$$= -\frac{z_{12}^2}{12\pi^2z_{13}^2z_{23}^2}\Big[\ln\frac{z_{13}^2z_{23}^2\mu^2}{z_{12}^2}+\frac{8}{3}\Big] + \frac{1}{12\pi^2z_{13}^2}\ln z_{13}^2\mu^2 + \frac{1}{12\pi^2z_{23}^2}\ln z_{23}^2\mu^2 \quad (217)$$

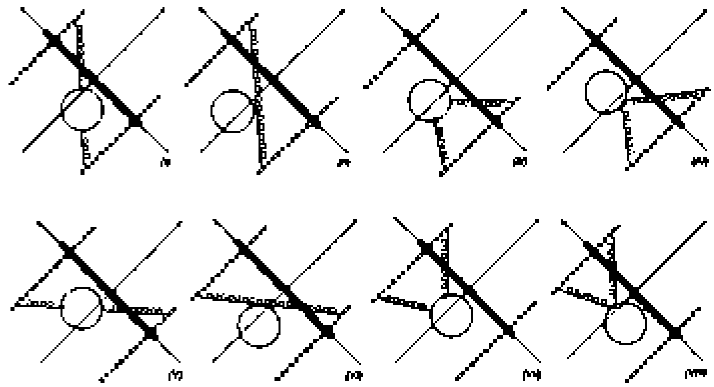

FIG. 19: Diagrams with the bare scalar loop.

and therefore

$$\sigma\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.18}\ z_4\to z_3}$$
$$=\frac{\alpha_s^2N_c}{2\pi^3}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{z_{13}^2z_{23}^2\mu^2}{z_{12}^2}+\frac{8}{3}\Big]\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^bU_{z_3}U^\dagger_{z_2}\} \tag{218}$$

Combining Eqs. (213) and (218) we obtain the full contribution of diagrams in Fig. 18 to the NLO kernel in the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{scalars}}=-\frac{3\alpha^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big[-2+\frac{z_{13}^2z_{24}^2+z_{14}^2z_{23}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big]$$
$$\times(U^{aa'}_{z_3}U^{bb'}_{z_4}-z_4\to z_3)\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}[T^{a'},T^{b'}]\hat U_{z_2}\}$$
$$+\frac{\alpha_s^2N_c}{2\pi^3}\int d^2z\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{z_{13}^2z_{23}^2\mu^2}{z_{12}^2}+\frac{8}{3}\Big]\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} \tag{219}$$

where we have promoted the shock-wave Wilson lines to operators.

Scalar loop

Besides diagrams with the scalar loop bisected by the shock wave calculated above, there are diagrams with the ordinary scalar loop shown in Fig. 19. The integral for the scalar loop has the form

$$\int d^{4-\epsilon}k'\ \frac{(2k'-k)_\mu(2k'-k)_\nu}{(k'^2+i\epsilon)[(k-k')^2+i\epsilon]}=\frac{2i\Gamma(\frac{\epsilon}{2})\Gamma(1-\frac{\epsilon}{2})\Gamma(2-\frac{\epsilon}{2})}{(4\pi)^{2-\frac{\epsilon}{2}}\Gamma(4-\epsilon)(-k^2)^{\frac{\epsilon}{2}}}(k_\mu k_\nu-g_{\mu\nu})$$
$$\simeq\frac{i}{48\pi^2}\Big[\frac{2}{\epsilon}+\ln\frac{1}{-k^2}+\frac{8}{3}\Big](k_\mu k_\nu-\delta_{\mu\nu}) \tag{220}$$

and therefore the contribution of the diagram shown in Fig. 19 takes the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.\ 19}} = -\frac{\alpha_s^2N_c}{\pi}\int d^2z_3[\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^aU_{z_3}U^\dagger_{z_2}\} - \frac{1}{N_c}\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}]$$
$$\times\int d^2kd^2k'd^2q[e^{i(q,z_{13})} - e^{i(q,z_{23})}][e^{-i(k,z_{13})} - e^{-i(k,z_{23})}]\frac{(q,k)}{k^2q^2}\Big\{\ln\frac{\mu^2}{k^2} + \frac{8}{3}\Big\} \quad (221)$$

As usual we should add diagrams obtained by the reflection of diagrams shown in Fig. (19) with respect to the shock-wave line. Their contribution is obtained from Eq. (221) by the replacement $q \leftrightarrow k$ in the logarithm so the final result for the sum of all diagrams of Fig. 19 type has the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{Fig.\ 19\ +refl.}} = -\frac{\alpha_s^2N_c}{\pi}\int d^2z_3\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^aU_{z_3}U^\dagger_{z_2}\} \quad (222)$$
$$\times\int d^2kd^2k'd^2q\ [e^{i(q,z_{13})} - e^{i(q,z_{23})}][e^{-i(k,z_{13})} - e^{i(k,z_{23})}]\frac{(q,k)}{k^2q^2}\Big\{\ln\frac{\mu^4}{q^2k^2} + \frac{16}{3}\Big\}$$
$$= -\frac{\alpha_s^2N_c}{4\pi^3}\int d^2z_3\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}\Big\{\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln\frac{z_{13}^2z_{23}^2}{\mu^{-4}} + \frac{16}{3}\Big] + \Big[\frac{1}{z_{13}^2} - \frac{1}{z_{23}^2}\Big]\ln\frac{z_{13}^2}{z_{23}^2}\Big\}$$

The total contribution of scalar particles to the NLO kernel from the diagrams in Fig. 18 and 19 is a sum of Eqs. (222) and (219)

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{scalars}} = -\frac{3\alpha^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big[-2 + \frac{z_{13}^2z_{24}^2 + z_{14}^2z_{23}^2}{z_{13}^2z_{24}^2 - z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big] \quad (223)$$
$$\times\ (U^{aa'}_{z_3}U^{bb'}_{z_4} - z_4 \to z_3)\mathrm{Tr}\{[T^a, T^b]\hat U^\eta_{z_1}[T^{a'}, T^{b'}]\hat U^{\dagger\eta}_{z_2}\}$$
$$-\frac{\alpha_s^2N_c}{4\pi^3}\int d^2z_3\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_3}T^a\hat U_{z_3}U^\dagger_{z_2}\}\Big\{\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln z_{12}^2\mu^2 + \frac{8}{3}\Big] + \Big[\frac{1}{z_{13}^2} - \frac{1}{z_{23}^2}\Big]\ln\frac{z_{13}^2}{z_{23}^2}\Big\}$$

Finally one needs to add the contribution of diagrams without scalar-shockwave intersection shown in Fig. 20. They are proportional to the "parent dipole" $\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}$, and their contribution can be found from Eq. (223) using the requirement that the r.h.s. of the evolution equation must vanish as $z_1 \to z_2$ (since $\lim_{z_1\to z_2}\hat U^\eta_{z_1}\hat U^{\eta\dagger}_{z_2} = 1$, see the discussion below Eq. (45)). It is easy to see that the following formula for the total contribution of scalar particle fulfills this requirement:

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\rangle_{\mathrm{scalars}} = -\frac{3\alpha^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big[-2 + \frac{z_{13}^2z_{24}^2 + z_{14}^2z_{23}^2}{z_{13}^2z_{24}^2 - z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big] \quad (224)$$
$$\times(\hat U^{\eta,aa'}_{z_3}\hat U^{\eta,bb'}_{z_4} - z \to z_4)\mathrm{Tr}\{[T^a, T^b]\hat U^\eta_{z_1}[T^{a'}, T^{b'}]\hat U^{\dagger\eta}_{z_2}\}$$
$$-\frac{\alpha_s^2N_c}{4\pi^3}\int d^2z_3[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$\times\Big\{\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln z_{12}^2\mu^2 + \frac{8}{3}\Big] + \Big[\frac{1}{z_{13}^2} - \frac{1}{z_{23}^2}\Big]\ln\frac{z_{13}^2}{z_{23}^2}\Big\}$$

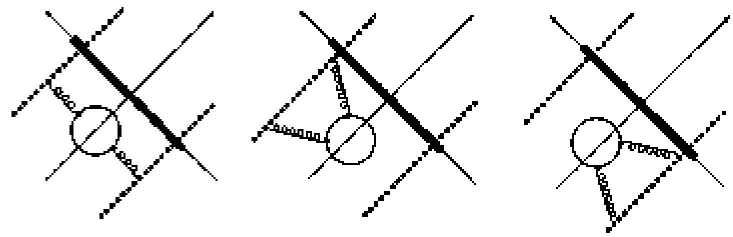

FIG. 20: Diagrams without scalar-shockwave intersection.

Note that we have written this equation in the operator form by promoting the shock-wave fields in the r.h.s. of Eq. (223) to operators.

V.2.3 Gluino contribution

The diagrams for the gluino contribution to the NLO kernel are shown in Fig. 21. The gluino propagator in the shock-wave background has the form

$$\langle \mathrm{T}\lambda^{aI}_{\beta}(x)\,\bar\lambda^{bJ}_{\dot\alpha}(y)\rangle = \theta(x_*y_*)\int d^4k\; e^{-ik\cdot(x-y)}\;\frac{ik_\mu\,\bar\sigma^{\mu}_{\beta\dot\alpha}}{k^2+i\epsilon}\delta^{ab}\delta^{IJ}$$
$$+\delta_{IJ}\theta(x_*)\theta(-y_*)\int_0^\infty d\alpha\;\frac{e^{-i\alpha(x-y)_\bullet}}{2\alpha^2 s}(x_\perp|(\alpha\bar p_1+\bar p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}x_*}\bar p_2 U^{ab}e^{i\frac{p_\perp^2}{\alpha s}y_*}(\alpha\bar p_1+\bar p_\perp)|y_\perp)$$
$$+\delta_{IJ}\theta(-x_*)\theta(y_*)\int_0^\infty d\alpha\;\frac{e^{i\alpha(x-y)_\bullet}}{2\alpha^2 s}(x_\perp|(\alpha\bar p_1+\bar p_\perp)e^{i\frac{p_\perp^2}{\alpha s}x_*}\bar p_2 U^{\dagger ab}e^{-i\frac{p_\perp^2}{\alpha s}y_*}(\alpha\bar p_1+\bar p_\perp)|y_\perp)$$

This propagator has the same form as the quark propagator in the shock-wave background in QCD so one can use the known result for the quark part of the NLO BK kernel in QCD. We replace $\mathrm{tr}\{t^aU_{z_3}t^bU^\dagger_{z_4}\}$ in the fundamental representation by $2\mathrm{Tr}\{T^aU_{z_3}T^bU^\dagger_{z_4}\}$ in the adjoint representation (and $\mathrm{tr}\{t^aU_{z_1}t^bU^\dagger_{z_2}\}$ by $\mathrm{Tr}\{T^aU_{z_1}T^bU^\dagger_{z_2}\}$ as usual) and obtain

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}^{\mathrm{gluino}} = \frac{2\alpha_s^2N_c}{3\pi^3}\int d^2z_3\,[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$\times\Big[-\frac{z_{12}^2}{z_{13}^2z_{23}^2}(\ln z_{12}^2\mu^2+\frac{5}{3})+\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}\Big]$$
$$-\frac{2\alpha_s^2}{\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\Big\{1-\frac{z_{14}^2z_{23}^2+z_{24}^2z_{13}^2-z_{12}^2z_{34}^2}{2(z_{13}^2z_{24}^2-z_{14}^2z_{23}^2)}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$
$$\times(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}[T^{a'},T^{b'}]\hat U^{\dagger\eta}_{z_2}\}$$

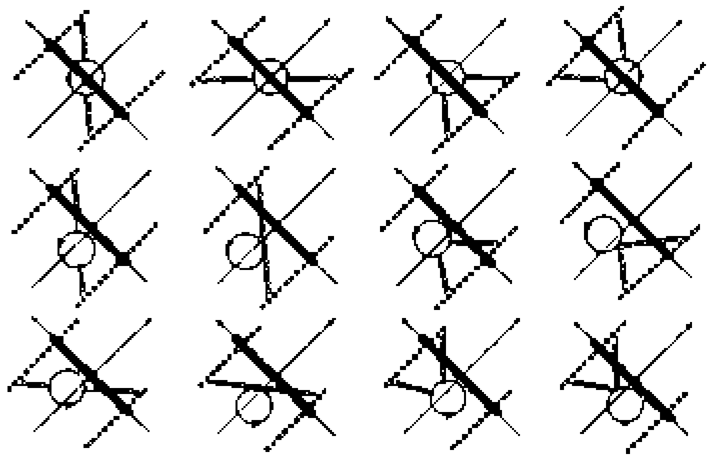

FIG. 21: Diagrams with the gluino loop.

V.2.4 The N=4 kernel

Now we are in a position to assemble the NLO BK kernel in $\mathcal{N}=4$ SYM. Adding the gluon contribution (203), the scalar part (224), and the gluino term (225) we obtain

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} \qquad (225)\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{8\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ (\hat U^\eta_{z_3})^{aa'}\Big\{\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}+z_3\leftrightarrow z_4\Big]\\
&\qquad\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}\\
&+\Big[\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}-z_3\leftrightarrow z_4\Big]\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_4})^{bb'}\Big\}
\end{aligned}
$$

Using Eq. (330) from Appendix B.5 one can rewrite this equation as follows:

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}=\frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'} \qquad (226)
\end{aligned}
$$

All terms in the r.h.s. of this equation are Möbius invariant except the double-log term proportional to $\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}$. As we discussed in the Introduction, the reason for this non-invariance is the cutoff in the longitudinal direction which violates the formal invariance of the non-cut Wilson lines.

It is worth noting that conformal and non-conformal terms come from graphs with different topology: the conformal terms come from 1→3 dipoles diagrams (see Fig. 6 in Ref. [40]) which describe the dipole creation while the non-conformal double-log term comes from the 1→2 dipole transitions (Fig.9 in Ref. [40])) which can be regarded as a combination of dipole creation and dipole recombination.

V.3 CONFORMAL DIPOLE AND CONFORMAL NLO KERNEL

As we discussed in the Introduction, it is possible to define the composite conformal dipole operator order by order in perturbation theory in such a way that the evolution equation for this operator would be Möbius invariant. The form of this operator can be guessed from the expression (316) (see the discussion in the Appendix B.2)

$$
\begin{aligned}
&[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&+\frac{\alpha_s}{2\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}
\end{aligned}
\tag{227}
$$

Let us find the NLO evolution kernel for this operator and demonstrate that it is conformal.

For the evolution of the composite operator (227) we obtain

$$
\begin{aligned}
&\frac{d}{d\eta}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&+\frac{\alpha_s}{2\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\frac{d}{d\eta}\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}
\end{aligned}
\tag{228}
$$

Writing down the evolution of the four-Wilson-line operator (322) calculated in Appendix B.3 we obtain

$$
\begin{aligned}
\frac{d}{d\eta}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} &= \frac{\alpha_s}{\pi^2}\int d^2z_3\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1+\frac{\alpha_sN_c}{4\pi}\Big[-\frac{\pi^2}{3}-2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
+\frac{\alpha_s}{8\pi^4}\int d^2z_3d^2z_4(\hat U^\eta_{z_3})^{aa'}\Big[&\frac{z_{12}^2}{z_{14}^2z_{34}^2}\mathrm{Tr}\{T^aT^b\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}T^{b'}T^{a'}\hat U^{\dagger\eta}_{z_2}\}\\
&\times(2\hat U^\eta_{z_4}-\hat U^\eta_{z_1}-\hat U^\eta_{z_2})^{bb'}\\
-\frac{z_{13}^2}{z_{14}^2z_{34}^2}\mathrm{Tr}\{T^aT^b\hat U^\eta_{z_1}[T^{a'},T^{b'}]\hat U^{\dagger\eta}_{z_2}&+[T^b,T^a]\hat U^\eta_{z_1}T^{b'}T^{a'}\hat U^{\dagger\eta}_{z_2}\}(2\hat U^\eta_{z_4}-\hat U^\eta_{z_1}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
\tag{229}
$$

$$-\frac{z_{23}^2}{z_{24}^2 z_{34}^2}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}$$

$$\times(2\hat U^\eta_{z_4}-\hat U^\eta_{z_2}-\hat U^\eta_{z_3})^{bb'}\Big]\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}$$

$$-\frac{\alpha_sN_c}{2\pi^4}\int d^2z_3d^2z_4(\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_4}T^a\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\})\frac{z_{12}^2}{z_{13}^2z_{23}^2}\frac{z_{12}^2}{z_{14}^2z_{24}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}+\ldots$$

where the dots stand for the last (conformal) term in the r.h.s. of Eq. (226).

Next we need the "counterterms" converting the four-Wilson-line operator $\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}$ into the conformal operator. In principle, this should be done similarly to obtaining the "conformal dipole" (200) in Appendix B.3: one should expand the T-product of conformal operators in the next (α_s^2) order in perturbation theory and rearrange the 6-Wilson-line operators in such a way that the coefficient in front of the combination $[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}$ is conformal. Since it means the calculation of the NNLO impact factor which probably is a formidable task, we will use another method to get this four-Wilson-line conformal operator. We will make a guess

$$\begin{aligned}
&[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}\\
&=\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&+\frac{\alpha_s}{8\pi^2}\int d^2z_4(\hat U^\eta_{z_3})^{aa'}\Big\{\mathrm{Tr}\{T^aT^b\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}T^{b'}T^{a'}\hat U^{\dagger\eta}_{z_2}\}\\
&\qquad(2\hat U^\eta_{z_4}-\hat U^\eta_{z_1}-\hat U^\eta_{z_2})^{bb'}\frac{z_{12}^2}{z_{14}^2z_{24}^2}\ln\Big(\frac{az_{12}^2}{z_{14}^2z_{24}^2}\Big)\\
&-\mathrm{Tr}\{T^aT^b\hat U^\eta_{z_1}[T^{a'},T^{b'}]\hat U^{\dagger\eta}_{z_2}+[T^b,T^a]\hat U^\eta_{z_1}T^{b'}T^{a'}\hat U^{\dagger\eta}_{z_2}\}\\
&\qquad(2\hat U^\eta_{z_4}-\hat U^\eta_{z_1}-\hat U^\eta_{z_3})^{bb'}\frac{z_{13}^2}{z_{14}^2z_{34}^2}\ln\Big(\frac{az_{13}^2}{z_{14}^2z_{34}^2}\Big)\\
&-\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\\
&\qquad(2\hat U^\eta_{z_4}-\hat U^\eta_{z_2}-\hat U^\eta_{z_3})^{bb'}\frac{z_{23}^2}{z_{24}^2z_{34}^2}\ln\Big(\frac{az_{23}^2}{z_{24}^2z_{34}^2}\Big)\Big\}\\
&-\frac{\alpha_sN_c}{2\pi^4}\int d^2z_4\frac{z_{12}^2}{z_{14}^2z_{24}^2}(\mathrm{Tr}\{T^aU_{z_1}U^\dagger_{z_4}T^aU_{z_4}U^\dagger_{z_2}\}-N_c\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\})\ln\frac{az_{12}^2}{z_{14}^2z_{24}^2}
\end{aligned}\qquad(230)$$

and check that it leads to the conformal evolution equation (199).

