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WINGS AND PERFECT GRAPHS

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An edge uv of a graph G is called a wing if there exists a chordless path with vertices u, v, x, y and edges uv, vx, xy. The wing-graph $W(G)$ of a graph G is a graph having the same vertex set as G; uv is an edge in $W(G)$ if and only if uv is a wing in G. A graph G is saturated if G is isomorphic to $W(G)$. A star-cutset in a graph G is a non-empty set of vertices such that $G - C$ is disconnected and some vertex in C is adjacent to all the remaining vertices in C. V. Chvátal proposed to call a graph unbreakable if neither G nor its complement contain a star-cutset. We establish several properties of unbreakable graphs using the notions of wings and saturation. In particular, we obtain seven equivalent versions of the Strong Perfect Graph Conjecture.

0. Introduction

Claude Berge proposed to call a graph G *perfect* if for every induced subgraph H of G the chromatic number of H equals the size of the largest clique in H. He conjectured that a graph is perfect if and only if its complement is perfect. This conjecture was proved by Lovász [4] and is known as the Perfect Graph Theorem.

A graph G is *minimal imperfect* if G itself is imperfect but every proper induced subgraph of G is perfect.

The only known minimal imperfect graphs are the odd chordless cycles of length at least five (also called *odd holes*) and their complements (termed *odd anti-holes*). Berge conjectured that these are the only minimal imperfect graphs. This conjecture is the celebrated Strong Perfect Graph Conjecture (SPGC, for short) and it is still open.

An edge uv of a graph G will be called a *wing* if there exists a P_4 (standing for the chordless path with three edges) in G, with vertices u, v, x, y and edges uv, vx, xy. The *wing-graph* $W(G)$ of a graph G is a graph having the same vertex-set as G; uv is an edge in $W(G)$ if and only if uv is a wing in G.

Obviously, if the SPGC holds true, then $W(G)$ is an odd hole whenever G is a minimal imperfect graph. It was this link between perfection and wings that motivated the work presented in this paper: in fact, we shall prove several equivalent versions of the SPGC. One of them states that the SPGC holds true *if and only if* the wing-graph of every minimal imperfect graph is an odd hole.

Some of the results established here for minimal imperfect graphs hold for a larger class of graphs.

A *star-cutset* in a graph G is a non-empty set C of vertices such that $G - C$ is disconnected and some vertex of C is adjacent to all the remaining vertices in C. Chvátal proposed to call a graph *unbreakable* if neither G nor its complement $\bar{G}$ contains a star-cutset. He also showed that every minimal imperfect graph is unbreakable (see Chvátal [1]).

Our first two results, which we call Theorem 1 and Theorem 2 play a key role in the rest of the paper. The first one shows that in an unbreakable graph a wing is *two-sided,* meaning that it extends from each side to a P_4. The second gives a characterization of wings in an unbreakable graph.

Next, Theorem 3 asserts that in an unbreakable graph every vertex is endpoint of at least two wings. If the SPGC is true, then every vertex of a minimal imperfect graph is endpoint of precisely two wings. We prove that the converse implication holds as well.

Bruce Reed conjectured that $W(G)$ is connected whenever G is a minimal imperfect graph. In fact, this conjecture is an easy corollary of the following theorem of Chvátal and Hoang [2]:

> *If the vertices of a minimal imperfect graph G are coloured red and white in such a way that every colour appears on at least one vertex, then at least one P_4 in G has one vertex of one colour and three of the other.*

(To settle Reed's conjecture, we only need observe that one wing of this P_4 has endpoints of different colours).

The Chvátal–Hoang theorem also implies that $W(G)$ is non-bipartite whenever G is a minimal imperfect graph. (Here, we only need observe that the other wing of the P_4 has endpoints of the same colour.)

We prove that for an unbreakable graph G, $W(G)$ is disconnected if and only if $\bar{G}$ is bipartite. This result also implies Reed's conjecture. Furthermore, if G is an unbreakable graph, then at most one of $W(G)$, $W(\bar{G})$ is disconnected.

We prove a stronger statement than Reed's conjecture, namely that in every minimal imperfect graph G, the wing-graph $W(G)$ is 2-connected. It turns out that the SPGC is true if and only if in every minimal imperfect graph G, the wing-graph $W(G)$ is minimally 2-connected.

Throughout this paper we shall use the symbol N for "neighbourhood":

> $N(u)$ stands for the set of vertices adjacent to u;
>
> $N'(u)$ stands for the set of vertices adjacent to u in the complement.

We shall rely on the following known properties of unbreakable graphs:

(P1) Every unbreakable graph contains a P_4.

