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A. V. Radyushkin
Old Dominion University, aradyush@odu.edu

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Structure of parton quasi-distributions and their moments

A.V. Radyushkin [a,b,*]

[a] *Physics Department, Old Dominion University, Norfolk, VA 23529, USA*
[b] *Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA*

ABSTRACT

We discuss the structure of the parton quasi-distributions (quasi-PDFs) $Q(y, P_3)$ outside the "canonical" $-1 \leq y \leq 1$ support region of the usual parton distribution functions (PDFs). Writing the y^n moments of $Q(y, P_3)$ in terms of the combined $x^{n-2l}k_\perp^{2l}$-moments of the transverse momentum distribution (TMD) $\mathcal{F}(x, k_\perp^2)$, we establish a connection between the large-$|y|$ behavior of $Q(y, P_3)$ and large-$k_\perp^2$ behavior of $\mathcal{F}(x, k_\perp^2)$. In particular, we show that the $1/k_\perp^2$ hard tail of TMDs in QCD results in a slowly decreasing $\sim 1/|y|$ behavior of quasi-PDFs for large $|y|$ that produces infinite y^n moments of $Q(y, P_3)$. We also relate the $\sim 1/|y|$ terms with the $\ln z_3^2$-singularities of the Ioffe-time pseudo-distributions $\mathfrak{M}(\nu, z_3^2)$. Converting the operator product expansion for $\mathfrak{M}(\nu, z_3^2)$ into a matching relation between the quasi-PDF $Q(y, P_3)$ and the light-cone PDF $f(x, \mu^2)$, we demonstrate that there is no contradiction between the infinite values of the y^n moments of $Q(y, P_3)$ and finite values of the x^n moments of $f(x, \mu^2)$.

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1. Introduction

In the original Feynman approach [1], the parton distribution functions (PDFs) $f(x)$ were introduced as the infinite momentum $P_3 \to \infty$ limit of distributions in the longitudinal $k_3 = yP_3$ momentum of partons. These distributions basically coincide with the quasi-PDFs $Q(y, P_3)$ introduced more recently by X. Ji [2].

As is well-known, "x" of the parton model corresponds to the ratio $x = k_+/P_+$ of the light-cone-plus components of the parton and hadron momenta, rather than the ratio $y = k_3/P_3$ of their third Cartesian components. However, in the $P_3 \to \infty$ limit, the difference between y and x disappears.

In the parton model, $f(x)$'s were treated as $k_\perp$-integrals of more detailed $f(x, k_\perp)$ distributions that involve also the transverse momentum $k_\perp$. From the start, it was understood by Feynman that the $P_3 \to \infty$ limit exists only if $f(x, k_\perp)$ rapidly decreases with $k_\perp$, so that the integral over $k_\perp$ does not diverge. This happens, in particular, in the theories/models with transverse momentum cut-off $k_\perp \lesssim \Lambda$, e.g., in super-renormalizable models, but not in QED and other renormalizable field theories.

One may ask two natural questions. First, why the shape of $Q(y, P_3)$ for a finite P_3 differs from that of $f(x)$? Second, how does the shape of $Q(y, P_3)$ convert into that of $f(x)$ when

$P_3 \to \infty$? A qualitative answer is that the parton's longitudinal momentum $k_3 = yP_3$ comes from two sources: from the motion of the hadron as a whole (xP_3) and from a Fermi motion of quarks inside the hadron, so that $(y - x)P_3 \sim 1/R_{hadr}$. As $P_3 \to \infty$, the role of the $y - x \sim 1/P_3 R_{hadr}$ fraction decreases and $Q(y, P_3) \to f(x)$.

In this picture, the $(y - x)P_3$ part has the same physical origin as the parton's transverse momentum. Hence, one should be able to relate quasi-PDFs to the transverse momentum distributions (TMDs) and quantify the difference between $Q(y, P_3)$ and $f(x)$ in terms of TMDs $f(x, k_\perp)$.

An important point is that the components of $k_\perp$ may take any values from $-\infty$ to ∞, even when the distribution in $k_\perp$ is mostly restricted to a limited range, like in a Gaussian $e^{-k_\perp^2/\Lambda^2}$. Similarly, the $(y - x)P_3$ part of the k_3-distribution may take any values. As a result, $Q(y, P_3)$ formally has the $-\infty < y < \infty$ support region, though possibly with a rapid decrease (say, like $e^{-y^2 P_3^2/\Lambda^2}$) for large y.

In other words, for a finite P_3, there is no requirement that the fraction y is smaller than 1 or positive. Even in a fast-moving hadron, there is some probability that a parton moves in the opposite direction, and hence, that some other parton has the momentum k_3 larger than P_3. Still, with increasing P_3, the chances for fractions outside the [0, 1] segment decrease rapidly, reflecting the large-$k_\perp$ dependence of the relevant TMD $f(x, k_\perp)$.

When $Q(y, P_3) \sim e^{-y^2 P_3^2/\Lambda^2}$, one may consider y^n moments of quasi-PDFs $Q(y, P_3)$ calculated over the whole $-\infty < y < \infty$ axis

and study their relation to the x^n moments of the light-cone PDFs $f(x)$.

Still, starting with the first papers [2,3] on quasi-PDFs, it was known that the simplest perturbative calculations produce $\sim 1/|y|$ behavior for quasi-PDFs at large $|y|$. Such a behavior reflects a slow $\sim 1/k_\perp^2$ decrease of the perturbative hard tail of TMDs in renormalizable theories. Clearly, if $Q(y, P_3) \sim 1/|y|$, then even the zeroth moment of $Q(y, P_3)$ diverges, so that it apparently makes no sense to consider y^n moments of $Q(y, P_3)$. Since the standard procedures of extracting PDFs from the lattice [4–6] do not involve a calculation of the moments, the divergence of these moments did not attract much attention.

However, recently it was argued by G.C. Rossi and M. Testa [7,8] that the divergence of the y^n moments of $Q(y, P_3)$ poses a serious problem for extraction of PDFs from lattice QCD simulations. The basic claim is that the infinite values of $\langle y^n \rangle_Q$ quasi-PDF moments are in conflict with the finite values of the $\langle x^n \rangle_f$ moments of the usual PDFs.

Irrespectively of these claims, we find that the structure of quasi-PDFs $Q(y, P_3)$ outside the central $|y| \leq 1$ region is an interesting problem on its own, and we analyze it in the present paper. Our study is based on the concept [9] of the Ioffe-time pseudo-distributions (pseudo-ITDs) $\mathcal{M}(\nu, -z^2)$. They are basically the matrix elements $M(z, p)$ of bilocal operators $\sim \phi(0)\phi(z)$ treated as functions of the Lorentz invariants, the Ioffe time $\nu = -(zp)$ [10, 11] and the invariant interval z^2. [Our convention is to add "pseudo" to the name of distributions defined for nonzero z^2, and skip it for their light-cone analogs.]

While $\mathcal{M}(\nu, -z^2)$ does not involve momentum fraction variables like y and x, quasi-PDFs $Q(y, P_3)$ and pseudo-PDFs $\mathcal{P}(x, z_3^2)$ may be obtained [9] from $\mathcal{M}(\nu, -z^2)$ as Fourier transforms. The advantage of this approach is a direct use of the coordinate representation that greatly simplifies further considerations of pseudo-PDFs, TMDs and quasi-PDFs.

