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Algorithms for the Numerical Solution of a Finite-Part Integral Equation

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Abstract—The authors investigate a hypersingular integral equation which arises in the study of acoustic wave scattering by moving objects. A Galerkin method and two collocation methods are presented for solving the problem numerically. These numerical techniques are compared and contrasted in three test problems. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Investigations into the scattering of acoustic waves by moving objects have led the authors to consider the integral equation [1]

$$\int_a^b \frac{f(s)}{\Delta(s)} \left\{ \frac{1}{(s-x)^2} + C \ln|s-x| + D(s,x) \right\} ds = g(x), \qquad x \in (a,b), \tag{1}$$

where C is a constant, $D(s,x)$ is a nonsingular kernel, and

$$\Delta(s) = \sqrt{(b-s)(s-a)}. \tag{2}$$

The integral equation is accompanied by two subsidiary conditions (linear functionals)

$$W_k(f) = g_k, \qquad k = 0, 1, \tag{3}$$

either or both of which may be of the integral type

$$W_k(f) = \int_a^b \frac{w_k(s)}{\Delta(s)} f(s)\, ds = g_k \tag{4}$$

in which the $w_k(s)$ are prescribed functions, or of the end point type

$$W_k(f) = f(c) = g_k \tag{5}$$

in which $c = a$ or b.

The first integral in (1) is a finite-part integral defined by

$$\int_a^b \frac{F(s)}{(s-x)^2}\, ds = \frac{d}{dx} \int_a^b \frac{F(s)}{s-x}\, ds, \tag{6}$$

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where the integral on the right of (6) is a Cauchy principal value integral. Equations involving such integrals have been investigated by a number of authors [2–6] but not that given by (1) and (3).

It is well known that the shifted Chebyshev polynomials of the first and second kind, respectively,

$$t_n(s) = T_n\left(\frac{2s-b-a}{b-a}\right), \quad u_n(s) = U_n\left(\frac{2s-b-a}{b-a}\right), \qquad a \le s \le b \tag{7}$$

are related via the Cauchy type integral [7]

$$\frac{b-a}{2\pi}\int_a^b \frac{t_n(s)}{\Delta(s)(s-x)}\,ds = \begin{cases} 0, & n=0, \\ u_{n-1}(x), & n \ge 1. \end{cases} \tag{8}$$

Therefore, by differentiation, we find that

$$\frac{(b-a)^2}{8\pi}\int_a^b \frac{t_n(s)}{\Delta(s)(s-x)^2}\,ds = \begin{cases} 0, & n=0,1, \\ c^2_{n-2}(x), & n \ge 2, \end{cases} \tag{9}$$

where

$$c^2_{n-2}(s) = C^2_{n-2}\left(\frac{2s-b-a}{b-a}\right) \tag{10}$$

and the $C^\lambda_m(y)$ are Gegenbauer polynomials of degree m and parameter λ. Equation (9) suggests that (1) and (3) may be solved by expanding $f(s)$ in a series of shifted Chebyshev polynomials of the first kind. We may therefore seek to approximate $f(s)$ by the finite sum

$$f(s) = \sum_{n=0}^{N} f_n t_n(s). \tag{11}$$

With this approximation the subsidiary conditions (3) take the form

$$\sum_{n=0}^{N} f_n w_{nk} = g_k, \qquad k = 0, 1, \tag{12}$$

where

$$w_{nk} = W_k(t_n). \tag{13}$$

In the case of an integral condition (4), we have

$$w_{nk} = \int_a^b \frac{w_k(s)}{\Delta(s)} t_n(s)\,ds, \tag{14}$$

while, for a left end condition, we have

$$w_{nk} = (-1)^n, \tag{15}$$

and for a right end condition

$$w_{nk} = 1, \tag{16}$$

$k = 0, 1$; $n = 0, 1, \ldots, N$.

On substituting from (11) into (1) and making use of (9) and of the result [7]

$$\int_a^b \frac{t_n(s)}{\Delta(s)} \ln|s-x|\,ds = -\pi\mu_n t_n(x) \tag{17}$$

in which

$$\mu_0 = \ln\left(\frac{4}{b-a}\right), \quad \mu_n = \frac{1}{n}, \qquad (n \geq 1), \tag{18}$$

we find that the coefficients f_n, $n = 0, 1, 2, \dots, N$ must satisfy the equation

$$\sum_{n=0}^{N} f_n K_n(x) = g(x), \qquad x \in (a, b), \tag{19}$$

where

$$K_n(x) = \frac{8\pi}{(b-a)^2}(1-\delta_{n0})(1-\delta_{n1})c_{n-2}^2(x) - \pi C \mu_n t_n(x) + D_n(x) \tag{20}$$

and

$$D_n(x) = \int_a^b \frac{t_n(s)}{\Delta(s)} D(s, x)\, ds. \tag{21}$$

2. THE GALERKIN METHOD

In the Galerkin Method, $g(x)$ is approximated by the sum

$$g(x) = \sum_{m=2}^{N} g_m c_{m-2}^2(x). \tag{22}$$

The orthogonality conditions [7]

$$\int_a^b \Delta^3(x) c_{p-2}^2(x) c_{q-2}^2(x)\, dx = \frac{\pi}{8}\left(\frac{b-a}{2}\right)^4 \left(p^2 - 1\right) \delta_{pq}, \tag{23}$$

