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THE MORPHOLOGY OF CONVEX POLYGONS

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Abstract—A simple polygon P is said to be unimodal if for every vertex of P, the Euclidian distance function to the other vertices of P is unimodal. The study of unimodal polygons has emerged as a fruitful area of computational and discrete geometry. We study unimodality properties of a number of special convex polygons from the morphological point of view. In particular, we establish a hierarchy among three classes of convex polygons in terms of their unimodality properties.

1. INTRODUCTION

In pattern recognition and classification, the shape of an object is routinely represented by a polygon obtained from an image processing device [1,2]. One of the fundamental features that contributes to a morphological description useful in shape analysis is the distance properties among vertices of the polygon [3]. Traditionally, convexity has played a central role in analysing relevant features of the shape of a set of points.

Recently, Toussaint [4] pointed out that the notions of convexity and unimodality are quite different: convex polygons need not be unimodal, and unimodal polygons need not be convex. Furthermore, in [4] it is argued that the key factor for obtaining very efficient algorithms for a large number of problems in computational geometry is not convexity, but rather unimodality.

It is not surprising, therefore, that unimodality and multimodality have received considerable attention in the literature [4–8]. In [7] it is shown that every convex polygon with at most five vertices must contain a unimodal vertex; [6] exhibits examples of n-vertex ($n \geq 6$) convex polygons none of whose vertices are unimodal. An interesting question is to investigate unimodality properties of special convex polygons. The purpose of this note is to study unimodality properties of a number of special convex polygons from the morphological point of view. In particular, we establish a hierarchy among three classes of convex polygons in terms of their unimodality properties.

2. THE RESULT

Formaly, a simple n-vertex polygon $P = p_0 \ldots p_{n-1}$ ($n \geq 3$) in the plane is specified by the list of its vertices along with their Cartesian coordinates; P is said to be in ***standard form*** whenever its vertices are enumerated in clockwise order, with all the vertices distinct, and no three consecutive vertices collinear.

A pair p_i, p_j of distinct vertices defines an edge of P if, and only if, $|i - j| = 1 \pmod n$; otherwise, p_i, p_j defines a diagonal of P. A polygon P is termed *convex* if all its diagonals lie entirely within P. The *diameter* of P is defined as $\max_{0 \leq i \neq j \leq n-1} d(p_i, p_j)$.

The author is indebted to Professor Toussaint for reading an earlier version of the manuscript, and for many useful discussions. I would also like to thank an anonymous referee for many constructive comments.

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Avis *et al.* [6] proposed to call a vertex p_i of P *unimodal* with respect to the Euclidian distance, if there exists a subscript j $(0 \leq j \leq n-1)$ such that $d(p_i, p_k)$ is *non-decreasing* for $k = i+1, i+2, \ldots, j$ and *non-increasing* for $k = j+1, j+2, \ldots, i-1$. (Here, as usual, subscript arithmetic is modulo n.) A non-unimodal vertex is termed *multimodal*. The polygon P itself is termed *unimodal* if all its vertices are unimodal.

A convex polygon is termed *cigar-shaped* if all its vertices lie inside the diameter circle and the diameter itself is not an edge in the polygon (see Figure 1).

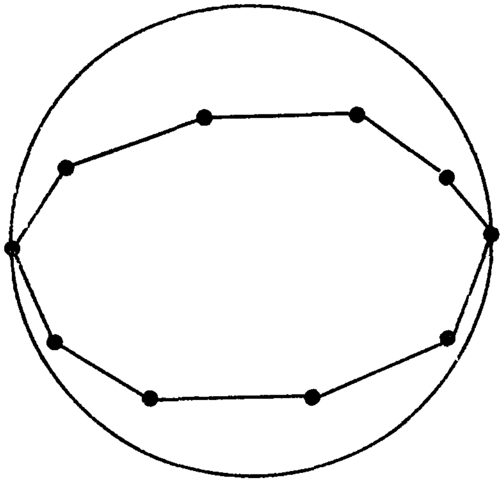

Figure 1. A cigar-shaped polygon.