Furthermore, as we will show, the fact that the quasi-PDFs $Q(y, P_3)$ do not vanish outside the $|y| \leq 1$ region, is directly connected with the presence of a non-trivial z_3^2- dependence in the relevant pseudo-PDFs $\mathcal{P}(x, z_3^2)$.

The paper is organized as follows. In Section 2, we start with reminding the definition of the pseudo-ITDs and their relation to pseudo-PDFs, quasi-PDFs and TMDs. We write a formal $1/P_3^{2l}$ series expansion for the $\langle y^n \rangle_Q$ moments of the quasi-PDFs in terms of the combined $\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}$ moments of TMDs $\mathcal{F}(x, k_\perp^2)$. In the case of "very soft" TMDs, i.e., those vanishing faster than any inverse power of $k_\perp^2$ for large $k_\perp$, this expansion allows to study $\langle y^n \rangle_Q$ moments (which are finite in this case) and their relation to $\langle x^n \rangle_f$ moments of the usual PDFs.

In Section 3, we study the consequences of having a hard $\sim 1/k_\perp^2$ tail of TMDs, present in renormalizable theories, including QCD. In this case, the combined $\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}$ moments diverge. For $l = 0$, one has a logarithmic divergence corresponding to the usual perturbative evolution. For $l \geq 1$, one faces power divergences equivalent to those discussed in Refs. [7,8]. We show that they reflect the slowly $\sim 1/|y|$ decreasing perturbative contributions to $Q(y, P_3)$. We also show that the $|y| > 1$ parts of $Q(y, P_3)$ are generated by the z_3^2-dependence of the pseudo-PDFs $\mathcal{P}(x, z_3^2)$. In Section 4, we study possible forms of the z_3^2-dependence.

In Section 5, we discuss the matching relations connecting the lightcone PDFs to pseudo-ITDs and quasi-PDFs. According to the operator product expansion (OPE), the reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ is given by the $\overline{\text{MS}}$-ITD $\mathcal{I}(\nu, \mu^2)$ plus $\mathcal{O}(\alpha_s)$ perturbative contribution that contains the $\sim \ln z_3^2$ term responsible for the slowly varying $\sim 1/|y|$ terms in the $|y| > 1$ part of the quasi-PDF $Q(y, P_3)$. The latter, hence, is given by the $\overline{\text{MS}}$-PDF $f(x, \mu^2)$ plus

$\mathcal{O}(\alpha_s)$ perturbative contribution that contains the slowly varying $\sim 1/|y|$ terms in the $|y| > 1$ part.

Vice versa, $f(x, \mu^2)$ is given by the difference between the lattice quasi-PDF $Q_L(y, P_3)$ and that $\mathcal{O}(\alpha_s)$ perturbatively calculable contribution. This means that the implementation of the matching condition includes a subtraction, though not of the kind discussed by Rossi and Testa in Refs. [7,8]. The final point is that, for large P_3, the quasi-PDF $Q_L(y, P_3)$ must be purely perturbative in the $|y| > 1$ region. Hence, the above difference vanishes outside the $|y| \leq 1$ segment, and the moments of the light-cone PDF $f(x, \mu^2)$ extracted in this way are finite.

Section 6 contains summary and conclusions.

2. Parton distributions

2.1. Ioffe-time distributions and pseudo-PDFs

Defining a parton distribution either in a continuum theory or on the lattice, one starts with a matrix element $\langle p|\phi(0)\phi(z)|p \rangle \equiv M(z, p)$ of a product of two parton fields. We use here simplified scalar notations, since the details of parton spin structure are not central to the concept of parton distributions, and may be added, if needed, at later stages.

By Lorentz invariance, $M(z, p)$ is a function of two scalars, the *Ioffe time* [10,11] $(pz) \equiv -\nu$ and the interval z^2

$$M(z, p) = \mathcal{M}(-(pz), -z^2) . \tag{2.1}$$

As shown in Refs. [12,13], for any contributing Feynman diagram, the Fourier transform of $\mathcal{M}(\nu, -z^2)$ with respect to the Ioffe time ν has the $-1 \leq x \leq 1$ support, familiar from the studies of the usual parton densities,

$$\mathcal{M}(\nu, -z^2) = \int_{-1}^{1} dx \, e^{ix\nu} \, \mathcal{P}(x, -z^2) . \tag{2.2}$$

When z is on the light cone, $z^2 = 0$, we deal with the ordinary (or light-cone) parton distributions

$$\mathcal{M}(\nu, 0) = \int_{-1}^{1} dx \, e^{ix\nu} \, f(x) . \tag{2.3}$$

Thus, $\mathcal{P}(x, 0) = f(x)$, and the function $\mathcal{P}(x, -z^2)$ generalizes the concept of PDFs onto the case of non-lightlike intervals z. Following Ref. [9], we will refer to it as *pseudo-PDF* or parton *pseudo-distribution* function.

2.2. Quasi-PDFs

The simplest example of a spacelike interval is obtained when just one component is nonzero, $z = \{0, 0, 0, z_3\}$. Choosing $p = (E, \mathbf{0}_\perp, P)$, one can define the quasi-PDF [2] as the Fourier transform of $M(z_3, P)$ with respect to z_3

$$Q(y, P) = \frac{P}{2\pi} \int_{-\infty}^{\infty} dz_3 \, e^{-iyPz_3} \, M(z_3, P). \tag{2.4}$$

Combining Eqs. (2.2) and (2.4) gives a relation between the quasi-PDF $Q(y, P)$ and the pseudo-PDF $\mathcal{P}(x, z_3^2)$ corresponding to the $z = z_3$ separation

$$Q(y, P) = \frac{P}{2\pi} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dz_3 \, e^{-i(y-x)Pz_3} \, \mathcal{P}(x, z_3^2) . \tag{2.5}$$

One can see that though the pseudo-PDFs have the $-1 \leq x \leq 1$ support, the quasi-PDFs $Q(y, P)$ are defined for all real y.

Another observation is that if the pseudo-PDF does not depend on z_3^2, i.e., if $\mathcal{P}(x, z_3^2) = f(x)$, then the quasi-PDF $Q(y, P)$ does not depend on P, and $Q(y, P) = f(y)$.

Thus, it is the dependence of $\mathcal{P}(x, z^2)$ (or, equivalently, of $\mathcal{M}(\nu, z^2)$) on z^2 that determines the deviation of quasi-PDFs from PDFs. In particular, it generates the parts of $Q(y, P)$ outside the PDF support region $|y| \leq 1$.

In QCD and other renormalizable theories, the presence of the z_3^2-dependence is unavoidable, because $\mathcal{M}(\nu, z^2)$ has $\sim \ln z^2$ contributions for small z^2. Furthermore, these terms are singular in the $z^2 \to 0$ limit which complicates the definition of the light-cone PDFs.

2.3. Transverse momentum dependent PDFs

A very convenient way to parametrize the z^2-dependence for a space-like z is provided by a description in terms of the transverse momentum dependent PDFs. Using again $p = (E, \mathbf{0}_\perp, P)$ and choosing z that has only z_- and $z_\perp = \{z_1, z_2\}$ components, while $z_+ = 0$, we have $\nu = -p_+ z_-$, $z^2 = -z_\perp^2$, and the TMD is defined by

$$\mathcal{P}(x, z_\perp^2) = \int d^2 k_\perp \, e^{-i(k_\perp \cdot z_\perp)} \mathcal{F}(x, k_\perp^2) \,. \tag{2.6}$$

Due to the rotational invariance, this TMD depends on $k_\perp^2$ only. Integrating over the angle between $\mathbf{k}_\perp$ and $\mathbf{z}_\perp$ gives

$$\mathcal{P}(x, z_\perp^2) = 2\pi \int_0^\infty dk_\perp \, k_\perp J_0(k_\perp z_\perp) \, \mathcal{F}(x, k_\perp^2) \,, \tag{2.7}$$

where J_0 is the Bessel function.

