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WEAK BIPOLARIZABLE GRAPHS

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We characterize a new class of perfectly orderable graphs and give a polynomial-time recognition algorithm, together with linear-time optimization algorithms for this class of graphs.

1. Introduction

A linear order $<$ on the set of vertices of a graph G is *perfect* in the sense of Chvátal [3] if no induced path with vertices a, b, c, d and edges ab, bc, cd has $a<b$ and $d<c$. Graphs which admit a perfect order are termed *perfectly orderable.*

Recognizing perfectly orderable graphs in polynomial time seems to be a difficult problem. Quite naturally, this motivated the study of particular classes of perfectly orderable graphs. Such classes have been studied by Golumbic, Monma and Trotter [7], Chvátal, Hoang, Mahadev, and de Werra [4], Hoang and Khouzam [9], and Preissmann, de Werra and Mahadev [12].

Recently, Hertz and de Werra [8] proposed to call a graph G *bipolarizable* if G admits a linear order $<$ on the set V of its vertices such that $b<a$ and $c<d$ whenever $\{a, b, c, d\}$ induces a path in G with edges ab, bc, cd.

They characterize bipolarizable graphs by forbidden subgraphs and prove that both bipolarizable graphs and their complements are perfectly orderable.

In this paper we first define and characterize the class of weak bipolarizable graphs which properly contain the class of bipolarizable graphs. This characterization can be exploited to obtain a polynomial-time recognition algorithm for weak bipolarizable graphs. Finally, given a weak bipolarizable graph G, we show how an algorithm of Rose, Tarjan and Lueker [13] can be used to obtain efficiently a linear order on the vertices of G. As soon as this is done, an algorithm of Chvátal, Hoang, Mahadev and de Werra [4] can be used to optimize weak bipolarizable graphs in linear time.

Given a graph G, we shall let $\bar{G}$ denote the complement of G; if x is a vertex in G, then $N_G(x)$ stands for the set of all the vertices in G which are adjacent to x; $N'_G(x)$ denotes the set of all the vertices in G which are adjacent to x in $\bar{G}$ (whenever possible, we shall write simply $N(x)$ and $N'(x)$). We shall let G_H stand for the subgraph of G induced by H; $C_k(P_k)$ will stand for an induced chordless cycle (path) with k vertices.

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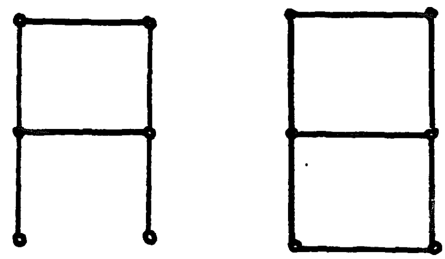

F_1 F_2

Fig. 1.

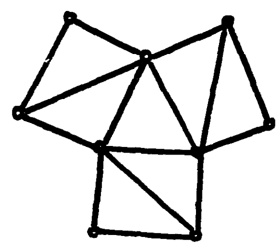

Fig. 2.

A graph G is called *triangulated* if every cycle of length greater than three in G has a chord. Dirac [5] proved that every triangulated graph contains a *simplicial* vertex: this is a vertex w such that $N(w)$ is a clique.

A proper subset H ($|H| \geq 2$) of vertices of G will be referred to as *homogeneous* if every vertex outside H is either adjacent to all the vertices in H or to none of them.

A graph G will be called a *weak bipolarizable graph* if G has no induced subgraph isomorphic to C_k ($k \geq 5$), $\bar{P}_5$ or to one of the graphs F_1, F_2 in Fig. 1.

Since every forbidden subgraph of a weak bipolarizable graph is also a forbidden subgraph of a bipolarizable graph it follows that every bipolarizable graph is also weak bipolarizable. In addition, note that the graph in Fig. 2 is a weak bipolarizable graph but not a bipolarizable graph.

Therefore, the class of weak bipolarizable graphs properly contains the class of bipolarizable graphs. As it turns out, the class of weak bipolarizable graphs also contains all triangulated graphs, all Welsh–Powell opposition graphs (see Olariu [10]), all superbrittle graphs (see Preissmann, de Werra, and Mahadev [12]) and all superfragile graphs (see Preissmann, de Werra, and Mahadev [12]).

2. The results

The following theorem provides a characterization of the class of weak bipolarizable graphs.