(P2) No unbreakable graph contains two vertices x, y such that

$$N(x) \subseteq \{y\} \cup N(y).$$

(P3) No unbreakable graph contains a set H of at least two vertices such that all vertices outside H are either adjacent to all vertices of H or to none of them. (A set H with the property described above is often referred to as a *homogeneous set*).

(P4) In every minimal imperfect graph G, every vertex is contained in exactly ω cliques of size ω. (Here ω denotes the largest size of a set of pairwise adjacent vertices in G).

(P5) If G is a minimal imperfect graph, then for every vertex w of G, $\omega(G - w) = \omega(G)$.

((P1) follows from a result of Seinsche [8]; (P2) is immediate; (P3) is a restatement of Theorem 1 in Lovász [3]; (P4), (P5) are included in a result of Padberg [6]).

1. Basics

Theorem 1. *In an unbreakable graph every wing is two-sided.*

Proof. Let $G = (V, E)$ be an unbreakable graph and let vertices a, b, c, d induce a P_4 with edges ab, bc, cd. We only need find vertices v, w such that $\{a, b, v, w\}$ induces a P_4 with edges $ba, av, vw \in E$.

For this purpose, write $C = \{b\} \cup N(b)$; $A = N(a) - C$.

Since G is unbreakable, we have $A \neq \emptyset$, and $G - C$ is connected. Since $d \notin A \cup C$, it follows that some v in A is adjacent so some w in $G - (A \cup C)$, as claimed. This completes the proof of the theorem. □

Note. When referring to unbreakable graphs we shall use the term wing as a synonym for two-sided wing, as justified by Theorem 1.

Theorem 2. *For an unbreakable graph G the following two statements are equivalent*:

(i) *the edge uv is a wing*

(ii) *there exists a vertex w in G distinct from u and v and adjacent to neither of them.*

Proof. Let $G = (V, E)$ be an unbreakable graph.
The implication (i)→(ii) is trivial.
To prove the implication (ii)→(i), let uv be an edge, and let w be a vertex satisfying (ii). Write

$t \in A$ whenever $tu \in E$, $tv \notin E$, $t \neq v$,

$t \in B$ whenever $tu \in E$, $tv \in E$,

$t \in A'$ whenever $tu \notin E$, $tv \in E$, $t \neq u$,

$t \in B'$ whenever $tu \notin E$, $tv \notin E$.

By our assumption B' is non-empty. Since G is unbreakable, there must exist a path in $G-(\{u\}\cup A\cup B)$ from v to some vertex in B'. The shortest such path contains two edges. To put it differently, there exist vertices x in A' and y in B' such that $xy\in E$. Now $\{u, v, x, y\}$ induces a P_4 in G, and so uv is a wing and the proof is complete. □

Theorem 3. *In an unbreakable graph every vertex is endpoint of at least two wings.*

Proof. Let $G=(V, E)$ be an unbreakable graph and let w be an arbitrary vertex in G. Write $N(w)=N_0\cup N_1$ such that wt is a wing if and only if $t\in N_0$.

If N_1 is empty, then we are done: since G is unbreakable, w must be endpoint of at least two edges, both wings.

Now $N_1\neq\emptyset$. We note that since $\bar{G}$ is connected, N_0 must be non-empty: otherwise by Theorem 2, every vertex in $N(w)$ would be adjacent to all vertices in $N'(w)$,

We claim that

> *If N_1 is non-empty then every vertex in $N'(w)$ is adjacent to at least one vertex in N_0.*

[If a vertex z in $N'(w)$ is adjacent to no vertex in N_0 then, in $\bar{G}$, $\{z\}\cup N_0$ is a star-cutset; since $\bar{G}$ is unbreakable, such a vertex z cannot exist.]

By this claim and Theorem 2 combined, $|N_0|=1$ implies that every vertex w' in $N'(w)$ satisfies $N(w)\subseteq\{w'\}\cup N(w')$; since G is unbreakable this cannot happen. Hence N_0 contains at least two vertices, as claimed.

This completes the proof of the theorem. □

Theorem 3 implies the following result of Chvátal [9]:

Corollary 3a. *In an unbreakable graph every vertex is endpoint of at least two P_4's and midpoint of at least two P_4's.*

Proof. Follows from Theorem 3 together with Theorem 1. □

Lemma 4.1. *Let $G=(V, E)$ be an unbreakable graph and let C be a proper subset of V such that $V-C$ splits into disjoint subsets U and B, satisfying:*

(*) $|B|\geqslant 2$,

(**) *$uv\in E$ for all $u\in U$, $v\in C$, and*

(***) *at most one vertex in B is endpoint of wings joining vertices from B and C.*

Then C induces a clique in G.