Now recall that $\mathcal{P}(x, -z^2)$ is a function defined in a covariant way by Eq. (2.2). This implies that this *TMD representation* [14] may be written for a general spacelike z. One should just change $z_\perp \to \sqrt{-z^2}$ and $k_\perp \to k$ in Eq. (2.7). In particular, one may take $z = \{0, 0, 0, z_3\}$, i.e., choose z in the purely longitudinal direction, and write

$$\mathcal{P}(x, z_3^2) = 2\pi \int_0^\infty dk \, k \, J_0(k z_3) \, \mathcal{F}(x, k^2) \,. \tag{2.8}$$

While $\mathcal{F}(x, k^2)$ is a function that coincides with the TMD, one does not need to specify a "transverse" plane and treat k as the magnitude of a 2-dimensional momentum in that plane.

2.4. Support mismatch

Using the TMD parametrization (2.8) in the quasi/pseudo-PDF relation (2.5), and expanding $J_0(k z_3)$ into the Taylor series, we get a formal $1/P^{2l}$ expansion for the quasi-PDF $Q(y, P)$

$$Q(y, P) = \sum_{l=0}^\infty \int d^2 k_\perp \frac{k_\perp^{2l}}{4^l P^{2l} (l!)^2} \frac{\partial^{2l}}{\partial y^{2l}} \mathcal{F}(y, k_\perp^2) \,. \tag{2.9}$$

To shorten formulas, we have switched here back $k \to k_\perp$ in the *notation* for the integration variable of the TMD representation (2.8), and also wrote the resulting $2\pi k_\perp dk_\perp$ as $d^2 k_\perp$. We can do this because the TMD $\mathcal{F}(x, k_\perp^2)$ does not depend on angles. As a matter of caution, we repeat again that k or $k_\perp$ should be understood simply as scalar variables of the TMD parametrization. There is no need to specify in which plane $k_\perp$ is.

According to Eq. (2.5), the quasi-PDF $Q(y, P)$ has the $-\infty < y < \infty$ support region. However, the quasi-PDF $Q(y, P)$ in Eq. (2.9) is given by a sum of terms involving the TMD $\mathcal{F}(y, k_\perp^2)$ that has the $-1 \leq y \leq 1$ support. The explanation of the apparent discrepancy is that the innocently-looking derivatives of $\mathcal{F}(y, k_\perp^2)$ in the expansion (2.9) may generate an infinite tower of singular functions like $\delta(y)$, $\delta(y \pm 1)$ and their derivatives. To this end, we recollect that, even when a function $f(y)$ has a nontrivial support Ω (say, $-1 \leq y \leq 1$), one may formally represent it by a series

$$f(y) = \sum_{N=0}^\infty \frac{(-1)^N}{N!} M_N \, \delta^{(N)}(y) \tag{2.10}$$

over the functions $\delta^{(N)}(y)$ with an apparent support at one point $y = 0$ only. Here, M_N are the moments of $f(y)$,

$$M_N = \int_\Omega dy \, y^N f(y) \,. \tag{2.11}$$

Hence, the support mismatch may be explained by the fact that the delta-function and its derivatives are integration prescriptions (mathematical distributions) rather than ordinary functions. But this also means that while the difference between $Q(y, P)$ and $f(y)$ is formally given by a series in powers of $1/P^2$, its coefficients are not the ordinary functions of y.

2.5. Moments of very soft quasi-PDFs

In order to get relations involving usual functions, one may wish to integrate the equations in which these distributions enter, e.g., to take moments. Indeed, the derivatives disappear if we calculate the y^n moments $\langle y^n \rangle_Q$ of the quasi-PDFs

$$\langle y^n \rangle_Q \equiv \int_{-\infty}^\infty dy \, y^n Q(y, P)$$

$$= \sum_{l=0}^{[n/2]} \frac{n!}{(n - 2l)!(l!)^2} \frac{\langle x^{n-2l} k_\perp^{2l} \rangle_\mathcal{F}}{4^l P^{2l}} \,, \tag{2.12}$$

where $\langle x^{n-2l} k_\perp^{2l} \rangle_\mathcal{F}$ are the combined moments of TMDs

$$\langle x^{n-2l} k_\perp^{2l} \rangle_\mathcal{F} \equiv \int_{-1}^1 dx \, x^{n-2l} \int d^2 k_\perp \, k_\perp^{2l} \mathcal{F}(x, k_\perp^2) \,. \tag{2.13}$$

In the case of *very soft distributions* which vanish faster than any power of $1/k_\perp^2$ for large $k_\perp$, all the combined moments $\langle x^{n-2l} k_\perp^{2l} \rangle_\mathcal{F}$ are finite and Eq. (2.12) tells us that then $\langle y^n \rangle_Q$ differs from $\langle x^n \rangle_f$ by terms having the $(\langle k_\perp^2 \rangle_\mathcal{F} / P^2)^l$ structure.

Two lowest moments $n = 0$ and $n = 1$ do not involve $l \geq 1$ terms. For the normalization integral, Eq. (2.12) gives

$$\int_{-\infty}^\infty dy \, Q(y, P) = \int_{-1}^1 dx \int d^2 k_\perp \, \mathcal{F}(x, k_\perp^2)$$

$$= \int_{-1}^1 dx \, f(x) \,. \tag{2.14}$$

Thus, the area under $Q(y, P)$ does not change with P and is equal to the area under $f(x)$, the phenomenon corresponding to the quark number conservation.

Similarly, the first y-moment is given by

$$\int_{-\infty}^{\infty} dy \, y \, Q(y, P) = \int_{-1}^{1} dx \, x \, f(x) \,, \tag{2.15}$$

which corresponds to the momentum conservation. These two sum rules have been originally derived in our paper [12].

3. Hard part

3.1. Perturbative evolution

In renormalizable theories (most importantly, in QCD, but also in models with Yukawa gluons), i.e., theories having a dimensionless coupling constant g, the perturbative corrections to all "twist-2" $\phi(0)\phi(z)$-type correlators (in QCD we have in mind $\bar{\psi}(0)\Gamma\psi(z)$ quark and $G(0)G(z)$ gluon operators) unavoidably contain terms that are logarithmic in z^2 for small z^2, e.g. $\sim g^2 \ln(-z^2 m^2)$ at one-loop level, m being some infrared cut-off. For DIS structure functions $F(x_{\rm Bj}, Q^2)$, such terms produce the logarithms $\sim g^2 \ln(Q^2/m^2)$ generating their perturbative evolution [15–17] with Q^2.

For pseudo-PDFs $\mathcal{P}(x, z_\perp^2)$ that define TMDs through Eq. (2.6), the $\sim g^2 \ln(-z^2 m^2)$ terms result in the $\sim g^2 \ln(z_\perp^2 m^2)$ contributions for small $z_\perp$. The 2-dimensional Fourier transform with respect to $z_\perp$ converts such terms into contributions with a $\sim 1/k_\perp^2$ "hard tail" for large $k_\perp$ (see, e.g., Ref. [12]).