Rewriting the evolution equation (229) in terms of conformal operators (227) and (230) we obtain

$$\frac{d}{d\eta}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}=\frac{\alpha_s}{\pi^2}\int d^2z\;\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1$$

$$+\frac{\alpha_s N_c}{4\pi}\Big[-\frac{\pi^2}{3}-2\ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2}\Big]\Big\}[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\frac{\alpha_s^2}{8\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{(\hat U^\eta_{z_3})^{aa'}\mathrm{Tr}\{T^aT^b\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}T^{b'}T^{a'}\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_1}+\hat U^\eta_{z_2})^{bb'}\frac{z_{12}^2}{z_{14}^2z_{24}^2}\ln\big(\frac{z_{13}^2z_{23}^2}{z_{14}^2z_{24}^2}\big)$$

$$+4(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\ \frac{z_{23}^2}{z_{24}^2z_{34}^2}\big(\frac{z_{14}^2z_{23}^2}{z_{12}^2z_{34}^2}\big)$$

$$+4(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\frac{z_{23}^2}{z_{24}^2z_{34}^2}\ln\big(\frac{z_{14}^2z_{23}^2}{z_{12}^2z_{34}^2}\big)$$

$$-(\hat U^\eta_{z_1}+\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\frac{z_{13}^2}{z_{14}^2z_{34}^2}\ln\big(\frac{z_{13}^4z_{23}^2}{z_{12}^2z_{14}^2z_{34}^2}\big)$$

$$-(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_2}+\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\frac{z_{23}^2}{z_{24}^2z_{34}^2}\ln\big(\frac{z_{23}^4z_{13}^2}{z_{12}^2z_{24}^2z_{34}^2}\big)\Big\}$$

$$+\frac{\alpha_s^2N_c}{2\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{23}^2}\frac{z_{12}^2}{z_{14}^2z_{24}^2}\mathrm{Tr}\{T^a\hat U_{z_1}\hat U^\dagger_{z_3}T^a\hat U_{z_3}\hat U^\dagger_{z_2}\}\ln\frac{z_{14}^2z_{24}^2}{z_{13}^2z_{23}^2}+\ldots \tag{231}$$

Note that with our accuracy we do not need to specify the form of "counterterms" for the conformal composite operators in the α_s^2 term in the r.h.s. of this equation.

Using Eq. (324) and the integral (332) from Appendix we get

$$\frac{d}{d\eta}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}=\frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1+ \tag{232}$$

$$\frac{\alpha_sN_c}{4\pi}\Big[-\frac{\pi^2}{3}-2\ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2}\Big]\Big\}\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$-\frac{\alpha_s}{2\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}$$

$$+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\hat U^{\dagger\eta}_{z_2}\}\ln\frac{z_{14}^2z_{23}^2}{z_{12}^2z_{34}^2}\Big\}$$

$$+\frac{\alpha_sN_c}{2\pi^3}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2}$$

$$\times\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-\frac{\alpha_sN_c^2}{\pi^3}2\zeta(3)\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}+\ldots$$

$$=\frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big\}[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\frac{\alpha_s}{2\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}$$

$$\times \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}\hat U^{\dagger\eta}_{z_4}\}\ln\frac{z_{14}^2z_{23}^2}{z_{12}^2z_{34}^2}\}+...$$

We see that the non-conformal term double-log term $\sim \ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2}$ is canceled with the correction coming from the subtitution of the dipole by the composite operator (200). This confirms our expression (200) for the conformal dipole and justifies our guess for the conformal composite 4-Wilson-line operator (230).

Substituting now the dots in the r.h.s. of this equation for the last (conformal) term in Eq. (226) we get the final evolution equation for the conformal composite operator cited in the Introduction (199):

$$\begin{aligned}
&\frac{d}{d\eta}[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \qquad (233)\\
&=\frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big][\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{Tr}\{T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf}\\
&-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]
\end{aligned}$$

At this point we would like to discuss the gauge invariance of our evolution equation (233). The Wilson-line operator is gauge invariant up to gauge rotations at $\pm\infty$. As it was discussed in Refs. [45, 17], the evolution equation should be reformulated in terms of gauge-invariant Wilson loops. In particular, $\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$ in the l.h.s. of this equation should be promoted to

$$\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}=\mathrm{Tr}\{[\infty p_1+z_1,-\infty p_1+z_1][z_1,z_2]_{-\infty}[-\infty p_1+z_2,\infty p_1+z_2][z_2,z_1]_\infty\} \qquad (234)$$

where we use the notation $[z_1,z_2]_{\pm\infty}\equiv[z_1\pm\infty p_1,z_2\pm\infty p_1]$ and the precise form of contours connecting these points does not matter since the fields at infinity are pure gauges. We do not have a simple way to introduce these gauge links at infinity to the r.h.s. of Eq. (233) in the adjoint representation, but it can be easily done of one rewrites the adjoint traces in terms of traces in the fundamental representation using Eq. (323) from the Appendix B.4. For example, $\mathrm{tr}\{U_{z_4}U^\dagger_{z_1}\}\mathrm{tr}\{U_{z_2}U^\dagger_{z_3}U_{z_1}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\}$ should be replaced by

$$\begin{aligned}
&\mathrm{Tr}\{[\infty p_1+z_4,-\infty p_1+z_4][z_4,z_1]_{-\infty}[-\infty p_1+z_1,\infty p_1+z_1][z_1,z_4]_\infty\}\\
&\times\mathrm{Tr}\{[\infty p_1+z_2,-\infty p_1+z_2][z_2,z_3]_{-\infty}\}
\end{aligned}$$

$$\times[-\infty p_1 + z_3, \infty p_1 + z_3][z_3, z_1]_\infty[z_1, z_2]_{-\infty}[-\infty p_1 + z_2, \infty p_1 + z_2]$$
$$\times\ [z_2, z_3]_\infty[\infty p_1 + z_3, -\infty p_1 + z_3][z_3, z_4]_{-\infty}[-\infty p_1 + z_4, \infty p_1 + z_4][z_4, z_2]_\infty\}$$

and similarly for other traces in the r.h.s. of Eq. (324). With this replacement, the evolution equation (233) is gauge invariant.

V.4 COMPARISON TO NLO BFKL

In this section we compare our kernel with the forward NLO BFKL results for $\mathcal{N} = 4$ SYM [47]. To compare to the BFKL amplitude of gluon-gluon scattering at high energies we need to expand our Wilson lines up to two-gluon accuracy. We define the analog of Eq. 33 in the adjoint approximation:

$$\hat{\mathcal{V}}^\eta(x, y) = 1 - \frac{1}{N_c^2 - 1}\mathrm{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\} = \frac{N_c^2}{N_c^2 - 1}[\hat{\mathcal{U}}(x, y) + \hat{\mathcal{U}}(y, x) - \hat{\mathcal{U}}(x, y)\hat{\mathcal{U}}(x, y)]$$
$$\simeq \frac{N_c^2}{N_c^2 - 1}[\hat{\mathcal{U}}(x, y) + \hat{\mathcal{U}}(y, x)] \tag{235}$$

The corresponding conformal dipole operator in the BFKL approximation has the form

$$\hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_2) = \hat{\mathcal{V}}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{V}}^\eta(z_1, z_3) + \hat{\mathcal{V}}^\eta(z_2, z_3) - \hat{\mathcal{V}}^\eta(z_1, z_2)] \tag{236}$$

Using color traces (326), (327), and (329) from Appendix B.4 it is possible to demonstrate that the conformal 4-Wilson-line operator (230) reduces to the sum of three conformal dipoles:

$$[\mathrm{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \mathrm{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\mathrm{conf}}$$
$$= -\frac{N_c}{2}(N_c^2 - 1)[\hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_3) + \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_2, z_3) - \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_1)] \tag{237}$$

and therefore the evolution equation (233) turns into

$$\frac{d}{d\eta}\hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_2) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 - \frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big] \tag{238}$$
$$\times[\hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_3) + \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_2, z_3) - \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_1, z_3)]$$
$$+\frac{\alpha_s^2 N_c^2}{8\pi^4}\int \frac{d^2z_3 d^2z_4}{z_{34}^4}\ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}[\hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_3, z_4) + \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_2, z_4) - \hat{\mathcal{V}}^\eta_{\mathrm{conf}}(z_2, z_3)]$$
$$\times\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}$$

It is convenient to change $z_4 \leftrightarrow z_3$ in the second term in square brackets and to perform the integral over z_4 for the second and the third terms. One obtains:

$$\begin{aligned}
&\int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} [\hat{\mathcal{V}}^\eta_{\rm conf}(z_2, z_4) - \hat{\mathcal{V}}^\eta_{\rm conf}(z_2, z_3)] \\
&\times\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\} \\
&= \int d^2 z_3\, \hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_3)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int d^2 z_4 \Big\{2\Big[\frac{z_{13}^2}{z_{14}^2 z_{34}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} - \frac{z_{23}^2}{z_{24}^2 z_{34}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}\Big] \\
&\qquad -\Big[\frac{z_{13}^2}{z_{14}^2 z_{34}^2} + \frac{z_{23}^2}{z_{24}^2 z_{34}^2} - \frac{z_{12}^2}{z_{14}^2 z_{24}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\} \\
&= 12\pi\zeta(3)\hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_2) \qquad (239)
\end{aligned}$$

because

$$\begin{aligned}
&\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int d^2 z_4 \Big\{2\Big[\frac{z_{13}^2}{z_{14}^2 z_{34}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} - \frac{z_{23}^2}{z_{24}^2 z_{34}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}\Big] \\
&\qquad -\Big[\frac{z_{13}^2}{z_{14}^2 z_{34}^2} + \frac{z_{23}^2}{z_{24}^2 z_{34}^2} - \frac{z_{12}^2}{z_{14}^2 z_{24}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\} \\
&= 12\pi\zeta(3)[\delta(z_{13} - \delta(z_{23})]
\end{aligned}$$

as it is easily seen from Eqs. (330) and (341) from Appendix B.5.

Using Eq. (239) one can rewrite the linearized equation (238) in the following way:

$$\begin{aligned}
&\frac{d}{d\eta}\hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_2) = \frac{\alpha_s N_c}{2\pi^2}\int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1 - \frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big] \\
&\qquad \times[\hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_3) + \hat{\mathcal{V}}^\eta_{\rm conf}(z_2, z_3) - \hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_2)] \qquad (240) \\
&+\frac{\alpha_s^2 N_c^2}{8\pi^4}\int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \\
&\qquad \Big[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}\hat{\mathcal{V}}^\eta_{\rm conf}(z_3, z_4) + \frac{3\alpha_s^2 N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{V}}^\eta_{\rm conf}(z_1, z_2)
\end{aligned}$$

In this form the evolution equation (240) coincides with the conformal NLO BFKL equation which we have restored [53] from the eigenvalues calculated in Ref. [47] .

V.5 EVOLUTION EQUATION IN THE FUNDAMENTAL REPRESENTATION

V.5.1 $\mathcal{N} = 4$ SYM

For comparison with QCD let us calculate the evolution equation for color dipoles in the fundamental representation. The gluon part can be taken from Ref. [40]

$$\Big(\frac{d}{d\eta}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\Big)_{\rm gluon} \qquad (241)$$

$$= \frac{\alpha_s}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1+\frac{\alpha_sN_c}{4\pi}\Big[\frac{11}{3}\ln z_{12}^2\mu^2-\frac{11}{3}\frac{z_{13}^2-z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{64}{9}-\frac{\pi^2}{3}$$

$$-2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\ [\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z_3d^2z_4\Big[\Big\{-\frac{4}{z_{34}^4}+2\frac{z_{13}^2z_{24}^2+z_{14}^2z_{23}^2-4z_{12}^2z_{34}^2}{z_{34}^4[z_{13}^2z_{24}^2-z_{14}^2z_{23}^2]}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$

$$+\Big[\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}+z_3\leftrightarrow z_4\Big]\Big\}$$

$$\times\ [\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^{\dagger\eta}_{z_2}\}-\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U_{z_4}U^\dagger_{z_2}\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}-(z_4\to z_3)]$$

$$+\Big\{\frac{z_{12}^2}{z_{34}^2}\Big[\frac{1}{z_{13}^2z_{24}^2}+\frac{1}{z_{23}^2z_{14}^2}\Big]-\frac{z_{12}^4}{z_{13}^2z_{24}^2z_{14}^2z_{23}^2}\Big\}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\ \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^{\dagger\eta}_{z_2}\}\Big]$$

The scalar part can be obtained from Eq. (224) by the replacement $T^a\to t^a$ and Tr $\to$ tr:

$$\Big(\frac{d}{d\eta}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\Big)_{\rm scalars}=\frac{6\alpha^2}{16\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \Big[-2+\frac{z_{13}^2z_{24}^2+z_{14}^2z_{23}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\}-\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}-(z_4\to z_3)]$$

$$-\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z_3[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$\times\Big\{\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[\ln z_{12}^2\mu^2+\frac{8}{3}\Big]+\Big[\frac{1}{z_{13}^2}-\frac{1}{z_{23}^2}\Big]\ln\frac{z_{13}^2}{z_{23}^2}\Big\} \tag{242}$$

Similarly, for gluino contribution one obtains from Eq. (225)

$$\Big(\frac{d}{d\eta}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\Big)_{\rm gluino}=\frac{\alpha_s^2N_c}{3\pi^3}\int d^2z_3\ [\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$\times\Big[-\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\ln z_{12}^2\mu^2+\frac{5}{3}]+\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}\Big]$$

$$+\frac{\alpha_s^2}{\pi^4}\int d^2z_3d^2z_4\ \frac{1}{z_{34}^4}\Big\{1+\frac{z_{14}^2z_{23}^2+z_{24}^2z_{13}^2-z_{12}^2z_{34}^2}{2(z_{14}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{24}^2z_{13}^2}{z_{14}^2z_{23}^2}\Big\}$$

$$\times[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\}-\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}-z_4\to z_3] \tag{243}$$

Adding together Eq. (245), (242) and (243) we obtain the evolution equation for the color dipole with longitudinqal cutoff (45) in the fundamental representation:

$$\frac{d}{d\eta}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}=\frac{\alpha_s}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1 \tag{244}$$

$$+\frac{\alpha_sN_c}{4\pi}\Big[-\frac{\pi^2}{3}-2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z_3d^2z_4\ \Big[\Big\{\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}+z_3\leftrightarrow z_4\Big\}$$

$$\times[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^{\dagger\eta}_{z_2}\} - \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U_{z_4}U^\dagger_{z_2}\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\} - (z_4\to z_3)]$$
$$+\Big\{\frac{z_{12}^2}{z_{34}^2}\Big[\frac{1}{z_{13}^2z_{24}^2}+\frac{1}{z_{23}^2z_{14}^2}\Big] - \frac{z_{12}^4}{z_{13}^2z_{24}^2z_{14}^2z_{23}^2}\Big\}\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^{\dagger\eta}_{z_2}\}\Big]$$

which can be rewritten as

$$\frac{d}{d\eta}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1$$
$$-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z_3d^2z_4\,\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$\times[2\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U_{z_4}\hat U^{\dagger\eta}_{z_2}\}$$
$$-\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U_{z_4}U^\dagger_{z_2}\hat U_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_4}\hat U_{z_3}U^\dagger_{z_2}\hat U_{z_4}\hat U^{\dagger\eta}_{z_3}\} - (z_4\to z_3)] \tag{245}$$

due to Eq. (330) from Appendix B.4. The composite conformal dipole in the fundamental representation can be obtained from Eq. (200)) by the usual substitution $T^a\to t^a$ and $\mathrm{Tr}\to\mathrm{tr}$:

$$[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} \tag{246}$$
$$-\frac{\alpha_s}{4\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$

Similarly, the conformal 4-Wilson-line operator (230) turns to

$$[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = [\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]$$
$$+\frac{\alpha_s^2}{4\pi^2}\int d^2z_4\Big\{\frac{z_{13}^2}{z_{14}^2z_{34}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_3}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}]\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{13}^2}{z_{14}^2z_{34}^2}$$
$$+\frac{z_{23}^2}{z_{34}^2z_{24}^2}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}[\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{23}^2}{z_{24}^2z_{34}^2}$$
$$-\frac{z_{12}^2N_c}{z_{14}^2z_{24}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{14}^2z_{24}^2}$$
$$+\frac{1}{2}\Big[\frac{z_{13}^2}{z_{14}^2z_{34}^2}\ln\frac{az_{13}^2}{z_{14}^2z_{34}^2}+\frac{z_{23}^2}{z_{24}^2z_{34}^2}\ln\frac{az_{23}^2}{z_{34}^2z_{24}^2}-\frac{z_{12}^2}{z_{14}^2z_{24}^2}\ln\frac{az_{12}^2}{z_{14}^2z_{24}^2}\Big]$$
$$\times[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\hat U^\eta_{z_4}\hat U^{\dagger\eta}_{z_2}\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_4}\} + (z_3\leftrightarrow z_4) - 2\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\Big\} \tag{247}$$

Repeating the steps which lead us to the Eq. (233) in Sect. V.3 we obtain the conformal evolution equation in the fundamental representation

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \tag{248}$$

$$= \frac{\alpha_s}{2\pi^2}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big]\ [\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}}$$
$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z_3 d^2z_4\ \frac{z_{12}^2}{z_{13}^2 z_{34}^2 z_{24}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \Big[1+\frac{z_{12}^4 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}$$
$$\times[2\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_4}\}\mathrm{tr}\{\hat{U}^\eta_{z_4}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\hat{U}^\eta_{z_4}U^{\dagger\eta}_{z_2}\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_4}\}$$
$$-\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_4}\hat{U}^\eta_{z_3}U^{\dagger\eta}_{z_2}\hat{U}^\eta_{z_4}\hat{U}^{\dagger\eta}_{z_3}\} - (z_4\to z_3)]$$

Note that it can be obtained from the equation in the adjoint representation (233) by same replacement $T^a \to t^a$, $\mathrm{Tr} \to \mathrm{tr}$.