Theorem 1. *For a graph G the following three statements are equivalent*:

(i) *G is a weak bipolarizable graph*

(ii) *Every induced subgraph H of G is triangulated, or H contains a homogeneous set which induces a connected subgraph of $\bar{G}$*

(iii) *Every induced subgraph H of G is triangulated or H contains a homogeneous set.*

Proof. To prove the implication (i)$\rightarrow$(ii), consider a graph $G=(V,E)$ that satisfies (i). Assuming the implication (i)$\rightarrow$(ii) true for graphs with fewer vertices than G, we only need prove that G itself satisfies (ii).

If G contains a homogeneous set with the property mentioned in (ii), then we are done. We shall assume, therefore, that G contains no such homogeneous set. We want to show that, with this assumption, G is triangulated.

For this purpose, we only need show that G has no induced C_4.

Suppose not; now some vertices x, y, z, t induce a C_4 with edges xy, yz, zt, $tx \in E$. Consider the component F of the subgraph of $\bar{G}$ induced by $N(y) \cap N(t)$, containing x and z. By assumption, F is not a homogeneous set, and thus there exists a vertex u in $V-F$, adjacent to some but not all vertices in F. By connectedness of F in $\bar{G}$, we find non-adjacent vertices x', z' in F such that $ux' \in E$ and $uz' \notin E$.

Trivially, u is not in $N(y) \cap N(t)$, and hence u is adjacent to at most one of y, t. If u is adjacent to precisely one of y, t then $\{u, x', y, z', t\}$ induces a $\bar{P}_5$, a contradiction.

Now u is adjacent to neither y nor t. Write $N(x') \cap N(z') = U_0 \cup U_1$ in such a way that

every vertex in U_1 is adjacent to u, and

no vertex in U_0 is adjacent to u.

By the above argument, y and t belong to U_0 and thus $|U_0| \geqslant 2$. Observe that every vertex in U_1 is adjacent to every vertex in U_0, for otherwise $\{u, p, q, x', z'\}$ induces a $\bar{P}_5$, for any non-adjacent vertices p in U_0 and q in U_1.

Consider the connected component H of the subgraph of $\bar{G}$ induced by U_0 that contains the vertices y and t.

Since H is not homogeneous, there must exist a vertex v in $V-H$ adjacent to some but not all vertices in H. Trivially, v is not in $\{x', z', u\} \cup U_0 \cup U_1$. By connectedness of H in $\bar{G}$, we find non-adjacent vertices y', t' in H such that $vy' \in E$, $vt' \notin E$. Now v is adjacent to at most one of the vertices x' and z'. If v is adjacent to precisely one of them, then $\{v, x', z', y', t'\}$ induces a $\bar{P}_5$, a contradiction. Thus, v is adjacent to neither x' nor z'. By definition of U_0, u is adjacent to neither y' nor t'.

However, this implies that $\{u, v, x', y', z', t'\}$ induces either an F_2 or an F_1, depending on whether or not $uv \in E$.

This proves that G is triangulated, as claimed.

The implication (ii)→(iii) is trivial. To prove (iii)→(i) we only need observe that if a graph G does not satisfy (i), then (iii) fails.

This completes the proof of the theorem. □

Consider a graph G_1 and a graph G_2 containing at least two vertices, and let v be an arbitrary vertex in G_1.

It is customary to say that a graph G arises from G_1 and G_2 by *substitution* if G is obtained as follows:

(*) delete the vertex v from G_1, and

(**) join each vertex in G_2 by an edge to every neighbour of v in G_1.

If G arises by substitution from graphs G_1 and G_2, then we shall say that G is *substitution-composite*. It is a simple observation that a graph G is substitution-composite if and only if G contains a homogeneous set. Now the equivalence (i)⇔(iii) in Theorem 1 can be rephrased as follows.

Corollary 1a. *A graph G is weak bipolarizable if and only if every induced subgraph of G is either triangulated or substitution-composite.*

Let Ψ be the class of graphs defined as follows:

(ψ1) if G is triangulated, then G is in Ψ.

(ψ2) if G' is obtained from a graph G_1 in Ψ and a triangulated graph G_2 by substitution, then G' is in Ψ.

Theorem 2. *Ψ is precisely the class of weak bipolarizable graphs.*

Proof. To begin, we claim that

$$\text{every graph in } \Psi \text{ is weak bipolarizable.} \tag{1}$$

For this purpose, let G be an arbitrary in Ψ. Assuming (1) to be true for all graphs with fewer vertices than G, we only need prove that G itself is weak bipolarizable. This, however, follows immediately from the observation that G is either triangulated or it contains a homogeneous set. Now Theorem 1 guarantees that G is weak bipolarizable.