Thus, in general, TMDs $\mathcal{F}(x, k_\perp^2)$ in renormalizable theories *must* have a hard part that has the $1/k_\perp^2$ behavior for large $k_\perp$. For non-singlet densities in QCD, it is given at one loop by

$$\mathcal{F}^{\rm hard}(x, k_\perp^2) = \frac{\Delta(x)}{\pi k_\perp^2} \,, \tag{3.1}$$

where $\Delta(x)$ is obtained from the PDF $f^{\rm soft}(x)$ (corresponding to a primordial soft TMD) through

$$\Delta(x) = \frac{\alpha_s}{2\pi} C_F \int_x^1 \frac{du}{u} B(u) f^{\rm soft}(x/u) \,, \tag{3.2}$$

and $B(u)$ is the Altarelli–Parisi (AP) evolution kernel [15]

$$B(u) = \left[\frac{1+u^2}{1-u} \right]_+ \,. \tag{3.3}$$

Since the parton densities $f(x, \mu^2)$ are obtained from the TMDs by a $d^2k_\perp$ integration, the well-known logarithmic evolution of $f(x, \mu^2)$ with a cut-off μ, is a direct consequence of the $1/k_\perp^2$ behavior of the relevant TMDs in QCD.

If one calculates the combined moments $\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}$ for the hard term, they diverge, starting from the lowest $l = 0$ moment in $k_\perp^2$. In the $l = 0$ case, the divergence is logarithmic. Let us see that it just reflects the fact that the quasi-PDF $Q(y, P)$ for large P in this case has the logarithmic perturbative evolution with respect to P^2. To begin with, we write the hard part in the coordinate representation

$$\mathcal{P}^{\rm hard}(x, z_3^2) = -\ln(z_3^2 m^2) \Delta(x) \,, \tag{3.4}$$

where m is some infrared regularization scale. Rewriting the quasi-PDF definition in terms of the pseudo-ITD as

$$Q(y, P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \, e^{-iy\nu} \mathcal{M}(\nu, \nu^2/P^2) \tag{3.5}$$

we find that

$$\mathcal{M}^{\rm hard}(\nu, \nu^2/P^2) = -\frac{\alpha_s}{2\pi} C_F \ln(\nu^2 m^2/P^2)$$
$$\times \int_0^1 du \, B(u) \int_{-1}^1 dx \, e^{-iux\nu} f^{\rm soft}(x) \,. \tag{3.6}$$

As a result, the hard part of the quasi-PDF $Q(y, P)$ has the evolution $\ln P^2$ part

$$Q^{\rm ev}(y, P) = \ln(P^2/m^2) \Delta(y) \,. \tag{3.7}$$

Comparing with Eq. (3.1), we conclude that, calculating the evolution part, one should cut-off the $k_\perp$ integral at $|k_\perp| \sim P$ values, so that it is given by

$$Q^{\rm ev}(y, P) = \int_{|k_\perp| \lesssim P} d^2k_\perp \, \mathcal{F}^{\rm hard}(y, k_\perp^2) \simeq \ln(P^2) \Delta(y) \,. \tag{3.8}$$

3.2. Two lowest moments

As we have seen, for very soft distributions, the $n = 0$ and $n = 1$ moments of quasi-PDF $Q(y, P)$ coincide with these moments of the PDF $f(x)$. To proceed with the hard part, we use

$$\int_0^1 dx \, x^n \Delta(x) = -\frac{\alpha_s}{2\pi} C_F \gamma_n \int_0^1 d\zeta \, \zeta^n f^{\rm soft}(\zeta) \,, \tag{3.9}$$

where γ_n's are related to anomalous dimensions of operators with n derivatives,

$$\gamma_n = -\int_0^1 du \, u^n B(u) \,. \tag{3.10}$$

Thus, for the zeroth moment of $Q^{\rm ev}(y, P)$, the coefficient in front of $\ln P^2$ is proportional to the anomalous dimension γ_0 of the vector current. Since γ_0 vanishes, the area under $Q(y, P)$ does not change with P and is equal to the area under $f(x)$, the phenomenon corresponding to the quark number conservation.

Similarly, the first y-moment of the hard part of $Q(y, P)$ has the $\ln P^2$ part proportional to the anomalous dimension $\gamma_1 = 4/3$ that is nonzero. This reflects the fact that the quark-gluon interactions change the momentum carried by the quarks, and only the total momentum of quarks plus gluons is conserved in the evolution process.

3.3. Higher moments and large-$|y|$ behavior

According to the general formula (2.12), the y^2-moment is given by

$$\langle y^2 \rangle_Q = \langle x^2 \rangle_{\mathcal{F}} + \frac{\langle k_\perp^2 \rangle_{\mathcal{F}}}{2P^2} \,, \tag{3.11}$$

(see also Ref. [18]), where

$$\langle k_\perp^2 \rangle_{\mathcal{F}} = \int_{-1}^1 dx \int d^2k_\perp \, k_\perp^2 \, \mathcal{F}(x, k_\perp^2) \,. \tag{3.12}$$

When $\mathcal{F}(x, k_\perp^2)$ vanishes faster than $1/k_\perp^4$ for large $k_\perp$, the $k_\perp$-integral converges. Then the difference between $\langle y^2 \rangle_Q$ and $\langle x^2 \rangle_{\mathcal{F}}$ decreases as $\langle k_\perp^2 \rangle_{\mathcal{F}}/2P^2$ for large P.[1]

However, for a hard $\sim 1/k_\perp^2$ TMD, the $\langle k_\perp^2 \rangle_{\mathcal{F}}$ integral diverges quadratically. If, by analogy with Eq. (3.8), we would set the upper limit of $k_\perp$ integration to be proportional to P, the $k_\perp^2$-weighted integral (3.12) would be proportional to P^2.

Because of the compensation of the initial $1/P^2$ suppression factor by the P^2 factor resulting from the quadratic divergence of the $k_\perp$-integral, the contribution of the $\langle k_\perp^2 \rangle_{\mathcal{F}}/2P^2$ term does not disappear in the $P \to \infty$ limit. One may also argue that, on the lattice, the upper limit on the $k_\perp$ integral may be set by the lattice spacing. Then, a cut-off for the $k_\perp$ integral at the $\sim 1/a$ value would result in a $\sim 1/a^2 P^2$ contribution.

These worries have been formulated in recent papers by G.C. Rossi and M. Testa [7,8], who warned that one might need to perform a nonperturbative *subtraction* of such terms in lattice calculations. The questions raised in Ref. [7] have been subsequently addressed in Ref. [20] by X. Ji et al., who stated that the extraction of PDFs does not involve taking moments of quasi-PDFs. It was also argued that the moments of quasi-PDFs do not exist because $Q(y, P)$ decreases as $1/|y|$ for large y. While we agree with these statements in general, we think that the problem deserves a more detailed investigation.

4. Sources of z^2 dependence

As we discussed already, the $|y| > 1$ parts of quasi-PDFs $Q(y, P)$ are generated by the z_3^2-dependence of the ITD $\mathcal{M}(\nu, z_3^2)$. In particular, for large z_3^2, $\mathcal{M}(\nu, z_3^2)$ has a fast decrease with z_3. This reflects a finite size of the system. Such a behavior should appear in any reasonable theory/model used to describe hadrons. The second type of the z_3^2-dependence appears in renormalizable theories. As already mentioned, then $\mathcal{P}(x, -z^2)$ and $\mathcal{M}(\nu, -z^2)$ contain, for small $-z^2$, the terms $\sim \ln(-z^2)$ corresponding to the $\sim 1/k_\perp^2$ hard tail of $\mathcal{F}(x, k_\perp^2)$. The tail is generated by hard gluon exchanges and is proportional to a small parameter $\alpha_s/\pi \sim 0.1$.