CHAPTER VI

CONCLUSIONS AND OUTLOOK

We have calculated the NLO kernel for the evolution of the color dipoles. It consists of three parts: the running-coupling part proportional to β-function (see diagrams shown in Fig. 13), the conformal part describing $1 \to 3$ dipoles transition (diagrams in Fig. 11) and the non-conformal term coming from the diagrams in Fig. (14). The result agrees with the forward NLO BFKL kernel [29].

There is a recent result in [35] where the dipole form of the non-forward NLO BFKL kernel is calculated using the non-forward NLO BFKL kernel [36]. The kernel obtained in [35] is different from our result (and not conformally invariant). We think that at least part of the difference is coming from the fact that the evolution kernel (44) should be compared to the non-symmetric "evolution" NLO BFKL kernel $K^{\rm evol}(q,p)$ rather that to the symmetric kernel $K(q,p)$ defined by Eq. (167). The kernel $K^{\rm evol}$ corresponds to the Green function $\tilde{G}_\omega$ defined by Eq. (167) with different lower cutoff for the longitudinal integration

$$\langle\hat{\mathcal{U}}(x,0)\rangle = \frac{1}{4\pi^2}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}\alpha_s(q)(e^{iqx}-1)(e^{-iqx}-1)\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{q'^2}\Big)^{\omega}\tilde{G}_\omega(q,q') \tag{249}$$

The $\tilde{G}_\omega(q,q')$ satisfies the equation (168) with the kernel $K^{\rm evol}$

$$\omega\tilde{G}_\omega(q,q') = \delta^{(2)}(q-q') + \int d^2p K^{\rm evol}(q,p)\tilde{G}_\omega(p,q') \tag{250}$$

and the relation between $K^{\rm evol}(q,p)$ and $K(q,p)$ has the form (cf. Ref. [29])

$$K^{\rm evol}(q,p) = K(q,p) - \frac{1}{2}\int d^2q' K(q,q')\ln\frac{q^2}{q'^2}K(q',p) \tag{251}$$

It is easy to see that the structure (249) repeats itself after differentiation with respect to s so it can be rewritten as an evolution equation for $\mathcal{U}(x)$ (whereas the derivative of the original formula (167) does not have the structure of the evolution equation due to an extra $\frac{1}{|q|^\omega}$). In terms of eigenvalues, the modified kernel (251) lead to the shifts of the type $\chi(n,\gamma) \to \chi(n,\gamma-\frac{\omega}{2})$ which we saw in Sect. V.4B.

It should be emphasized that the conformally invariant NLO kernel describes the evolution of the light-like Wilson lines with the "rigid" cutoff in the longitudinal momenta (45). On the contrary, for dipoles with the non-light-like slope the sum of the diagrams in Fig. 11 is not conformally invariant (see Appendix). The reason is that a general Wilson line is a non-local operator which is not conformally invariant to begin with - for example, the non-light-like Wilson line turns into a circle under the inversion $x^\mu \to x^\mu/x^2$. With the light-like Wilson lines, the situation is different. Formally, a Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \; A_+(x^+, x_\perp)\right\} \tag{252}$$

is invariant under the inversion $x^\mu \to x^\mu/x^2$ (with respect to the point with zero (-) component). Indeed, $(x^+, x_\perp)^2 = -x_\perp^2$ so after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$ and therefore

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2} \; A_+(\frac{x^+}{x_\perp^2}, x_\perp)\right\} = [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \tag{253}$$

Thus, it is not surprising that the bulk of our NLO kernel for the light-like dipoles is conformally invariant in the transverse space. The part proportional to the β-function is not conformally invariant and should not be, but there is another term $\sim \ln\frac{(x-y)^2}{(x-z)^2}\ln\frac{(x-y)^2}{(y-z)^2}$ which is not invariant. The reason for that is probably the cutoff $|\alpha| < \sigma$ which can be expressed as a cutoff in longitudinal coordinate x^+, and therefore under the inversion $x^+ \to x^+/x_\perp^2$ the cutoff can pick up some logs of transverse separations. It is worth noting that conformal and non-conformal terms come from graphs with different topology: the conformal terms come from 1→3 dipoles diagrams in Fig. (11) which describe the dipole creation while the non-conformal double-log term comes from the1→2 dipole transitions (see Fig. 14) which can be regarded as a combination of dipole creation and dipole recombination.

The second conclusion of this thesis regards the conformal NLO BFKL kernel which we have restored out of the eigenvalues known from the forward NLO BFKL result by using the requirement of Möbius invariance of $\mathcal{N} = \Delta$ SYM amplitudes in the Regge limit.

Then we have calculated the NLO kernel of the BK equation in the $N = 4$ SYM theory. The amplitudes in this gauge theory are conformally invariant and therefore in the Regge limit (298) they must be invariant with respect to Möbius transformations of the transverse plane. If we want to use the operator expansion to find this amplitude, we better expand in operators which are Möbius invariant. As we demonstrate in Appendix B.1, light-like Wilson lines are formally invariant, but they are divergent in the longitudinal direction, and at present the regularization of this rapidity divergence which respects conformal invariance is not known. We managed to circumvent this problem by using the non-invariant "rigid cutoff" (45) and restoring the conformal invariance order by order in perturbation theory by subtracting the proper counterterms (made again of Wilson lines). The resulting NLO evolution equation for "composite conformal dipoles" is Möbius invariant and agrees with forward NLO BFKL calculations in $\mathcal{N}=4$ SYM.

Let us comment on the non-conformal result of the calculation of NLO BFKL kernel in $\mathcal{N}=4$ SYM carried out in Ref. [46]. We think that the difference between our kernel and that of Ref. [46] is due to different cutoffs for longitudinal integrations. The authors of Ref. [46] propose that the transformation of their kernel of the type $\hat{K}_{\rm NLO} \to \hat{K}_{\rm NLO} - \alpha_s[\hat{K}_{\rm LO}, \hat{O}]$ with some suitable operator $\hat{O}$ will restore conformal invariance. This is exactly what happens in our case of the kernel (226) with the "rigid cutoff" (45) of the rapidity divergence. Let us discuss the transformation proposed in Ref. [46]. If we define $\hat{\tilde{\mathcal{V}}}(z)$ as

$$\hat{\tilde{\mathcal{V}}}^{\eta}(z) = \hat{\mathcal{V}}^{\eta}(z) + \alpha_s \int d^2z' f(z,z')\hat{\mathcal{V}}^{\eta}(z') \tag{254}$$

we see that the (linear) evolution equations for the operators $\hat{\mathcal{V}}^{\eta}$ and $\hat{\tilde{\mathcal{V}}}^{\eta}$ are related by

$$\hat{\tilde{K}}_{\rm NLO} = \hat{K}_{\rm NLO} - \alpha_s[\hat{K}_{\rm LO}, \hat{f}] \tag{255}$$

Indeed, differentiating the operator (254) with respect to η we obtain

$$\begin{aligned}
\frac{d}{d\eta}\hat{\tilde{\mathcal{V}}}^{\eta}(z) &= \frac{d}{d\eta}\hat{\mathcal{V}}^{\eta}(z) + \alpha_s \int d^2z' f(z,z')\frac{d}{d\eta}\hat{\mathcal{V}}^{\eta}(z') \\
&= \int dz' K_{\rm LO}(z,z')\hat{\mathcal{V}}^{\eta}(z') + \alpha_s \int dz' K_{\rm NLO}(z,z')\hat{\mathcal{V}}^{\eta}(z') + \\
&\qquad \alpha_s \int dz'dz'' f(z,z')K_{\rm LO}(z',z'')\hat{\mathcal{V}}^{\eta}(z') \\
&= \int dz' K_{\rm LO}(z,z')\hat{\tilde{\mathcal{V}}}^{\eta}(z) + \alpha_s \int dz' K_{\rm NLO}(z,z')\hat{\mathcal{V}}^{\eta}(z') \\
&+\alpha_s \int dz'dz''[f(z,z')K_{\rm LO}(z',z'') - K_{\rm LO}(z,z')F(z',z'')]\hat{\tilde{\mathcal{V}}}^{\eta}(z) + O(\alpha_s^3)
\end{aligned} \tag{256}$$

which corresponds to Eq. (255). Our transition between $\hat{\tilde{\mathcal{V}}}^{\eta}$ and $\hat{\tilde{\mathcal{V}}}^{\eta}_{\rm conf}$ is of the type (254) so it is not surprising that the kernels (226) and (199) are different. We think that one can recover the conformal kernel (199) from the kernel of Ref. [46] as long as one finds the appropriate $f(z,z')$. It should be also mentioned that the conformal NLO kernel is unique - the transformation (254) with both $\hat{\mathcal{V}}^{\eta}$ and $f(z,z')$ conformally invariant does not change $K_{\rm NLO}$ as can be easily seen from Eq. (256).

Let us discuss now possible generalizations of our method. The operator expansion of the type (299) is relevant for processes like deep inelastic scattering where the strong gluon fields come from the nucleon (or nucleus) target and the spectator (virtual photon) is a weak source of the gluon field. For the processes like heavy-ion collisions, the projectile-target symmetric language of 2+1 - dimensional effective action seems more adequate (here 2 is the number of transverse dimension and 1 stands for rapidity). There are many attempts in the literature to find comprehensive high-energy effective action but the answer for the ultimate high-energy effective action eludes us so far. It is possible that considering this problem in $\mathcal{N} = 4$ SYM, where we have the additional requirement of conformal (Möbius) symmetry to restrict the effective action, will help us to find the correct effective action at high energies.

We have already mentioned in the introduction that the results presented in this thesis will be relevant in future experiments like the Electron Ion Collider (EIC) which will be the future upgrade of Jefferson National Laboratory or of Brookhaven National Laboratory, and of the Large Hadron electron Collider(LHeC) in Geneva.

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APPENDIX A

APPENDIX FOR NLO EVOLUTION OF COLOR DIPOLES

A.1 UV PART OF THE ONE-TO-THREE DIPOLES KERNEL

As we mentioned above, it is convenient to separate the UV-divrgent and UV-finite parts of the Eq. (74) by writing down $U_z^{mm'}U_{z'}^{nn'} = (U_z^{mm'}U_{z'}^{nn'} - U_z^{mm'}U_z^{nn'}) + U_z^{mm'}U_z^{nn'}$. The contribution of the first part leads to Eq. (89) while the second UV-divergent term have the same color structure as the leading-order BK equation. After replacing $U_z^{mm'}U_{z'}^{nn'}$ by $U_z^{mm'}U_z^{nn'}$, integrating over u with the prescription (78) and changing variables to $k_2 = q_2 = k'$, $p = q_1 + q_2$, $l = q_1 - k_1$ (so that $q_1 = p - k'$, $k_1 = p - l - k'$ and $k_1 + k_2 = p - l$) the Eq. (74) turns into

$$
\begin{aligned}
&\langle \mathrm{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.11}\ z'-z} = \frac{g^4}{8\pi^2}\int_0^\sigma \frac{d\alpha}{\alpha}\int d^2z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - \frac{1}{2}\mathrm{Tr}\{U_x U_y^\dagger\}\Big)\\
&\times\Bigg[\Big\{\int d^{2-\epsilon}p\, d^{2-\epsilon}\ F_1(p,l) + \int d^2p\, d^2l\, F_2(p,l)\Big\}(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})\\
&+(e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)})\\
&\times\frac{(k',p-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\Bigg]\\
&+\int d^2p\, d^2l\, d^2k'\Big\{(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l-k',X)-i(k',Y)} - e^{-i(p-l-k',Y)-i(k',X)})\\
&\times\frac{(k',p-l-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-k')}{p^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2}
\end{aligned}
\tag{257}
$$

where

$$
\begin{aligned}
F_1(p,l) &= \int d^{2-\epsilon}k'\ \Bigg(\frac{1-\frac{\epsilon}{2}}{p^2(p-l)^2}\Bigg\{-2 - \frac{(p-k')^2 + (p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2}\ln\frac{(p-k')^2}{(p-k'-l)^2}\\
&+\frac{k'^2 + (p-k')^2}{(p-k')^2 - k'^2}\ln\frac{(p-k')^2}{k'^2} + \frac{(p-k'-l)^2 + k'^2}{(p-k'-l)^2 - k'^2}\ln\frac{(p-k'-l)^2}{k'^2}\Bigg\}\\
&+\frac{2(p,p-l)}{p^2(p-l)^2}\Bigg\{\Big(\frac{(p-k',p-k'-l)}{(p-k')^2(p-k'-l)^2} - \frac{1}{k'^2}\Big)\ln\frac{(p-k')^2(p-k'-l)^2}{k'^4}\\
&-\Big(\frac{(p-k',p-k'-l)}{(p-k')^2} + \frac{(p-k',p-k'-l)}{(p-k'-l)^2} + 2\Big)\frac{\ln(p-k')^2/(p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2}\Bigg\}\\
&+\frac{2}{p^2}\frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{p^2}{k'^2} + \frac{2}{(p-l)^2}\frac{(p-l,p-k')}{(p-k')^2k'^2}\ln\frac{(p-l)^2}{k'^2}\Bigg)
\end{aligned}
\tag{258}
$$

and

$$F_2(p,l) = \mu^{2\epsilon}\int d^2k' \left(2l_i\left[\left(-\frac{p_i}{p^2}+\frac{(p-k')_i}{(p-k')^2}\right)\frac{\ln (p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2-(p-k'-l)^2]}\right.\right. \tag{259}$$

$$+\left(\frac{(p-l)_i}{(p-l)^2}-\frac{(p-l-k')_i}{(p-k'-l)^2}\right)\frac{1/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}\Bigg]$$

$$+\frac{2}{p^2}\Bigg[\frac{2(l,p-k'-l)(p,k')/k'^2}{(p-k'-l)^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-k'-l)^2}$$

$$-\frac{(p-k',p-l-k')(p,k')}{k'^2(p-k')^2(p-k'-l)^2}\ln\frac{(p-k')^2}{k'^2}+\frac{(l,k')/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}$$

$$-\frac{(p-k'-l,k')}{2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2}+\frac{(p-l,k')/k'^2}{(p-l-k')^2-k'^2}\ln\frac{(p-k'-l)^2}{k'^2}$$

$$+\frac{2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}-$$

$$\frac{2\ln(p-l-k')^2/k'^2}{(p-l-k')^2-k'^2}+\frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{(p-k')^2}{p^2}\Bigg]$$

$$+\frac{2}{(p-l)^2}\Bigg[\frac{-2(l,p-k')(p-l,k')/k'^2}{(p-k')^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-k'-l)^2}$$

$$-\frac{(p-k',p-l-k')(p-l,k')}{k'^2(p-k')^2(p-k'-l)^2}\ln\frac{(p-k'-l)^2}{k'^2}$$

$$-\frac{(l,k')/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}+\frac{(p,k')/k'^2}{(p-k')^2-k'^2}\ln\frac{(p-k')^2}{k'^2}$$

$$-\frac{(p-k',k')}{2k'^2(p-k')^2}\ln\frac{(p-l-k')^2}{k'^2}+\frac{2}{(p-k')^2-(p-k'-l)^2}$$

$$\left.\left.\ln\frac{(p-k')^2}{(p-l-k')^2}-\frac{2\ln(p-k')^2/k'^2}{(p-k')^2-k'^2}+\frac{(p-l,p-k')}{(p-k')^2k'^2}\ln\frac{(p-l-k')^2}{(p-l)^2}\right]\right) \tag{260}$$

We need to perform the integration over k'. Let us start with the UV-divergent term $\sim F_1$. Using the integrals

$$4\pi\int d^dk' \frac{(l-k')_i}{k'^2(l-k')^2}\ln\frac{p^2}{k'^2}$$

$$= l_i\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)}\frac{\Gamma(2-\frac{d}{2})}{l^{4-d}}\left[\ln\frac{p^2}{l^2}+\frac{2}{d-2}+\psi(2-\frac{d}{2})+\psi(d-1)-\psi(\frac{d}{2})-\psi(1)\right]$$

$$4\pi\int d^dk' \frac{(k',k'-l)}{k'^2(l-k')^2}\ln\frac{(p-k')^2}{k'^2}=\frac{1}{2}\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2}$$

$$4\pi\int d^dk' \frac{(p-k',p-k'-l)}{(p-k')^2(p-k'-l)^2}\ln\frac{(p-k')^2(p-k'-l)^2}{k'^4} = -\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2}$$

$$4\pi\int d^dk' \frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{p^2}{k'^2}$$

$$= (p, p-l)\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)}\frac{\Gamma(2-\frac{d}{2})}{|p-l|^{4-d}}$$
$$\times\Big[\ln\frac{p^2}{(p-l)^2}+\frac{2}{d-2}+\psi(2-\frac{d}{2})+\psi(d-1)-\psi(\frac{d}{2})-\psi(1)\Big] \tag{261}$$

one obtains

$$F_1(p,l) = \frac{1}{4\pi}\Big\{\frac{2(p,p-l)}{p^2(p-l)^2}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Gamma(\varepsilon/2)\Big(-4+\frac{1-\frac{\varepsilon}{2}}{3-\varepsilon}\Big)+\frac{2(p,p-l)}{p^2(p-l)^2}\Big(\frac{11}{3}\ln\frac{l^2}{\mu^2}$$
$$-\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2}-\ln^2\frac{(p-l)^2}{p^2}+\frac{\pi^2}{3}\Big)-\frac{\ln p^2/l^2}{3(p-l)^2}-\frac{\ln(p-l)^2/l^2}{3p^2}+O(\varepsilon)\Big\} \tag{262}$$

