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Characterization of the Local Lipschitz Constant

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A characterization, using polynomials introduced by A. V. Kolushov, is given for the local Lipschitz constant for the best approximation operator in Chebyshev approximation from a Haar set. The characterization is then used to study the existence of uniform local Lipschitz constants. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let X be a closed subset of $[a, b]$ with at least $n+1$ points, and let $C(X)$ denote the space of continuous real valued functions on X endowed with the uniform norm $\|\cdot\|$. Let H_n denote a Haar set of dimension n, and let the best approximation of f in $C(X)$ from H_n be $B_n(f)$. Freud [12] showed that B_n is Lipschitz continuous at f, i.e., B_n has a global Lipschitz constant at f, $\lambda_n(f)$, defined by

$$\lambda_n(f) = \sup\{\|B_n(f) - B_n(g)\|/\|f-g\|,\ g \in C(X),\ g \neq f\}. \tag{1.1}$$

For $\delta > 0$, define

$$\lambda_n(f, \delta) = \sup\{\|B_n(f) - B_n(g)\|/\|f-g\| : 0 < \|f-g\| < \delta,\ g \in C(X)\}. \tag{1.2}$$

Then the local Lipschitz constant, introduced in [1], is defined by

$$\lambda_n^l(f) = \lim_{\delta \to 0^+} \lambda_n(f, \delta). \tag{1.3}$$

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It was introduced because of the difficulty of dealing with $\lambda_n(f)$ and because one is interested in the behavior of B near f. Knowledge of $\lambda_n^l(f)$ will help us know better the properties of the best approximation operator, one of the more difficult problems in approximation theory.

The behavior of the global Lipschitz constant and its dependence on X, n, and f have often been investigated (see, e.g., [7, 11, 16–19]). The local Lipschitz constant has been investigated in [1, 2, 8], and the relationship between the local and global Lipschitz constants has been explored in [1–3].

A significant difficulty in studying the local and global Lipschitz constants has been the lack of characterizations of them in a general setting. The Lipschitz constants are intimately related to the strong unicity constant $\gamma_n(f)$; in particular, it is known [10] that $\lambda_n(f) \leqslant 2/\gamma_n(f)$. Studies of the strong unicity constants have frequently been based on characterizations of them, some of which appear in [4, 5, 21]. These characterizations show that the strong unicity constant depends only on the signs of the error $e_n(f)$ on the extreme points $E_n(f)$, where $E_n(f)$ is defined by

$$E_n(f) = \{x : |e_n(f)(x)| = \|e_n(f)\|\}, \tag{1.4}$$

$e_n(f)$ is defined by

$$e_n(f)(x) = (f - B_n(f))(x), \tag{1.5}$$

and $|E_n(f)|$ denotes the cardinality of $E_n(f)$.

In the special case when $E_n(f)$ has minimal cardinality, i.e., $|E_n(f)| = n+1$, a characterization of $\lambda_n^l(f)$ was given in [1] and then used to study $\lambda_n^l(f)$ and the Gateaux derivative of B_n.

Theorem 1 characterizes the local Lipschitz constant without any assumption on $|E_n(f)|$, using polynomials introduced by A. V. Kolushov [14]. The characterization shows that $\lambda_n^l(f)$, like the strong unicity constant, depends only on $E_n(f)$ and the signs of $e_n(f)$ at the points of $E_n(f)$. Theorem 1 also generalizes the result in [1] about the relationship between $\lambda_n^l(f)$ and the Gateaux derivative $D_f B_n$. A Kroo [13] showed that the right Gateaux derivative of B_n at f in the direction ϕ,

$$D_f B_n^+(\phi) = \lim_{t \to 0^+} \frac{B_n(f + t\phi) - B_n(f)}{t} \tag{1.6}$$

exists for each ϕ in $C(X)$. With the assumption of minimal cardinality, i.e., $|E_n(f)| = n+1$, it was shown in [1] that

$$\lambda_n^l(f) = \|D_f^+ B_n\|. \tag{1.7}$$

Theorem 1 shows that (1.7) holds with no restriction on $|E_n(f)|$.

The problem of determining whether a set $S \subseteq C(X)$ has a uniform Lipschitz constant, i.e., $\sup\{\lambda_n(f) : f \in S\}$ is bounded, was first studied in [11] and further studied in [6, 16–18]. The characterization of $\lambda_n^l(f)$ in Theorem 1 and results from [7] are used to study the boundedness of $\sup\{\lambda_n^l(f) : f \in S\}$ in Section 3. Unlike the situation for strong unicity constants, Example 2 shows that a set can have a uniform local Lipschitz constant while not having a uniform global Lipschitz constant.

Theorem 2 determines precisely when a set of functions, each with the minimal number of extreme points, has a uniform local Lipschitz constant. Theorems 3, 4, and 5 generalize Theorem 2. Theorem 7 and Example 3 give specific families of functions which have uniform local Lipschitz constants but not uniform strong unicity constants.

Remark. The study of the existence of a uniform global Lipschitz constant for S is affected by the uniform boundedness of the functions in S [11] and by the presence of almost alternation sets. As defined in [7], a sequence $S = \{f_k\}_{k=1}^{\infty}$ does not have an almost alternation set, if whenever a sequence $\{g_k\}_{k=1}^{\infty}$ satisfies $\lim_{k \to \infty} \|f_k - g_k\| = 0$, there is a constant M such that for all $k = 1, ..., d_0(A(f_k), A(g_k)) \leqslant M \, \|f_k - g_k\|$ where $A(f_k) = \{x_i\}_{i=1}^{n+1}$ and $A(g_k) = \{y_i\}_{i=1}^{n+1}$ are any alternation sets for f_k and g_k respectively. Here $d_0(A(f_k), A(g_k)) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|$ when $x_1 \in E^+(f_k)$ and $y_1 \in E^+(g_k)$, or when $x_1 \in E^-(f_k)$ and $y_1 \in E^-(g_k)$; otherwise, $d_0(A(f_k), A(g_k))$ is set equal to $b - a$. The characterization of $\lambda_n^l(f)$ in Theorem 1 shows that it is determined by the extreme points $E_n(f)$, and hence uniform boundedness of $\lambda_n^l(f)$ for f in S is unaffected by almost alternation sets or by the uniform boundedness of S.