Finally, in QCD (and other gauge theories), there is the third source of the z^2-dependence related to some special contributions originating from the gauge link. These contributions vanish on the light cone $z^2 = 0$, but do not vanish for spacelike z^2. Moreover, they contain link-specific UV divergencies, similar to those one encounters in the heavy-quark effective theory (HQET). Let us discuss these types of z^2-dependence.

4.1. Long-distance z^2-dependence

To begin with, $\mathcal{P}(x, z_3^2)$ describes a finite-size system (moreover, a system of confined quarks). Hence, it should rapidly decrease for large z_3, say, like a Gaussian $\sim e^{-z_3^2/R^2}$ or an exponential $\sim e^{-z_3/R}$, where R characterizes the size of the system. A finite size of the system imposes no restrictions on the behavior of $\mathcal{P}(x, z_3^2)$ for small z_3^2. Such a behavior is determined by the short-distance dynamics. In models involving just soft interactions, one would expect that $\mathcal{P}(x, z_3^2)$ is finite in the $z_3 \to 0$ limit, like in the Gaussian and exponential cases. Then one may simply take $z_3 = 0$ in $\mathcal{P}(x, z_3^2)$ to get $f(x)$. In terms of TMDs, soft models usually are chosen to have a Gaussian $e^{-k_\perp^2/\Lambda^2}$ or a power-law $\sim 1/(k_\perp^2 + \Lambda^2)^n$ behavior for large $k_\perp$. If $n > 1$, then the relevant pseudo-PDFs are finite for $z_3^2 = 0$.

4.2. Evolution-related z_3^2-dependence

Since the small-z^2 limit in QCD is perturbative, one would expect that the only singularities of $\mathcal{P}(x, -z^2)$ for $z^2 = 0$ are those generated by perturbative corrections. As already mentioned, at one loop one gets $\sim \alpha_s \ln(-z^2)$ terms. Hence, it makes sense to treat $\mathcal{P}(x, -z^2)$ as a sum of a "primordial" soft part $\mathcal{P}^{\text{soft}}(x, -z^2)$ that has a finite $z^2 \to 0$ limit, and a logarithmically singular *hard part* reflecting the evolution, and generated by hard gluon corrections to the original purely soft function. The same applies to $\mathcal{M}(\nu, -z^2)$.

A singularity at $z^2 = 0$ means that the lightcone object $\mathcal{M}(\nu, -z^2 = 0)$ is a divergent quantity. In perturbative calculations of the lightcone matrix element, the $\ln(-z^2)$ singularities convert into ultraviolet logarithmic divergences. These UV divergences are then additional to the usual UV divergences related to the propagator and vertex renormalization.

Still, as far as z^2 is kept finite, one does not have these additional UV divergences, and does not need to introduce a regularization for the $\bar{\psi}(0) \dots \psi(z)$ operator. One should deal with the usual UV divergences and their renormalization only. Such a renormalization (characterized by some parameter λ) would produce (in a covariant gauge, say) just a trivial $Z_\psi(\lambda/m)$ renormalization factor for the ψ-fields (m being an infrared cut-off, e.g., a mass of the ψ field). This factor is the same whether z is on the light cone or not.

Except for this trivial dependence on the UV cut-off λ, the pseudo-ITDs $\mathcal{M}(\nu, -z^2)$ in a general renormalizable (but non-gauge) theory, depend on ν and z^2 only. The $\ln(-z^2)$ terms are just a particular form of the z^2-dependence, and they do not require any regularization as far as z^2 is finite, which is the case in lattice simulations.

Theoretically, one may take z on the light cone. Then one should regularize the resulting extra UV divergences in some way, e.g., by imposing a momentum cut-off or by incorporating the $\overline{\text{MS}}$ scheme, etc. The resulting *lightcone* ITD $\mathcal{I}(\nu, \mu^2)$

$$\mathcal{I}(\nu, \mu^2) = \int_{-1}^{1} dx \, e^{ix\nu} f(x, \mu^2) \tag{4.1}$$

introduced in Ref. [11] naturally depends on the parameter μ involved in the regularization of these ultraviolet divergences generated by taking $\ln z^2$ for $z^2 = 0$.

4.3. UV singular terms generated by the gauge link

Furthermore, in QCD, the gauge link factor connecting $\bar{\psi}(0)$ and $\psi(z)$ generates contributions that are absent on the light cone, and moreover, are ultraviolet divergent. These divergences may be regularized using, e.g., the Polyakov prescription [21] $1/z^2 \to 1/(z^2 - a^2)$ for the gluon propagator in the coordinate space. Then one finds that, for a fixed UV cut-off a, these terms vanish in the $z_3^2 \to 0$ limit, like $|z_3|/a$ for the linear UV divergence and like $\ln(1 + z_3^2/a^2)$ for the logarithmic one. That is why such terms are invisible on the light cone. Hence, we must make an effort to completely exclude these terms from $\mathcal{M}(\nu, z_3^2)$. We emphasize that we need to *eliminate* the terms invisible in the light-cone limit even if they are UV finite.

As a matter of fact, in QCD they are UV divergent, and this fact has shifted the whole subject to the discussion of the UV divergences. These UV divergences were considered as the main problem in many recent papers [23,22,24,25]. Having UV singularities, one should add the regularization parameter (a in this case) to the argument of the regularized pseudo-ITD: $\mathcal{M}(\nu, -z^2) \to \mathcal{M}(\nu, -z^2; a)$. These UV divergences are similar to those known

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from the HQET studies, and are multiplicatively renormalizable [22–24].

Since the parameter a appears only in the combination z_3/a, the UV-sensitive terms form a factor $Z(z_3^2/a^2)$. As discussed above, this factor is an artifact of having a non-lightlike z, and has nothing to do with the lightcone PDFs. Thus, constructing the latter, we should *exclude* $Z(z_3^2/a^2)$ from the pseudo-ITD $\mathcal{M}(\nu, z_3^2; a)$. In other words, one should build quasi-PDFs from the modified function $Z^{-1}(z_3^2/a^2)\mathcal{M}(\nu, z_3^2; a)$.

By construction, $Z^{-1}(z_3^2/a^2)\mathcal{M}(\nu, z_3^2; a)$ does not have $a \to 0$ UV divergences. However, if the goal is just to remove the divergences, then one may use any combination of the $Z^{-1}(1/\mu_{\mathrm{UV}}^2 a^2) \times \mathcal{M}(\nu, z_3^2; a)$ type for the renormalized ITD. But the result then will have the dependence on the renormalization scale μ_{UV}. The renormalized ITD will also contain the z_3^2-dependence of the $Z(z_3^2/a^2)$-factor, that should be excluded in the construction of the light-cone PDFs. In the approaches of Refs. [23–25], this is done at the final stage, when the matching conditions are applied.