Let us at first consider the UV-divergent contribution

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{UV}}$$
$$=\frac{\alpha_s^2}{\pi}\mu^{2\varepsilon}\int d^{2-\varepsilon}z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big)\int d^{2-\varepsilon}p\; d^{2-\varepsilon}l$$
$$\times\,(e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2}$$
$$\times\Big[\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Gamma(\varepsilon/2)\Big(-4+\frac{1-\frac{\varepsilon}{2}}{3-\varepsilon}\Big)+\frac{11}{3}\ln\frac{l^2}{\mu^2}+O(\varepsilon)\Big] \tag{263}$$

To this contribution we should add the counterterm corresponding to quark and gluon loops lying inside the shock wave. The rigorous calculation of the counterterm was performed in Ref. [24] and the result is

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{CT}}=-b\frac{\alpha_s^2}{\pi}\frac{2}{\varepsilon}\int d^{2-\varepsilon}z_\perp\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big)$$
$$\times\int d^dp\; d^dl\;(e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2} \tag{264}$$

where we need the gluon part of b $(=\frac{11}{3}N_c)$. After subtraction of the counterterm (264) the UV-divergent contribution (263) reduces to

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{UV-CT}}=\frac{\alpha_s^2}{\pi}\int d^2z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big)$$
$$\times\int d^2p\; d^2l\;(e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2}\Big[\frac{11}{3}\ln\frac{l^2}{\mu^2}-\frac{67}{9}\Big] \tag{265}$$

so one obtains the regularized F_1 in the form

$$F_1^{\mathrm{reg}}(p,l)=\frac{1}{4\pi}\Big[\frac{2(p,p-l)}{p^2(p-l)^2}\Big(\frac{11}{3}\ln\frac{l^2}{\mu^2}-\frac{67}{9}-\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2}-\ln^2\frac{(p-l)^2}{p^2}+\frac{\pi^2}{3}\Big)$$
$$-\frac{\ln p^2/l^2}{3(p-l)^2}-\frac{\ln(p-l)^2/l^2}{3p^2}\Big] \tag{266}$$

It is convenient to calculate first the Fourier transform with $e^{i(p,X)-i(p-l,Y)}$. Using the integrals

$$\int d^2p\, d^2l\; e^{i(p,\Delta)+i(l,Y)}\,\frac{(p,p-l)}{p^2(p-l)^2}\ln\frac{l^2}{\mu^2} = -\frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln\frac{X^2Y^2}{\Delta^2}\mu^2$$
$$\int d^2p\, d^2l\; e^{i(p,\Delta)+i(l,Y)}\,\frac{(p,p-l)}{p^2(p-l)^2}\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2} = \frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}$$
$$\int d^2p\, d^2l\; e^{i(p,\Delta)+i(l,Y)}\,\frac{(p,p-l)}{p^2(p-l)^2}\ln^2\frac{(p-l)^2}{p^2} = \frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln^2\frac{X^2}{Y^2} \tag{267}$$

we get

$$\int d^2p\, d^2l\; e^{i(p,\Delta)+i(l,Y)}F_1^{reg}(p,l) = -\frac{11}{24\pi^3}\frac{(X,Y)}{X^2Y^2}\Big(\ln\frac{X^2Y^2}{\Delta^2}\mu^2+\frac{67}{33}\Big) \tag{268}$$
$$-\frac{1}{16\pi^3}\frac{(X,Y)}{X^2Y^2}\Big[\ln^2\frac{X^2}{\Delta^2}+\ln^2\frac{Y^2}{\Delta^2}+\ln^2\frac{X^2}{Y^2}-\frac{2\pi^2}{3}\Big]$$
$$-\frac{1}{48\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}+\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big]$$

Hereafter we use the notation $\Delta \equiv X - Y = x - y$.

Next we calculate the F_2 contribution. We need the following Fourier integrals:

$$\int d^2p d^2l\; e^{i(p,\Delta)+i(l,Y)}\int d^2k'\; 2l_i\Big[\Big(-\frac{p_i}{p^2}+\frac{(p-k')_i}{(p-k')^2}\Big)\frac{\ln(p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2-(p-k'-l)^2]}$$
$$+\Big(\frac{(p-l)_i}{(p-l)^2}-\frac{(p-l-k')_i}{(p-k'-l)^2}\Big)\frac{1/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}\Big]$$
$$=\frac{(X,Y)}{16\pi^3X^2Y^2}\Big[\ln^2\frac{X^2}{Y^2}+\frac{2\pi^2}{3}\Big] \tag{269}$$

$$\int d^2p\, d^2l\; e^{i(p,\Delta)+i(l,Y)}$$
$$\times\Big[\frac{4}{p^2}\int dk'\frac{(l,p-l-k')(p,k')}{k'^2(p-k'-l)^2((p-k')^2-(p-k'-l)^2)}\ln\frac{(p-k')^2}{(p-k'-l)^2}$$
$$-\frac{4}{(p-l)^2}\int dk'\frac{(l,p-k')(p-l,k')}{k'^2(p-k')^2((p-k')^2-(p-k'-l)^2)}\ln\frac{(p-k')^2}{(p-k'-l)^2}\Big]$$
$$=\frac{1}{16\pi^3}\Big\{-\frac{\pi^2}{3}\frac{(X+Y)^2}{X^2Y^2}+\frac{2}{Y^2}\int_0^1 du\frac{\ln u}{u-\frac{X^2}{X^2-Y^2}}+\frac{2}{X^2}\int_0^1 du\frac{\ln u}{u+\frac{Y^2}{X^2-Y^2}}$$
$$+\frac{i\kappa}{X^2Y^2}\Big(2\int_0^1 du\Big[\frac{\ln u}{u-\frac{(X,\Delta)-i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(Y,\Delta)+i\kappa}{\Delta^2}}-c.c.\Big]$$
$$+\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big)\Big\} \tag{270}$$

where $\kappa = \sqrt{X^2Y^2 - (X,Y)^2}$, and

$$\int d^2p d^2l\ e^{i(p,\Delta)+i(l,Y)} \int d^2k' \Big[-\frac{2(p-k',p-l-k')(p,k')}{p^2k'^2(p-k')^2(p-k'-l)^2} \ln\frac{(p-k')^2}{k'^2} \tag{271}$$

$$-\frac{2(p-k',p-l-k')(p-l,k')}{(p-l)^2k'^2(p-k')^2(p-k'-l)^2}\ln\frac{(p-k'-l)^2}{k'^2}\Big]$$

$$= \frac{1}{32\pi^3X^2Y^2}\Big[X^2\ln^2\frac{X^2}{\Delta^2} + Y^2\ln^2\frac{Y^2}{\Delta^2} + 2(X,Y)\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}\Big]$$

$$+\frac{i\kappa}{16\pi^3X^2Y^2}\int_0^1 du\Big[\frac{\ln u}{u - \frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.}$$

$$-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big]$$

$$\int d^2p d^2l\ e^{i(p,\Delta)+i(l,Y)}\int d^2k'\Big(\frac{2(l,k')p^{-2}}{k'^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2}$$

$$+\frac{2(p-l,k')p^{-2}}{k'^2[(p-k'-l)^2-k'^2]}\ln\frac{(p-l-k')^2}{k'^2}$$

$$-\frac{2(l,k')(p-l)^{-2}}{k'^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2}$$

$$+\frac{2(p,k')(p-l)^{-2}}{k'^2[(p-k')^2-k'^2]}\ln\frac{(p-k')^2}{k'^2}\Big)$$

$$= -\frac{1}{16\pi^3}\Big(\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2} - \frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}\Big)\ln\frac{X^2}{Y^2}$$

$$+\frac{1}{8\pi^3}\Big[\frac{\pi^2}{6X^2}+\frac{\pi^2}{6Y^2}-\frac{1}{X^2}\int_0^1\frac{du\ln u}{u+\frac{Y^2}{X^2-Y^2}} - \frac{1}{Y^2}\int_0^1\frac{du\ln u}{u-\frac{X^2}{X^2-Y^2}}\Big] \tag{272}$$

$$\int d^2p d^2l\ e^{i(p,\delta)+i(l,Y)}\int d^2k'$$

$$\times\Big[-\frac{(p-k'-l,k')}{p^2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2} - \frac{(p-k',k')}{(p-l)^2k'^2(p-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\Big]$$

$$= \frac{1}{32\pi^3}\ln\frac{X^2}{Y^2}\Big[\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2} - \frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}\Big] \tag{273}$$

$$\int d^2p\ d^2l\ e^{i(p,\Delta)+i(l,Y)}\int d^2k'\ \Big[\frac{4/p^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2} \tag{274}$$

$$-\frac{4\ln(p-l-k')^2/k'^2}{p^2[(p-l-k')^2-k'^2]} - \frac{4\ln(p-k')^2/k'^2}{(p-l)^2[(p-k')^2-k'^2]}$$

$$+\frac{4}{(p-l)^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2}$$

$$= \frac{1}{4\pi^3Y^2}\ln\frac{X^2}{\Delta^2} + \frac{1}{4\pi^3X^2}\ln\frac{Y^2}{\Delta^2}$$

$$\int d^2p d^2l\, e^{i(p,\Delta)+i(l,Y)} \int d^2k' \left[\frac{2(p,p-l-k')}{p^2(p-l-k')^2k'^2}\ln\frac{(p-k')^2}{p^2} + \frac{2(p-l,p-k')}{(p-k')^2k'^2(p-l)^2}\ln\frac{(p-l-k')^2}{(p-l)^2}\right]$$
$$= \frac{i\kappa}{16\pi^3X^2Y^2}\Big\{2\int_0^1 du\Big[\frac{\ln u}{u-\frac{(X,\Delta)+i\kappa}{\Delta^2}} + \frac{\ln u}{u+\frac{(Y,\Delta)-i\kappa}{\Delta^2}} - \text{c.c.}\Big] - \ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\} + \frac{(X,Y)}{16\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2} \tag{275}$$

Adding the integrals (269) - (273) we obtain

$$\int d^2p\, d^2l\, e^{i(p,\Delta)+i(l,Y)}F_2(p,l) \tag{276}$$
$$= \frac{1}{4\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2} + \frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big] + \frac{(X+Y)^2}{32\pi^3X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2} + \frac{(X,Y)}{8\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2}$$
$$+\frac{i\kappa}{16\pi^3X^2Y^2}\Big\{\int_0^1 du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.}\Big] - \frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\}$$

and therefore

$$\int d^2p\, d^2l\, e^{i(p,\Delta)+i(l,Y)}[F_1^{\text{reg}}(p,l) + F_2(p,l)]$$
$$= -\frac{1}{8\pi^3}\frac{(X,Y)}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] + \frac{1}{32\pi^3}\frac{\Delta^2}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2} + \frac{11}{48\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2} + \frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big]$$
$$+\frac{i\kappa}{16\pi^3X^2Y^2}\Big\{\int_0^1 du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.}\Big] - \frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\} \tag{277}$$

Note that the r.h.s. of this equation is finite as $X \to Y$ (taken separately, the contributions of F_1 and F_2 are singular in this limit):

$$\int d^2p\, d^2l\, e^{i(l,X)}[F_1^{\text{reg}}(p,l) + F_2(p,l)] = -\frac{1}{8\pi^3}\frac{1}{X^2}\Big[\frac{11}{3}\ln X^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] \tag{278}$$

Using Eqs. (277) and (278) we obtain

$$\int d^2p\, d^2l\, e^{i(p,\Delta)+i(l,Y)}[F_1^{\text{reg}}(p,l) + F_2(p,l)](e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})$$
$$= -\frac{1}{8\pi^3}\frac{\Delta^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] - \frac{1}{16\pi^3}\frac{\Delta^2}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}$$

$$-\frac{i\kappa}{8\pi^3X^2Y^2}\Big\{\int_0^1du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-c.c.\Big]-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\} \tag{279}$$

Now we turn our attention to last two terms in Eq. (257). Using Fourier transformation

$$\int d^2k_1d^2k_2\ e^{-i(k_1,x_1)-i(k_2,x_2)}\frac{k_{2i}}{(k_1+k_2)^2k_2^2}\ln\frac{k_1^2}{k_2^2} \tag{280}$$
$$=\frac{i}{8\pi^2}\Big(x_{1i}-\frac{(x_1,x_{12})}{x_{12}^2}x_{12i}\Big)\frac{1}{i\kappa_{12}}\Big\{\int_0^1du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_1^2}}-c.c.\Big]$$
$$+\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_2,x_{12})+i\kappa_{12}}{(x_2,x_{12})-i\kappa_{12}}\Big\}+\frac{ix_{12i}}{16\pi^2x_{12}^2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{x_1^2}{x_2^2}$$

(where $x_{12}\equiv x_1-x_2$ and $\kappa_{12}\equiv\sqrt{x_1^2x_2^2-(x_1,x_2)^2}$) one easily obtains

$$\int d^2pd^2ld^2k'(e^{-i(p-l,X)}-e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)}-e^{i(p-k',Y)+i(k',X)})$$
$$\times\frac{(k',p-k')}{(p-l)^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$=\frac{i\kappa}{16\pi^3X^2Y^2}\Big[\int_0^1du\Big(\frac{\ln u}{u-\frac{(X,Y)-i\kappa}{X^2}}+\frac{\ln u}{u-\frac{(X,Y)-i\kappa}{Y^2}}-c.c.\Big)$$
$$+\frac{1}{2}\ln\frac{X^2}{Y^2}\ln\frac{[(\Delta,X)+i\kappa][(\Delta,Y)+i\kappa]}{[(\Delta,X)-i\kappa][(\Delta,Y)-i\kappa]}\Big]$$
$$=\frac{i\kappa}{16\pi^3X^2Y^2}\Big[-\int_0^1du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-c.c.\Big]$$
$$-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big]-\frac{(X,Y)}{32\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2} \tag{281}$$

Similarly,

$$\int d^2k_1d^2k_2\ e^{-i(k_1,x_1)-i(k_2,x_2)}\frac{(k_1,k_2)k_{1i}}{(k_1+k_2)^2k_1^2k_2^2}\ln\frac{k_1^2}{k_2^2} \tag{282}$$
$$=\frac{i}{16\pi^2}\Big(x_{1i}-\frac{(x_1,x_{12})}{x_{12}^2}x_{12i}\Big)\frac{1}{i\kappa}\Big[\int_0^1du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_1^2}}-c.c.\Big]$$
$$+\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_2,x_{12})+i\kappa}{(x_2,x_{12})-i\kappa}\Big]$$
$$-\frac{i}{16\pi^2}\Big(x_{2i}-\frac{(x_2,x_{12})}{x_{12}^2}x_{12i}\Big)\frac{1}{i\kappa}\Big[\int_0^1du\Big[\frac{\ln u}{u+\frac{(x_2,x_{12})-i\kappa}{x_2^2}}-c.c.\Big]$$

$$+\frac{1}{2}\ln\frac{x_2^2}{x_{12}^2}\ln\frac{(x_1,x_{12})+i\kappa}{(x_1,x_{12})-i\kappa}\Big]+\frac{ix_{12i}}{32\pi^2x_{12}^2}\ln\frac{x_1^2x_2^2}{x_{12}^4}\ln\frac{x_1^2}{x_2^2}$$
$$+\frac{i}{16\pi^2}\Big(x_{12i}-\frac{(x_1,x_{12})}{x_1^2}x_{1i}\Big)\frac{1}{i\kappa}\Big[\int_0^1 du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_{12}^2}}-c.c.\Big]$$
$$-\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_1,x_2)+i\kappa}{(x_1,x_2)-i\kappa}\Big]-\frac{ix_{1i}}{32\pi^2x_1^2}\ln\frac{x_1^2}{x_2^2}\ln\frac{x_2^2}{x_{12}^2}$$

and therefore

$$\int d^2p d^2l d^2k'(e^{-i(p-l,X)}-e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)}-e^{i(p-k',Y)+i(k',X)})$$
$$\times\frac{2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$=\frac{i\kappa Y^{-2}}{16\pi^3X^2}\Big[\int_0^1 du\Big(-2\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}-2\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-c.c.\Big)$$
$$+\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big]$$
$$-\frac{(X,Y)}{32\pi^3X^2Y^2}\ln\frac{X^2\Delta^2}{Y^4}\ln\frac{X^2}{\Delta^2}-\frac{(X,Y)}{32\pi^3X^2Y^2}\ln\frac{Y^2\Delta^2}{X^4}\ln\frac{Y^2}{\Delta^2}$$
$$-\frac{1}{32\pi^3}\Big(\frac{1}{X^2}+\frac{1}{Y^2}\Big)\ln\frac{X^2}{Y^2}\ln\frac{Y^2}{\Delta^2}\qquad(283)$$

Adding the equations (281) and (283) we obtain

$$\int d^2p d^2l d^2k'(e^{-i(p-l,X)}-e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)}-e^{i(p-k',Y)+i(k',X)})\qquad(284)$$
$$\times\frac{(k',p-k')(p-k')^2-2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$=\frac{i\kappa}{16\pi^3X^2Y^2}\Big[\int_0^1 du\Big(\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-c.c.\Big)$$
$$-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big]+\frac{\Delta^2}{32\pi^3X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}$$

It is easy to see that the contribution of the last term in Eq. (257) is equal to (284) so we get

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.11}\ z\to z'}=\frac{g^4}{8\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int d^2z\ [\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\Big[\int d^2p d^2l\ [F_1^{\mathrm{reg}}(p,l)+F_2(p,l)](e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})$$
$$+2\int d^2p d^2l d^2k'(e^{-i(p-l,X)}-e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)}-e^{i(p-k',Y)+i(k',X)})$$
$$\times\frac{(k',p-k')(p-k')^2-2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\Big]$$

$$= -\frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]$$
$$\times\frac{\Delta^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] \tag{285}$$

Note that the dilogarithms and products of logarithms have canceled. The simplicity of the final result indicates that there should be a less tedious derivation but we were not able to find it.