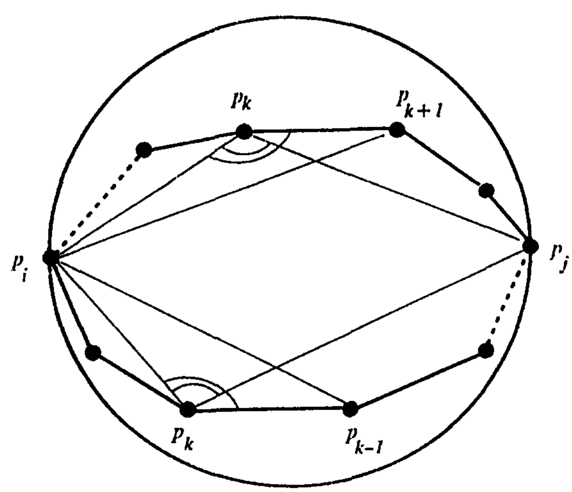

Figure 2.

Let $P = p_0 p_1, \ldots, p_{n-1}$ be a cigar-shaped polygon, and let p_i and p_j be the vertices of P which realize the diameter; we will refer to p_i and p_j as the *tips* of P.

LEMMA 1. *In a cigar-shaped polygon, every tip is unimodal.*

PROOF. Let k be an arbitrary subscript with $i+1 \leq k \leq j-1$. We note that the angle $\sphericalangle p_i p_k p_{k+1}$ is greater than $\pi/2$; to see that this is the case, consider the angle $\sphericalangle p_i p_k p_j$ and refer to Figure 2.

By the convexity of P, $\pi/2 < \sphericalangle p_i p_k p_j < \sphericalangle p_i p_k p_{k+1}$. Now in the triangle $p_i p_k p_{k+1}$ the angle $\sphericalangle p_i p_k p_{k+1} > \pi/2$ implies that $d(p_i, p_k) < d(p_i, p_{k+1})$. Since k was arbitrary, it follows that p_i is unimodal with respect to the chain $p_{i+1}, p_{i+2}, \ldots, p_j$.

Similarly, let k be an arbitrary subscript in the range $j+1 \leq k \leq i-1$. We claim that the angle $\sphericalangle p_i p_k p_{k-1}$ is greater than $\pi/2$: this follows, instantly, from the convexity of P. Now in the triangle $p_i p_k p_{k-1}$ we have $d(p_i, p_k) < d(p_i, p_{k-1})$. Since k was arbitrary it follows that the vertex p_i is unimodal, as claimed.

The proof of the fact that p_j is also unimodal follows by a mirror argument and is, therefore, omitted. ∎

The following result shows that every cigar-shaped polygon must have at least two unimodal vertices. As it turns out, some cigar-shaped polygons contain exactly two unimodal vertices.

Theorem 2. *Cigar-shaped polygons need not have more than two unimodal vertices.*

Proof. Let P be a cigar-shaped polygon; Lemma 1 guarantees the existence of at least two unimodal vertices of P. To complete the proof of the theorem we show that no other vertices of a cigar-shaped polygon need to be unimodal.

Consider two points x_0 and y_0 in the plane. We propose to construct a cigar-shaped polygon having x_0 and y_0 as tips and such that no other vertices of the polygon are unimodal. For this purpose, draw the circle having x_0 and y_0 as a diameter and refer to Figure 3.

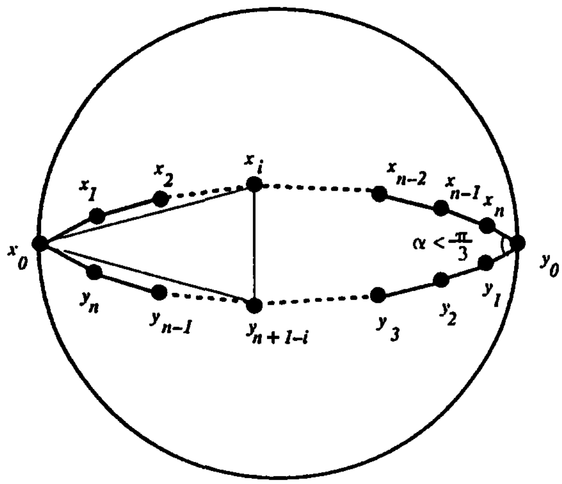

Figure 3.