2. Kolushov Polynomials

A. V. Kolushov [14] established a representation of $D_f B_n^+(\phi)$ with no restrictions on $|E_n(f)|$ in terms of polynomials $p_n(f, \phi)$, hereafter called Kolushov polynomials.

An alternant $A_n(f)$ of the error function is any set $\{x_0, ..., x_n\} \subseteq E_n(f)$ such that $e_n(f)(x_i) = \gamma(-1)^i \, \|e_n(f)\|$, $i = 0, ..., n$ where $\gamma = \pm 1$.

THEOREM I [14]. *Given ϕ in $C(X)$ and f in $C(X) \setminus H_n$, there exists a unique constant α and a unique polynomial $p_n(f, \phi)$ in H_n such that*

$$(\phi(x) - p_n(f, \phi)(x)) \operatorname{sgn} e_n(f)(x) \leqslant \alpha \tag{2.1}$$

for every x in $E_n(f)$; *in addition, there exists an alternant* $A_{n,\phi}(f)$, *not necessarily unique, on which equality holds in* (2.1). *Furthermore,*

$$D_f B_n^+(\phi) = p_n(f, \phi), \tag{2.2}$$

where if $f \in H_n$, *then* $p_n(f, \phi) = B_n(\phi)$.

Denote $A_{n,\phi}(f)$ by A_ϕ. The Kolushov Lipschitz constant $K\lambda_n(f)$, introduced in [8], is defined by

$$K\lambda_n(f) = \sup\{\|p_n(f, \phi)\| : \|\phi\| \leqslant 1\}. \tag{2.3}$$

Thus,

$$K\lambda_n(f) = \|D_f B_n^+\|. \tag{2.4}$$

It is known [8] that $K\lambda_n(f) = \lambda_n^l(f)$ if X is finite and in general [8] that

$$K\lambda_n(f) \leqslant \lambda_n^l(f). \tag{2.5}$$

Also if $|E_n(f)| = n+1$, so that $E_n(f)$ consists of a single alternant of $e_n(f)$, then $p_n(f, \phi)$ is just the best approximate to ϕ on $E_n(f)$ [7]; in this case it was shown in [1] that $\lambda_n^l(f) = K\lambda_n(f)$. Notice that $K\lambda_n(f)$ depends only on $E_n(f)$ and the sign of the error function $e_n(f)$ on $E_n(f)$.

3. Characterization of $\lambda_n^l(f)$

First observe that for a given function f in $C(X)$, $\lambda_n^l(cf+h) = \lambda_n^l(f)$ for any non-zero constant c and any function h in H_n. Since the unit ball of a Haar space H_n is compact it has a uniform modulus of continuity

$$\omega(\delta) = \sup\{\omega_h(\delta) : h \in H_n,\ \|h\| = 1\}, \tag{3.1}$$

where $\omega(\delta) \downarrow 0$ as $\delta \downarrow 0$. Given any positive integer J and a function f in $C(X)$, let δ_J satisfy $\omega(\delta_J) \leqslant (J\lambda(f))^{-1}$. We require some additional notation. Let

$$\begin{aligned} E_n^+(f) &= \{x : e_n(f) = \|f - B_n(f)\|\}, \\ E_n^-(f) &= \{x : e_n(f) = -\|f - B_n(f)\|\}, \end{aligned} \tag{3.2}$$

and

$$d(f) = \inf\{|x-y| : x \in E_n^+(f),\ y \in E_n^-(f)\}. \tag{3.3}$$

Also, given sets U and V, the density of U in V is defined by

$$\delta(U, V) = \sup_{v \in V} \inf_{u \in U} |u - v|, \tag{3.4}$$

and the Hausdorff distance is

$$\rho(U, V) = \max\{\delta(U, V), \delta(V, U)\}. \tag{3.5}$$

Convergence of sets will refer to the Hausdorff metric.

LEMMA 1. *Let $f \in C(X)$, $\|f\| = 1$, $B_n(f) = 0$, and J be a positive integer. Suppose $g \in C(X)$ and $e_n(g)$ has an alternant $\{y_0, ..., y_n\}$ such that there is an alternant $\{x_0, ..., x_n\}$ of f with $|x_i - y_i| < \delta_J$, $i = 0, ..., n$ and* $\operatorname{sgn} e_n(g)(y_i) = \operatorname{sgn} f(x_i)$, *$i = 0, ..., n$. Then there exists a $\bar{g} \in C(X)$ such that $\{x_0, ..., x_n\}$ is an alternant of $e_n(\bar{g})$,* $\operatorname{sgn} e_n(\bar{g})(x_i) = \operatorname{sgn} f(x_i)$, *$i = 0, ..., n$, $B_n(\bar{g}) = B_n(g)$, and $\|f - \bar{g}\| \leqslant (1 + 1/J)\,\|f - g\|$.*

Proof. For $i = 0, ..., n$ define

$$\bar{g}(x_i) = B_n(g)(x_i) + (g - B_n(g))(y_i). \tag{3.6}$$

We carry out the details for the case $e_n(f)(x_i) > 0$; the proof in the case $e_n(f)(x_i) < 0$ is similar. Thus, let $f(x_i) = 1$ so that $(g - B_n(g))(y_i) = \|e_n(g)\|$. Let $\|f - g\| = \beta$. Then

$$\begin{aligned} |B_n(g)(x_i) - B_n(g)(y_i)| &\leqslant \omega(\delta_J)\,\|B_n(g)\| \\ &\leqslant \omega(\delta_J)\,\lambda_n(f)\,\|f - g\| \\ &\leqslant \beta/J. \end{aligned} \tag{3.7}$$