A.2 CUTOFF DEPENDENCE OF THE NLO KERNEL

We will repeat the procedure from Sect. (III.1.3), this time using the cutoff by the slope.

$$(K_{\mathrm{NLO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\})_{\mathrm{shockwave}} = \frac{d}{d\eta}(\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\})_{\mathrm{shockwave}} - (K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\})_{\mathrm{shockwave}} \tag{286}$$

Instead of Eq. (46) we get

$$g^4\int_0^\infty du\int_{-\infty}^0 dv\langle\hat A_\bullet^a(un+x_\perp)\hat A_\bullet^b(vn+y_\perp)\rangle = \frac{1}{2}g^2\frac{s^2}{4}f^{and}f^{bm'l'}\int d\!\!\!{}^-\alpha\, d\!\!\!{}^-\alpha_1 \tag{287}$$
$$\times\int d\!\!\!{}^-\beta\, d\!\!\!{}^-\beta'\, d\!\!\!{}^-\beta_1\, d\!\!\!{}^-\beta_1'\, d\!\!\!{}^-\beta_2\, d\!\!\!{}^-\beta_2'\int d^2z d^2z'\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2\; e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp}$$
$$\frac{4\alpha_1(\alpha-\alpha_1)U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp}}{(\beta-\beta_1-\beta_2+i\epsilon)(\beta'-\beta_1'-\beta_2'+i\epsilon)(\beta+\xi\alpha-i\epsilon)(\beta'+\xi\alpha'-i\epsilon)}$$
$$\frac{d_{\bullet\lambda}(\alpha p_1+\beta p_2+q_{1\perp}+k_{1\perp})d_{\lambda'\bullet}(\alpha p_1+\beta' p_2+q_{2\perp}+k_{2\perp})}{[\alpha\beta s-(q_1+q_2)_\perp^2+i\epsilon][\alpha\beta' s-(k_1+k_2)_\perp^2+i\epsilon]}$$
$$\frac{d_{\mu\xi}(\alpha_1p_1+\beta_1p_2+q_{1\perp})}{\alpha_1\beta_1 s-q_{1\perp}^2+i\epsilon}\frac{d^\xi{}_{\mu'}(\alpha_1p_1+\beta_1'p_2+k_{1\perp})}{\alpha_1\beta_1's-k_{1\perp}^2+i\epsilon}\frac{d_{\nu\eta}((\alpha-\alpha_1)p_1+\beta_2p_2+q_{2\perp})}{(\alpha-\alpha_1)\beta_2s-q_{2\perp}^2+i\epsilon}$$
$$\frac{d^\eta{}_{\nu'}((\alpha-\alpha_1)p_1+\beta_2'p_2+k_{2\perp})}{(\alpha-\alpha_1)\beta_2's-k_{2\perp}^2+i\epsilon}$$
$$\Gamma^{\mu\nu\lambda}(\alpha p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\Gamma^{\mu'\nu'\lambda'}(\alpha p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})$$

where $\xi = e^{-2\eta_1}$. In this formula $\frac{1}{\beta+\xi\alpha-i\epsilon}$ comes from the integration over u parameter in the l.h.s. and $\frac{1}{\beta'+\xi\alpha'-i\epsilon}$ from the integration over v parameter.

Taking residues at $\beta=-\xi\alpha$ and $\beta'=-\xi\alpha'$ and $\beta_2=-\beta_1$, $\beta_2'=-\beta_1'$ we obtain

$$\int_0^\infty du\int_{-\infty}^0 dv\langle\hat A_\bullet^a(un+x_\perp)\hat A_\bullet^b(vn+y_\perp)\rangle \tag{288}$$
$$= \frac{1}{2}g^2\frac{s^2}{4}f^{and}f^{bm'l'}\int d\!\!\!{}^-\alpha\, d\!\!\!{}^-\alpha_1\, d\!\!\!{}^-\beta_1\, d\!\!\!{}^-\beta_1'\int d^2zd^2z'\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2\; e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp}$$

$$4\frac{\alpha_1(\alpha-\alpha_1)}{\alpha^2}U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1)z-i(q_2-k_2)z'}\ \frac{(q_{1\perp}+q_{2\perp})_\lambda}{(q_1+q_2)_\perp^2+\xi\alpha^2}\frac{(k_{1\perp}+k_{2\perp})_{\lambda'}}{(k_1+k_2)_\perp^2+\xi\alpha^2}$$
$$\frac{d_\mu^{\ \xi}(\alpha_1p_1+q_{1\perp})}{\alpha_1\beta_1s-q_{1\perp}^2+i\epsilon}\frac{d_{\xi\mu'}(\alpha_1p_1+k_{1\perp})}{\alpha_1\beta_1's-k_{1\perp}^2+i\epsilon}\frac{d^\eta_{\ \eta}((\alpha-\alpha_1)p_1+q_{2\perp})}{-(\alpha-\alpha_1)\beta_1s-q_{2\perp}^2+i\epsilon}\frac{d_{\eta\nu'}((\alpha-\alpha_1)p_1+k_{2\perp})}{-(\alpha-\alpha_1)\beta_1's-k_{2\perp}^2+i\epsilon}$$
$$\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\Gamma^{\mu'\nu'\lambda'}(\alpha_1p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})$$

which leads to (cf. Eq. (51)

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.8a}}=\frac{g^4}{4\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'} \tag{289}$$
$$\times\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1du\ \bar uu\int d^2q_1d^2q_2d^2k_1d^2k_2$$
$$\frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2\bar u+q_2^2u)(k_1^2\bar u+k_2^2u)}$$
$$\times\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar u}(q_1+q_2)_iq_{2j}\Big]$$
$$\times\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar u}(k_1+k_2)_ik_{2j}\Big]$$
$$\times=\frac{g^4}{4\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$
$$\times\times\ \xi\frac{d}{d\xi}\int_0^\infty d\alpha\int_0^\alpha d\alpha'\int d^2q_1d^2q_2d^2k_1d^2k_2$$
$$\times\frac{(\alpha-\alpha')\alpha'e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,y-z)_\perp-i(k_2,y-z')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2(\alpha-\alpha')+q_2^2\alpha')(k_1^2(\alpha-\alpha')+k_2^2\alpha')}$$
$$\times\Big[\frac{\delta_{ij}}{\alpha}(q_1^2-q_2^2)-\frac{2}{\alpha'}q_{1i}(q_1+q_2)_j+\frac{2}{\alpha-\alpha'}(q_1+q_2)_iq_{2j}\Big]$$
$$\times\Big[\frac{\delta_{ij}}{\alpha}(k_1^2-k_2^2)-\frac{2}{\alpha'}k_{1i}(k_1+k_2)_j+\frac{2}{\alpha-\alpha'}(k_1+k_2)_ik_{2j}\Big]$$

(recall that $\frac{d}{d\eta}=-2\xi\frac{d}{d\xi}$). The contribution which is sensitive to the subtraction of $(\mathrm{LO})^2$ is

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.8a}}=\frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$
$$\times\xi\frac{d}{d\xi}\int_0^\infty d\alpha\int_0^\alpha d\alpha'\int d^2q_1d^2q_2d^2k_1d^2k_2\Big[\frac{\alpha}{\alpha'}(q_1,k_1)+\frac{\alpha}{\alpha-\alpha'}(q_2,k_2)\Big](q_1+q_2,k_1+k_2)$$
$$\frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2(\alpha-\alpha')+q_2^2\alpha')(k_1^2(\alpha-\alpha')+k_2^2\alpha')} \tag{290}$$

The "+"-prescription (78) leads to the subtraction

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.8a}}=\xi\frac{d}{d\xi}\frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$

$$\times\int_0^\infty\frac{d\alpha}{\alpha}\int d^2q_1d^2q_2d^2k_1d^2k_2\ (q_1+q_2,k_1+k_2)\frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}$$
$$\times\Big\{\int_0^\alpha\frac{d\alpha'}{\alpha'}\Big[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha')[k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\Big]$$
$$+\int_0^\alpha\frac{d\alpha'}{\alpha-\alpha'}\Big[\frac{\alpha^2(q_2,k_2)}{[q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_2,k_2)}{q_2^2k_2^2}\Big]\Big\}\tag{291}$$

The details of the upper cutoff in α do not matter since they correspond to changes in the impact factor which do not affect the evolution. For example,

$$-2\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\frac{1}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}$$
$$\times\int_0^\alpha\frac{d\alpha'}{\alpha'}\Big[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\Big]$$
$$=\frac{1}{(q_1+q_2)^2(k_1+k_2)^2}\int_0^1\frac{du}{u}\Big[\frac{(q_1,k_1)}{(q_1^2\bar u+q_2^2u][k_1^2\bar u+k_2^2u]}-\frac{(q_1,k_1)}{q_1^2k_1^2}\Big]$$
$$=\frac{d}{d\sigma}\int_0^\sigma\frac{d\alpha}{\alpha}\frac{1}{(q_1+q_2)^2(k_1+k_2)^2}$$
$$\times\int_0^\alpha\frac{d\alpha'}{\alpha'}\Big[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\Big]\tag{292}$$

where the last line is exactly our "rigid cutoff" with "+" subtraction (78).

On the contrary, the details of the upper cutoff in α' are essential for the evolution equation (286). The contribution to $\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle$ corresponding to the "slope" cutoff (33) has the form

$$\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle^{\rm slope}_{\rm Fig.8a}=-\frac{g^4}{2\pi^2}{\rm Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$
$$\times\int d^2q_1d^2q_2d^2k_1d^2k_2\ \frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2}e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}$$
$$\times\int_0^\infty\frac{d\alpha'}{\alpha'}\ \Big[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\Big]$$

and therefore the difference between the subtractions in "rigid cutoff" (291) and "slope cutoff" (292) prescriptions can be written as

$$\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm rigid}-K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm slope}\rangle_{\rm Fig.8a}$$
$$=\frac{g^4}{2\pi^2}{\rm Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$
$$\int d^2q_1d^2q_2d^2k_1d^2k_2\ e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}$$
$$\times\Big\{\frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2}\int_0^\infty\frac{d\alpha'}{\alpha'}\ \Big[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\Big]$$

$$+2\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\,\frac{(q_1+q_2,k_1+k_2)}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}\Big[\frac{(q_1,k_1)}{q_1^2k_1^2}+\frac{(q_2,k_2)}{q_2^2k_2^2}\Big]\int_0^\alpha\frac{d\alpha'}{\alpha'}\Big\}$$

$$=-\frac{g^4}{2\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}$$

$$\times\int d^2q_1d^2q_2d^2k_1d^2k_2\; e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}$$

$$\times\int_0^\infty\frac{d\alpha'}{\alpha'}\,\Big\{\frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2}\Big[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\Big]$$

$$-\frac{(q_1+q_2,k_1+k_2)}{[(q_1+q_2)^2+\xi\alpha'^2][(k_1+k_2)^2+\xi\alpha'^2]}\Big[\frac{(q_1,k_1)}{q_1^2k_1^2}+\frac{(q_2,k_2)}{q_2^2k_2^2}\Big]\Big\}$$

It is instructive to rewrite this result in Schwinger's notations

$$\langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}^{\mathrm{rigid}}-K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}^{\mathrm{slope}}\rangle_{\mathrm{Fig.8a}}=\frac{g^4}{2\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2z$$

$$\times\int_0^\infty\frac{d\alpha'}{\alpha'}\Big[(x|\frac{p_i}{p^2+\xi\alpha'^2}|z)U_z^{nn'}(z|\frac{p_i}{p^2+\xi\alpha'^2}|y)(z|\frac{p_j}{p^2}U^{ll'}\frac{p_j}{p^2}|y)$$

$$-(x|\frac{p_i}{p^2}|z)U_z^{nn'}(z|\frac{p_i}{p^2}|y)(z|\frac{p_j}{p^2+\xi\alpha'^2}U^{ll'}\frac{p_j}{p^2+\xi\alpha'^2}|y)\Big]$$

We see now that the difference between the two regularizations of the longitudinal divergence is given by the difference of $(\mathrm{LO})^2$ contributions with cutoffs in α determined by the momenta on the first and on the second step of $(\mathrm{LO})^2$ evolution.

It is easy to see that for the sum of all diagrams this yields (see eq. (86))

$$\langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}^{\mathrm{rigid}}-K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}^{\mathrm{slope}}\rangle=\alpha_s^2\int d^2zd^2z'$$

$$\times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}+(z\leftrightarrow z')]\int_0^\infty\frac{dt}{t}$$

$$\times\Big\{\Big[(x|\frac{p_i}{p^2+t}|z)-(y|\frac{p_i}{p^2+t}|z)\Big]^2\Big[(z|\frac{p_i}{p^2}|z')-(y|\frac{p_i}{p^2}|z')\Big]^2$$

$$-\Big[(x|\frac{p_i}{p^2}|z)-(y|\frac{p_i}{p^2}|z)\Big]^2\Big[(z|\frac{p_i}{p^2+t}|z')-(y|\frac{p_i}{p^2+t}|z')\Big]^2$$

$$+\Big[(x|\frac{p_i}{p^2+t}|z')-(y|\frac{p_i}{p^2+t}|z')\Big]^2\Big[(z'|\frac{p_i}{p^2}|z)-(x|\frac{p_i}{p^2}|z)\Big]^2$$

$$-\Big[(x|\frac{p_i}{p^2}|z')-(y|\frac{p_i}{p^2}|z')\Big]^2\Big[(z'|\frac{p_i}{p^2+t}|z)-(x|\frac{p_i}{p^2+t}|z)\Big]^2\Big\}$$

Using the integral

$$\int_0^\infty\frac{dt}{t^{1-\epsilon}}(x|\frac{p_i}{p^2+t}|z)(y|\frac{p_i}{p^2+t}|z)=-\frac{(X,Y)}{4\pi^2}\int_0^1du\;\frac{\Gamma(\epsilon)\Gamma(2-\epsilon)(4\bar uu)^{-\epsilon}}{(X^2\bar u+Y^2u)^{2-\epsilon}}$$

$$=\frac{(X,Y)}{4\pi^2X^2Y^2}\Big(-\frac{1}{\epsilon}-\ln4+\frac{X^2\ln Y^2-Y^2\ln X^2}{X^2-Y^2}-\ln X^2Y^2\Big)+O(\epsilon)$$

$$\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U_y^\dagger\}^{\rm rigid} - K_{\rm LO}{\rm Tr}\{\hat U_x\hat U_y^\dagger\}^{\rm slope}\rangle = -\frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z'\ \frac{1}{(z-z')^2X^2Y^2}$$
$$\times[{\rm Tr}\{U_xU_z^\dagger\}{\rm Tr}\{U_zU_{z'}^\dagger\}{\rm Tr}\{U_{z'}U_y^\dagger\} - {\rm Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} + z\leftrightarrow z']$$
$$\times\Big(\frac{Y^2}{Y'^2}\Big\{(X,Y)\Big[\frac{X^2+Y^2}{X^2-Y^2}\ln\frac{X^2}{Y^2}+2\Big] + (\Delta,Y)\ln X^2 - (\Delta,X)\ln Y^2\Big\}$$
$$-\frac{\Delta^2}{Y'^2}\Big\{(z-z',Y')\Big[\frac{(z-z')^2+Y'^2}{(z-z')^2-Y'^2}\ln\frac{(z-z')^2}{Y'^2}+2\Big] -$$
$$(Y,Y')\ln(z-z')^2 + (Y,z-z')\ln Y'^2\Big\} + x\leftrightarrow y\Big) \tag{293}$$

The NLO kernel for the evolution of color dipoles with respect to the slope is the sum of Eq. (44) and the correction (293). Note that the correction term (293) is not conformally invariant (cf. Ref. [38]). This is hardly surprising since the non-light-like Wilson line turns into a circle under the inversion $x_\mu \to x_\mu/x^2$. Also, the forward kernel of the correction term will probably spoil the agreement with the NLO BFKL. We were not able to compute the corresponding eigenvalues but it is hard to imagine how the integral of the r.h.s. of Eq. (293) with $(z-z')^{2\gamma}$ can be a number independent of γ (recall that our discrepancy with NLO BFKL is $2\zeta(3)$).

APPENDIX B

APPENDIX FOR NLO EVOLUTION IN $\mathcal{N}=4$ SYM

B.1 CONFORMAL PROPERTIES OF THE LIGHT-LIKE WILSON LINES

In this Section we demonstrate that the light-like Wilson lines are invariant under the conformal (Möbius) group SL(2,C). It is easy to demonstrate that the set of transverse-space operators

$$\hat{S}_- \equiv \frac{i}{2}(K^1+iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D+iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1-iP^2)$$
$$\bar{\hat{S}}_- \equiv \frac{i}{2}(K^1-iK^2), \quad \bar{\hat{S}}_0 \equiv \frac{i}{2}(D-iM^{12}), \quad \bar{\hat{S}}_+ \equiv \frac{i}{2}(P^1+iP^2) \qquad (294)$$

form an SL(2,C) algebra:

$$[\hat{S}_0,\hat{S}_\pm]=\pm\hat{S}_\pm, \quad [\hat{S}_+,\hat{S}_-]=2\hat{S}_0, \qquad [\bar{\hat{S}}_0,\bar{\hat{S}}_\pm]=\pm\bar{\hat{S}}_\pm, \quad [\bar{\hat{S}}_+,\bar{\hat{S}}_-]=2\bar{\hat{S}}_0 \quad (295)$$

Here we use standard textbook definitions of the momentum operator $\hat{P}$, angular mpmentum operator $\hat{M}$, dilatation operator $\hat{D}$, and special conformal generator $\hat{K}$. Using the conventional commutators of these generators with the gluon field

$$i[\hat{P}^\mu,\hat{A}^\alpha]=\partial^\mu\hat{A}^\alpha, \quad i[\hat{D},\hat{A}^\alpha]=(x_\mu\partial^\mu+1)\hat{A}^\alpha,$$
$$i[\hat{M}^{\mu\nu},\hat{A}^\alpha]=(x^\mu\partial^\nu-x^\nu\partial^\mu)\hat{A}^\alpha-(g^{\nu\alpha}\hat{A}^\mu-g^{\mu\alpha}\hat{A}^\nu)$$
$$i[K^\mu,A^\alpha] = (2x^\mu x_\nu\partial^\nu-x^2\partial^\mu+2x^\mu)A^\alpha-2x_\nu(g^{\nu\alpha}A^\mu-g^{\mu\alpha}A^\nu) \qquad (296)$$

one can easily obtain the action of these operators on light-like Wilson lines. In complex notations

$$z\equiv z^1+iz^2, \quad \bar{z}\equiv z^1-iz^2, \quad \frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial z^1}-i\frac{\partial}{\partial z^2}\Big), \quad \frac{\partial}{\partial\bar{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial z^1}+i\frac{\partial}{\partial z^2}\Big)$$

these commutators take the form

$$[\hat{S}_-,\hat{U}(z,\bar{z})]=z^2\partial_{z_3}\hat{U}(z,\bar{z}), \quad [\hat{S}_0,\hat{U}(z,\bar{z})]=z\partial_{z_3}\hat{U}(z,\bar{z}), \quad [\hat{S}_+,\hat{U}(z,\bar{z})]=-\partial_{z_3}\hat{U}(z,\bar{z})$$
$$[\bar{\hat{S}}_-,\hat{U}(z,\bar{z})]=\bar{z}^2\partial_{\bar{z}}\hat{U}(z,\bar{z}), \quad [\bar{\hat{S}}_0,\hat{U}(z,\bar{z})]=\bar{z}\partial_{\bar{z}}\hat{U}(z,\bar{z}), \quad [\bar{\hat{S}}_+,\hat{U}(z,\bar{z})]=-\partial_{\bar{z}}\hat{U}(z,\bar{z})$$

These equations mean that the operators $U(z,\bar{z})$ lie in the standard representation of conformal group $SL(2,C)$ with the conformal spin 0.