Imagine symmetric arcs through x_0 and y_0 such that their angle α is less than $\pi/3$. Now take n points $x_1, x_2, \ldots, x_n$ in clockwise order on the upper arc and take n points $y_1, y_2, \ldots, y_n$ on the lower arc such that

$$d(x_0, x_i) = d(x_0, y_{n-i+1}), \quad (i = 1, 2, \ldots, n).$$

We claim that none of the x_i's and y_i's ($i = 1, 2, \ldots, n$) can be unimodal. For this purpose, let i be an arbitrary subscript in the range $1 \leq i \leq n$. Clearly, in the isosceles triangle $x_0\, x_i\, y_{n-i+1}$, we must have

$$d(x_i,\, x_0) > d(x_i,\, y_{n-i+1}),$$

because the angle $\sphericalangle x_i\, x_0\, y_{n-i+1} < \pi/3$ by construction.

Similarly, in the isosceles triangle $y_0\, x_i\, y_{n-i+1}$, we can write

$$d(x_i,\, y_0) > d(x_i,\, y_{n-i+1})$$

and so neither x_i nor y_i are unimodal. Since i was arbitrary, the conclusion follows. ∎

Call a convex polygon *semi-circle* if all its vertices lie inside the diameter circle, and the diameter itself is an edge in the polygon (see Figure 4). As it turns out, every vertex of a semi-circle polygon is unimodal. Specifically, we state the following result.

Theorem 3. *Every semi-circle polygon is unimodal.*

Proof. Let $P = p_0\, p_1 \ldots p_{n-1}$ be a semi-circle polygon and assume without loss of generality that the diameter is realized by p_0 and p_{n-1}. Referring to Figure 5, let p_i ($0 \leq i \leq n-1$) be an arbitrary vertex of P.

We shall prove that $d(p_i,\, p_k)$ is non-decreasing for $k = i+1, i+2, \ldots, n-1$ and non-increasing for $k = 0, 1, \ldots, i-1$. First, consider a vertex p_j with $j \in \{i+1, i+2, \ldots, n-2\}$. We claim that

$$\sphericalangle p_i\, p_j\, p_{j+1} > \frac{\pi}{2}.$$

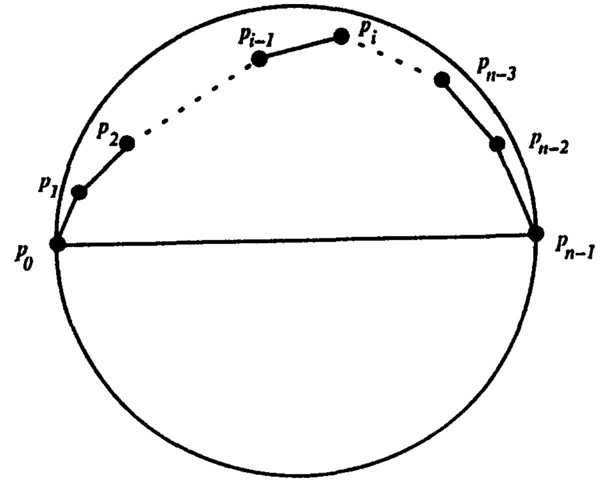

Figure 4. A semi-circle polygon.

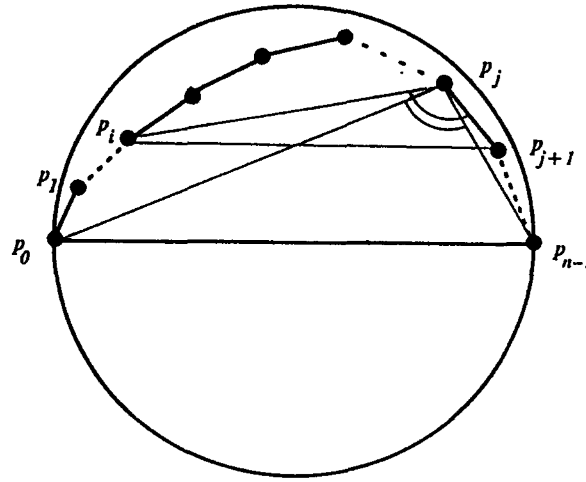

Figure 5.