$p, q = 2, 3, \dots$ are then used to show that (19) is equivalent to the linear algebraic system

$$\sum_{n=0}^{N} f_n K_{nm} = g_m, \qquad m = 2, 3, \dots, N, \tag{24}$$

where

$$K_{nm} = \alpha_{nm} + C\gamma_{nm} + D_{nm}, \tag{25}$$

$$g_m = \frac{8}{\pi(m^2-1)}\left(\frac{2}{b-a}\right)^4 \int_a^b \Delta^3(x) g(x) c_{m-2}^2(x)\, dx, \tag{26}$$

$$\begin{aligned} \alpha_{nm} &= (1-\delta_{n0})(1-\delta_{n1})\frac{16}{m^2-1}\left(\frac{2}{b-a}\right)^6 \int_a^b \Delta^3(x) c_{n-2}^2(x) c_{m-2}^2(x)\, dx \\ &= \frac{8\pi}{(b-a)^2}(1-\delta_{n0})(1-\delta_{n1})\delta_{nm}, \end{aligned} \tag{27}$$

$$\gamma_{nm} = \frac{-8\mu_n}{m^2-1}\left(\frac{2}{b-a}\right)^4 \int_a^b \Delta^3(x) t_n(x) c_{m-2}^2(x)\, dx, \tag{28}$$

$$D_{nm} = \frac{8}{\pi(m^2-1)}\left(\frac{2}{b-a}\right)^4 \int_a^b \int_a^b \frac{\Delta^3(x)}{\Delta(s)} t_n(s) c_{m-2}^2(x) D(s, x)\, ds\, dx, \tag{29}$$

$n = 0, 1, \dots, N$; $m = 2, 3, \dots, N$.

Now, by means of the change of variable $x = ((b-a)/2)y + (b+a)/2$, we readily see that $\gamma_{nm} = (-8\mu_n/(m^2-1))J_{nm}$, where

$$\begin{aligned} J_{nm} &= \int_{-1}^{1} \left(1-y^2\right)^{3/2} T_n(y) C_{m-2}^2(y)\, dy \\ &= \frac{1}{2}\int_{-1}^{1} \left(1-y^2\right)^{1/2} T_n(y)\left[yU_{m-1}(y) - mT_m(y)\right] dy \\ &= \frac{1}{2}\int_0^{\pi} \sin\theta \cos n\theta \left[\sin m\theta \cos\theta - m \sin\theta \cos m\theta\right] d\theta \\ &= \frac{\pi}{16}\left[(m+1)(\delta_{n\,m-2} + \delta_{n0}\delta_{0\,m-2}) - 2m\delta_{n\,m} + (m-1)(\delta_{n\,m+2} + \delta_{n0}\delta_{0\,m+2})\right]. \end{aligned} \tag{30}$$

Thus

$$\gamma_{0m} = -\pi \ln\left(\frac{4}{b-a}\right)\delta_{m2}, \quad \gamma_{1m} = -\frac{\pi}{4}\delta_{m3}, \quad \gamma_{nm} = \frac{\pi}{2}\left[\frac{2\delta_{nm}}{n^2-1} - \frac{\delta_{n+2\,m}}{n(n+1)} - \frac{\delta_{n-2\,m}}{n(n-1)}\right], \tag{31}$$

$n = 2,3,\ldots,N$; $m = 2,3,\ldots,N$.

In order to evaluate the elements g_m and D_{nm}, we recall the Gaussian quadrature formula [8]

$$\int_a^b \frac{F(s)}{\Delta(s)}\,ds \approx \frac{\pi}{M}\sum_{k=1}^{M} F(s_k), \tag{32}$$

where M is the number of Gaussian points and

$$s_k = \frac{b-a}{2}\cos\left[\frac{(2k-1)\pi}{2M}\right] + \frac{b+a}{2}, \qquad k = 1,2,\ldots,M. \tag{33}$$

An application of this formula to (26) and (29) yields the results

$$g_m = \frac{8}{m^2-1}\left(\frac{2}{b-a}\right)^4 \frac{1}{M}\sum_{k=1}^{M}\Delta^4(s_k)g(s_k)c_{m-2}^2(s_k), \tag{34}$$

$$D_{nm} = \frac{8\pi}{m^2-1}\left(\frac{2}{b-a}\right)^4 \frac{1}{M^2}\sum_{k,l=1}^{M}\Delta^4(s_k)t_n(s_l)c_{m-2}^2(s_k)D(s_l,s_k), \tag{35}$$

$n = 0,1,\ldots,N$; $m = 2,3,\ldots,N$.