B.2 NLO IMPACT FACTOR

As we demonstrated in the Appendix B.1 the light-like Wilson lines $U(x_\perp)$ are formally Möbius invariant and this is the reason why the leading-order BK equation is conformal. However, because our cutoff of the rapidity divergence is not invariant, the NLO evolution kernel (226) has the non-invariant double-log term. To illustrate the non-invariance of the dipole with the cutoff (45) let us consider the operator expansion of two BPS-protected currents $\mathcal{O} \equiv \frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{Z^2\}$ $(Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2))$. The Regge limit of the amplitude

$$A(x,y;x',y') = (x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(x')\mathcal{O}(y')\rangle \tag{297}$$

can be achieved by the rescaling [45]

$$x \;\rightarrow\; \lambda x_* p_1 + \frac{1}{\lambda}x_\bullet p_2 + x_\perp, \qquad y \;\rightarrow\; \lambda y_* + \frac{1}{\lambda}y_\bullet p_2 + y_\perp \tag{298}$$

with $\lambda \to \infty$. In this regime, the T-product of the currents $\mathcal{O}(x)$ and $\mathcal{O}(y)$ can be expanded in color dipoles:

$$\begin{aligned}T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} &= \int d^2z_1 d^2z_2\; I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\ &+ \int d^2z_1 d^2z_2 d^2z_3\; I^{\rm NLO}(z_1,z_2,z_3)[\mathrm{Tr}\{T^n\hat{U}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\end{aligned} \tag{299}$$

(structure of the NLO contribution is clear from the topology of diagrams in the shock-wave background, see Fig. 315 below)

Let us calculate the impact factor taking $x_\bullet = y_\bullet = 0$ for simplicity. The leading-order impact factor is proportional to the product of two propagators (302)

$$\begin{aligned}\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\rangle_A^{\rm LO} &= \frac{16\pi^4}{(N_c^2-1)}\int d^4z_1 d^4z_2\; \delta(z_{1*})\delta(z_{2*})U^{ab}_{z_1}U^{ab}_{z_2}\\ &\times\Big(\frac{1}{4\pi^2[(x-z_1)^2-i\epsilon]}(2i)\partial_*^{(z_1)}\frac{1}{4\pi^2[(y-z_1)^2-i\epsilon]}\Big)\\ &\times\Big(\frac{1}{4\pi^2[(x-z_2)^2-i\epsilon]}(2i)\partial_*^{(z_2)}\frac{1}{4\pi^2[(y-z_2)^2-i\epsilon]}\Big)\\ &= \frac{1}{\pi^2(N_c^2-1)}\int d^2z_{1\perp}d^2z_{2\perp}\frac{(x_*y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\;\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}\end{aligned} \tag{300}$$

where $\mathcal{Z}_i \equiv \frac{(x-z_i)_\perp^2}{x_*} - \frac{(y-z_i)_\perp^2}{y_*}$ and the color trace is taken in the adjoint representation. This expressions coincides with the result of Ref. [41]. It is easy to see that under

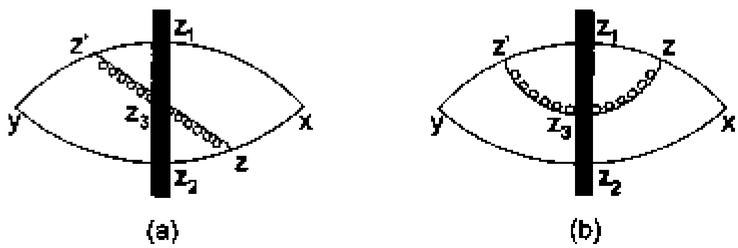

FIG. 22: Diagrams for the NLO impact factor.

the inversion $x_\perp \to x_\perp/x_\perp^2, x_* \to x_*/x_\perp^2$

$$x_\perp \to x_\perp/x_\perp^2, x_* \to x_*/x_\perp^2, \qquad y_\perp \to y_\perp/x_\perp^2, y_* \to y_*/y_\perp^2, \qquad z_{i\perp} \to z_{i\perp}/z_{i\perp}^2 \tag{301}$$

$\mathcal{Z}_i$ is transformed as $\mathcal{Z}_i \to z_i^{-4}\mathcal{Z}_i$ so the leading-order impact factor is invariant.

The NLO impact factor for two Z^2 currents is given by the two diagrams shown in Fig. 22. To calculate them we will use the following representation of scalar and gluon propagators in the shock-wave background (at $x_* > 0 > y_*$)

$$\begin{aligned}\langle\hat\Phi_I(x)\hat\Phi_J(y)\rangle_A =&\ 2i\delta^{IJ}\int d^4z\delta(z_*)\frac{1}{4\pi^2[(x-z)^2-i\epsilon]}\ U^{ab}_{z_\perp}\ \partial_*^{(z)}\frac{1}{4\pi^2[(z-y)^2-i\epsilon]}\\ =& -\frac{\delta_{IJ}}{4\pi^3}\int d^2z_\perp\frac{x_*y_*U^{ab}_{z_3}}{\left[\frac{4}{s}(x-y)_\bullet x_*y_* - (x-z)_\perp^2y_* + (y-z)_\perp^2x_* + i\epsilon\right]^2}\end{aligned} \tag{302}$$

and

$$\begin{aligned}&\langle\hat A^a_\mu(x)\hat A^b_\nu(y)\rangle_A =\\ &-\frac{i}{2}\int d^4z\ \delta(z_*)\ \frac{x_*g^\perp_{\mu\xi}-p_{2\mu}(x-z)^\perp_\xi}{\pi^2[(x-z)^2-i\epsilon]^2}\ U^{ab}_{z_\perp}\frac{1}{\partial_*^{(z)}}\ \frac{y_*\delta^{\perp\xi}_\nu - p_{2\nu}(y-z)^\xi_\perp}{\pi^2[(z-y)^2-i\epsilon]^2}\end{aligned} \tag{303}$$

where $\frac{1}{\partial_*}$ can be either $\frac{1}{\partial_*+i\epsilon}$ or $\frac{1}{\partial_*-i\epsilon}$ which leads to the same result after subtraction of the leading-order contribution (see below).

To calculate the next-to-leading impact factor we need the three-point scalar-scalar-gluon vertex Green function (vertex with tails):

$$\begin{aligned}&\int d^4z\ \theta(z_*)\Big[\frac{1}{4\pi^2[(x-z)^2-i\epsilon]}\ \overleftrightarrow{\partial}^z_\mu\ \frac{1}{4\pi^2[(z-z_1)^2-i\epsilon]}\Big]\frac{z_*\delta^{\perp\xi}_\mu - p_{2\mu}(z-z_3)^\xi_\perp}{2\pi^2[(z-z_3)^2-i\epsilon]^2}\\ &= \frac{ix_*z_{13}^{\perp\xi}}{8\pi^4z_{13}^2[(x-z_1)^2-i\epsilon][(x-z_3)^2-i\epsilon]}\end{aligned} \tag{304}$$

Since $z_{1*} = z_* = 0$ one can easily check that $\theta(z_-)$ in the l.h.s. of this integral can be erased since the contribution from $z_* < 0$ vanishes. After that, one easily gets Eq. (304) from the conformal integral

$$\int d^4z \,\Big[\frac{1}{(x-z)^2}\,\overleftrightarrow{\partial}_\mu\,\frac{1}{(z-y)^2}\Big]\frac{z_\nu}{z^4} - \mu \leftrightarrow \nu \;=\; 4i\pi^2\frac{x_\mu y_\nu - x_\nu y_\mu}{x^2y^2(x-y)^2} \tag{305}$$

which is easily calculated by inversion $z_\mu \to z_\mu/z^2$.

Now we are in position to calculate the contribution of the diagram in Fig. 22a:

$$\begin{aligned}
&\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\rm Fig.22a} = g^2\frac{32\pi^4}{N_c^2-1}\int d^4z_1d^4z_2d^4z_3\;\delta(z_{3*}){\rm Tr}\{T^nU_{z_1}T^{n'}U^\dagger_{z_2}\}U_{z_3}^{nn'}\\
&\Big(2i\,\partial_{1*}\frac{\delta(z_{1*})}{4\pi^2[(y-z_1)^2-i\epsilon]}\Big)\Big(2i\,\partial_{2*}\frac{\delta(z_2*)}{4\pi^2[(x-z_2)^2-i\epsilon]}\Big)\\
&\times\int d^4z\Big(\frac{1}{4\pi^2[(x-z)^2-i\epsilon]}\,\overleftrightarrow{\partial}^z_\mu\,\frac{\delta(z_1*)}{4\pi^2[(z-z_1)^2-i\epsilon]}\Big)\\
&\times\frac{z_*\theta(z_*)}{2\pi^2[(z-z_3)^2-i\epsilon]^2}\Big[g^\perp_{\mu\rho}-p_{2\mu}\frac{(z-z_3)^\rho_\perp}{z_*}\Big]\,\frac{2i}{\partial_{3*}}\\
&\times\int d^4z'\Big(\frac{\delta(z_2*)}{4\pi^2[(z_2-z')^2-i\epsilon]}\,\overleftrightarrow{\partial}^{z'}_\mu\,\frac{1}{4\pi^2[(y-z')^2-i\epsilon]}\Big)\\
&\times\frac{z'_*\theta(-z'_*)}{2\pi^2[(z_3-z')^2-i\epsilon]^2}\Big[g_\nu^{\perp\rho}-p_{2\nu}\frac{(z'-z_3)^\rho_\perp}{z'_*}\Big]
\end{aligned} \tag{306}$$

Using the Eq. (304) one can reduce this equation to

$$\begin{aligned}
&\times\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\rm Fig.22a} = -g^2\frac{32i\pi^4}{N_c^2-1}\int d^4z_1d^4z_2d^4z_3\;\delta(z_{3*}){\rm Tr}\{T^nU_{z_1}T^{n'}U^\dagger_{z_2}\}U_{z_3}^{nn'}\\
&\times\frac{1}{4\pi^2[(x-z_1)^2-i\epsilon]}\Big(2i\,\partial_{1*}\frac{\delta(z_{1*})}{4\pi^2[(y-z_1)^2-i\epsilon]}\Big)\\
&\times\Big(2i\,\partial_{2*}\frac{\delta(z_2*)}{4\pi^2[(x-z_2)^2-i\epsilon]}\Big)\frac{1}{4\pi^2[(y-z_2)^2-i\epsilon]}\;\frac{(z_{13},z_{23})_\perp}{z_{13}^2z_{23}^2}\\
&\times\frac{x_*}{2\pi^2[(x-z_3)^2-i\epsilon]}\frac{2}{\partial_{3*}}\;\frac{y_*}{2\pi^2[(y-z_3)^2-i\epsilon]}
\end{aligned} \tag{307}$$

Performing integration with respect to z_{*i} we get

$$\begin{aligned}
&\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\rm Fig.22a} = \frac{2\alpha_s}{(N_c^2-1)\pi^4x_*^2y_*^2}\int\frac{d^2z_1d^2z_2}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\int d^2z_3\frac{(z_{13},z_{23})_\perp}{z_{13}^2z_{23}^2}\\
&\times{\rm Tr}\{T^nU_{z_1}U^\dagger_{z_3}T^nU_{z_3}U^\dagger_{z_2}\}\int_0^\infty\frac{d\alpha}{\alpha}\;e^{i\alpha\frac{s}{4}\mathcal{Z}_3}
\end{aligned} \tag{308}$$

(recall that $\mathcal{Z}_i = \frac{(x-z_i)^2_\perp}{x_*} - \frac{(y-z_i)^2_\perp}{y_*}$). Similarly, one can demonstrate that the diagram shown in Fig. 22b yields

$$\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\rm Fig.22b} = \frac{2\alpha_s}{(N_c^2-1)\pi^4x_*^2y_*^2}\int\frac{d^2z_1d^2z_2}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\int d^2z_3\frac{1}{z_{13}^2}$$

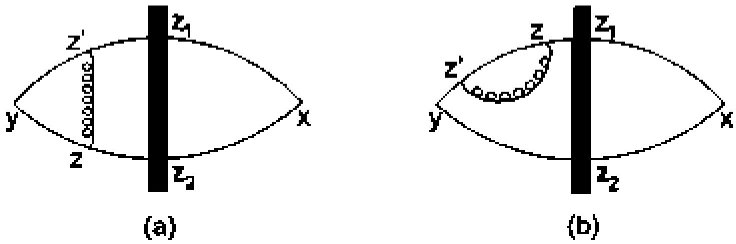

FIG. 23: Diagrams for the NLO impact factor without gluon-shockwave intersection.

$$\times \mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} \int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha\frac{s}{4}Z_3} \tag{309}$$

Using the $z_1 \leftrightarrow z_2$ symmetry in this equation one can write down the sum of Eq. (308) and Eq. (309) as

$$\begin{aligned} \langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\mathrm{Fig.22}} &= \frac{\alpha_s}{(N_c^2-1)\pi^4 x_*^2 y_*^2} \int \frac{d^2z_1 d^2z_2}{Z_1^2 Z_2^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \\ &\times \mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} \int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha\frac{s}{4}Z_3} \end{aligned} \tag{310}$$

For future use it is convenient to rewrite this in the following form

$$\begin{aligned} \langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\mathrm{Fig.22}} &= \frac{\alpha_s}{(N_c^2-1)\pi^4 x_*^2 y_*^2} \int \frac{d^2z_1 d^2z_2}{Z_1^2 Z_2^2} \\ &\times \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} - N_c \mathrm{Tr}\{U_{z_1} U^\dagger_{z_2}\} \int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha\frac{s}{4}Z_3} \\ &+ \frac{2\alpha_s N_c}{(N_c^2-1)\pi^4 x_*^2 y_*^2} \int \frac{d^2z_1 d^2z_2}{Z_1^2 Z_2^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \mathrm{Tr}\{U_{z_1} U^\dagger_{z_2}\} \int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha\frac{s}{4}Z_3} \end{aligned} \tag{311}$$

Let us discuss now the contribution of Fig. 23 diagrams. Since this contribution is proportional to $\mathrm{Tr}\{U_{z_1} U^\dagger_{z_2}\}$ it can be restored from the comparison of Eq. (311) with the pure perturbative series for the correlator $\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle$. Indeed, if we switch off the shock wave the contribution of the Fig. 22 diagrams is given by the second term in Eq. (311) (with $U, U^\dagger$ repaced by 1). On the other hand, perturbative series for the correlator $\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle$ vanishes and therefore the contribution of the Fig. 23 diagrams should be equal to the second term in the r.h.s. Eq. (311) with opposite sign. Thus, the first term in the r.h.s. Eq. (311) gives the total contribution to the impact factor:

$$\langle T\{\hat{O}(x)\hat{O}(y)\}\rangle_A^{\mathrm{Fig.22+Fig.23}} = \frac{\alpha_s}{(N_c^2-1)\pi^4 x_*^2 y_*^2} \int \frac{d^2z_1 d^2z_2}{Z_1^2 Z_2^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}$$

$$\times \mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} - N_c \mathrm{Tr}\{U_{z_1} U^\dagger_{z_2}\} \int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha \frac{s}{4} \mathcal{Z}_3} \tag{312}$$

The integral over α in the r.h.s. of Eq. (312) diverges. This divergence reflects the fact that the r.h.s. of Eq. (312) is not exactly the NLO impact factor since we must subtract the matrix element of the leading-order contribution. Indeed, the NLO impact factor is a coefficient function defined according to Eq. (299). To find the NLO impact factor, we consider operator equation (299) in the shock-wave background (in the leading order $\langle \hat{U}_{z_3}\rangle_A = U_{z_3}$):

$$\begin{aligned}
&\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\rangle_A - \int d^2z_1 d^2z_2\ I^{\mathrm{LO}}(x,y;z_1,z_2)\langle \mathrm{Tr}\{\hat{U}^\eta{}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A \qquad (313)\\
&= \int d^2z_1 d^2z_2 d^2z_3\ I^{\mathrm{NLO}}(x,y;z_1,z_2,z_3;\eta)[\mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} - N_c \mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}]
\end{aligned}$$