Proof. The proof is by induction on the cardinality of C. Suppose the statement true for sets C with $1 \leqslant |C| < k$ and let $|C| = k > 1$.

We claim that

$$C \text{ induces a disconnected subgraph of } \bar{G}. \tag{1}$$

[Suppose not; let B' be the set of all the vertices in B that have at least one neighbour in C. Since C is not homogeneous, we must have $B' \neq \emptyset$. We note that if every edge joining a vertex x in B' to a vertex in C is a non-wing, then x is adjacent to all vertices in C. (This follows from Theorem 2 together with the assumption that C induces a connected subgraph of $\bar{G}$).

Thus, if no wing has an endpoint in B and the other endpoint in C, then C is a homogeneous set.

Now there exists a vertex b in B and some vertex c in C such that bc is a wing. (Recall that by (***), b is unique.)

Note that $B' = B$. (Else, since every vertex in $B' - \{b\}$ is adjacent to all vertices in C, it follows that $\{c\} \cup B' \cup U$ is a star-cutset in G, a contradiction.

But now, $\{b\}$ is a star-cutset in $\bar{G}$. Hence, C induces a disconnected subgraph of $\bar{G}$, as claimed.]

By virtue of (1), there exists a partition of C into non-empty, vertex-disjoint sets C_1, C_2 such that every vertex in C_1 is adjacent to all vertices in C_2.

By the induction hypothesis, (with the other C_i adjoined to U) C_1 and C_2 are cliques and therefore C is a clique, as claimed.

This completes the proof of the lemma. □

Theorem 4. *For an unbreakable graph G the following three statements are equivalent*:

(i) *$W(G)$ is disconnected*

(ii) *the set of vertices of G partitions into sets B and C with $|B| \geqslant 3$, $|C| \geqslant 3$ such that no wing joins a vertex in B to a vertex in C.*

(iii) *$\bar{G}$ is bipartite.*

Proof. Let $G = (V, E)$ be an unbreakable graph.

The implication (i)$\rightarrow$(ii) is immediate: $|B| \geqslant 3$, $|C| \geqslant 3$ are implied by Theorem 2.

To prove the implication (ii)$\rightarrow$(iii), note that by Lemma 4.1 with $U = \emptyset$, it follows that both B and C induce cliques. Thus $\bar{G}$ is bipartite.

Finally, to prove the implication (iii)$\rightarrow$(i), we note that if $\bar{G}$ is bipartite, then we can write $V = V_1 \cup V_2$ such that V_1, V_2 induce complete subgraphs in G. Trivially, no edge joining vertices from V_1 and V_2 can be a wing. Hence $W(G)$ is disconnected. □

Bruce Reed's conjecture mentioned in the introduction is implied by Theorem 4. More precisely,

Corollary 4a. *In a minimal imperfect graph G, $W(G)$ is connected.*

Proof. Let G be a minimal imperfect graph. Since $\bar{G}$ cannot be bipartite, it follows (by Theorem 4) that $W(G)$ is connected. □

Corollary 4b. *If G is an unbreakable graph, then at most one of $W(G)$, $W(\bar{G})$ is disconnected.*

Proof. If both $W(G)$, $W(\bar{G})$ were disconnected then by Theorem 4 it must be that both $G, \bar{G}$ are bipartite. However, this implies that G has at most four vertices, and we are done. □

Theorem 5. *If G is an unbreakable graph, then every component of $W(G)$ is 2-connected.*

Proof. Let $G = (V, E)$ be an unbreakable graph, let A be a component of $W(G)$ and let A' stand for the set $V - A$.

If A is not 2-connected, then there exist distinct vertices x, y in A and a vertex z in A such that all paths joining x and y and consisting of wings only contain z.

Let X stand for the component of $W(G) - \{z\}$ containing x, and let Y stand for $A - (X \cup \{z\})$. Lemma 4.1 with $B = X \cup \{z\}$, $C = Y \cup A'$, $U = \emptyset$, guarantees that

$$Y \cup A' \text{ is a clique.} \tag{2}$$

Lemma 4.1 with $B = Y \cup \{z\}$, $C = X \cup A'$, $U = \emptyset$, guarantees that

$$X \cup A' \text{ is a clique.} \tag{3}$$

Let vertices u in X and v in Y be such that the edges uz and vz are both wings. By Theorem 2, there exist vertices w, w' such that $w \neq u$, $w \neq z$ and wu, $wz \notin E$, $w' \neq v$, $w' \neq z$ and $w'v$, $w'z \notin E$. By (2), $w \in Y$; by (3), $w' \in X$.