The justification of this claim relies on the observation that $\sphericalangle p_0\, p_j\, p_{n-1}$ must be greater than $\pi/2$ (because p_j is inside the semi-circle). Now the convexity of P guarantees that $\sphericalangle p_i\, p_j\, p_{j+1} > \sphericalangle p_0\, p_j\, p_{n-1}$. Therefore, in the triangle $p_i\, p_j\, p_{j+1}$, we must have $d(p_i, p_j) < d(p_i, p_{j+1})$. Since p_j was arbitrary, it follows that $d(p_i, p_k)$ is non-decreasing for $k = i+1, i+2, \ldots, n-1$.

Similarly, for $j \in \{0, 1, 2, \ldots, i-2\}$ we claim that

$$\sphericalangle p_i\, p_{j+1}\, p_j > \frac{\pi}{2}.$$

As before, the convexity of P together with the fact that P is semi-circle guarantees that

$$\sphericalangle p_i\, p_{j+1}\, p_j > \sphericalangle p_n\, p_{j+1}\, p_0 > \frac{\pi}{2}.$$

Now in the triangle $p_i\, p_{j+1}\, p_j$, $d(p_i, p_{j+1}) > d(p_i, p_j)$ and the conclusion follows. ∎

A convex polygon P is said to be *weakly semi-circle* if the diameter of P is an edge in the polygon (see Figure 6). Let $P = p_0\, p_1 \ldots p_{n-1}$ be a weakly semi-circle polygon. Without loss of generality, we let p_0 and p_{n-1} realize the diameter.

We note that, by definition, all the vertices of P must lie in the region $S(p_0, p_n)$ delimited by the intersection of:

- the left half-plane generated by $p_0\, p_{n-1}$;
- the circle drawn with p_0 as center and of radius $d(p_0, p_{n-1})$;
- the circle drawn with p_{n-1} as center and of radius $d(p_0, p_{n-1})$.

The region $S(p_0, p_{n-1})$ defined above will be referred to as the *semi-lune* of p_0 and p_{n-1}.

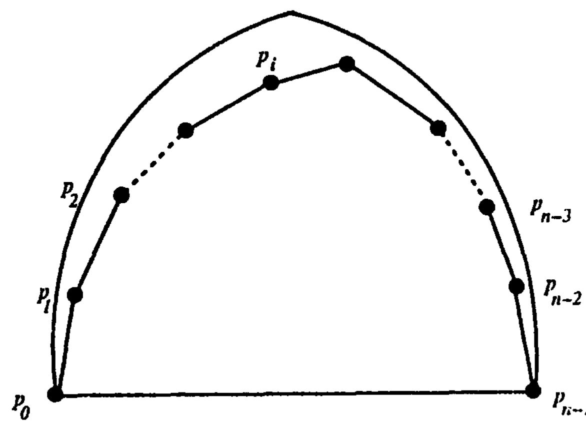

Figure 6. A weakly semi-circle polygon.

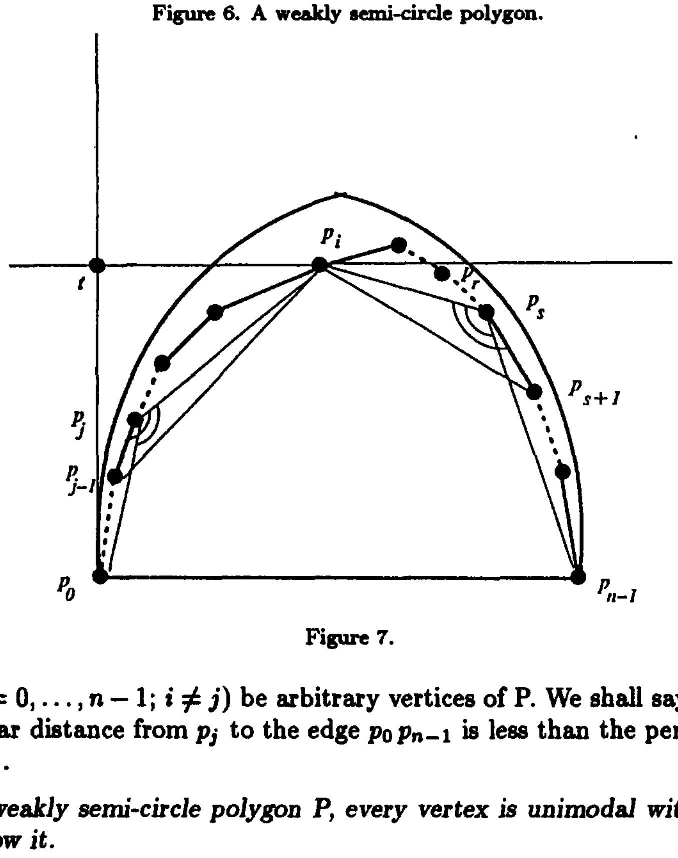

Figure 7.