Hence by (3.6) and since $f(y_i) \leqslant 1 = f(x_i)$ we have

$$\begin{aligned} \bar{g}(x_i) - f(x_i) &= g(y_i) - f(x_i) + B_n\,g(x_i) - B_n\,g(y_i) \\ &\leqslant f(y_i) + \beta - f(x_i) + \beta/J \\ &\leqslant \beta(1 + 1/J). \end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned} \bar{g}(x_i) - f(x_i) &= B_n\,g(x_i) + \|e_n(g)\| - f(x_i) \\ &= B_n(g)(x_i) - g(x_i) + \|e_n(g)\| + g(x_i) - f(x_i) \\ &\geqslant g(x_i) - f(x_i) \\ &\geqslant -\beta, \end{aligned} \tag{3.9}$$

and hence by (3.7) and (3.8), it follows that

$$|\bar{g}(x_i) - f(x_i)| \leqslant (1 + 1/J)\,\beta.$$

The inequality

$$\max\{f(x)-(1+1/J)\beta,\ B_n(g)(x)-\|e_n(g)\|\}$$
$$\leqslant \min\{f(x)+(1+1/J)\beta,\ B_n(g)(x)+\|e_n(g)\|\}$$

holds for each x in X, and $\bar{g}(x_i)$ is between these bounds for $i=0, ..., n$. So extend $\bar{g}$ continuously by the Tietze extension Theorem [9] to X to satisfy the bounds and the lemma is proven.

THEOREM 1. *If* $f\in C(X)$, *then*

$$\lambda_n^l(f)=K\lambda_n(f)=\|D_f B_n^+\|. \tag{3.10}$$

Proof. Without loss of generality it can be assumed that $\|f\|=1$ and $B_n(f)=0$. By (2.4) and (2.5) it suffices to show that $\lambda_n^l(f)\leqslant K\lambda_n(f)$. Let $0<\varepsilon<\delta_f$. A proof similar to that employed in Lemma 3 of [1] shows that we can assume $\delta(E_n^+(f), E_n^+(g))<\varepsilon$ and $\delta(E_n^-(f), E_n^-(g))<\varepsilon$ for δ small enough. Consequently, if $0<\|f-g\|<\delta$, then for some alternant $\{y_0, ..., y_n\}$ of g, there is an alternant $\{x_0, ..., x_n\}$ of f such that $|x_i-y_i|<\varepsilon$ and $\operatorname{sgn} e_n(g)(y_i)=\operatorname{sgn} e_n(f)(x_i)$, $i=0, ..., n$. By Lemma 1 choose $\bar{g}\in C(X)$ such that $\{x_0, ..., x_n\}$ is an alternant of $\bar{g}$, $\operatorname{sgn} e_n(\bar{g})(x_i)=\operatorname{sgn} e_n(f)(x_i)$, $i=0, ..., n$, $B_n(\bar{g})=B_n(g)$ and $\|f-\bar{g}\|\leqslant(1+1/J)\ \|f-g\|$. If $B_n(g)=0$ we have immediately that $K\lambda_n(f)\geqslant\|B_n(g)\|/((1+J^{-1})\|g-f\|)$. Now assume $B_n(g)=B_n(\bar{g})\neq 0$. Thus $\bar{g}\neq f$ and we can define $\phi=(\bar{g}-f)/\|\bar{g}-f\|$. Then by choosing $p_n(f,\phi)=B_n(\bar{g})/\|\bar{g}-f\|$, (2.1) is satisfied with $\alpha=(\|e_n(\bar{g})\|-1)/\|\bar{g}-f\|$ and there is equality in (2.1) on the alternant $\{x_i\}_{i=0}^n$. Since $\|\phi\|=1$, we have

$$K\lambda_n(f)=\|p_n(f,\phi)\|=\frac{\|B_n(\bar{g})\|}{\|\bar{g}-f\|}\geqslant\frac{\|B_n(g)\|}{(1+1/J)\ \|g-f\|},$$

and hence $\lambda_n^l(f)\leqslant(1+1/J)\ K\lambda_n(f)$ for any positive integer J and the theorem is established.

The results which follow use not only Theorem 1 as stated but also

$$\lambda_n^l(f)\leqslant\sup\{\sup\{\|B_n(\phi, A(f))\| : \|\phi\|\leqslant 1\},\ A(f) \text{ an alternation set of } f\}. \tag{3.11}$$

This holds because $p_n(f,\phi)=B_n(\phi, A_\phi)$ [8].

4. UNIFORM LOCAL LIPSCHITZ CONSTANTS

In this section we restrict our attention to $C[a,b]$. Following [7], a set $S\subseteq C[a,b]$ is said to have a uniform global Lipschitz constant if

$$\sup\{\lambda_n(f) : f\in S\}<\infty.$$

Similarly S is said to have a uniform local Lipschitz constant if

$$\sup\{\lambda_n^l(f) : f \in S\} < \infty.$$

Since $\lambda_n^l(f) \leqslant \lambda_n(f)$, the circumstances in [6] which guarantee a uniform global Lipschitz constant also guarantee a uniform local Lipschitz constant.

Clearly a set S has a uniform local Lipschitz constant if and only if every sequence of functions in S does. Since $\lambda_n^l(f) \leqslant 2$ for every f in H_n we only study uniform local Lipschitz constants for sets S in $C(X)\backslash H_n$. In some of the examples, approximation is from Π_n, the algebraic polynomials of degree n or less. Since there is a uniform strong unity constant and hence a uniform global Lipschitz and a uniform local Lipschitz constant for all of $C(X)$ when $n = 1$ (cf. [6]), we assume henceforth that $n > 1$.

One of the usual special cases considered for a set S in $C(X)$ is when each f in S is assumed to have the minimal number of extreme points. The following theorem, which follows immediately from the results in [7], describes when such a set S has a uniform global Lipschitz constant.

THEOREM II [7]. *Let $S \subseteq C(X)\backslash H_n$ and let $|E_n(f)| = n+1$, $f \in S$. Assume S has no almost alternation sets. Then S has a uniform Lipschitz constant if and only if $|E^0| \geqslant n$ for every cluster point E^0 of $\{E_n(f) : f \in S\}$.*

This result does not hold without the assumptions about the non-existence of almost alternation sets; however, assuming there are no almost alternation sets excludes consideration of differentiable functions, as the following proposition shows.