If A' is not empty, then by (2) and (3) combined, $\{z\}$ is a star-cutset in the complement of G.

Now we may assume that A' is empty. We claim that

$$\text{every path in } G \text{ joining } w \text{ and } w' \text{ contains } z \text{ or a neighbour of } z. \tag{4}$$

To see that this is the case, let N_X, N_Y stand for $N(z) \cap X$, $N(z) \cap Y$ respectively. Clearly, both $X - N_X$ and $Y - N_Y$ are non-empty, and no edge in G has one endpoint in $X - N_X$ and the other in $Y - N_Y$, for otherwise, by Theorem 2, we contradict that z is an articulation vertex. Now (4) follows by connectedness of G.

However, (4) implies that $\{z\} \cup N(z)$ is a star-cutset in G, a contradiction. This completes the proof of the theorem. □

Corollary 5a. *The wing-graph of every minimal imperfect graph is 2-connected.*

Proof. Let G be a minimal imperfect graph. By Corollary 4a, $W(G)$ is connected. Now the conclusion follows by Theorem 5. □

2. Equivalent versions of the *SPGC*

The following result gives seven equivalent versions of the Strong Perfect Graph Conjecture involving the notion of wings.

Theorem 6. *The following seven statements are equivalent*:

(i) *every minimal imperfect graph is either an odd hole or an odd anti-hole*
(ii) *the wing-graph of every minimal imperfect graph is an odd hole*
(iii) *in every minimal imperfect graph, every wing extends to precisely one P_4 in each direction*
(iv) *in every minimal imperfect graph, every vertex is endpoint of at most two wings.*
(v) *in every minimal imperfect graph, every vertex is endpoint of exactly two wings*
(vi) *the wing-graph of every minimal imperfect is minimally 2-connected*
(vii) *the wing-graph of every minimal imperfect graph is triangle-free.*

Proof of Theorem 6. The implications (i)→(ii) and (ii)→(iii) are immediate. The implication (iii)→(iv) follows from the following stronger statement.

Lemma 6.1. *If in an unbreakable graph G every wing extends to precisely one P_4 in each direction, then every vertex in G is endpoint of at most two wings.*

Proof of Lemma 6.1. Let $G=(V, E)$ be an unbreakable graph satisfying the hypothesis of the Lemma, and let u be an arbitrary vertex in G.

Theorem 3 guarantees the existence of a vertex v such that uv is a wing. Write

$$V=\{u, v\} \cup A \cup B \cup A' \cup B'$$

with

$$A=N(u)-(\{v\} \cup N(v)),$$
$$A'=N(v)-(\{u\} \cup N(u)),$$
$$B=N(u) \cap N(v),$$
$$B'=V-(N(u) \cup N(v)).$$

By Theorem 2, B' is not empty. By assumption, there exist vertices x in A, x' in A' and y, y' in B' such that $\{u, v, x, y\}$ induces a P_4 with edges vu, ux, xy and $\{u, v, x', y'\}$ induces a P_4 with edges uv, vx', $x'y'$.

For further reference we make the following observations:

Observation 1. No vertex in $A \cup A' - \{x, x'\}$ is adjacent to a vertex in B'. (Else, the wing uv would extend to more than one P_4 in some direction.)

Observation 2. Vertices y and y' are adjacent to all the vertices in B. (Else, let $yb \notin E$ for some $b \in B$. Now yx extends to either $yxbv$ or $yxub$ in addition to $yxuv$, a contradiction.)

Observation 3. x is adjacent to all vertices in $A - \{x\}$, x' is adjacent to all vertices in $A' - \{x'\}$. (Else, by Observation 1 the wing yx (or $y'x'$) would extend to more than one P_4 in the same direction.)

Observation 4. Both x and x' are non-adjacent to at most one vertex in B. (Assume x is non-adjacent to different vertices t, t' in B. This implies that both $\{x, y, t, v\}$ and $\{x, y, t', v\}$ induce a P_4, and thus xy extends to more than one P_4 in the same direction, contrary to our assumption. The same argument with x' in place of x and y' in place of y shows that x' is non-adjacent to at most one vertex in B.)

Observation 5. If $y = y'$ then $B' = \{y\}$. (Else, by Observation 2, $\{y\} \cup B \cup \{x, x'\}$ is a star-cutset in G.)

Fact 1. $|A| = |A'| = 1$.

Proof of Fact 1. Let t be a vertex of A distinct from x. By Observation 3 we have $tx \in E$; by Observation 1, t is adjacent to no vertices in B'. We claim that

all paths joining t to a vertex in B' contain u or a neighbour of u.