Let p_i, p_j $(i, j = 0, \dots, n-1;\ i \neq j)$ be arbitrary vertices of P. We shall say that p_j is *below* p_i if the perpendicular distance from p_j to the edge $p_0\,p_{n-1}$ is less than the perpendicular distance from p_i to $p_0\,p_{n-1}$.

Lemma 4. *In a weakly semi-circle polygon P, every vertex is unimodal with respect to all the vertices lying below it.*

Proof. Draw an infinite ray from p_0 perpendicular to $p_0\,p_{n-1}$ and wholly contained in the left half-plane determined by $p_0\,p_{n-1}$ and refer to Figure 7. Draw an infinite line parallel to $p_0\,p_{n-1}$ passing through p_i and denote by t the intersection point with the ray drawn from p_0.

Consider an arbitrary vertex p_j $(j = 1, 2, \dots, i-1)$; by the convexity of P,

$$\sphericalangle p_i\,p_j\,p_{j-1} > \sphericalangle p_i\,p_j\,p_0.$$

At the same time,

$$\sphericalangle p_i\,p_j\,p_0 > \sphericalangle p_i\,t\,p_0 = \frac{\pi}{2}$$

(because p_j lies inside the semi-circle passing through p_i, t, p_0 and having $p_0\,p_i$ as a diameter).

Combining the two inequalities above, we get $\sphericalangle p_i\,p_j\,p_{j-1} > \pi/2$, and so $d(p_i, p_j) < d(p_i, p_{j-1})$. Now it follows that p_i is unimodal with respect to all the vertices p_k $(k = 0, 1, \dots, i-1)$.

Similarly, let p_r be the first vertex below p_i on the path from p_i to p_{n-1} in the clockwise direction. A mirror argument shows that for any subscript $s \in \{r, r+1, \ldots, n-2\}$ we have $d(p_i, p_s) < d(p_i, p_{s+1})$. The conclusion follows. ∎

COROLLARY 4.1. *Any antipodal vertex corresponding to the diameter $p_0 p_{n-1}$ is unimodal.*

PROOF. Clearly, all the vertices of the polygon lie below an antipodal vertex p_j of the diameter $p_0 p_{n-1}$. Now Lemma 4 guarantees that p_j is unimodal. ∎

Although every vertex of a semi-circle polygon is unimodal, surprisingly, weakly semi-circle polygons need not have more than two unimodal vertices. The details are contained in the following theorem.

THEOREM 5. *Weakly semi-circle polygons need not contain more than two unimodal vertices.*

PROOF. Let $P = p_0 p_1 \ldots p_{n-1}$ be a weakly semi-circle polygon, and let p_0 and p_{n-1} realize the diameter of P. Refer to Figure 8. Let p_m be the antipodal vertex corresponding to $p_0 p_{n-1}$. By Corollary 4.1, p_m is unimodal.

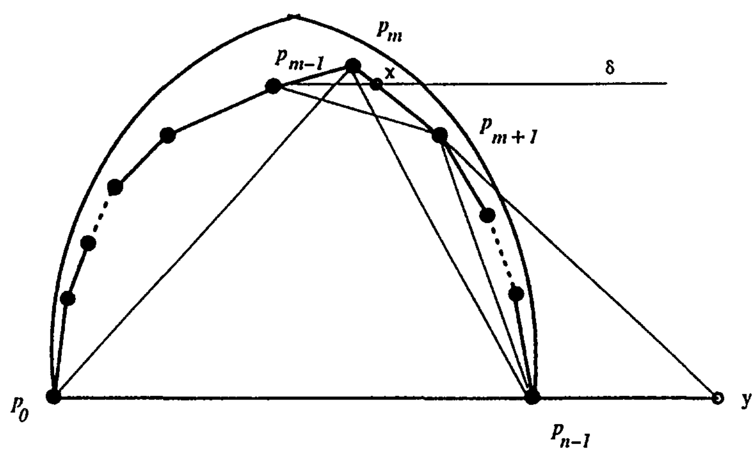

Figure 8.