The NLO matrix element $\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\}\rangle_A$ is given by Eq. (312) while

$$\begin{aligned}
&\int d^2z_1 d^2z_2\ I^{\mathrm{LO}}(x,y;z_1,z_2)\langle \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\\
&= \frac{(x_*y_*)^{-2}}{\pi^2(N_c^2-1)}\int d^2z_1 d^2z_2\ \frac{1}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\frac{\alpha_s}{\pi^2}\int_0^\sigma \frac{d\alpha}{\alpha}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\\
&\quad\times[\mathrm{Tr}\{T^n U_{z_1} U^\dagger_{z_3} T^n U_{z_3} U^\dagger_{z_2}\} - N_c \mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}] \qquad (314)
\end{aligned}$$

as follows from Eq. (41). The α integration is cut above by $\sigma = e^\eta$ in accordance with the definition of operators $\hat{U}^\eta$ (45). Subtracting (314) from Eq. (312) we get

$$\begin{aligned}
I^{\mathrm{NLO}}(x,y;z_1,z_2,z_3;\eta) &= \frac{\alpha_s(x_*y_*)^{-2}}{\pi^4(N_c^2-1)}\frac{z_{13}^2}{z_{12}^2 z_{23}^2 \mathcal{Z}_1^2\mathcal{Z}_2^2}\Big[\int_0^\infty \frac{d\alpha}{\alpha}\, e^{i\alpha\frac{s}{4}\mathcal{Z}_3} - \int_0^\sigma \frac{d\alpha}{\alpha}\Big]\\
&= -\frac{\alpha_s(x_*y_*)^{-2}}{\pi^4(N_c^2-1)}\frac{z_{13}^2}{z_{12}^2 z_{23}^2 \mathcal{Z}_1^2\mathcal{Z}_2^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big] \qquad (315)
\end{aligned}$$

Let us rewrite the operator expansion (299) in the explicit form

$$\begin{aligned}
(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} &= \frac{(x-y)^4}{\pi^2(N_c^2-1)}\int d^2z_{1\perp} d^2z_{2\perp}\frac{(x_*y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\ \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} \qquad (316)\\
&- \frac{\alpha_s(x-y)^4}{\pi^4(N_c^2-1)}\int d^2z_{1\perp} d^2z_{2\perp} d^2z_3 \frac{z_{12}^2(x_*y_*)^{-2}}{z_{13}^2 z_{23}^2 \mathcal{Z}_1^2\mathcal{Z}_2^2}\\
&\times\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big][\mathrm{Tr}\{T^n \hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3} T^n \hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]
\end{aligned}$$

It is easy to see now that under the inversion (301) the leading-order impact factor is invariant while the NLO impact factor is not because of the non-invariant logarithmic term $\ln\frac{\sigma s}{4}\mathcal{Z}_3$. Since the original T-product of the currents in the l.h.s. of the Eq. (316) is conformal, it indicates that our operators $\hat{U}^\eta$ with the "rigid cutoff"

(45) are not Möbius invariant. However, if we expand the original T-product in composite conformal operators (200) instead, the resulting impact factor is conformally invariant:

$$(x-y)^4T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x-y)^4}{\pi^2(N_c^2-1)}\int d^2z_{1\perp}d^2z_{2\perp}\frac{(x_*y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\,[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf} \quad (317)$$
$$-\frac{\alpha_s(x-y)^4}{2\pi^4(N_c^2-1)}\int d^2z_{1\perp}d^2z_{2\perp}d^2z_3\frac{z_{12}^2(x_*y_*)^{-2}}{z_{13}^2z_{23}^2\mathcal{Z}_1^2\mathcal{Z}_2^2}\Big(\ln\frac{a\sigma^2s^2z_{12}^2}{16z_{13}^2z_{23}^2}\Big[\frac{(x-z_3)^2}{x_*}-\frac{(y-z_3)^2}{y_*}\Big]^2$$
$$-i\pi+2C\Big)[\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

The arbitrary dimensional constant a should be chosen in such a way that the impact factor in the r.h.s of Eq. (318) does not change under the rescaling (298). The proper choice for our T-product is $a = \frac{x_*y_*}{s^2(x-y)^2}$ so our final operator expansion takes the form

$$(x-y)^4T\{\hat{O}(x)\hat{O}(y)\} = \frac{(x-y)^4}{\pi^2(N_c^2-1)}\int d^2z_{1\perp}d^2z_{2\perp}\frac{(x_*y_*)^{-2}}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\,[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$
$$-\frac{\alpha_s(x-y)^4}{2\pi^4(N_c^2-1)}\int d^2z_{1\perp}d^2z_{2\perp}d^2z_3\frac{z_{12}^2(x_*y_*)^{-2}}{z_{13}^2z_{23}^2\mathcal{Z}_1^2\mathcal{Z}_2^2}$$
$$\times\Big(\ln\frac{x_*y_*z_{12}^2e^{2\eta}}{16(x-y)_\perp^2z_{13}^2z_{23}^2}\Big[\frac{(x-z_3)^2}{x_*}-\frac{(y-z_3)^2}{y_*}\Big]^2-i\pi+2C\Big)$$
$$\times[\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}] \quad (318)$$

Now it is evident that the impact factor in the r.h.s. of this equation is Möbius invariant and does not scale with λ so Eq. (227) gives conformally invariant operator up to α_s^2 order. In higher orders, one should expect the correction terms with more Wilson lines. This procedure of finding the dipole with conformally regularized rapidity divergence is analogous to the construction of the composite renormalized local operator by adding the appropriate counterterms order by order in perturbation theory.

B.3 LEADING-ORDER EVOLUTION OF THE FOUR-WILSON-LINE OPERATOR

In this Appendix we derive the evolution equation for the four-Wilson-line operator $\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}$ in the leading order in perturbation theory. As a first step, we rewrite the hierarchy equations of Ref. [45] in the adjoint representation:

$$\frac{d}{d\eta}(\hat{U}^\eta_{z_1})_{ij}(\hat{U}^\eta_{z_2})_{kl} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{(z_{13},z_{23})}{z_{13}^2z_{23}^2}$$
$$\times[(T^a\hat{U}^\eta_{z_1})_{ij}(\hat{U}^\eta_{z_2}T^b)_{kl}+(\hat{U}^\eta_{z_1}T^b)_{ij}(T^a\hat{U}^\eta_{z_2})_{kl}](2\hat{U}^\eta_{z_3}-\hat{U}^\eta_{z_1}-\hat{U}^\eta_{z_2})^{ab}, \quad (319)$$

for the pair-wise interaction and

$$\frac{d}{d\eta}(\hat{U}^{\eta}_{z_1})_{ij} = \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{1}{z_{13}^2}(T^a\hat{U}^{\eta}_{z_1}T^b)_{ij}(\hat{U}^{\eta}_{z_3}-\hat{U}^{\eta}_{z_1})^{ab} \tag{320}$$

for the self-interaction. Using the above equations we get

$$\begin{aligned}
&\frac{d}{d\eta}[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}] = \frac{\alpha_s}{2\pi^2}\int d^2z_4(\hat{U}^{\eta}_{z_3})^{aa'}\\
&\times\Big[\frac{(z_{14},z_{34})}{z_{14}^2z_{34}^2}\mathrm{Tr}\{T^aT^b\hat{U}^{\eta}_{z_1}[T^{a'},T^{b'}]\hat{U}^{\dagger\eta}_{z_2} + [T^b,T^a]\hat{U}^{\eta}_{z_1}T^{b'}T^{a'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_1}-\hat{U}^{\eta}_{z_3})^{bb'}\\
&+\frac{(z_{24},z_{34})}{z_{24}^2z_{34}^2}\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_2}-\hat{U}^{\eta}_{z_3})^{bb'}\\
&-\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}\mathrm{Tr}\{T^aT^b\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^{\eta}_{z_1}T^{b'}T^{a'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_1}-\hat{U}^{\eta}_{z_2})^{bb'}\\
&-\frac{1}{z_{34}^2}\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}[T^{a'},T^{b'}]\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-2\hat{U}^{\eta}_{z_3})^{bb'}\\
&+\frac{1}{z_{14}^2}\mathrm{Tr}\{T^aT^b\hat{U}^{\eta}_{z_1}T^{b'}T^{a'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-2\hat{U}^{\eta}_{z_1})^{bb'}\\
&+\frac{1}{z_{24}^2}\mathrm{Tr}\{T^bT^a\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-2\hat{U}^{\eta}_{z_2})^{bb'}\Big]\\
&-\frac{\alpha_sN_c}{\pi^2}\int d^2z_4\frac{z_{12}^2}{z_{14}^2z_{24}^2}(\mathrm{Tr}\{T^aU_{z_1}U^{\dagger}_{z_4}T^aU_{z_4}U^{\dagger}_{z_2}\} - N_c\mathrm{Tr}\{U_{z_1}U^{\dagger}_{z_2}\})
\end{aligned} \tag{321}$$

After some algebra, the r.h.s. of Eq. (321) reduces to

$$\begin{aligned}
&\frac{d}{d\eta}[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}] = \frac{\alpha_s}{4\pi^2}\int d^2z_4(\hat{U}^{\eta}_{z_3})^{aa'}\\
&\Big[\frac{z_{12}^2}{z_{14}^2z_{24}^2}\mathrm{Tr}\{T^aT^b\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^{\eta}_{z_1}T^{b'}T^{a'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_1}-\hat{U}^{\eta}_{z_2})^{bb'}\\
&-\frac{z_{13}^2}{z_{14}^2z_{34}^2}\mathrm{Tr}\{T^aT^b\hat{U}^{\eta}_{z_1}[T^{a'},T^{b'}]\hat{U}^{\dagger\eta}_{z_2} + [T^b,T^a]\hat{U}^{\eta}_{z_1}T^{b'}T^{a'}\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_1}-\hat{U}^{\eta}_{z_3})^{bb'}\\
&-\frac{z_{23}^2}{z_{24}^2z_{34}^2}\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2} + T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(2\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_2}-\hat{U}^{\eta}_{z_3})^{bb'}\Big]\\
&-\frac{\alpha_sN_c}{\pi^2}\int d^2z_4\frac{z_{12}^2}{z_{14}^2z_{24}^2}(\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_4}T^a\hat{U}^{\eta}_{z_4}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{U_{z_1}U^{\dagger}_{z_2}\})
\end{aligned} \tag{322}$$

B.4 TRACES

In this section we rewrite the adjoint traces in our evolution equation (199) in terms of the fundamental traces. The master formula for traces has the form (in this section we use the space-saving notations $U_i \equiv \hat{U}_{z_i}$)

$$\mathrm{Tr}\{[T^a,T^b]U_1^{\eta}T^{a'}T^{b'}U_2^{\dagger} + T^bT^aU^{\eta}_{z_1}[T^{b'},T^{a'}]U^{\dagger}_{z_2}\}U_3^{aa'}U_4^{bb'} \tag{323}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big[- \mathrm{tr}\{U_1U_2^\dagger\}\mathrm{tr}\{U_2U_4^\dagger\}\mathrm{tr}\{U_4U_3^\dagger\}\mathrm{tr}\{U_3U_1^\dagger\} - \mathrm{tr}\{U_1^\dagger U_2\}\mathrm{tr}\{U_2^\dagger U_4\}\mathrm{tr}\{U_4^\dagger U_3\}\mathrm{tr}\{U_3^\dagger U_1\} \\
&+\mathrm{tr}\{U_1U_2^\dagger\}\mathrm{tr}\{U_2U_4^\dagger U_3U_1^\dagger U_4U_3^\dagger\} + \mathrm{tr}\{U_1U_3^\dagger\}\mathrm{tr}\{U_3U_2^\dagger U_4U_1^\dagger U_2U_4^\dagger\} \\
&+\mathrm{tr}\{U_3U_4^\dagger\}\mathrm{tr}\{U_4U_2^\dagger U_1U_3^\dagger U_2U_1^\dagger\} + \mathrm{tr}\{U_2U_4^\dagger\}\mathrm{tr}\{U_4U_1^\dagger U_3U_2^\dagger U_1U_3^\dagger\} \\
&-\mathrm{tr}\{U_1U_4^\dagger\}\mathrm{tr}\{U_4U_2^\dagger U_3U_1^\dagger U_2U_3^\dagger\} - \mathrm{tr}\{U_2U_3^\dagger\}\mathrm{tr}\{U_3U_1^\dagger U_4U_2^\dagger U_1U_4^\dagger\} \\
&-\mathrm{tr}\{U_1U_2^\dagger U_4U_3^\dagger\}\mathrm{tr}\{U_1^\dagger U_2U_4^\dagger U_3\} + \mathrm{tr}\{U_1^\dagger U_2\}\mathrm{tr}\{U_2^\dagger U_4U_3^\dagger U_1U_4^\dagger U_3\} \\
&+\mathrm{tr}\{U_1^\dagger U_3\}\mathrm{tr}\{U_3^\dagger U_2U_4^\dagger U_1U_2^\dagger U_4\} + \mathrm{tr}\{U_3^\dagger U_4\}\mathrm{tr}\{U_4^\dagger U_2U_1^\dagger U_3U_2^\dagger U_1\} \\
&+\mathrm{tr}\{U_2^\dagger U_4\}\mathrm{tr}\{U_4^\dagger U_1U_3^\dagger U_2U_1^\dagger U_3\} - \mathrm{tr}\{U_1^\dagger U_4\}\mathrm{tr}\{U_4^\dagger U_2U_3^\dagger U_1U_2^\dagger U_3\} \\
&-\mathrm{tr}\{U_2^\dagger U_3\}\mathrm{tr}\{U_3^\dagger U_1U_4^\dagger U_2U_1^\dagger U_4\} - \mathrm{tr}\{U_1^\dagger U_2U_4^\dagger U_3\}\mathrm{tr}\{U_1U_2^\dagger U_4U_3^\dagger\}\Big]
\end{aligned}
$$

where Tr and tr stand for the trace in the adjoint and the fundamental representation, respectively. We will also need traces

$$
\begin{aligned}
&\mathrm{Tr}\{T^aU_1T^bU_2^\dagger\}U_3^{ab} = \mathrm{Tr}\{T^aU_1U_3^\dagger T^bU_3U_2^\dagger\} \\
&= \frac{1}{2}\mathrm{tr}[\{U_1U_2^\dagger\}\mathrm{tr}\{U_2U_3^\dagger\}\mathrm{tr}\{U_3U_1^\dagger\} + \mathrm{tr}\{U_1U_3^\dagger\}\mathrm{tr}\{U_3U_2^\dagger\}\mathrm{tr}\{U_2U_1^\dagger\} \\
&-\mathrm{tr}\{U_1U_2^\dagger U_3U_1^\dagger U_2U_3^\dagger\} - \mathrm{tr}\{U_1U_3^\dagger U_2U_1^\dagger U_3U_2^\dagger\}], \\
&\mathrm{Tr}\{[T^a,T^b]U_1T^{a'}T^{b'}U_2^\dagger + T^bT^aU_1[T^{b'},T^{a'}]U_2^\dagger\}U_3^{aa'}U_3^{bb'} \\
&= \mathrm{Tr}\{[T^a,T^b]U_1^\eta T^{a'}T^{b'}U_2^\dagger + T^bT^aU_{z_1}^\eta[T^{b'},T^{a'}]U_{z_2}^\dagger\}U_1^{aa'}U_3^{bb'} \\
&= \mathrm{Tr}\{[T^a,T^b]U_1^\eta T^{a'}T^{b'}U_2^\dagger + T^bT^aU_{z_1}^\eta[T^{b'},T^{a'}]U_{z_2}^\dagger\}U_3^{aa'}U_2^{bb'} \\
&= -N_c\mathrm{Tr}\{T^aU_1U_3^\dagger T^aU_3U_2^\dagger\}, \\
&\mathrm{Tr}\{T^aT^bU_1T^{a'}T^{b'}U_2^\dagger + T^bT^aU_1T^{b'}T^{a'}U_2^\dagger\}U_1^{aa'}U_3^{bb'} \\
&= \mathrm{Tr}\{T^aT^bU_1T^{a'}T^{b'}U_2^\dagger + T^bT^aU_1T^{b'}T^{a'}U_2^\dagger\}U_2^{aa'}U_3^{bb'} = N_c\mathrm{Tr}\{T^aU_1U_3^\dagger T^aU_3U_2^\dagger\}
\end{aligned}
$$

which easily follow from Eq. (323).

Let us now find the master trace (323) in the two-gluon (BFKL) approximation. First, note that in this approximation the trace of six Wilson lines reduces to one dipole. Indeed,

$$
\begin{aligned}
&\mathrm{tr}\{U_1U_3^\dagger U_4U_2^\dagger U_3U_4^\dagger\} = \mathrm{tr}\{(U_1U_3^\dagger - 1)U_4(U_2^\dagger U_3 - 1)U_4^\dagger\} + \mathrm{tr}\{(U_1U_3^\dagger - 1\} \\
&+\mathrm{tr}\{(U_2^\dagger U_3 - 1\} + N_c^2 \\
&= \mathrm{tr}\{(U_1U_3^\dagger - 1)U_3(U_2^\dagger U_3 - 1)U_3^\dagger\} + \mathrm{tr}\{(U_1U_3^\dagger - 1\} + \mathrm{tr}\{(U_2^\dagger U_3 - 1\} + N_c^2 \\
&= \mathrm{tr}\{U_1U_2^\dagger\} \qquad (324)
\end{aligned}
$$

where we replaced by U_4 by U_3 in the first term in the second line since it does not matter in the two-gluon approximation (both of them should be replaced by 1). All other six-Wilson-line terms in the r.h.s. of Eq. (323) can be similarly reduced to single-dipole terms.

Using the same trick we can reduce the trace of four Wilson lines to the sum of dipoles:

$$\begin{aligned}
&\mathrm{tr}\{U_1U_2^\dagger U_4U_3^\dagger\} = \mathrm{tr}\{(U_1U_2^\dagger - 1)(U_4U_3^\dagger - 1)\} + \mathrm{tr}\{U_1U_2^\dagger - 1)\} + \mathrm{tr}\{U_4U_3^\dagger - 1)\} + N_c^2\\
&= \mathrm{tr}\{(U_1U_2^\dagger - 1)U_2U_4^\dagger(U_4U_3^\dagger - 1)\} + \mathrm{tr}\{U_1U_2^\dagger - 1)\} + \mathrm{tr}\{U_4U_3^\dagger - 1)\} + N_c^2\\
&= \mathrm{tr}\{(U_1U_2^\dagger - 1)\} + \mathrm{tr}\{U_2U_4^\dagger - 1\} + \mathrm{tr}\{U_4U_3^\dagger - 1)\} + \mathrm{tr}\{U_1U_3^\dagger - 1)\}\\
&-\mathrm{tr}\{U_1U_4^\dagger - 1)\} - \mathrm{tr}\{U_2U_3^\dagger - 1)\} + N_c^2 \qquad (325)
\end{aligned}$$

where in the first term in the second line we have inserted $U_2U_4^\dagger$ since it does not matter in the two-gluon approximation.