[If not, then t has a neighbour t' in A'. Clearly, $t'y \notin E$, for otherwise yt' would extend to two P_4's in the same direction, namely $yt'vu$ and $yt'tu$, a contradiction.

Note that $xt' \notin E$, for otherwise yx would extend to $yxt'v$ in addition to $yxuv$. But now, $yxtt'$ is a P_4, contrary to our assumption.]

This completes the proof of Fact 1. □

Fact 1 allows us to write $A = \{x\}$, $A' = \{x'\}$. If B is empty, then we are done by Fact 1.

Now B is non-empty.

Fact 2. *Every vertex in B' is adjacent to all vertices in B.*

Proof of Fact 2. First, we note that if y and y' coincide, then we are done by Observation 2 and Observation 5.

Now, $y \neq y'$. Clearly, xy', $x'y \notin E$. Let z be a vertex in B' non-adjacent to some vertex t in B. Since G is unbreakable, there must exist a path

$$z = w_0, \quad w_1, \ldots, w_p = x' \qquad (p \geq 2)$$

with w_i in $V - (\{y\} \cup N(y))$ for all $i = 1, 2, \ldots, p$, joining z to x'.

By taking p as small as possible, we ensure that this path is chordless. Let j be the first subscript such that $w_jt \in E$. But now, the wing $w_{j-1}w_j$ extends to two different P_4's in the same direction, namely $w_{j-1}w_jtu$ and $w_{j-1}w_jtv$, a contradiction.

This completes the proof of Fact 2. □

Fact 3. *If $|B'| \geqslant 2$, then x' is adjacent to all vertices in B.*

Proof of Fact 3. Note that by Observation 5, y and y' are distinct. Suppose that there exists a vertex t in B non-adjacent to x'.

It is easy to see that $yy' \in E$, for if not then yt extends to the P_4's $ytvx'$ and $yty'x'$, a contradiction.

Next, observe that x is not adjacent to x', for otherwise $y'x'$ would extend to the P_4's $y'x'xu$ and $y'x'vu$.

But now, $x'y'$ extends to $x'y'yx$ and $x'y'tu$. Since this cannot happen, the conclusion follows. □

Fact 4. *If $|B'| = 1$, then x' is non-adjacent to at most one vertex in $A \cup B$.*

Proof of Fact 4. Now y, y' coincide; by Observation 4, x' is non-adjacent to at most one vertex in B. The only way the statement can fail is to have $xx' \notin E$ and $x't \notin E$ for some t in B.

But this implies that the wing $x'y$ extends to two P_4's in the same direction, namely $x'ytu$ and $x'yxu$.

The conclusion follows. □

To complete the proof of Lemma 6.1, we note that by Observation 4, Fact 1, Fact 2, Fact 3, Fact 4 and Theorem 2 combined, it follows that u is endpoint of at most two wings, as claimed. □

The implication of (iv)$\rightarrow$(v) follows by Theorem 3.
To prove the implication (v)$\rightarrow$(vi), consider a minimal imperfect graph G. By Theorem 5, $W(G)$ is 2-connected. Since (v) is satisfied, by removing any edge in $W(G)$ two vertices of degree one are obtained. Thus (vi) holds.
The implication (vi)$\rightarrow$(vii) follows from a result of Plummer [10].
For the proof of the implication (vii)$\rightarrow$(i) we need the following result.

Lemma 6.2. *Let G be a minimal imperfect graph with $\omega(G) \geqslant 3$. If $W(G)$ contains no triangle then every edge in $\bar{G}$ is a wing.*

Proof of Lemma 6.2. By Theorem 2, we only need prove that every two non-adjacent vertices in G have a common neighbour. For this purpose, consider any two non-adjacent vertices x and y. If x and y have no common neighbour then, by Theorem 2, y is endpoint of wings only; since $W(G)$ contains no triangle, no triangle in G contains y and so $\omega(G) = 2$, a contradiction.

This completes the proof of Lemma 6.2. □

Consider a minimal imperfect graph G that does not satisfy (i). Now $\omega(G) \geq 3$ and $\omega(\bar{G}) \geq 3$. By the Perfect Graph Theorem, both G and $\bar{G}$ are minimal imperfect; hence we only need show that at least one of $W(G)$ or $W(\bar{G})$ contains a triangle.

If $W(G)$ contains no triangle, then by Lemma 6.2, every edge in $\bar{G}$ is a wing and so, having $\omega(\bar{G}) \geq 3$, $W(\bar{G})$ must contain a triangle.