Conversely, we claim that

$$\text{every weak bipolarizable graph is in } \Psi. \tag{2}$$

Let G be a weak bipolarizable graph. Assume that (2) holds for all graphs with fewer vertices than G. If G is triangulated, then G is in Ψ by (ψ1). Now we may assume that G is not triangulated. Theorem 1 guarantees that G contains a homogeneous set. Let H be a minimal homogeneous set in G (here, minimal is meant with respect to set inclusion, not cardinality). By Theorem 1, H must be triangulated. By the induction hypothesis, the graph induced by $(V-H)\cup\{h\}$ is in Ψ, for any choice of h in H. Hence, by (ψ2), G itself is in Ψ, as claimed. □

We shall refer to a graph G which contains no homogeneous set as *substitution-prime*. For later reference we shall make the following simple observation, whose justification is immediate.

Observation 1. *If a graph G with a homogeneous set H contains an induced substitution-prime subgraph F, then either every vertex of F belongs to H or else F and H have at most one vertex in common.*

Let Σ be a class of graphs such that all forbidden graphs for Σ are substitution-prime.

Theorem 3. *If G arises by substitution from graphs G_1 and G_2 in Σ, then G is also in Σ.*

Proof. Suppose not; now G must contain an induced subgraph F isomorphic to a forbidden graph for the class Σ. By assumption, F is an induced subgraph of neither G_i $(i = 1, 2)$.

By Observation 1, F has precisely one vertex in common with G_2. However, this implies that G_1 has an induced subgraph isomorphic to F, a contradiction. □

Theorem 1 and Theorem 3 provide the basis for a polynomial-time recognition algorithm for weak bipolarizable graphs. In addition, we shall rely on algorithms to recognize triangulated graphs (see, for example, Rose, Tarjan and Leuker [13]), as well as polynomial time algorithms to detect the presence of a homogeneous set in a graph (see Spinrad [11]).

The following two-step algorithm recognizes weak bipolarizable graphs.

Algorithm Recognize(G);
{Input: A graph $G = (V, E)$.
Output: 'Yes' if G is weak bipolarizable; 'No' otherwise.}
Step 1. Call Check(G)
Step 2. Return('Yes'); stop.

Procedure Check(G);
begin
  if G is not triangulated **then**
    if G is not substitution-composite **then begin**
      return('No');
      stop
    end

else begin
{now G contains a homogeneous set H; let H' stand for the set of all the remaining vertices in G.}
Check(G_H);
pick an arbitrary vertex h in H;
Check($G_{\{h\}\cup H'}$)
end
end; {Check}

The correctness of this algorithm follows directly from Theorem 1 and Theorem 2. Furthermore, its running time is clearly bounded by $O(n^3)$: to see this, note that Check is invoked $O(n)$ times for a graph G with n vertices. Each invocation of Check runs in $O(n^2)$ time since the recognition of triangulated graphs [13] and the detection of a homogeneous set [11] are both performed in $O(n^2)$ time.

Given a P_4 with vertices a, b, c, d and edges ab, bc, cd, the vertices a and d are called *endpoints* and the vertices b, c are called *midpoints* of the P_4.

We shall say that a vertex x in a graph G is *semi-simplicial* if x is midpoint of no P_4 in G. Trivially, every simplicial vertex is also semi-simplicial, but not conversely.

A linear order $<$ on the vertex-set V of G is said to be a *(semi-)perfect elimination* if the corresponding ordering $x_1, x_2, \ldots, x_n$ of the vertices of G with $x_i < x_j$ iff $i < j$ satisfies

$$x_i \text{ is a (semi-)simplicial vertex in } G_{\{x_i, x_{i+1}, \ldots, x_n\}} \text{ for every } i. \tag{3}$$

It is immediate that every graph G with a semi-perfect elimination is *brittle* in the sense of Chvátal [2]: every induced subgraph H of G contains a vertex which is either midpoint or endpoint of no P_4 in H. Furthermore, it is an easy observation that every brittle graph is also perfectly orderable.

Hertz and De Werra [8] demonstrated that bipolarizable graphs are brittle; we extend their result by showing that weak bipolarizable graphs are also brittle. Actually, we also exhibit a linear-time (and thus optimal) algorithm that finds a perfect order for any weak bipolarizable graph. The details are spelled out in Theorem 4.