Without loss of generality, let p_{m-1} be the vertex immediately below p_m; we claim that p_{m-1} is also unimodal.

To justify this claim, note that by Lemma 4, p_{m-1} is unimodal with all the vertices of P except, perhaps, p_m. In order to establish the unimodality of p_{m-1}, we only need show that

$$d(p_{m-1}, p_m) < d(p_{m-1}, p_{m+1}). \tag{1}$$

Let δ be an infinite ray originating at p_{m-1} and parallel to $p_0 p_{n-1}$; δ intersects the edge $p_m p_{m+1}$ at x. Note that by our assumption that p_{m-1} is the first vertex below p_m, x must belong to the interior of $p_m p_{m+1}$.

Now $\sphericalangle p_{m-1} x p_m > \sphericalangle p_{m-1} p_{m+1} p_m$ and therefore in the triangle $p_{m-1} x p_{m+1}$

$$d(p_{m-1}, x) < d(p_{m-1}, p_{m+1}). \tag{2}$$

Consider the triangle $p_0 p_m p_{n-1}$; since p_0 and p_{n-1} realize the diameter of P, we have $d(p_0, p_n) \leq d(p_0, p_m)$, which implies that

$$\sphericalangle p_0 p_{n-1} p_m \leq \sphericalangle p_0 p_m p_{n-1}. \tag{3}$$

Let y stand for the intersection point of the extensions of the lines $p_0 p_{n-1}$ and $p_m p_{m+1}$: by convexity, y must lie outside $p_0 p_{n-1}$. Trivially,

$$\sphericalangle p_0 p_{n-1} p_m > \sphericalangle p_n y p_{m+1}. \tag{4}$$

On the other hand, since δ is parallel to $p_0 p_{n-1}$, we have

$$\sphericalangle p_{n-1}\, y\, p_{m+1} = \sphericalangle p_{m-1}\, x\, p_m. \tag{5}$$

By the convexity of P, we can write

$$\sphericalangle p_{m-1}\, p_m\, p_{m+1} > \sphericalangle p_0\, p_m\, p_{n-1}. \tag{6}$$

Now combining (3)–(6) we get

$$\sphericalangle p_{m-1}\, x\, p_m = \sphericalangle p_{n-1}\, y\, p_{m+1} < \sphericalangle p_0\, p_{n-1}\, p_m \leq \sphericalangle p_0\, p_m\, p_{n-1} < \sphericalangle p_{m-1}\, p_m\, p_{m+1}.$$

Consequently, in the triangle $p_{m-1}\, p_m\, x$, we have

$$d(p_{m-1}, p_m) < d(p_{m-1}, x). \tag{7}$$

Finally, note that (2) and (7) combined imply (1) which proves the unimodality of p_{m-1}. To complete the proof of Theorem 5, we shall exhibit an instance of a weakly semi-circle polygon featuring exactly two unimodal vertices.

Let p_1 and p_5 be two arbitrary points in the plane and refer to Figure 9. We propose to construct a weakly semi-circle pentagon with vertices p_1, p_2, p_3, p_4, p_5 having $p_1 p_5$ as a diameter and such that exactly two of the vertices are unimodal.

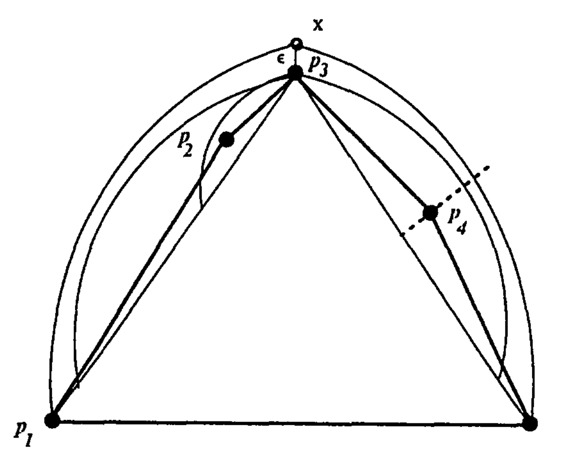

Figure 9.