Using formulas (324) and (329) it is easy to demonstrate that all the terms containing traces of four and six Wilson lines in the r.h.s. of the Eq. (323) sum to $2N_c^2$ in the two-gluon approximation so one gets

$$\begin{aligned}
&\mathrm{Tr}\{[T^a,T^b]U_1^\eta T^{a'}T^{b'}U_2^\dagger + T^bT^aU_{z_1}^\eta[T^{b'},T^{a'}]U_{z_2}^\dagger\}U_3^{aa'}U_4^{bb'}\\
&\stackrel{2-\mathrm{gluon}}{=} -N_c^2(N_c^2-1) - \frac{1}{2}N_c^3[\mathrm{tr}\{U_1U_2^\dagger - 1\} + \mathrm{tr}\{U_1^\dagger U_2 - 1\} + \mathrm{tr}\{U_2U_4^\dagger - 1\}\\
&+\mathrm{tr}\{U_2^\dagger U_4 - 1\} + \mathrm{tr}\{U_4U_3^\dagger - 1\} + \mathrm{tr}\{U_4^\dagger U_3 - 1\} + \mathrm{tr}\{U_3U_1^\dagger - 1\} + \mathrm{tr}\{U_3^\dagger U_1\}]\\
&= -N_c^2(N_c^2-1) + \frac{1}{2}N_c^2(N_c^2-1)[\hat{\mathcal{V}}(z_1,z_2) + \hat{\mathcal{V}}(z_1,z_3) + \hat{\mathcal{V}}(z_2,z_4) + \hat{\mathcal{V}}(z_3,z_4)] \qquad (326)
\end{aligned}$$

where we use the definition (236). For completeness, let us present also $\mathrm{Tr}\{T^aU_1U_3^\dagger T^bU_3U_2^\dagger\}$ in the two-gluon approximation. From the first line in Eq. (324) one easily obtains

$$\begin{aligned}
&\mathrm{Tr}\{T^aU_1U_3^\dagger T^bU_3U_2^\dagger\}\\
&\stackrel{2-\mathrm{gluon}}{=} N_c(N_c^2-1) - \frac{1}{2}N_c(N_c^2-1)[\hat{\mathcal{V}}(z_1,z_2) + \hat{\mathcal{V}}(z_1,z_3) + \hat{\mathcal{V}}(z_2,z_3)] \qquad (327)
\end{aligned}$$

To describe the conformal operator (230) we need one more trace (which eventually drops out after the integration of the operator (230) over z_3 with the weight $\frac{z_{12}^2}{z_{13}^2z_{23}^2}$)

$$\begin{aligned}
&\mathrm{Tr}\{T^aT^bU_1T^{a'}T^{b'}U_2^\dagger\}U_3^{aa'}U_4^{bb'} = \frac{1}{4}\Big[\mathrm{tr}\{U_1U_3^\dagger\}\mathrm{tr}\{U_3U_2^\dagger\}\mathrm{tr}\{U_2U_4^\dagger\}\mathrm{tr}\{U_4U_1^\dagger\}\\
&+\mathrm{tr}\{U_3U_1^\dagger\}\mathrm{tr}\{U_2U_3^\dagger\}\mathrm{tr}\{U_4U_2^\dagger\}\mathrm{tr}\{U_4^\dagger U_1\} + \mathrm{tr}\{U_3U_1^\dagger U_4U_2^\dagger\}\mathrm{tr}\{U_1U_4^\dagger U_2U_3^\dagger\}\\
&+\mathrm{tr}\{U_2U_4^\dagger U_1U_3^\dagger\}\mathrm{tr}\{U_1^\dagger U_3U_2^\dagger U_4\} + \mathrm{tr}\{U_4U_3^\dagger\}\mathrm{tr}\{U_3U_1^\dagger U_2U_4^\dagger U_1U_2^\dagger\}\\
&+\mathrm{tr}\{U_1U_2^\dagger\}\mathrm{tr}\{U_4^\dagger U_3U_1^\dagger U_4U_3^\dagger U_2\} + \mathrm{tr}\{U_4^\dagger U_3\}\mathrm{tr}\{U_2U_1^\dagger U_4U_2^\dagger U_1U_3^\dagger\}
\end{aligned}$$

$$+\mathrm{tr}\{U_1^\dagger U_2\}\mathrm{tr}\{U_3U_4^\dagger U_1U_3^\dagger U_4U_2^\dagger\} - \mathrm{tr}\{U_1U_3^\dagger\}\mathrm{tr}\{U_2U_1^\dagger U_4U_2^\dagger U_3U_4^\dagger\}$$
$$-\mathrm{tr}\{U_1^\dagger U_4\}\mathrm{tr}\{U_1U_3^\dagger U_2U_4^\dagger U_3U_2^\dagger\} - \mathrm{tr}\{U_3U_2^\dagger\}\mathrm{tr}\{U_1^\dagger U_2U_4^\dagger U_1U_3^\dagger U_4\}$$
$$-\mathrm{tr}\{U_2U_4^\dagger\}\mathrm{tr}\{U_1U_2^\dagger U_3U_1^\dagger U_4U_3^\dagger\} - \mathrm{tr}\{U_3U_1^\dagger\}\mathrm{tr}\{U_4U_3^\dagger U_2U_4^\dagger U_1U_2^\dagger\}$$
$$-\mathrm{tr}\{U_4U_2^\dagger\}\mathrm{tr}\{U_1^\dagger U_3U_4^\dagger U_1U_3^\dagger U_2\} - \mathrm{tr}\{U_2U_3^\dagger\}\mathrm{tr}\{U_1U_4^\dagger U_3U_1^\dagger U_4U_2^\dagger\}$$
$$-\mathrm{tr}\{U_4^\dagger U_1\}\mathrm{tr}\{U_2U_3^\dagger U_4U_2^\dagger U_3U_1^\dagger\}\Big] \qquad (328)$$

In the two-gluon approximation this yields

$$\mathrm{Tr}\{T^aT^bU_1T^{a'}T^{b'}U_2^\dagger\}U_3^{aa'}U_4^{bb'} \qquad (329)$$
$$= \frac{1}{2}N_c^2(N_c^2-1) - \frac{1}{4}N_c^2(N_c^2-1)[\hat{\mathcal{V}}(z_1,z_3) + \hat{\mathcal{V}}(z_2,z_3) + \hat{\mathcal{V}}(z_1,z_4) + \hat{\mathcal{V}}(z_2,z_4)]$$

B.5 INTEGRALS

In this section we calculate two basic integrals which we have used for our result. The first one is

$$\frac{z_{12}^2}{z_{13}^2z_{23}^2}\int\frac{d^2z_4}{\pi}\Big[\frac{z_{13}^2}{z_{14}^2z_{34}^2} + \frac{z_{23}^2}{z_{24}^2z_{34}^2} - \frac{z_{12}^2}{z_{14}^2z_{24}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2} = 4\zeta(3)[\delta(z_{23}) - \delta(z_{13})] \qquad (330)$$

The easiest way to prove this at $z_3 \neq z_1, z_2$ is to set $z_2 = 0$ and make an inversion $x \to 1/\tilde{x}$ so the integral (330) reduces to

$$2(\tilde{z}_1 - \tilde{z}_2)^2\int d^2\tilde{z}_4\frac{(\tilde{z}_1 - \tilde{z}_3, \tilde{z}_1 - \tilde{z}_4)}{(\tilde{z}_1 - \tilde{z}_4)^2(\tilde{z}_3 - \tilde{z}_4)^2}\ln\frac{(\tilde{z}_1 - \tilde{z}_3)^2}{(\tilde{z}_1 - \tilde{z}_4)^2} = 0$$

The δ-function terms in the r.h.s. of Eq. (330) can be restored from the formula

$$\int\frac{d^2z_3d^2z_4}{\pi^2}\Big[\frac{(z_{12},z_{23})}{z_{34}^2}\Big(\frac{1}{z_{13}^2z_{24}^2} + \frac{1}{z_{23}^2z_{14}^2}\Big) - \frac{z_{12}^2(z_{12},z_{23})}{z_{13}^2z_{24}^2z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{23}^2z_{14}^2} = 4\zeta(3) \qquad (331)$$

which follows from the integral

$$\frac{1}{\pi}\int d^dzd^dz'\Big[\frac{(x,z)}{(x-z)^2(z-z')^2z'^2} + \frac{(x,z)}{(x-z')^2(z-x')^2z^2}$$
$$-\frac{x^2(x,z)}{(x-z)^2z^2(x-z')^2z'^2}\Big]\ln\frac{(x-z)^2z'^2}{(x-z')^2z^2} =$$
$$= B(\frac{d}{2},\frac{d}{2}-1)B(d-1,\frac{d}{2}-1)\frac{\Gamma(3-d)}{(x^2)^{2-d}}\Big\{3\psi(\frac{d}{2}-1)$$
$$-2\psi(d-2) + 2\psi(1) - 2\psi(2-\frac{d}{2}) - \psi(\frac{d}{2})$$
$$+\frac{4}{d-2}\frac{\Gamma(\frac{d}{2})\Gamma(3\frac{d}{2}-2)\Gamma^2(2-\frac{d}{2})}{\Gamma^2(d-1)\Gamma(3-d)}\Big\} \stackrel{d\to 2}{\to} -4\zeta(3)$$

The second integral is somewhat more tedious

$$\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int\frac{d^2 z_4}{\pi}\Big[\frac{z_{23}^2}{z_{34}^2 z_{24}^2}\ln\frac{z_{12}^2 z_{13}^2 z_{34}^2}{z_{14}^4 z_{24}^2}+\frac{z_{13}^2}{z_{14}^2 z_{34}^2}\ln\frac{z_{13}^4 z_{23}^2}{z_{12}^2 z_{14}^2 z_{34}^2}+\frac{z_{12}^2}{z_{14}^2 z_{24}^2}\ln\frac{z_{14}^2 z_{24}^2}{z_{13}^2 z_{23}^2}\Big]$$
$$= 2\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2}{z_{13}^2}\ln\frac{z_{12}^2}{z_{23}^2} - 8\zeta(3)\delta(z_{13}) \tag{332}$$

First we prove this equation at $z_3 \neq z_1, z_2$. Let us start from the first term in the square brackets in the l.h.s.

$$\int d^d z'\ \frac{z^2}{(z-z')^2 z'^2}\ln\frac{x^2(x-z)^2(z-z')^2}{(x-z')^4 z'^2} = -2\int d^2 z'\ \Big[\frac{1}{(z-z')^2} \tag{333}$$
$$-\frac{1}{z'^2}\Big]\ln\frac{(x-z')^2}{x^2} - \ln\frac{x^2}{(x-z)^2}\int d^2 z'\ \frac{z^2}{(z-z')^2 z'^2} - 4\int d^2 z'\ \frac{(z,z-z')}{(z-z')^2 z'^2}\ln\frac{(x-z')^2}{x^2}$$

We get

$$-2\int d^2 z'\ \Big[\frac{1}{(z-z')^2}-\frac{1}{z'^2}\Big]\ln\frac{(x-z')^2}{x^2} - \ln\frac{x^2}{(x-z)^2}\int d^2 z'\ \frac{z^2}{(z-z')^2 z'^2}$$
$$= \lim_{d\to 2}\Big(-2\int d^d z'\ \Big[\frac{1}{(z-z')^2}-\frac{1}{z'^2}\Big]\ln\frac{(x-z')^2}{x^2} - \ln\frac{x^2}{(x-z)^2}\int d^d z'\ \frac{z^2}{(z-z')^2 z'^2}\Big)$$
$$= \lim_{d\to 2} 2\pi\Big[\frac{\Gamma(1-\frac{d}{2})}{((x-z)^2)^{1-\frac{d}{2}}} - \frac{\Gamma(1-\frac{d}{2})}{(x^2)^{1-\frac{d}{2}}} - \frac{\Gamma(2-\frac{d}{2})}{(z^2)^{1-\frac{d}{2}}}\ln\frac{x^2}{(x-z)^2}\Big]B\Big(\frac{d}{2},\frac{d}{2}-1\Big)$$
$$= -\pi\ln\frac{(x-z)^2 x^2}{z^4}\ln\frac{(x-z)^2}{x^2} \tag{334}$$

and

$$-4\int d^d z'\ \frac{(z,z-z')}{(z-z')^2 z'^2}\ln\frac{(x-z')^2}{x^2}$$
$$= \lim_{d\to 2} 4\frac{\pi^{d/2}}{d-2}\frac{\Gamma(d/2)}{\Gamma(d-1)}\Gamma\Big(\frac{d}{2}-1\Big)\Gamma\Big(2-\frac{d}{2}\Big)\Big[1+\Big(\frac{d}{2}-1\Big)\ln z^2$$
$$+\Big(\frac{d}{2}-1\Big)\ln z^2 + \frac{1}{2}\Big(\frac{d}{2}-1\Big)^2\Big(\ln^2\frac{z^2 x^2}{(x-z)^2} - \ln^2\frac{x^4}{(x-z)^2} + \ln^2 x^2\Big)$$
$$-1-\Big(\frac{d}{2}-1\Big)\ln z^2 - \frac{1}{2}\Big(\frac{d}{2}-1\Big)^2\ln^2 z^2\Big] = 2\pi\ln\frac{x^2}{z^2}\ln\frac{(x-z)^2}{x^2} \tag{335}$$

so

$$\int d^d z'\ \frac{z^2}{(z-z')^2 z'^2}\ln\frac{x^2(x-z)^2}{(x-z')^4} = -\pi\ln^2\frac{(x-z)^2}{x^2} \tag{336}$$

Similarly

$$\int d^2 z'\ \Big[-\frac{(x-z)^2}{(x-z')^2(z-z')^2}\ln\frac{x^2(x-z')^2(z-z')^2}{(x-z)^4 z^2} + \frac{x^2}{(x-z')^2 z'^2}\ln\frac{(x-z')^2 z'^2}{(x-z)^2 z^2}\Big]$$

$$= \lim_{d\to 2} 2\pi \frac{\Gamma(2-\frac{d}{2})\Gamma^2(\frac{d}{2})}{(\frac{d}{2}-1)\Gamma(d-1)}\Big\{|x-z|^{d-2}\ln\frac{(x-z)^4z^2}{x^2}$$
$$-2|x-z|^{d-2}\Big[-\frac{1}{d-2}+\ln(x-z)^2-\psi(2-\frac{d}{2})+\psi(\frac{d}{2})$$
$$+\psi(1)-\psi(d-1)\Big]+2|x|^{d-2}\Big[-\frac{1}{d-2}+\ln x^2-\psi(2-\frac{d}{2})+\psi(\frac{d}{2})$$
$$+\psi(1)-\psi(d-1)\Big]-|x|^{d-2}\ln(x-z)^2z^2\Big\}$$
$$= \pi\ln^2\frac{x^2}{(x-z)^2}+2\pi\ln\frac{x^2}{z^2}\ln\frac{x^2}{(x-z)^2} \tag{337}$$

and therefore

$$\int d^2z'\,\Big[-\frac{z^2}{(z-z')^2z'^2}\ln\frac{(x-z')^4}{x^2(x-z)^2}$$
$$+\frac{(x-z)^2}{(x-z')^2(z-z')^2}\ln\frac{(x-z)^4z^2}{x^2(x-z')^2(z-z')^2}+\frac{x^2}{(x-z')^2z'^2}\ln\frac{(x-z')^2z'^2}{(x-z)^2z^2}\Big]$$
$$= 2\pi\ln\frac{x^2}{z^2}\ln\frac{x^2}{(x-z)^2} \tag{338}$$

It is easy to see however that some sort of δ-function contribution to the r.h.s. is necessary. If we integrate the l.h.s. over z with the weight $\frac{x^2}{(x-z)^2z^2}$ we get zero because of the antisymmetry of the integrand with respect to $z\leftrightarrow z'$. On the other hand, the integral of the r.h.s. does not vanish because

$$\int d^2z\,\frac{x^2}{(x-z)^2z^2}\ln\frac{x^2}{z^2}\ln\frac{x^2}{(x-z)^2} = 4\pi\zeta(3) \tag{339}$$

To fix the coefficients in front of possible δ-function contributions $\sim \delta(z)$ and/or $\delta(x-z)$ we calculate the integral of the l.h.s. of Eq. (338) with the trial function $\frac{(x,x-z)}{z^2(x-z)^2}$. We get

$$\frac{1}{\pi^2}\int d^2zd^2w\frac{(x,x-z)}{z^2(x-z)^2}\Big[\frac{(x-z)^2\ln\frac{z^2x^2}{w^4}}{(x-w)^2(w-z)^2}-\frac{x^2}{(x-w)^2w^2}\ln\frac{z^2(x-z)^2}{w^2(x-w)^2}$$
$$+\frac{z^2}{(z-w)^2w^2}\ln\frac{z^4(x-z)^2}{x^2w^2(z-w)^2}\Big]$$
$$= \lim_{d\to 2}B(\frac{d}{2},\frac{d}{2}-1)B(d-1,\frac{d}{2}-1)\frac{\Gamma(3-d)}{(x^2)^{2-d}}\Big\{\psi(1)+\psi(\frac{d}{2})+\psi(3-d)$$
$$+\psi(3\frac{d}{2}-2)-2\psi(d-1)-2\psi(2-\frac{d}{2})\Big\} = -4\zeta(3) \tag{340}$$

where $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$. It is clear now that the result for the integral in the l.h.s. of the formula (332) should be as cited in the r.h.s. of Eq. (332) - it satisfies both the Eq. (340) and the requirement that the integral of the l.h.s. with the trial function $\frac{x^2}{z^2(x-z)^2}$ vanishes.

We will need one more integral which is obtained by antisymmetrization of Eq. (332) with respect to $z_1 \leftrightarrow z_2$

$$\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int \frac{d^2 z_4}{\pi}\ \Big[\frac{z_{13}^2}{z_{14}^2 z_{34}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} - \frac{z_{23}^2}{z_{34}^2 z_{24}^2}\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}\Big] = 4\zeta(3)[\delta(z_{13}) - \delta(z_{23})] \tag{341}$$

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