PROPOSITION 1. *If S has no almost alternation set then no f in S is differentiable in a neighborhood of any point $x \in E(f)$.*

Proof. Fix $f \in S_n$, $\|f\| = 1$, $B_n f = 0$ and let $f_k = f$, $k = 1, \ldots$. Fix $x \in E(f)$, $f(x) = 1$ and assume $x < b$. Define functions $g_k \in C(X)$ by $g_k(x + 1/k) = 1$, $g_k = f$ on $(x + 1/2k,\ x + 2/k)^c$, and $\|g_k - f\| = 1 - f(x + 1/k)$ by the Tietze Extension Theorem. Then if there is no almost alternation set, there exists a constant M such that for all k,

$$d(A(f_k),\ A(g_k)) \leqslant M\ \|f_k - g_k\|.$$

Thus

$$\frac{1}{k} \leqslant M\left(1 - f\left(x + \frac{1}{k}\right)\right),$$

and

$$\frac{-1}{M} \geqslant \frac{f(x+1/k)-f(x)}{1/k},$$

and

$$\frac{-1}{M} \geqslant f'(x).$$

But if f is differentiable in a neighborhood of x, $f'(x)=0$.

Remark. The characterization in the next Theorem for local Lipschitz constants resembles the result which follows from Theorems 4 and 6 in [6]: If every f in S satisfies $|E_n(f)| = n+1$, then S has a uniform strong unicity constant if and only if $|E_n^0| = n+1$ for every cluster point E_n^0 of $\{E_n(f) : f \in S\}$.

THEOREM 2. *Let $S \subseteq C(X) \backslash H_n$ and let every $f \in S$ satisfy $|E_n(f)| = n+1$. Then S has a uniform local Lipschitz constant if and only if $|E_n^0| \geqslant n$ for every cluster point E_n^0 of $\{E_n(f) : f \in S\}$.*

Proof. To prove sufficiency suppose $E_n(f_k) = \{x_{0,k}, ..., x_{n,k}\}$. Set $D_i = \det\{x_{j,k}^s\}$, $0 \leqslant s \leqslant n-1$, $0 \leqslant j \leqslant n$, $j \neq i$, and $D_{i,r}$ results from D_i by replacing $x_{r,k}$ with x $(r \neq i)$. Note that $D_{i,r}(x_i) = (-1)^{r+i+1} D_r$. Then we have for the Cline polynomials

$$q_i(x) = \sum_{\substack{r=0 \\ r \neq 1}}^{n} (-1)^r D_{i,r}(x)/D_i \tag{4.1}$$

and

$$q_i(x_i) = (-1)^{i+1} \sum_{\substack{r=0 \\ r \neq i}}^{n} D_r/D_i. \tag{4.2}$$

Thus $|q_i(x)| \leqslant M/D_i$ and

$$\frac{|g_i(x)|}{|+|q_i(x_i)|} \leqslant M\left(\sum_{r=0}^{n} D_r\right)^{-1} \leqslant M_1 \tag{4.3}$$

because at least one of the D_r's is bounded away from zero. Thus by Theorem 1 in [2] we are done.

To prove necessity let $E_n(f_k) = \{x_{0,k}, ..., x_{n,k}\}$, $x_{j,k} \to x^*$, $j = i, i+1, i+2$ but $\lambda_n^l(f_k) \leqslant M$ $(k = 1, 2, ...)$. Note that $\lambda_n^l(f_k) = \|B_n^k\|$, where B_n^k is the operator of best approximation on $E_n(f_k)$. Let ϕ_k in $C(X)$ be such that $\phi_k(x_{i,k})=1$, $\phi_k(x_{i+1,k})=-1$, $\phi_k(x_{j,k})=0$, $j\neq i$, $i+1$. Then $e_k=\max_j |B_n^k(\phi_k)-\phi_k|\,(x_{j,k})$ satisfies $0\leqslant e_k\leqslant 1$. Since $\lambda_n^l(f_k)=\|B_n^k\|$, the polynomials $B_n(\phi_k)=p_k$ in H_n are uniformly bounded and a subsequence converges to p in H_n. Then we have

$$|p_k(x_{i,k})-1|=e_k=|p_k(x_{i+1,k})+1|,$$

and letting $k\to\infty$, we obtain

$$|p(x^*)-1|=e=|p(x^*)+1|,$$

where a subsequence of $\{e_k\}$ converges to e. Thus $p(x^*)=0$ and $e=1$. On the other hand, $|p_k(x_{i+2,k})|=e_k$ leads to $|p(x^*)|=e=1$, which is a contradiction.

Suppose now three neighboring points do not coalesce but two pairs do. Let $|x_{i,k}-x_{i+1,k}|\to 0$ and $|x_{s,k}-x_{s+1,k}|\to 0$. Define $\phi_k(x_{i,k})=1=-\phi_k(x_{i+1,k})$ and $\phi_k(x_{s,k})=1$ and $\phi_k(x_{s+1,k})=0$. Then as above using $x_{i,k}$ and $x_{i+1,k}$ we obtain $e=1$. Now $|p_k(x_{s,k})-1|=e_k=|p_k(x_{s+1,k})|$ leads to $|p(\bar{x})-1|=e=|p(\bar{x})|$ where $\{x_{s,k}\}\to\bar{x}$, which is a contradiction.

We now use Theorems 1 and 2 to study uniform local Lipschitz constants on sets S which need not contain functions with the minimal number of extreme points. Theorems 3, 4, and 5 generalize Theorem 2.