This completes the proof of Theorem 6. □

Remark. The graph with vertices a, b, c, d and edges ab, ac, ad is usually referred to as the *claw*. A graph is termed claw-free if it contains no induced subgraph isomorphic to the claw.

The implication (vii)$\rightarrow$(i) can also be established using the notion of claw-free graph. Parthasarathy and Ravindra [7] proved that the SPGC is true for claw-free graphs.

If (i) is false, then some minimal imperfect graph G has $\omega(\bar{G}) \geq 3$; by the result of Parthasarathy and Ravindra [7] combined with the Perfect Graph Theorem, $\bar{G}$ contains a claw.

To put it differently, G contains a triangle abc and a vertex d adjacent to neither of a, b or c. Since every edge of the triangle abc is a wing by Theorem 2, the proof is completed.

3. Saturation

Call a graph *saturated* if $G = W(G)$. The motivation for this concept comes from the observation that if the SPGC holds true, then for every minimal imperfect graph G, either G or $\bar{G}$ is saturated.

Remark. Theorem 2 implies that

> *An unbreakable graph G is saturated if and only if for every edge there exists a vertex of G non-incident with that edge and adjacent to neither of its endpoints.*

The next result gives a characterization of saturated unbreakable graphs.

Theorem 7. *For an unbreakable graph G the following two statements are equivalent*:

(i) *G is saturated*

(ii) *all induced P_4's are saturated.*

Proof.
The implication (i)$\rightarrow$(ii) is immediate.
To prove the implication (ii)$\rightarrow$(i), let G be an unbreakable graph and assume that all induced P_4's in G are saturated. Consider an arbitrary vertex u in G; as usual, write $N(u) = N_0 \cup N_1$ such that uu' is a non-wing if and only if $u' \in N_1$. We only need prove that $N_1 = \emptyset$. We claim that

every vertex in N_1 is adjacent to all vertices in N_0

[Let x be an arbitrary vertex in N_0 and let y be an arbitrary vertex in N_1. By Theorem 1, there are distinct vertices t, t' in $N'(u)$ such that $\{u, x, t, t'\}$ induced a P_4 with edges ux, xt, tt'. By Theorem 2, $t'y \in E$. Since every P_4 is saturated, $\{x, u, y, t'\}$ does not induce a P_4; hence $xy \in E$.]

Now Theorem 2 implies that every vertex in N_1 is adjacent to all vertices in $N'(u)$. Since $\bar{G}$ is not disconnected, it must be that $N_1 = \emptyset$, as claimed. $\square$

Remark. Theorem 7 would follow instantly if in every unbreakable graph every edge were in an induced P_4.

However, the latter statement is false: consider the graph $\bar{C}_9$.

Our next aim is to provide several sufficient conditions for an unbreakable graph to be saturated.

One of these results is Lemma 6.2. Other such conditions are given in Theorem 8 and Theorem 10.

Theorem 8. *If G is an unbreakable graph with no induced $\bar{P}_5$ then G is saturated.*

Theorem 8 follows from the following stronger statement.

Theorem 9. *In an unbreakable graph every non-wing is in a $\bar{P}_k$, for some $k \geq 5$.*

Proof. Let $G = (V, E)$ be an unbreakable graph and let uv be a non-wing in G. Write $N(u) = N_0 \cup N_1$ such that ut is a wing if and only if $t \in N_0$.

Let F be the subgraph of $\bar{G}$ induced by $N(u)$. Let S be the set of all the vertices w in N_1 for which there is no path in F from w to N_0. By definition, each vertex in S is adjacent to all vertices in $N(u) - S$; by Theorem 2, each vertex in S is adjacent to all the vertices in $N'(u)$. Since $\bar{G}$ is connected, S must be empty. In particular, there is a path $x_0, x_1, \ldots, x_j$ in F, with $x_0 \in N_0$ and $x_j = v$. By taking j as small as possible, we ensure that the path is chordless and that $x_1, x_2, \ldots, x_j \in N_1$. By Theorem 1, there are adjacent vertices t_1 and t_2 in $N'(u)$ such that $x_0t_1 \in E$, $x_0t_2 \notin E$. Now $\{t_1, u, t_2, x_0, x_1, \ldots, x_j\}$ induces the desired $\bar{P}_k$. $\square$

Remark. The converse of Theorem 8 is not true: there exist unbreakable graphs which are saturated and which contain a $\bar{P}_5$ (see Fig. 1).