Construct the semi-lune $S(p_1, p_5)$ and let x denote the tip (i.e., the vertex of $S(p_1, p_5)$ furthest away from the edge $p_1 p_5$). Let p_3 be a point on the perpendicular bisector of $p_1 p_5$ inside the semi-lune obtained by perturbing x by a very small $\varepsilon = 0$; write $d(p_3, x) = \varepsilon$. We define the following planar regions.

Region A_1: defined as the intersection of the circle centered at p_1 of radius $d(p_1, p_3)$ with the left half-plane determined by $p_3 p_5$;

Region A_2: defined as the intersection of the circle centered at p_5 of radius $d(p_5, p_3)$ with the left half-plane determined by $p_1 p_3$;

Let p_4 be a point on the open line segment determined by the intersection of A_1 with the perpendicular bisector of $p_3 p_5$.

Region A_3: defined as the intersection of A_2 with the circle centered at p_4 and of radius $d(p_4, p_3)$.

Take p_2 to be any point inside region A_3. Clearly, the points p_1, p_2, p_3, p_4, p_5 determine a convex polygon in the plane. We claim that

p_2 and p_3 are the only unimodal vertices of this polygon.

First, to argue that the points p_1, p_2, p_3, p_4, p_5 determine a weakly semi-circle polygon, we observe that the polygon lies completely inside the semi-lune $S(p_1, p_5)$. Next, note that, by construction,

$$d(p_1, p_4) < d(p_1, p_3) < d(p_1, p_5)$$

and, therefore, p_1 is not unimodal. Similarly,

$$d(p_5, p_2) < d(p_5, p_3) < d(p_5, p_1)$$

and so p_5 is not unimodal.

Finally, by construction,

$$d(p_4, p_1) > d(p_4, p_2) \text{ and } d(p_4, p_3) > d(p_4, p_2)$$

concluding the proof of the theorem. ∎

A natural generalization of the class of weakly semi-circle polygons is obtained as follows. We consider the class of convex polygons such that there exists an edge, say $p_0 p_{n-1}$, such that one can draw parallel lines of support δ_1 and δ_2 through p_0 and p_{n-1}, respectively, perpendicular to $p_0 p_{n-1}$. We call such a convex polygon *barn-shaped* (see Figure 10).

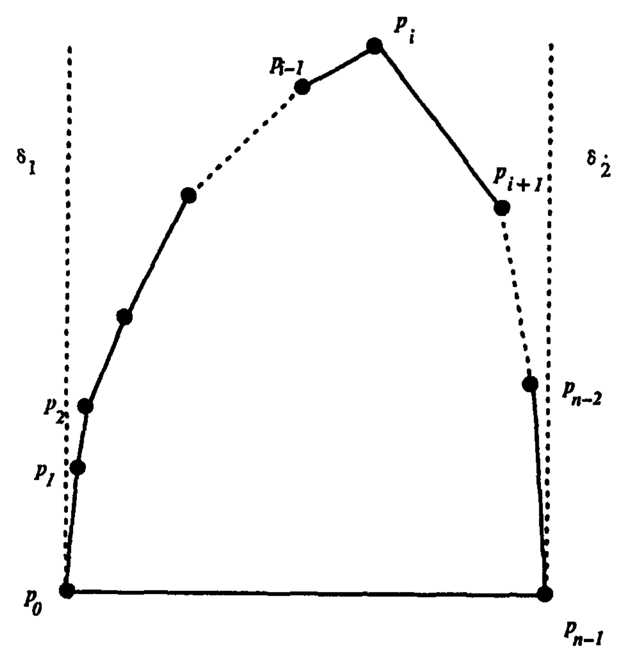

Figure 10. A barn-shaped polygon.

It is obvious that every weak semi-circle polygon is barned-shaped, but not conversely. We now study the unimodality properties of this new class of polygons. For this purpose, let $P = p_0 p_1 \dots p_{n-1}$ be a barn-shaped polygon; we inherit all the terminology established for weakly semi-circle polygons.

LEMMA 6. *In a barn-shaped polygon every vertex is unimodal with respect to all the vertices below it.*

PROOF. Follows directly from Lemma 4. ∎

Note that Lemma 6 implies that the vertex antipodal to the edge $p_0 p_{n-1}$ must be unimodal. As it turns out, the barn-shaped polygons need not have more than one unimodal vertex. Our next result asserts that this is the case.