THEOREM 3. *Let $\{f_k\}$ be a sequence of functions in $C(X)\backslash H_n$. If any sequence of alternants of $\{f_k\}$ has at most two points which coalesce, then*

$$\sup_k \lambda_n^l(f_k)<\infty.$$

Proof. Without loss of generality we can assume that $\|f_k\|=1$ and $B_n(f_k)=0$, $k=1,\ldots$. By Theorem 1, for any f_k there exists a ϕ_k such that

$$\|p_n(f_k,\phi_k)\|>\lambda_n^l(f_k)-1. \tag{4.4}$$

Now $p_n(f_k,\phi_k)=B_n(\phi_k,A(\phi_k))$ [7], where $A(\phi_k)$ is an alternation set for f_k. Define $\bar{f}_k$ by $\bar{f}_k=f_k$ on $A(\phi_k)$, $\|\bar{f}_k\|\leqslant 1$, $E_n(\bar{f}_k)=A(\phi_k)$, and $\bar{f}_k\in C(X)$. Then $B_n(\bar{f}_k)=0$ and by (4.4), for some alternation set $A(\bar{f}_k)$ of $\bar{f}_k$,

$$\begin{aligned}\lambda_n^l(\bar{f}_k)&\geqslant\|p_n(\bar{f}_k,\phi_k)\|=\|B_n(\phi_k,A(\bar{f}_k))\|\\&=\|B_n(\phi_k,A(\phi_k))\|\\&\geqslant\lambda_n^l(f_k)-1.\end{aligned}\tag{4.5}$$

By assumption, at most two points in the sequences of extreme points for $\{\bar{f}_k\}$ can coalesce and $|E(\bar{f}_k)|=n+1$. Hence by Theorem 2, $\sup_n\lambda_n^l(\bar{f}_k)<\infty$ and hence, by (4.5), $\sup_n\lambda_n^l(f_k)<\infty$.

The converse of Theorem 3 does not hold as shown by the following example.

EXAMPLE 1. Let $f_k(x) \in C[0, 3]$ be defined by

$$f_k(x) = \begin{cases} 1 & \text{if } x = 0, 2/k, 2, \\ -1 & \text{if } x = 1/k, 1, 3, \end{cases}$$

and be linear between the knots. Approximate from Π_1. Then $\{0, 1/k, 2/k\}$ is an alternation set for f_k which has $|A^0| = 1$. However, since $\{E(f_k)\} \to E^0 = \{0, 1, 2, 3\}$, which contains a limit alternation set, $\{f_k\}$ has a uniform strong unicity constant and thus a uniform local Lipschitz constant [6].

However, a partial converse of Theorem 3 holds. For the proof of the partial converse and subsequent results, we require the following three definitions.

DEFINITION 1. Given f in $C(X)$, define $L_n(f)(=L_n(E_n(f))$ by letting

$$E_n(f) = E_1 \cup E_2 \cup \cdots \cup E_L,$$

where $L = L_n(f)$ and where $\max\{x: x \in E_i\} < \min\{x: x \in E_{i+1}\}$, $i = 1, ..., L_n(f) - 1$, and for $x \in E_i$, $e_n(f)(x) = \gamma(-1)^i \|e_n(f)\|$, $\gamma = \pm 1$.

DEFINITION 2. For a sequence $\{f_k\}$ in $C(X)$ with $\{E_n(f_k)\} \to E_n^0$, let $L_n(E^0) \equiv L_n(E^0(f_k))$ denote the maximum number of points in E^0 such that $x_1 < \cdots < x_L$ where $x_{\text{odd}} \in E^0$ and $x_{\text{even}} \in E_0^-$, or $x_{\text{odd}} \in E^{0^-}$ and $x_{\text{even}} \in E^{0^+}$.

DEFINITION 3. If $u \in U$ and $v \in V$ implies $u < v$ for closed sets U and V then let the minimal distance between U and V be denoted by

$$d(U, V) = \min\{v: v \in V\} - \max\{u: u \in U\}.$$

THEOREM 4. *Let $S = \{f_k\}$ be a sequence in $C(X)\backslash H_n$, $|E_n(f_k)| < \infty$. If S has a uniform local Lipschitz constant then there exists a sequence of alternation sets which has at most two points which coalesce.*

Proof. Let $E_n(f_k) = E_1^k \cup \cdots \cup E_{L(f_k)}^k$. If $\lim_{k \to \infty} \delta(E_{i+1}^k, E_i^k) \neq 0$ for $i = 1, ..., n$, then clearly there exists an A^0 with $|A^0| = n + 1$. Thus there exists an I such that with $y_I^k = \min\{x \in E_I^k\}$ and $y_{I+1}^k = \min\{x \in E_{I+1}^k\}$ we have $\lim_{k \to \infty}(y_{I+1}^k - y_I^k) = 0$. Note that $\{E_I^k\} \to y_I$. Define ϕ_k in $C(X)$ by

$$\phi_k(x) = \begin{cases} 1 & \text{if } x \in E_I^k, \\ 0 & \text{if } x \in E_j^k, j \neq I, \end{cases}$$

and $\|\phi_k\| \leqslant 1$, where without loss of generality we assume $E_I^k \subset E^+(f_k)$. Let $p_{n,k}(x)$ denote the corresponding Kolushov polynomials and $A(\phi_k) = \{x_1^k, ..., x_{n+1}^k\}$ the associated alternation sets. It is not assumed that x_i^k is in E_i^k. For convenience let p_k denote $p_{n,k}$.

Now

$$(\phi_k - p_k)(y_I^k) \operatorname{sgn} e_n(f_k)(y_I^k) \leqslant \alpha_k \tag{4.6}$$

and

$$(\phi_k - p_k)(y_{I+1}^k) \operatorname{sgn} e_n(f_k)(y_{I+1}) \leqslant \alpha_k.$$

Since $\sup_k \lambda_n^I(f_k) < \infty$ it follows that $\sup_k \|p_k\| < \infty$. Also $|\alpha_k| \leqslant \|\phi_k\| \leqslant 1$ [14], so $\{\alpha_k\}$ is a bounded sequence. Let $\{p_k\}$ denote a subsequence converging to $q(x)$ and $\{\alpha_k\}$ a subsequence converging to α. Letting $k \to \infty$ in (4.6) we obtain

$$1 - q(y_I) \leqslant \alpha$$
$$-q(y_{I+1})(-1) \leqslant \alpha$$

and since $y_I = y_{I+1}$ we have $\alpha \geqslant 1/2$.