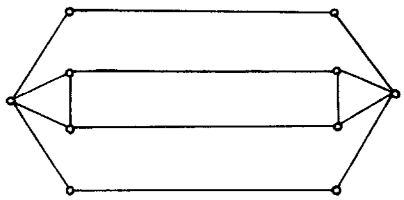

Fig. 1.

Is the converse true in the context of minimal imperfect graphs? More precisely, is it true that if a minimal imperfect graph is saturated then it contains no $\bar{P}_5$?

Obviously, if the SPGC is true, then the answer is yes.

Theorem 10. *Let G be an unbreakable graph. If no P_4 in $\bar{G}$ is saturated, then G is saturated.*

Proof. By Theorem 7, we only need prove that in G all induced P_4's are saturated. For this purpose, consider an arbitrary P_4 in G with vertices a, b, c, d and edges ad, bd, ac. Clearly, $\{a, b, c, d\}$ induce a P_4 in $\bar{G}$ with edges ab, bc, cd. By our assumption bc is not a wing. We claim that

a and d have a common neighbour in $\bar{G}$.

[Since $\bar{G}$ is unbreakable, there must exist a chordless path

$$a = w_0, w_1, \ldots, w_{p-1}, w_p = d \text{ with } w_i \notin \mathrm{N'}(b) \cup \{b\} \quad \text{for all } i = 1, 2, \ldots, p.$$

Trivially, $p \geq 2$. Note that if $p \geq 3$ then, by Theorem 2, $\{a, w_1, w_2, w_3\}$ induces a saturated P_4 in $\bar{G}$; since $\bar{G}$ is unbreakable, this eventuality cannot happen. Thus, $p = 2$ and the conclusion follows.]

Let e be a common neighbour of a and d. Note that e is adjacent in G to neither a nor d and thus $\{a, b, c, d\}$ induce a saturated P_4 in G. □

The Triangle Lemma. *In a minimal imperfect graph every non-wing belongs to a triangle.*

Proof. Olariu [6] has proved that in a minimal imperfect graph there cannot exist distinct vertices u and v such that every vertex in $G - \{u, v\}$ is adjacent to exactly one of u, v.

Let G be a minimal imperfect graph and let uv be an edge of G that belongs to no triangle. By Olariu's result, the set S of all the vertices adjacent to neither u nor v is not empty. Now Theorem 2 guarantees that uv is a wing.

This completes the proof of the Triangle Lemma. □

Remark. The Triangle Lemma is false in the context of unbreakable graphs which are not minimal imperfect. To see this, note that with G standing for the graph $\bar{C}_6$, the theorem fails.

Theorem 11. *If in a minimal imperfect graph the neighbourhood of a vertex u is disconnected then u is endpoint of wings only.*

Proof. Let $G=(V, E)$ be a minimal imperfect graph and let u be a vertex in V such that $N(u)$ is disconnected. We may assume that there is a non-wing uv, else we are done. By the Triangle Lemma, the connected component of $N(u)$ that includes v has at least two vertices; by Theorem 2, v is adjacent to all vertices in $N'(u)$. Since $N(u)$ is disconnected, $\{u, v\} \cup N'(u)$ is a star-cutset, and we are done. □

Remark. This statement is not true in the context of unbreakable graphs which are not minimal imperfect. A counterexample is, again, $\bar{C}_6$. However, the following statement holds for unbreakable graphs:

> *Let G be an unbreakable graph. If the neighbourhood of a vertex u is disconnected then uv is a wing whenever v belongs to a component H of* $N(u)$ *with* $|H| \geqslant 2$.

The proof follows easily from the proof of Theorem 11.

Theorem 12. *Let G be a minimal imperfect graph and let the edge uu' be a non-wing in G. There exist vertices x, x' (possibly* $x=x'$*) in* $N(u) \cap N(u')$ *such that both ux, u'x' are wings.*

Proof. Write $G=(V, E)$; write

$t \in A$ whenever $tu \in E$, $tu' \notin E$, $t \neq u'$,

$t \in B$ whenever $tu \in E$, $tu' \in E$,

$t \in A'$ whenever $tu \notin E$, $tu' \in E$, $t \neq u$.

Note that B is non-empty by the Triangle Lemma. If uv is a non-wing for all v in B, then by Theorem 2, $vz \in E$ whenever $z \in A'$. This, however, implies that $\{u'\} \cup A$ is a star-cutset in $\bar{G}$. Since G is unbreakable, it follows that there exists a vertex x in B such that ux is a wing.