Our point of view is that it is more beneficial to remove the UV divergences together with the associated z_3^2-dependence from the very beginning. This may be done by multiplying $\mathcal{M}(\nu, z_3^2; a)$ with the $Z^{-1}(z_3^2/a^2)$ factor. To do this, one should know the $Z(z_3^2/a^2)$ factor. Another possibility, proposed in our paper [9], is to use the reduced pseudo-ITD

$$\mathfrak{M}(\nu, z_3^2; a) \equiv \frac{\mathcal{M}(\nu, z_3^2; a)}{\mathcal{M}(0, z_3^2; a)} \ . \tag{4.2}$$

Then the UV-sensitive factor $Z(z_3^2/a^2)$ automatically cancels in the ratio (4.2), since it is ν-independent. So, there is no need to know it explicitly. The $Z_\psi(\lambda/m)$ factors reflecting the anomalous dimensions of the ψ fields also cancel in the ratio (4.2). The resulting function has a finite $a \to 0$ limit, which will be denoted by $\mathfrak{M}(\nu, z_3^2)$. This function *does not* depend on *any* UV cut-off or a UV renormalization scale like μ_{UV}.

We may say that $\mathfrak{M}(\nu, z_3^2)$ is a *physical observable*, just like the deep inelastic (DIS) structure functions $W(x_{\mathrm{Bj}}, Q^2)$. The latter depend on the external variables x_{Bj}, Q^2, but do not depend on any ultraviolet cut-off or a renormalization scale μ, even if they are calculated in a renormalizable theory.

A widespread statement is that $W(x_{\mathrm{Bj}}, Q^2)$ describes the hadron at the distance scale $\sim 1/Q$. In this sense, $\mathfrak{M}(\nu, z_3^2)$ and the pseudo-PDF $\mathcal{P}(x, z_3^2)$, by construction, describe a hadron at distance z_3, literally.

Thus, for the reduced ITD $\mathfrak{M}(\nu, z_3^2)$, there are just two sources of the z_3^2-dependence: the long-distance nonperturbative dependence reflecting the finite size of the system, and the short-distance perturbative $\sim \ln z_3^2$ dependence related to the usual perturbative evolution. In this respect, the reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ in QCD has the z_3^2-structure similar to that in non-gauge renormalizable theories, in which we also have just two first types of the z_3^2-dependence.

5. Matching

The relations for the moments, like the formula (3.11) for $\langle y^2 \rangle_Q$, and the general formula (2.12), that have been used in our preceding discussion, are based on the Taylor expansion of $\mathcal{P}(x, z_3^2)$ over z_3^2. Rossi and Testa in Refs. [7,8] also appeal to a Taylor expansion in z_3. The basic reason for using the Taylor expansion is that the z_3-dependence of the matrix element is, in general, unknown. So, a natural idea is to parametrize it through the values of the matrix elements of local operators.

While this may be reasonable in a *very soft* case (in which all the derivatives with respect to z_3^2 exist at $z_3^2 = 0$), it is clear that to use the Taylor expansion at $z_3^2 = 0$ for the *hard* logarithm $\ln z_3^2$ is problematic. Fortunately, the hard contribution also has an advantage: its z_3^2-dependence at small z_3^2 (unlike that of the soft contribution) is known: at one loop it is given by $\ln z_3^2$. Thus, if one needs to find a quasi-PDF corresponding to the $\ln z_3^2$ part of the matrix element, one can do this by simply calculating the Fourier transform of $\ln z_3^2$ dictated by the quasi-PDF definition (2.4) rather than to use a Taylor expansion at a singular point.

5.1. OPE and matching conditions for ITDs

When $\ln(-z^2)$ terms are present, a formal light-cone limit $z^2 \to 0$ is singular. Still, the PDF community wants lattice predictions for *the light cone* PDFs. In the continuum, the singular nature of the $z^2 \to 0$ limit is perceived as an ultraviolet divergence in the Feynman integrals for operators on the light cone. It is worth repeating once more that these UV divergences are just a consequence of our desire to take $z^2 = 0$. As far as z^2 is finite, these divergences are absent.

To work at $z^2 = 0$, we need to arrange an UV cut-off for these hand-made divergences. Using, say, the dimensional regularization and $\overline{\mathrm{MS}}$ scheme, one would define the light-cone ITD (4.1) $\mathcal{I}(\nu, \mu^2)$. Its connection to the pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ is given by the operator product expansion. At one loop in QCD, we have [20, 25–27]

$$\mathfrak{M}(\nu, z_3^2) = \mathcal{I}(\nu, \mu^2) - \frac{\alpha_s}{2\pi} C_F \int_0^1 du\, \mathcal{I}(u\nu, \mu^2)$$

$$\times \left\{ B(u) \left[\ln\left(z_3^2 \mu^2 \frac{e^{2\gamma_E}}{4}\right) + 1 \right] \right.$$

$$\left. + \left[4\frac{\ln(1-u)}{1-u} - 2(1-u) \right]_+ \right\} + \mathcal{O}(z_3^2) \ . \tag{5.1}$$

The OPE tells us that, for small z_3^2, the dependence of $\mathfrak{M}(\nu, z_3^2)$ on z_3^2 *must* be given by the $\ln z_3^2$ term on the right-hand side. Hence, to get the light-cone ITD $\mathcal{I}(\nu, \mu^2)$ from, say, lattice calculations of $\mathfrak{M}(\nu, z_3^2)$, one should *subtract* from the lattice pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ its perturbative $\ln z_3^2$ part present in the r.h.s. of Eq. (5.1). For an appropriately chosen/fitted α_s, the result of such a subtraction should be z_3^2-independent. Such a procedure of extracting $\mathcal{I}(\nu, \mu^2)$ from the lattice data of Ref. [28] was described in our Ref. [27].

5.2. Matching conditions for quasi-PDFs

Multiplying Eq. (5.1) by $Pe^{-iyz_3 P}$ and integrating over z_3, we get a relation between the quasi-PDF $Q(y, P)$ (obtained from the reduced pseudo-ITD) and the light cone PDF $f(x, \mu^2)$. It has the following structure

$$Q(y, P) = f(y, \mu^2) - \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} f(y/u, \mu^2)$$

$$\times \left\{ B(u) \ln\left(\mu^2/P^2\right) + C(u) \right\}$$

$$+ \frac{\alpha_s}{2\pi} C_F \int_{-1}^1 dx\, f(x, \mu^2) L(y, x) + \mathcal{O}(1/P^2) \ , \tag{5.2}$$

where the kernel $L(y, x; P)$ is formally given by

$$
L(y, x) = -\frac{P}{2\pi} \int_0^1 du \, B(u)
$$

$$
\times \int_{-\infty}^{\infty} dz_3 \, e^{-i(y - ux)z_3 P} \ln(z_3^2 P^2) . \tag{5.3}
$$

It involves the Fourier transform of $\ln z_3^2$ and, for large P, it is *the only* perturbative term that produces contributions in the $|y| > 1$ region. Eq. (5.2) tells us that the quasi-PDF $Q(y, P)$ *must* have $\mathcal{O}(\alpha_s)$ contributions in the $|y| > 1$ region. In actual lattice calculations it is desirable (though challenging) to try to check if the lattice quasi-PDF in the $|y| > 1$ region is indeed close to the convolution of the fitted PDF with the L-kernel.

For large P, the soft contributions disappear from the $|y| > 1$ region, and the perturbative terms are the only ones remaining for $|y| > 1$. This means that extracting the PDF $f(y, \mu^2)$ from the lattice data for $Q(y, P)$, one deals with the combination, the "reduced" quasi-PDF

$$
\widetilde{Q}(y, P) \equiv Q(y, P) - \frac{\alpha_s}{2\pi} C_F \int_{-1}^1 dx \, f(x, \mu^2) L(y, x) , \tag{5.4}
$$

that vanishes in the $|y| > 1$ region for large P (provided that we trust perturbative QCD!). We may say that the $f \otimes L$ contribution *cancels* the perturbative slow-decreasing terms of the $|y| > 1$ part of $Q(y, P)$. After that, all the remaining terms in Eq. (5.2) have the $|y| \leq 1$ support.