For ease of writing assume $x_1^k \in E^-(f_k)$ rather than $E^+(f_k)$. Using subsequences assume $\{x_j^k\} \to x_j$, $j = 1, ..., n+1$. At each $x_j^k \in E_j^k$ when $j \neq I$ we have by (4.6) $-p_k(x_j^k)(-1)^j = \alpha_k$ so letting $k \to \infty$ gives $q(x_j) = (-1)^{j+1}\alpha$, $x_j \in E_I$. Since $\alpha > 0$, $x_j \neq x_{j+1}$ unless $x_j \in E_I^k$ or $x_{j+1}^k \in E_I^k$. If none of the $\{x_j^k\}$ are from E_I^k this gives $n+1$ distinct alternating points $\{x_1, ..., x_{n+1}\}$. Suppose $x_J^k \in E_I^k$. Then we have points

$$x_1 < x_2 < \cdots < x_{J-2}^+ < x_{J-1}^- \leqslant x_J^+ \leqslant x_{J+1}^- < x_{J+2}^+ < \cdots < x_{n+1}, \tag{4.7}$$

where there are $(n+1) - 3 = n-2$ distinct points outside of $\{x_{J-1}, x_J, x_{J+1}\}$. If $I = 1$ then by (4.7) there are at least n points in A^0. (Note that $\{x_1, ..., x_{J-2}, x_{J-1}, x_{J+2}, ..., x_{n+1}\}$ alternates in sign so $L_n(E^0) \geqslant n-1$.) Now the reason A^0 might not satisfy $|A^0| \geqslant n$ is that there could be coalescence among $\{x_{J-1}, x_J, x_{J+1}\}$. Note that although $J = I$ and $x_J^k \in E_I^k$ we do not assume that x_{J-1}^k is in E_{I-1}^k.

Now consider the sets $\{E_1^k, ..., E_I^k, ..., E_{n+1}^k\}$ from which the points in (4.7) are obtained. Let s be the largest index $s > I$, such that $\delta(E_s^k, E_I^k) \to 0$, where we know $s \geqslant I+1$. Then E_s^k splits into two sets $E_{s,1}^k$ and $E_{s,2}^k$, one of which could be empty, such that $\max\{E_{s,1}^k\} < \min\{E_{s,2}^k\}$, $\lim_{k \to \infty}(\max E_{s,1}^k - \max E_I^k) = 0$ and $\lim_{k \to \infty}(\min E_{s,2}^k - \max E_I^k) \neq 0$. If in fact $\lim_{k \to \infty} d(E_J^k, E_{J+1}^k) \neq 0$ then $x_J \neq x_{J+1}$ and $|A_0| \geqslant n$ (since A_0 would contain $n-2$ points outside of $\{x_{J-1}, x_J, x_{J+1})$ and also x_J and x_{J+1}). Now also $\lim_{k \to \infty} d(E_{J-1}^k, E_J^k) = 0$ else again $x_{J-1} \neq x_J$ and $|A^0| \geqslant n$. So define sets $E_{t,1}^k$ and $E_{t,2}^k$ as before with $t < J$, $\lim_{k \to \infty} \delta(E_J^k, E_{t,2}^k) = 0$ and $\lim_{k \to \infty} \delta(E_{t,1}^k, E_J^k) \neq 0$.

Define functions ψ_k by

$$\psi_k(x)=\begin{cases} 1 & \text{if } x\in E_j^k,\ j=J+1, ..., s-1, \\ 1 & \text{if } x\in E_{s,1}^k, \\ 0 & \text{if } x\in E_J^k \cup E_{s,2}^k \cup E_{t,1}^k, \\ -1 & \text{if } x\in E_j^k,\ j=t+1, ..., J-1, \\ -1 & \text{if } x\in E_{t,2}^k, \end{cases}$$

and $\|\psi_k(x)\| \leqslant 1$. Let $A(\psi_k)=\{y_1^k, ..., y_{n+1}^k\}$ and $A_k=E_{t,2}^k \cup E_{t+1}^k \cup \cdots \cup E_{s,1}^k$. Then $\{A_k\} \to y_J$.

Let $\{y_j^k\} \to y_j$. We show that different sequences $\{y_j^k\}$ outside A_k converge to distinct points and hence there are at most $n-2$ of them. As before let p_k denote the Kolushov polynomial associated with ψ_k and assume $\{p_k\} \to \bar{q}$. Let $z_{J-1}^k=\max E_{J-1}^k$ and $z_J^k=\min E_J^k$ so that we know if $\{z_J^k\} \to z_J$ and $\{z_{J-1}^k\} \to z_{J-1}$ then $z_{J-1}=z_J$. We have

$$((\psi_k - p_k) \operatorname{sgn} e_n(f_k))(z_{J-1}^k) \leqslant \beta_k$$
$$((\psi_k - p_k) \operatorname{sgn} e_n(f_k))(z_J^k) \leqslant \beta_k$$

so

$$(-1-p_k(z_{J-1}^k))(-1) \leqslant \beta_k$$
$$-p_k(z_J^k) \leqslant \beta_k,$$

which implies

$$1+\bar{q}(z_J) \leqslant \beta$$
$$\bar{p}(z_J) \leqslant \beta,$$

so $\beta \geqslant 1/2$ and as before this implies that the points outside A_k are distinct.

Then there must be at least three sequences $\{y_j^k\}$ from points in A_k. For these sequences of points from A_k there are at most the following possibilities of alternating: $+, -, +$ or $-, +, -$, and ψ_k must have values $(-1, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, 1)$, $(0, 1, 1)$, or $(1, 1, 1)$. We shall see that none of these ten possibilities can occur. Hence there must have been at least $n-1$ points outside A_k and thus using one point from E_J we obtain $|A^0| \geqslant n$.