The same argument with u' playing the role of u guarantees the existence of a vertex x' in B such that $u'x'$ is a wing. □

Theorem 13. *Let u be a vertex of a minimal imperfect graph. If a vertex x in* $N(u)$ *is not endpoint of a* P_4 *in* $N(u)$ *then the edge ux is a wing.*

Proof. Write $G=(V, E)$. Let x be a vertex in $N(u)$ with the property specified in the theorem. By Theorem 12 there exists a vertex x' in $N(x) \cap N(u)$ such that xx' is a wing. By Theorem 1, this wing must extend symmetrically to a P_4 in G. Hence, there exist vertices y, z in V such that $\{x, x', y, z\}$ induces a P_4 in G with edges xx', $x'y$, yz. Since x is endpoint of no P_4 in $N(u)$, at least one of the vertices y and z must belong to $N'(u)$. Now Theorem 2 guarantees that ux is a wing. □

Remark. Theorem 13 is false in the context of unbreakable graphs which are not minimal imperfect. It is easy to see that with G standing for the graph $\bar{C}_6$, the theorem fails.

A triangle of a graph G will be called a *tent* if it contains precisely two wings. A chordless C_4 will be called a *shelter* if it contains two opposite edges which are wings, the other edges being non-wings.

Theorem 14. *For a minimal imperfect graph G, the following two statements are equivalent*:

(i) *G is not saturated*

(ii) *G contains a tent or a shelter.*

Proof. Let $G=(V, E)$ be a minimal imperfect graph.
The implication (ii)→(i) is immediate.
To prove the implication (i)→(ii), suppose the statement false. Now $G \neq W(G)$ and yet G contains neither a tent nor a shelter. Consider a non-wing uu' in G. Write

$x \in N_0$ if and only if ux is a wing;

$x \in N_1$ whenever ux is a non-wing and $xy \notin E$ for some $y \in N_0$.

Note that $N_1 \neq \emptyset$ or else $\bar{G}$ would be disconnected, a contradiction. [If $N_1=\emptyset$, then every vertex in $N(u)-N_0$ is adjacent to all vertices in N_0. On the other hand, by Theorem 2 every vertex in $N(u)-N_0$ is adjacent to all vertices in $\{u\} \cup N'(u)$].

For the proof of the implication (i)→(ii) we shall rely on the following facts:

Fact 1. $N(u)=N_0 \cup N_1$.

Proof of fact 1. If not, then let T stand for $N(u)-(N_0 \cup N_1)$, and let t be an arbitrary vertex in T. Note that t must be adjacent to all vertices in N_1 for otherwise there would exist vertices t' in N_1 and t'' in N_0 such that $tt' \notin E$ and (by the definition of N_1) $t't'' \notin E$, implying that $\{u, t, t''\}$ induces a tent, a contradiction.

However, now every vertex in T is adjacent to all vertices in $V-T$ and thus $\bar{G}$ is disconnected, a contradiction. Thus $T=\emptyset$, as claimed. □

Fact 2. *All edges xw with $x \in N_1$, $w \in N'(u)$ are wings.*

Proof of fact 2. By Theorem 2, every vertex x in N_1 is adjacent to all vertices in $N'(u)$. On the other hand, by the definition of N_1, for every x in N_1 there exists a vertex x' in N_0 such that $xx' \notin E$. By Theorem 2 for every x' in N_0 there exists a vertex x'' in $N'(u)$ such that $x'x'' \notin E$.

However, now again by Theorem 2 xx'' is a wing. Since $N'(u)$ is connected and since every edge in $N'(u)$ is a wing (Theorem 2, again) the result follows, or G contains a tent. □

Fact 3. *Every vertex in N_0 is either adjacent to all vertices in N_1 or to none of them.*

Proof of fact 3. Let t be a vertex in N_0 and let x, x' be vertices in N_1 such that, without loss of generality, $tx \in E$, $tx' \notin E$. Since G is unbreakable, $N(u)$ is a minimal cutset in G; hence t must have a neighbour t' in $N'(u)$. By Fact 2, xt', $x't'$ are both wings. Note that tt' must be a wing for otherwise $\{u, t, x', t'\}$ would induce a shelter in G, a contradiction.

However, this implies that either $\{u, t, x\}$ or $\{t, x, t'\}$ induces a tent in G, contradicting our assumption. Therefore, such a vertex t cannot exist. □

By Fact 3 together with the trivial observation that every vertex in N_1 is adjacent to all vertices in $\{u\} \cup N'(u)$ it follows that $|N_1| = 1$ or else N_1 would be a homogeneous set in G. Write $N_1 = \{x\}$.

By Theorem 12, $N(u) \cap N(x)$ contains a vertex $x' \in N_0$ such that xx' is a wing and thus the triangle uxx' is a tent.

This completes the proof of the theorem. □

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