In other words, the process of getting $\overline{\mathrm{MS}}$ PDFs from quasi-PDFs involves a *subtraction* of the perturbative $|y| > 1$ contributions generated by the $\ln z_3^2$ term.

5.3. Hard part of quasi-PDFs

An evident observation from the study of the hard contribution is that the quasi-PDFs do not simply convert into the usual PDFs in the large-P limit. They convert into PDFs only in the case of *soft* TMDs and quasi-PDFs generated from them.

When the hard part is included, $Q(y, P)$ contains the terms that are not present in the lightcone PDFs and which are, moreover, finite (for a fixed α_s) in the $P \to \infty$ limit. Such terms appear both in the "canonical" $-1 \leq y \leq 1$ region and, most importantly, outside it. The presence of such terms was known since the first papers on quasi-PDFs [2,3].

In the context of pseudo-PDFs, these terms are generated by the Fourier transform of the $\ln z_3^2$ hard term. In the momentum representation, $\ln z_3^2$ (equivalent to $\ln z_\perp^2$) corresponds to the $1/k_\perp^2$ behavior, which needs some infrared regularization. Let us choose the mass-type modification $1/k_\perp^2 \to 1/(k_\perp^2 + m^2)$. Then $\ln(z_3^2) \to -2K_0(z_3 m)$, and we have (see Ref. [26])

$$
Q^{\mathrm{hard}}(y, P) = C_F \frac{\alpha_s}{2\pi} \int_{-1}^1 \frac{dx}{|x|} R(y/x, m^2/x^2 P^2) \, f^{\mathrm{soft}}(x) , \tag{5.5}
$$

where the kernel $R(\eta, m^2/P^2)$ is given by

$$
R(\eta; m^2/P^2) = \int_0^1 du \, \frac{B(u)}{\sqrt{(\eta - u)^2 + m^2/P^2}} . \tag{5.6}
$$

In lattice extractions, the real part of the pseudo-ITD corresponds to an even function of y, while the imaginary part corresponds to an odd function of y. Hence, in both cases, it is sufficient to consider positive y only. For η, we need then to analyze three regions, $\eta < 0$, $0 \leq \eta \leq 1$ and $\eta > 1$.

In the central $0 \leq \eta \leq 1$ region, the $P \to \infty$ limit is singular, reflecting the presence of the evolution $\sim \ln P^2/m^2$ term (3.7). There are also terms [26]

$$
R^{\mathrm{middle}}(\eta) = \frac{1 + \eta^2}{1 - \eta} \ln[4\eta(1 - \eta)]
$$

$$
+ \frac{3/2}{1 - \eta} + 4\frac{\ln(1 - \eta)}{1 - \eta} - 1 + 2\eta \tag{5.7}
$$

that are independent of P in the $P \to \infty$ limit. For $|y| > 1$, we can neglect m^2/P^2 in the $P \to \infty$ limit and get

$$
Q^{\mathrm{hard,out}}(y, P \to \infty) = \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{dx}{x} R(y/x; 0) \, f^{\mathrm{soft}}(x) , \tag{5.8}
$$

with the kernel $R(\eta; 0) \equiv R(\eta)$ specified by

$$
R(\eta) = \int_0^1 \frac{du}{|\eta - u|} B(u) . \tag{5.9}
$$

At first sight, one would expect a $\sim 1/|\eta|$ behavior for large $|\eta|$ from Eq. (5.9). However, the $1/|\eta|$ term is accompanied by the integral of $B(u)$ which vanishes because of the plus-prescription structure of $B(u)$. This is also the reason why γ_0 in Eq. (3.10) vanishes. Hence, in the region $\eta > 1$, we can write the kernel as a series in $1/\eta$ starting with $n = 1$,

$$
R(\eta)|_{\eta > 1} = -\sum_{n=1}^{\infty} \frac{\gamma_n}{\eta^{n+1}} , \tag{5.10}
$$

or, in a closed form [26],

$$
R(\eta)|_{\eta > 1} \equiv R_>(\eta) = \frac{1 + \eta^2}{\eta - 1} \ln\left(\frac{\eta - 1}{\eta}\right) + \frac{3}{2(\eta - 1)} + 1 . \tag{5.11}
$$

Similarly, for negative values, we have the expansion

$$
R(\eta)|_{\eta < -1} = \sum_{n=1}^{\infty} \frac{\gamma_n}{\eta^{n+1}} , \tag{5.12}
$$

and a closed-form expression [26]

$$
R(\eta)|_{\eta < 0} \equiv R_<(\eta) = \frac{1 + \eta^2}{1 - \eta} \ln\left(\frac{1 - \eta}{-\eta}\right) + \frac{3}{2(1 - \eta)} - 1 . \tag{5.13}
$$

5.4. Large-$|y|$ behavior in QCD

According to Eq. (3.10), we have $\gamma_1 = 4/3$. Thus, the asymptotic behavior for large $|\eta|$ is given by

$$
R(\eta; 0)|_{|\eta| \gg 1} = -\frac{4}{3} \frac{\mathrm{sgn}(\eta)}{\eta^2} + \mathcal{O}(1/\eta^3) . \tag{5.14}
$$

The $\sim \mathrm{sgn}(\eta)/\eta^2$ behavior of $R(\eta)$ translates into the $\sim \mathrm{sgn}(y)/y^2$ behavior of the quasi-PDF $Q(y, P)$ for large values of $|y|$. As a result, the y^0 moment of $Q(y, P)$ converges for large $|y|$, while further moments involve divergences, in agreement with observations made in Sect. 3.2. In particular, the y^2 moment involves

a linear divergence. If $B(u)$ would not have the plus-prescription property, the divergence would be quadratic. This agrees with the estimate made in Sect. 3.3.

Hence, the divergences of the y^n integrals correspond to the presence of the P-independent terms $\sim 1/y^2$ in the hard part of the quasi-PDFs $Q(y, P)$ outside of the $0 \le y \le 1$ region.

As we discussed, the $\ln z_3^2$ part of the pseudo-ITDs contributes slowly-decreasing ($\sim 1/y$ or $\sim 1/y^2$) terms into the $|y| > 1$ part of quasi-PDFs. It is these terms that lead to the divergence of the y^n moments of the quasi-PDFs $Q(y, P)$.

5.5. Large-P matching

These terms are not eliminated by just taking the $P \to \infty$ limit. However, they disappear when one extracts $f(y, \mu^2)$ using the matching condition (5.2). Namely, we have

$$
f(y, \mu^2) = \widetilde{Q}(y, P) + \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} f(y/u, \mu^2)
$$
$$
\times \left\{ B(u) \ln\left(\mu^2/P^2\right) + C(u) \right\} + \mathcal{O}(1/P^2) . \quad (5.15)
$$

Since both the $\mathcal{O}(1/P^2)$ soft part and the $\widetilde{Q}(y, P)$ combination of Eq. (5.4) vanish for $|y| > 1$ in the $P \to \infty$ limit, Eq. (5.15) resolves the problem of the support mismatch between $f(y, \mu^2)$ and $Q(y, P)$. As a result, one can calculate the y^n moments of the light-cone PDFs $f(y, \mu^2)$ using Eq. (5.15) without getting divergences in its right-hand side.