If ψ has values -1 next to -1 (as in $(-1, -1, 0)$ or $(-1, -1, -1)$) then we obtain $(1-q)(-1)=\beta$ and $(1-q)(+1)=\beta$ so $\beta=0$ which contradicts $\beta \geqslant 1/2$. If ψ has value 1 next to 1 as in $(1, 1, 0)$ or $(1, 1, 1)$ then

$$(1-\bar{q})(-1)=\beta$$
$$(1-\bar{q})(+1)=\beta$$

so $\beta = 0$, which contradicts $\beta \geqslant 1/2$. Thus we need only consider $(-1, 0, 1)$. Since $E_J^k \subset E^+(f_k)$, the signs must be $-$, $+$, $-$. Thus we obtain

$$(-\bar{q})(+1) = \beta$$
$$(1-\bar{q})(-1) = \beta,$$

so $\beta = -1/2$, which again contradicts $\beta \geqslant 1/2$ and the proof is complete.

In [6] (in [7]) it was shown that if E^0 is a cluster point of $\{E(f) : f \in S\}$ and $|E^0| \leqslant n-1$ then S does not have a uniform strong unicity (respectively uniform global Lipschitz) constant. A similar result follows from the proof of Theorem 4 for local Lipschitz constants and implies the corresponding results for strong unicity and Lipschitz constants.

COROLLARY 1. *If* $S = \{f_k\}$ *is a sequence of functions in* $C(X)\backslash H_n$, $|E_n(f_k)| < \infty$, $\{E_n(f_k)\} \to E_n^0$ and $|E_n^0| \leqslant n-1$, *then* S *does not have a uniform local Lipschitz constant.*

In fact, the proof in [7] can be modified (just redefining p_{kj} to satisfy $\|p_{kj}\| = \delta_k/j$) to show that Corollary 1 holds without the assumption that $|E_n(f_k)| < \infty$.

Now if $\lim_{k \to \infty} \inf \delta(E_i^k, E_{i+1}^k) > 0$ for all i, or if $\lim_{k \to \infty} \inf \delta(E_{i+1}^k, E_i^k) > 0$ for all i, then $\lim_{k \to \infty} \inf p(E_i^k, E_{i+1}^k) > 0$ and $\sup_k \lambda_n(f_k) < \infty$, since it is easy to show that $|A_n^0| = n+1$ for some A_n^0 and hence there is a uniform strong unicity constant.

Next we consider what happens if $\lim_{k \to \infty} \inf p(E_I^k, E_{I+1}^k) = 0$ for some I with an assumption on $L_n(E_n(f_k))$ rather than on $|E_n(f_k)|$. Let $d_i^k = d(E_i^k, E_{i+1}^k)$.

THEOREM 5. *Let* $\{f_k\}$ *be a sequence in* $C(X)\backslash H_n$ *and assume* $L_n(f_k) = n+1, k = 1, \ldots$. *Suppose there exists an index* I *such that* $\lim_{k \to \infty} \inf \rho(E_I^k, E_{I+1}^k) = 0$. *Then* $\sup_k \lambda_n^l(f_k) < \infty$ *if and only if* $\lim_{k \to \infty} d_i^k > 0$.

Proof. To prove sufficiency observe that $\rho(E_I^k, E_{I+1}^k) \to 0$ implies E_I^k and E_{I+1}^k coalesce into a single point. Since $\lim_{k \to \infty} \inf d_i^k > 0$, $i \neq I$, any alternation set will have precisely two points which coalesce. Theorem 3 gives the desired conclusion. Suppose now that $\lim_{k \to \infty} \inf d_J^k = 0$, $J \neq I$ and assume $\sup_k \lambda_n^l(f_k) < \infty$. Let $y_J^k \in E_J^k$ and $y_{J+1}^k \in E_{J+1}^k$ such that $|y_J^k - y_{J+1}^k| \to 0$. Define $\phi_k \in C(X)$ by $\|\phi_k\| \leqslant 1$ and $\phi_k/E_I^k = -\mathrm{sgn}(f_k/E_I^k)$ and $\phi_k/E_i^k = 0$, $i \neq I$. (We assume $J+1 \neq I$. If $J+1 = I$ the proof is similar, just defining $\phi_k/E_{I+1}^k = -\mathrm{sgn}(f_k/E_{I+1}^k)$). Let $p_{k,n}(x)$ be the Kolushov polynomial for ϕ_k with associated alternant $\{x_i^k\}_{i=1}^{n+1}$. By assumption on $E_n(f_k)$, $A(\phi_k)$ contains one point in each set $\{E_i^k\}_{i=1}^{n+1}$. Then there exists α_k such that

$$(\phi_k(x) - p_{k,n}(x)) \text{ sgn } e_n f_k(x) \leqslant \alpha_k, \quad x \in E_n(f_k)$$

and

$$(\phi_k(x_i^k) - p_{k,n}(x_i^k)) \operatorname{sgn} e_n(f_k)(x_i^k) = \alpha_k, \qquad i = 1, ..., n+1.$$

Thus

$$(\phi_k(x_I^k) - p_{k,n}(x_I^k)) \operatorname{sgn} e_n f_k(x_I^k) = \alpha_k \tag{4.8}$$

$$p_{k,n}(x_{I+1}^k) \operatorname{sgn} e_n(f_k)(x_{I+1}^k) = \alpha_k \tag{4.9}$$

$$p_{k,n}(x_J^k) \operatorname{sgn} e_n(f_k)(x_J^k) \leqslant \alpha_k \tag{4.10}$$

$$p_{k,n}(x_{J+1}^k) \operatorname{sgn} e_n(f_k)(x_{J+1}^k) \leqslant \alpha_k. \tag{4.11}$$

Since $\sup_k \|p_{k,n}(x)\| < \infty$ we can assume that there exists a constant α and a polynomial $q(x)$ such that $\{\alpha_k\} \to \alpha$ and $\{p_{k,n}(x)\} \to q(x)$. In (4.8)–(4.11) letting $k \to \infty$ and then adding the resulting (4.8) and (4.9) and then the resulting (4.10) and (4.11), we obtain $-1 = 2\alpha$ and $0 \leqslant 2\alpha$ a contradiction which establishes the result.