Rose, Tarjan and Lueker [13] proposed a linear-time search technique which is referred to as Lexicographic Breadth-First Search (LBFS, for short). They prove that a graph G is triangulated if, and only if, any ordering of the vertices of G produced by LBFS is a perfect elimination.

We shall use their algorithm to obtain a perfect order on the set of vertices of a weak bipolarizable graph. To make our exposition self-contained, we give the details of LBFS.

```
procedure LBFS(G);
{Input: the adjacency list of G;
 Output: an ordering σ of the vertices of G}
begin
  for every vertex w in V do label(w) ← ∅;
  for i ← n downto 1 do begin
    pick an unnumbered vertex v with the largest label;
    σ(v) ← i; {assign to v number i}
    for each unnumbered w ∈ N(v) do
      add i to label (w)
  end
end;
```

Note that we can think of the output of LBFS as a linear order $<$ on V by setting

$$u < v \quad \text{whenever} \quad \sigma(u) < \sigma(v).$$

It is immediate (see Golumbic [6]) that every linear order produced by LBFS satisfies the following property.

(P) *$a < b$, $b < c$, $ac \in E$, and $bc \notin E$ imply the existence of a vertex b' with $bb' \in E$, $ab' \notin E$ and $c < b'$.*

We are now in a position to state our next result.

Theorem 4. *If G is a weak bipolarizable graph, then every ordering of the vertices of G produced by LBFS is a semi-perfect elimination.*

Our proof of Theorem 4 uses the following result of an independent interest.

Proposition 1. *Let G be a graph with no induced $\bar{P}_5$, C_k $(k \geq 5)$ and no F_2, and let $<$ be a linear order on the vertex-set of G satisfying the property (P). Then for every vertices a, b, c, d with*

$$a < b,\ b < c,\ a < d,\ ab,\ ac,\ bd \in E,\ bc,\ ad \notin E, \tag{4}$$

we have $cd \in E$.

Proof of Proposition 1. Write $G = (V, E)$, and let $<$ be a linear order on V satisfying the hypothesis of Proposition 1. If $<$ is a semi-perfect elimination, then the conclusion follows trivially.

We may, therefore, assume that $<$ is not a semi-perfect elimination. If the statement is false then we shall let a stand for the last vertex in the linear order $<$ for which there are vertices b, c, d with $cd \notin E$ satisfying (4). Next, we let c stand for the *largest* vertex in $N(a)$ for which there exist vertices b and d with $cd \notin E$

satisfying (4). Further, with a and c chosen as before, let b stand for the largest vertex in $<$ for which there is a vertex d, $cd \notin E$, such that (4) is satisfied. Finally, with a, b, c chosen as above, we let d be the largest vertex in the linear order $<$ for which (4) is satisfied.

For the proof of Proposition 1 we shall need the following intermediate results which we present as facts.

Fact 1. $c<d$.

Proof of Fact 1. Suppose not; apply property (P) to the vertices a, b, c: we find a vertex b' (which we choose as large as possible) with $ab' \notin E$, $bb' \in E$ and $c<b'$.

We must have $cb' \in E$, or else b' could play the role of d, contrary to our assumption. Note that $b'd \notin E$, for otherwise $\{a, b, c, b', d\}$ induces a $\bar{P}_5$.

Apply property (P) to the vertices a, d, c. We find a vertex d' with $ad' \notin E$, $dd' \in E$ and $c<d'$. We note that $cd' \notin E$, for otherwise $\{a, b, c, d, d'\}$ induces a C_5 or a $\bar{P}_5$.

If $bd' \in E$, then d' can play the role of d. Thus $bd' \notin E$.

Clearly, $b'd' \in E$, or else $\{b, b', d, d'\}$ induces a P_4, with b contradicting our choice of a. But now, $\{a, b, c, b', d, d'\}$ induces an F_2. □

Fact 2. *b and c have no common neighbour w with $a<w$ and $aw \notin E$.*

Proof of Fact 2. Let w be a common neighbour of b and c with $a<w$ and $aw \notin E$. We shall let w be as large as possible. Trivially, $dw \notin E$ (else $\{a, b, c, d, w\}$ induces a $\bar{P}_5$); Fact 1 implies $b<d$; furthermore,

$$d<w \tag{5}$$

[Otherwise, either b or w contradicts our choice of a.]

