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The Adjoint Alternative for Matrix Operators

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Abstract—The following inverse problem is considered: given a matrix B of rank r, does there exist a matrix A such that

$$B = T(A) = \text{adjoint } (A)$$

where the classical adjoint operation is intended? Conditions are determined on the rank of B which decides whether or not B lies in the range of the matrix adjoint operator.

Keywords—Adjoint alternative, Matrix range, Matrix operator, Adjoint range, Range characterization.

INTRODUCTION

Consider the following problem, posed by Wardlow [1] in *Mathematics Magazine.* Show that

$$E = \begin{bmatrix} 5 & 5 & 2 \\ 5 & 5 & 2 \\ 1 & 1 & 6 \end{bmatrix} \tag{1}$$

is not the classical adjoint of any matrix with real entries.

As shall be established, for a given matrix B whose rank is R_B, the inverse problem

$$B = T(A) = \text{adj}(A), \tag{2}$$

where A, B are $n \times n$ matrices, may have no solution, for A real or complex. Indeed, the counterexample (1) indicates that the range of T does not cover the whole space S^n of $n \times n$ matrices. The purpose of this note is to characterize the range of T, through a careful study of (2) by means of the adjoint property

$$AB = BA = \alpha I, \qquad \alpha = \det(A). \tag{3}$$

The general conclusion is that there are "more" B excluded from range (T) than are included.

THEOREM I. THE ADJOINT ALTERNATIVE. *As regards solutions of the inverse problem (2), the following trichotomy holds.*

(i) *If $R_B = n$, equation (2) has $n - 1$ solutions, some of which are complex.*
(ii) *If the* **incompatibility condition**

$$1 < R_B < n \tag{4}$$

holds, then equation (2) has no solution.

(iii) *If $R_B \leq 1$, equation (2) has infinitely many solutions, except when $B = 0$ and $n < 3$. In this case $A = 0$ is the only solution.*

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PROOF OF THE ADJOINT ALTERNATIVE.

I. Nonsingular B. In this case, equation (3) implies that

$$\det(A) = \det(B)^m, \qquad m = \frac{1}{n-1}. \tag{5}$$

For each root α_i, $i = 1, 2, \dots, n-1$ of $\det(B)$, equation (2) determines a unique matrix

$$A = \alpha_i B^{-1} \tag{6}$$

which satisfies equation (2). For complex α and real B, A is complex.

II. Incompatible B. For B satisfying the incompatibility condition (4), a search for solutions A of (3) which additionally satisfy (2) is necessary. However, if $BA = 0$, then A can have at most $n-2$ linearly independent columns. Therefore, $T(A) = 0$. Thus, equation (2) has no solutions.

IIIa. $B = 0$. If $n > 2$, any square matrix A of rank $R_A < n-1$ satisfies equation (2). However, if $n \geq 2$, $A = 0$ is the only solution of (2). Thus, an infinite number of solutions exist when $n > 2$.

IIIb. $\mathrm{Rank}(B) = 1$. This is the interesting case, as the proof is more difficult. Looking for solutions of $BA = 0$, the columns of A will be linear combinations of the $n-1$ linearly independent null vectors of B, which are denoted by $x_1, \dots, \bar{x}_{n_1}$. Each linear combination can involve $n-1$ free parameters. Thus the further requirement that $AB = 0$ appears to lead to a system of n^2 equations in $n^2 - n$ unknowns, about which little can be said. However, a closer look leads to more modest requirements.

LEMMA 1. *Let A, B denote $n \times n$ matrices, with $R_B = 1$. Then $AB = 0$ if and only if $A\bar{x} = 0$ for every invariant vector B which corresponds to a nonzero eigenvalue.*

PROOF. If $A = 0$, the result is trivial; therefore, assume $n > 1$. As $R_B = 1$, $\lambda = 0$ is an eigenvalue of geometric and algebraic multiplicity $n = 1$, with the null vectors $\bar{x}_i$ serving as a corresponding set of linearly independent eigenvectors. Complete this set with an eigenvector $\bar{x}_n$ which corresponds to the one nonzero eigenvalue of B. Writing a general vector $\bar{x}$ as a linear combination of the complete set $\bar{x}_1, \dots, \bar{x}_n$, with constants c_j, $j = 1, 2, \dots, n$, it follows that

$$AB\bar{x} = c_n \lambda_n A \bar{x}_n \tag{7}$$

Then $AB\bar{x} = 0$ for general $\bar{x}$ if and only if $A\bar{x}_n = 0$. ∎

The structure of the column vectors of A must now be taken into consideration. Let $\bar{b}$ with components $(b_1, b_2, \dots, b_n)$ be any nonzero row vector of B. It is just as general to assume that b_n is nonzero; otherwise, the structure of a matrix for which we aim in the sequel is merely row-shifted. Then, the typical column vector of the most general matrix A, satisfying $BA = 0$ is of the form

$$\bar{x}_j^{\mathsf{T}} = (\alpha_1^j, \alpha_2^j, \dots, \alpha_{n-1}^j, Q^j), \tag{8a}$$

where $j = 1, 2, \dots, n$ and

$$Q^j = \frac{-1}{b_n} \sum_{i=1}^{n-1} \alpha_i^j b_i. \tag{8b}$$

Now, for $j = 1, 2, \dots, n$, define vectors

$$\bar{\alpha}_j^{\mathsf{T}} = (\alpha_1^j, \alpha_2^j, \dots, \alpha_{n-1}^h), \tag{9}$$

and let

$$\bar{x}^{\mathsf{T}} = (c_1, c_2, \dots, c_n)$$

be any invariant vector of B corresponding to the nonzero eigenvalue (any nonzero column vector of B will serve). Invoking Lemma 1, the requirement $AB = 0$ leads to the system of equations

$$\sum_{j=1}^{n} c_j \bar{\alpha}_j = 0. \tag{10}$$

Here, one additional equation, which is a linear combination of the equation (10), has been discarded. (If $b_n = 0$ but $b_k \neq 0$, the equation to be discarded would come from row k versus row n of the coefficient matrix for the set of unknowns α_i^j resulting from $A\bar{x} = 0$.)

Since some c_k does not vanish, equation (10) expresses $\bar{\alpha}_k$ as a linear combination of the remaining $\bar{\alpha}_j$. The one unaddressed concern is that the resulting matrix A be of rank $n-1$. However, this clearly can be accomplished, nonuniquely, as many ways as one can determine a linearly independent set of $n-1$ vectors $\bar{\alpha}_j$.

Therefore, $AB = BA = 0$ has infinitely many solutions, A, of rank $n-1$. For each such A, $\operatorname{adj}(A) = 0$ or else $\operatorname{adj}(A)$ is a nonzero multiple of B. In this case, the relation

$$B = \frac{\operatorname{adj}(A)}{c} = \operatorname{adj}(qA), \qquad c = q^{1-n} \tag{11}$$

produces the desired result. The adjoint alternative is thus established.

AN EXAMPLE. Consider the matrices

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

and

$$A = \begin{bmatrix} 1 & 0 & \delta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix}. \tag{13}$$

For any finite values of γ, δ, $AB = BA = 0$ and $\operatorname{adj}(A) = \gamma B$. Therefore, $B = \operatorname{adj}(A/\sqrt{\gamma})$.

REFERENCE

1. W.P. Wardlow, Problem 1334, *Mathematics Magazine* **62** (5), 343, (December 1989).