THEOREM 7. *Barn-shaped polygon need not have more than one unimodal vertex.*

PROOF. By the previous argument, every barn-shaped polygon contains at least one unimodal vertex. We propose to exhibit a barn-shaped polygon with exactly one unimodal vertex. Our construction will involve a barn-shaped pentagon with vertices p_1, p_2, p_3, p_4, p_5.

Referring to Figure 11, let p_1 and p_5 be arbitrary points in the plane and write $d = d(p_1, p_5)$. Construct the semi-lune $S(p_1, p_5)$. Now take the vertex p_3 on the perpendicular bisector δ of $p_1 p_5$ at a distance of $d/2(4 - \sqrt{3})$ from $p_1 p_5$. It is easy to confirm that p_3 was chosen in such a way that the semi-lune $S(p_1, p_5)$ is seen from p_3 under an angle of $\pi/3$.

Let δ' stand for the perpendicular bisector of the segment $p_3 p_5$. Let p_2 be a point on the open line segment determined by the intersection of δ' with the area determined by $S(p_1, p_5)$ and the left half-plane determined by the infinite line collinear with $p_1 p_3$. Finally, let p_4 be the symmetric of p_2 with respect to δ.

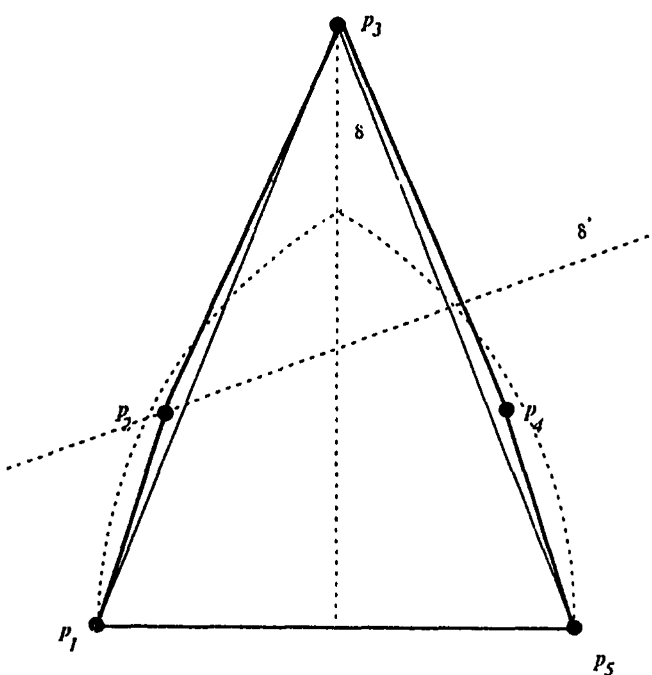

Figure 11.

It is easy to see that the pentagon with vertices p_1, p_2, p_3, p_4, p_5 is barn-shaped. In addition, we claim that none of the vertices p_1, p_2, p_4, p_5 are unimodal.

First, p_5 is not unimodal since by construction $d(p_5, p_2) < d(p_5, p_1)$ and $d(p_5, p_2) < d(p_5, p_3)$. The proof that p_1 is not unimodal follows by symmetry.

Next, by our choice of the vertex p_3, the angle $\sphericalangle p_2 p_3 p_4$ is less than $\pi/3$. Now the isosceles triangle $p_2 p - 3 p_4$ guarantees that $d(p_2, p_4) < d(p_2, p_3)$. Furthermore, since p_2 is on δ' we have $d(p_2, p_3) = d(p_2, p_5)$ and so p_2 is not unimodal. The fact that p_4 is not unimodal follows by a mirror argument, left as an exercise. ∎

We have established a hierarchy of three classes of convex polygon from the point of view of unimodality:

SEMI-CIRCLE ⊂ WEAKLY SEMI-CIRCLE ⊂ BARN-SHAPED.

Here, the semi-circle polygons are unimodal, that is, all their vertices are unimodal; weakly semi-circle polygons must have two unimodal vertices but not more than two; finally, the barn-shaped polygons must have one unimodal vertex.

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