Apply property (P) to the vertices b, d, w: by (5), we find a vertex d' with $bd' \notin E$, $dd' \in E$ and $w<d'$.

Now (5) implies that

$$d<d'. \tag{6}$$

Note that $cd' \notin E$, for otherwise $\{a, b, c, d, d'\}$ induces a $\bar{P}_5$ or a C_5. Next, $wd' \in E$, or else $\{b, w, d, d'\}$ induces a P_4, and so, by (5) and (6) combined, b contradicts our choice of a. Further, $ad' \in E$ for otherwise $\{a, b, c, d, d', w\}$ induces an F_2.

Apply property (P) to the vertices a, c, d': by (6), we find a vertex c' with $ac' \notin E$, $cc' \in E$ and $d'<c'$. Clearly, $dc' \notin E$, else $\{a, b, c, c', d\}$ would induce a C_5 or a $\bar{P}_5$. By the maximality of w, $bc' \notin E$.

Note that $c'd' \notin E$, else $\{a, b, c, c', d, d'\}$ induces an F_2. But now, with the assignment $d \leftarrow c'$, $b \leftarrow c$, $c \leftarrow d'$, (4) is still satisfied, contradicting our initial choice of c.

This completes the proof of Fact 2. □

Write $x \in B$ whenever there exists a path

$$b = w_0, w_1, \ldots, w_s = x \quad (s \geqslant 0)$$

joining b and x, with

$$w_{i-1} < w_i \quad \text{and} \quad aw_i \notin E, \quad (1 \leqslant i \leqslant s). \tag{7}$$

Similarly, write $y \in C$ whenever there exists a path

$$c = v_0, v_1, \ldots, v_t = y \quad (t \geqslant 0)$$

joining c and y, with

$$v_{i-1} < v_i \quad \text{and} \quad av_i \notin E, \quad (1 \leqslant i \leqslant t). \tag{8}$$

We note that Fact 1 implies that $B \neq \{b\}$. Furthermore, it is easy to see that we can apply property (P) to the vertices b, c, d; we obtain a vertex x adjacent to c but not to a, and such that $d < x$. By Fact 1, $c < d$ and so $c < x$. This shows that $C \neq \{c\}$.

Let b', c' stand for the largest vertex in $<$ which belongs to B, C, respectively. By the definition of B, we find a chordless path

$$b = b_0, b_1, \ldots, b_p = b'$$

in B, joining b and b', with the b_i's satisfying (7) in place of the w_i's.

Similarly, the definition of C guarantees the existence of a chordless path

$$c = c_0, c_1, \ldots, c_q = c'$$

in C, joining c and c', with the c_i's satisfying (8) in the place of the v_i's.

For further reference, we note that

$$cb_i \notin E, \quad (0 \leqslant i \leqslant p). \tag{9}$$

[To justify (9), let i stand for the smallest subscript for which $cb_i \in E$. Since $bc \notin E$, we have $i \geqslant 1$; by Fact 2, we have $i \geqslant 2$. But now, $\{a, c, b_0, b_1, \ldots, b_i\}$ induces a C_k with $k \geqslant 5$].

By Fact 1, $c < d \in B$, and so

$$c < b' \tag{10}$$

Now for the following Fact 3, symmetry allows us to assume that

$$b' < c'. \tag{11}$$

Fact 3. $B \cap C \neq \emptyset$.

Proof of Fact 3. Clearly, we may assume that no edge in G has one endpoint in B and the other in C, for otherwise we are done.

Let i be the subscript for which $c_{i-1} < b' < c_i$ (such a subscript must exist by virtue of (10) and (11) combined).

Property (P) applied to the vertices c_{i-1}, b', c_i guarantees the existence of a vertex b'' with $b'b'' \in E$, $c_{i-1}b'' \notin E$ and $c_i < b''$. We must have $ab'' \in E$, else we contradict the maximality of b'.

The shortest path joining b'' and b with all the internal vertices in B, together with $\{a, b, b''\}$ determines a chordless cycle Γ. By assumption, Γ contains at most four vertices.

Next, note that

$$c_0 b'' \in E$$

[If not, then $c_1 b'' \in E$, or else b'' contradicts our choice of c. But now, $\{a, c_0, c_1, b''\} \cup \Gamma$ induces a $\bar{P}_5$ or an F_2.]