Remark. Theorem 4 is a generalization of Theorem 2; to clarify the relationship between Theorem 4 and 5, assume that $L_n(f_k) = n+1$, $k = 1, ...,$ and $\lim_{k\to\infty} \inf \rho(E_I^k, E_{I+1}^k) = 0$. Then it follows that $\lim_{k\to\infty} \inf d_i^k > 0$ for $i \neq I$ if and only if $|A_n^0| \geqslant n$ for every n. Thus Theorem 5 under stronger assumptions than Theorem 4 has a stronger conclusion.

Example 1 in [7] can be modified to provide an example of a set S with a uniform local Lipschitz constant but not a uniform global Lipschitz constant.

EXAMPLE 2. Approximate by Π_1 on $[-1, 1]$. For $0 < \alpha < 1/2$, let $f_\alpha(-1) = f_\alpha(1) = -1$, $f_\alpha(-1+\alpha) = 1$, $f_\alpha(-1+2\alpha) = -1+\alpha^2$, $f_\alpha(0) = 1-\alpha$, and f_α be linear inbetween. Let $g_\alpha(-1) = g_\alpha(1) = -1$, $g_\alpha(-1+\alpha) = 1-\alpha^2$, $g_\alpha(-1+2\alpha) = -1-2\alpha^2$, $g_\alpha(0) = 1-\alpha$ and g_α be linear inbetween. Then $B_1(f_\alpha) = 0$ and $\|B_1(g_\alpha)\|/\|f_\alpha - g_\alpha\| \to \infty$ as $\alpha \to 0^+$. So $\{f_\alpha\}$ does not have a uniform global Lipschitz constant. But Theorem 2 implies that $\{f_\alpha\}$ has a uniform local Lipschitz constant.

It should be observed that, in the absence of almost alternation sets, uniformity for global Lipschitz constants occurs precisely when it occurs for local Lipschitz constants for bounded sets of functions.

THEOREM 6. *Let $S \subseteq C(X) \backslash H_n$ be a bounded set of functions with no almost alternation sets. Then S does not have a uniform local Lipschitz constant if and only if S does not have a uniform global Lipschitz constant (if and only if there is a sequence $\{f_k\} \subseteq S$ such that $\{E_n(f_k)\} \to E_n^0$ and $|E_n^0| \leqslant n-1$).*

Proof. If S does not have a uniform local Lipschitz constant then there exists an A_n^0 with $|A_n^0| \leqslant n-1$ by Theorem 3. The corresponding set E_n^0 can not satisfy $|E_n^0| \leqslant n+1$ since then by Theorem 4 in [7] there is either a limit alternation set or there is a uniform global Lipschitz constant. If $|E_n^0| = n$, then as observed in the proof of Theorem 7 in [7] there is an almost alternation set. Hence $|E_n^0| \leqslant n-1$ and by Theorem 7 in [7] there is no uniform global Lipschitz constant.

If S does not have a uniform global Lipschitz constant then there exists a sequence $\{f_k\} \subseteq S$ such that $\{E_n(f_k)\} \to E_n^0$ and $|E_n^0| \leqslant n-1$. Hence as observed after Corollary 1, there is no uniform local Lipschitz constant.

Finally, we describe some conditions which guarantee there will be a uniform local Lipschitz constant but not a uniform strong unicity constant and give examples.

THEOREM 7. *Let $S \subseteq C[-1, 1]$ be such that $f^{(n+1)}(x) > 0$, $x \in [-1, 1]$ for each $f \in S$. Approximate from Π_n on $[-1, 1]$. Suppose that S contains a sequence $\{f_k\}$ such that*

$$\lim_{k \to \infty} \left(\inf_x f_k^{(n+2)}(x)/f_k^{(n+1)}(x)\right) = \infty.$$

Then S has a uniform local Lipschitz constant and does not have a uniform strong unicity constant.

Proof. First consider the specific class of functions $S = \{g_a(x) : a > 1\}$ where $g_a(x) = 1/(a-x)$. Then $g_a^{(n)}(x) = (n+1)!\,(a-x)^{-(n+2)} > 0$, $x \in [-1, 1]$, and

$$\frac{g_a^{(n+2)}(x)}{g_a^{(n+1)}(x)} = \frac{n+2}{a-x} \leqslant \frac{n+2}{a-1}.$$

Let $x_n(a)$ denote the $(n+1)$st extreme point of $g_a(x)$. As observed in Theorem 4.1 in [20],

$$\lim_{a \to 1+} x_n(a) = 1.$$

Since $x_{n+1}(a) = 1$, E_n^0, the limit extreme point set for S satisfies $|E_n^0| \leqslant n+1$. Also there exist α and β, $\alpha \geqslant \beta > 0$ with

$$\frac{1}{\alpha} < \frac{g_a^{(n+2)}(x)}{g_a^{(n+1)}(x)} < \frac{1}{\beta},$$

and thus by Lemma 2 in [13] the first $n+1$ extreme points of $g_a(x)$ have no coalescence.

Thus $|E_n^0| = n+1$ and S has a uniform local Lipschitz constant by Theorem 2 and no uniform strong unicity constant by Theorem 4 in [6].

Now let S be any class satisfying the hypothesis. For any $a > 1$ there exists $k(a)$ such that

$$\frac{f_{k(a)}^{(n+2)}(x)}{f_{k(a)}^{(n+1)}(x)} \geqslant \frac{n+2}{a-1} \geqslant \sup_x \frac{g_a^{(n+2)}(x)}{g_a^{(n+1)}(x)}.$$

Thus by Theorem 3.1 and Corollary 3.2 in [20], $x_n(f_{k(a)}) > x_n(g_a)$ where $x_n(f_{k(a)})$ is the $(n+1)$st extreme point of $f_{k(a)}$. Thus $|E_n^0(f_{k(a)})| \leqslant n+1$ and as above $|E^0| = n+1$.

EXAMPLE 3. In addition to the above specific class $S = \{1/(a-x)\}_{a>1}$, the class $S = \{e^{mx}\}_{m>1}$ satisfies the conditions of the theorem.

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