As already noted, if we separate quasi-PDFs corresponding to the real $[Q_-(y, P)]$ and imaginary $[Q_+(y, P)]$ parts of the ITD, it is sufficient to consider positive y only. Using the fact that perturbative part of $\widetilde{Q}(y, P)$ vanishes outside the $|y| \le 1$ region, we may write the iterative solution of Eq. (5.15) for $y > 0$ as

$$
f_\mp(y, \mu^2) = Q_\mp(y, P)\,\theta(0 \le y \le 1)
$$
$$
- \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{dx}{x} \left[Q_\mp(x, P) - Q_\mp(y, P) \right]
$$
$$
\times \left[\theta(x \ge y) \left\{ \frac{1 + y^2/x^2}{1 - y/x} \left(\ln\left[4y(x - y)\frac{P^2}{\mu^2} \right] - 1 \right) \right. \right.
$$
$$
\left. \left. + \frac{3/2}{1 - y/x} + 1 \right\} + \theta(x \le y) R_>(y/x) \pm R_<(-y/x) \right]
$$
$$
+ \mathcal{O}(1/P^2) . \quad (5.16)
$$

Here the function $f_-(y)$ corresponds to the real part of the ITD and is given by $q(y) - \bar{q}(y)$, while $f_+(y)$ corresponds to the imaginary part of the ITD and is given by $q(y) + \bar{q}(y)$. The kernels $R_>(\eta), R_<(\eta)$ are given by Eqs. (5.11) and (5.13). The third line of Eq. (5.16) comes from $R^{\text{middle}}(\eta)$ of Eq. (5.7) and terms from Eq. (5.1). All the terms explicitly written in Eq. (5.16) involve quasi-PDFs in the $y < 1$ region only. The $y > 1$ part of $\widetilde{Q}(y, P)$ is included in $\mathcal{O}(1/P^2)$ term and vanishes in the $P \to \infty$ limit.

We remind that the starting point for the derivation of Eq. (5.16) is based on Eqs. (5.1) and (5.2). Hence, Eq. (5.16) applies to quasi-PDFs built from the *reduced* pseudo-ITDs (4.2).

6. Summary and conclusions

In this paper, we discussed a specific feature of the quasi-PDFs $Q(y, P_3)$ in which they differ from the usual PDFs $f(x)$, namely, the presence of terms outside the $|y| \le 1$ region.

In a model with a transverse momentum cut-off, such terms disappear in the $P \to \infty$ limit. However, in renormalizable theories, including QCD, one has $|y| > 1$ terms persisting (for a fixed α_s) even in the $P \to \infty$ limit. These terms have a perturbative origin that may be traced to the $\ln z_3^2$ singularities of the generating matrix element $\langle p|\bar{\psi}(0) \dots \psi(z_3)|p\rangle$.

Since one *knows* that such terms, absent in the light-cone PDFs $f(x)$, *must* be present in the quasi-PDFs $Q(y, P_3)$, one should just subtract them from $Q(y, P_3)$ obtained on the lattice. The resulting "reduced" quasi-PDF $\widetilde{Q}(y, P)$ for large P has support in the canonical region $|y| \le 1$ only. On a formal level, such a subtraction is automatically provided by implementing the matching conditions.

Eq. (5.16), that is given at the end of the paper, provides an explicit expression for the lightcone PDF $f(y, \mu^2)$ involving the quasi-PDF $Q(y, P)$ in the $|y| \le 1$ region. Hence, in actual lattice PDF extractions, one may ignore the $|y| > 1$ region altogether and operate with $Q(y, P)$ obtained in the $|y| \le 1$ region only.

A related practical question is if the complications with the $|y| > 1$ region may be avoided? Indeed, according to the OPE (5.1), the reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$, a function directly "coming out of the computer box", may be used, without intermediaries, to extract the lightcone ITDs $\mathcal{I}(\nu, \mu^2)$. The latter are the Fourier transforms of the lightcone PDFs $f(x, \mu^2)$, the functions that have the canonical $|x| \le 1$ support. Such an approach has been already applied in the exploratory lattice calculation [28] and in the construction [27] of $\overline{\text{MS}}$ ITD $\mathcal{I}(\nu, \mu^2)$ based on its results.

References

[1] R.P. Feynman, Photon–hadron interactions, W.A. Benjamin, Reading MA, 1972, 282 pp.
[2] X. Ji, Phys. Rev. Lett. 110 (2013) 262002.
[3] X. Xiong, X. Ji, J.H. Zhang, Y. Zhao, Phys. Rev. D 90 (1) (2014) 014051.
[4] H.W. Lin, J.W. Chen, S.D. Cohen, X. Ji, Phys. Rev. D 91 (2015) 054510.
[5] C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, K. Jansen, F. Steffens, C. Wiese, Phys. Rev. D 92 (2015) 014502.
[6] Y.Q. Ma, J.W. Qiu, Phys. Rev. Lett. 120 (2) (2018) 022003.
[7] G.C. Rossi, M. Testa, Phys. Rev. D 96 (1) (2017) 014507.
[8] G. Rossi, M. Testa, Phys. Rev. D 98 (5) (2018) 054028.
[9] A.V. Radyushkin, Phys. Rev. D 96 (3) (2017) 034025.
[10] B.L. Ioffe, Phys. Lett. B 30 (1969) 123.
[11] V. Braun, P. Gornicki, L. Mankiewicz, Phys. Rev. D 51 (1995) 6036.
[12] A. Radyushkin, Phys. Lett. B 767 (2017) 314.
[13] A.V. Radyushkin, Phys. Lett. B 131 (1983) 179.
[14] A. Radyushkin, Phys. Lett. B 770 (2017) 514.
[15] G. Altarelli, G. Parisi, Nucl. Phys. B 126 (1977) 298.
[16] V.N. Gribov, L.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438, Yad. Fiz. 15 (1972) 781.
[17] Y.L. Dokshitzer, Sov. Phys. JETP 46 (1977) 641, Zh. Eksp. Teor. Fiz. 73 (1977) 1216.
[18] W. Broniowski, E. Ruiz Arriola, Phys. Rev. D 97 (3) (2018) 034031.
[19] C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, F. Steffens, C. Wiese, PoS LATTICE 2016 (2016) 151.
[20] X. Ji, J.H. Zhang, Y. Zhao, Nucl. Phys. B 924 (2017) 366.
[21] A.M. Polyakov, Nucl. Phys. B 164 (1980) 171.
[22] T. Ishikawa, Y.Q. Ma, J.W. Qiu, S. Yoshida, Phys. Rev. D 96 (9) (2017) 094019.
[23] X. Ji, J.H. Zhang, Y. Zhao, Phys. Rev. Lett. 120 (11) (2018) 112001.
[24] J. Green, K. Jansen, F. Steffens, Phys. Rev. Lett. 121 (2) (2018) 022004.
[25] T. Izubuchi, X. Ji, L. Jin, I.W. Stewart, Y. Zhao, Phys. Rev. D 98 (5) (2018) 056004.
[26] A.V. Radyushkin, Phys. Lett. B 781 (2018) 433.
[27] A. Radyushkin, Phys. Rev. D 98 (2018) 014019.
[28] K. Orginos, A. Radyushkin, J. Karpie, S. Zafeiropoulos, Phys. Rev. D 96 (9) (2017) 094503.