Since $c_0 b'' \in E$ and $c_{i-1}b'' \notin E$, it follows that c_0 and c_{i-1} are distinct vertices. Let j be the first positive subscript such that $c_j b'' \notin E$ (such a subscript must exist since $c_{i-1}b' \notin E$). Note that $c_{j+1}b'' \in E$, for otherwise c_{j-1} contradicts our choice of the vertex c.

But now, $\{z, c_{j-1}, c_j, c_{j+1}, b''\}$ induces a $\bar{P}_5$ with $z = a$ or $z = c_{j-2}$. This completes the proof of Fact 3. □

Let w be the first vertex in the linear order $<$ which belongs to $B \cap C$. By the definition of B, there exists a chordless path Q_B in B joining w and b satisfying (7); similarly the definition of C implies the existence of a chordless path Q_C in C joining w and c, and satisfying (8).

By our choice of the vertex w, $Q_B \cap Q_C = \{w\}$. By Fact 2, w is adjacent to at most one of the vertices b and c, and thus G must contain a chordless cycle of length at least five induced by $\{a, b, c\}$ together with $Q_B \cup Q_C$.

With this the proof of Proposition 1 is complete. □

Proof of Theorem 4. Write $G = (V, E)$. If the statement is false, then some linear order $<$ on V produced by LBFS is not a semi-perfect elimination. We shall let a stand for the last vertex in the linear order $<$ which contradicts (3). Write $x \in A$ whenever $a < x$.

Let c be the *largest* vertex in $N(a) \cap A$ for which there exist a vertex b in $N(a) \cap A$ with $bc \notin E$, and a vertex in $N'(a) \cap A$ which is adjacent to precisely one of the vertices b and c. Our choice implies, trivially, that $b < c$.

Since every ordering produced by LBFS satisfies property (P), Proposition 1 guarantees that every vertex w in $N(b) \cap N'(a) \cap A$ is adjacent to c.

Therefore by our choice of a, we find a vertex d in A with $cd \in A$ and ad, $bd \notin E$. We shall let d be as large as possible.

Property (P) applied to vertices a, b, c guarantees the existence of a vertex b' such that $ab' \notin E$, $bb' \in E$ and $c < b'$. By Proposition 1, we must have $b'c \in E$. Obviously, $b'd \notin E$, or else $\{a, b, b', c, d\}$ would induce a $\bar{P}_5$.

We claim that

$$d < b'. \tag{12}$$

[To prove (12), assume $b' < d$, and apply property (P) to the vertices c, b', d; there exists a vertex b'' with $cb'' \notin E$, $b'b'' \in E$ and $d < b''$. Proposition 1 guarantees that $b''d \in E$. By Proposition 1, again, we must have $bb'' \notin E$. Clearly, $ab'' \in E$, else $\{a, b, b', b'', c, d\}$ induces an F_2. But now, b'' contradicts our choice of c.]

Next, we claim that

$$b < d. \tag{13}$$

[To justify (13), assume $d < b$, and apply property (P) to the vertices a, d, b. We find a vertex d' with $ad' \notin E$, $dd' \in E$ and $b < d'$. Note that $bd' \notin E$, for otherwise $\{a, b, c, d, d'\}$ induces a $\bar{P}_5$ or a C_5. Our choice of d guarantees that $cd' \notin E$. Further, $b'd' \in E$, or else d contradicts our choice of a. But now, $\{a, b, b', c, d, d'\}$ induces an F_2.]

By virtue of (12) and (13) combined, we can apply property (P) to the vertices b, d, b'. We find a vertex d' with $dd' \in E$, $bd' \notin E$ and $b' < d'$. Note that since $c < d'$, we must have $d' \neq c$. Clearly, $ad' \notin E$, for otherwise d' contradicts our choice of c. Furthermore, $cd' \notin E$, or else d' contradicts our choice of d.

Now $b'd' \in E$, for otherwise either c *or* d contradicts our choice of a.

It follows that $\{a, b, b', c, d, d'\}$ induces an F_2, a contradiction.

With this the proof of Theorem 4 is complete. □

Note. The proof of Theorem 4 does not use the forbidden graph F_1, and thus Theorem 4 provides a new proof of the main result of Hoang and Khouzam [9]. This result also characterizes the graphs for which the LBFS gives a semi-perfect elimination.

In the remainder of this paper we shall point out how Theorem 4 can be used to find linear-time solutions for the four classical optimization problems for weak bipolarizable graphs, namely:

- find a *minimum colouring* of G (a colouring of the vertices of G using the smallest number of colours),
- find a largest *clique* (standing for a set of pairwise adjacent vertices) in G,
- find a largest *stable set* (standing for a set of pairwise non-adjacent vertices) in G, and
- find a *minimum clique cover* of G (a partition of the vertices of G into the smallest number of cliques).

To solve all these problems in linear time, we shall rely on the following result.

Proposition 2 (Chvátal, Hoang, Mahadev, and de Werra [4]). *Given any graph $G = (V, E)$, along with a perfect order on G, one can find in time $O(|V| + |E|)$ a minimum colouring of G and a largest clique in G. Given any graph G, along with a perfect order on its complement $\bar{G}$, one can find in time $O(|V| + |E|)$ a minimum clique cover and a largest stable set in G.*

Furthermore, we shall need the following easy observations.

Observation 2. If $<$ is a semi-perfect elimination of a graph G, then the linear order $<'$ defined by

$$x <' y \quad \text{if, and only if } y < x$$

is a perfect order on G.

[To see this, consider vertices a, b, c, d with ab, bc, $cd \in E$, and such that $a <' b$ and $d <' c$. This implies that $b < a$ and $c < d$, and so either b or c contradicts the assumption that $<$ is a semi-perfect elimination.]

Observation 3. If $<$ is a semi-perfect elimination on graph G, then $<$ is a perfect order on the complement $\bar{G}$ of G.

[Let a, b, c, d be vertices of G with ab, bc, $cd \notin E$, and such that $a < b$ and $d < c$. But now, either a or d contradicts the assumption that $<$ is a semi-perfect elimination, depending on whether or not $a < d$.]

Let G be a weak bipolarizable graph. The following algorithm will produce a minimum colouring, a largest clique, a largest stable set and a maximum clique cover for G.

Step 1. Let $<$ be the linear order produced by LBFS with G as input.
Step 2. Call Colour($\bar{G}$, $<$);
Step 3. Call Max-Clique($\bar{G}$, $<$);
Step 4. Let $<'$ be obtained by reversing $<$;
Step 5. Call Colour(G, $<'$);
Step 6. Call Max-Clique(G, $<'$).

Here, Colour and Max-Clique are algorithms which, given a graph G along with a perfect order on G return a minimum colouring of G, and a largest clique in G, respectively. Their existence, as well as their running time, is guaranteed by Proposition 2. In addition LBFS takes linear-time to return an ordering of the vertices of an arbitrary graph. Theorem 4 guarantees that, with a weak bipolarizable graph G as input, LBFS will return a semi-perfect elimination. Hence the above algorithm correctly solves the four optimization problems in linear time.

References

[1] C. Berge and V. Chvátal, Topics on Perfect Graphs, Annals of Discrete Math 21 (Norh-Holland, Amsterdam, 1984).
[2] V. Chvátal, Perfect Graph Seminar (McGill University, Montreal, 1983).
[3] V. Chvátal, Perfectly ordered graphs, in Berge and Chvátal [1].
[4] V. Chvátal, C. Hoang, N.V.R. Mahadev and D. de Werra, Four classes of perfectly orderable graphs, J. Graph Theory 11 (1987) 481–495.
[5] G. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg. 25 (1961) 71–76.

[6] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, 1980, New York).
[7] M.C. Golumbic, C.L. Monma and W.T. Trotter Jr., Tolerance graphs, Disc. Appl. Math. 9 (1984) 157–170.
[8] A. Hertz and D. de Werra, Bipolarizable graphs, Départment de mathematiques, Ecole Polytechnique Fédérale de Lausanne, Suisse, O.R.W.P. 85/13, to appear in Discrete Mathematics.
[9] C. Hoang and N. Khouzam, A new class of brittle graphs, School of Computer Science, McGill University, Montreal, Que, Tech. Report SOCS-85-30.
[10] S. Olariu, Doctoral thesis, McGill University, Montreal (1986).
[11] J. Spinrad, P_4-trees and module detection, manuscript.
[12] M. Preissmann, D. de Werra and N.V.R. Mahadev, A note on superbrittle graphs, Disc. Math. 61 (1986) 259–267.
[13] D. Rose, R. Tarjan and G. Leuker, Algorithmic aspects of vertex elimination on graphs, SIAM J. Comput. 5 (1976) 266–283.