Lastly, we observe that the linear system (24) consists of $N-1$ equations for the $N+1$ unknowns f_n, $n = 0,1,\ldots,N$.

These are supplemented by the two subsidiary conditions (12) thus yielding the $(N+1)\times(N+1)$ system

$$\sum_{n=0}^{N} f_n W_{nk} = g_k, \qquad k = 0,1,\ldots,N, \tag{36}$$

where

$$W_{nk} = \begin{cases} w_{nk}, & n = 0,1,\ldots,N, \quad k = 0,1, \\ K_{nk}, & n = 0,1,\ldots,N, \quad k = 2,3,\ldots,N. \end{cases} \tag{37}$$

3. THE COLLOCATION METHOD

On approximating $f(s)$ by the finite sum (11), it was found that the integral equation (1) and its subsidiary conditions (3) yielded the equations

$$\sum_{n=0}^{N} f_n w_{nk} = g_k, \qquad k = 0,1 \tag{38}$$

and

$$\sum_{n=0}^{N} f_n K_n(x) = g(x), \qquad x \in (a,b) \tag{39}$$

for the expansion coefficients $f_0, f_1, f_2, \ldots, f_N$. In the Collocation Method, $N-1$ collocation points $x_2, x_3, x_4, \ldots, x_N \in (a,b)$ are chosen and the f_n are found by solving the $N+1$ simultaneous linear algebraic equations

$$\begin{aligned} \sum_{n=0}^{N} f_n w_{nk} &= g_k, && k = 0,1, \\ \sum_{n=0}^{N} f_n K_n(x_k) &= g(x_k), && k = 2,3,\ldots,N. \end{aligned} \tag{40}$$

In the numerical results which follow, two distinct collocation schemes are used. In the first of these, the collocation points x_k^0 are taken to be the zeros of $t_{N-1}(x)$ so that

$$x_k^0 = \frac{1}{2}(b-a)\cos\left[\frac{2k-3}{2N-2}\pi\right] + \frac{1}{2}(b+a), \qquad k = 2,3,\dots,N. \tag{41}$$

In the second scheme, the collocation points x_k^1 are taken to be the zeros of $u_{N-1}(x)$ so that

$$x_k^1 = \frac{1}{2}(b-a)\cos\left[(k-1)\frac{\pi}{N}\right] + \frac{1}{2}(b+a), \qquad k = 2,3,\dots,N. \tag{42}$$

4. NUMERICAL RESULTS

In this section, we illustrate the use of the above algorithms by displaying the results obtained from their application to three test problems.

PROBLEM 1. The first problem to be considered is that of solving the integral equation

$$\int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \left\{ \frac{1}{(s-x)^2} + \ln|s-x| + 2(s-x)^2 \right\} ds = g(x), \qquad x \in (-1,1), \tag{43}$$

with subsidiary conditions

$$f(-1) = 0 \quad \text{and} \quad f(1) = 0 \tag{44}$$

and right-hand side

$$g(x) = \frac{16\pi}{(5-4x)^2} + \pi \ln\left(\frac{5-4x}{8}\right) + \frac{\pi}{12}\left(20\ln 2 - 5 + 24x - 16x^2\right). \tag{45}$$

It is not too difficult to show that this problem has the exact solution

$$f(s) = \frac{3}{5-4s} - \frac{5+4s}{3}. \tag{46}$$

Table 1 below exhibits the results obtained for Problem 1. It shows the exact and computed values of $f(s)$ for several values of s when 25 terms of the expansion are taken and 25 quadrature points used. This table also shows the time taken for each of the numerical schemes used.

Table 1. Test results for Problem 1 with 25 series terms and 25 quadrature points.

s	Values of $f(s)$			
	Exact	Galerkin	Collocation with x_k^0	Collocation with x_k^1
−1.00	0.0000000	0.0000000	0.0000000	0.0000000
−0.80	−0.2341463	−0.2341463	−0.2341464	−0.2341463
−0.60	−0.4612613	−0.4612612	−0.4612613	−0.4612613
−0.40	−0.6787879	−0.6787878	−0.6787880	−0.6787879
−0.20	−0.8827586	−0.8827585	−0.8827586	−0.8827586
0.00	−1.0666667	−1.0666665	−1.0666665	−1.0666666
0.20	−1.2190476	−1.2190475	−1.2190476	−1.2190476
0.40	−1.3176471	−1.3176469	−1.3176472	−1.3176471
0.60	−1.3128205	−1.3128203	−1.3128207	−1.3128206
0.80	−1.0666667	−1.0666664	−1.0666668	−1.0666667
1.00	0.0000000	0.0000000	0.0000000	0.0000000
Elapsed Time (Sec)		12.76	0.12	0.12

PROBLEM 2. Next we consider the problem of solving the integral equation

$$\int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \left\{ \frac{1}{(s-x)^2} + \ln|s-x| + 2(s-x)^2 \right\} ds = g(x), \qquad x \in (-1,1), \tag{47}$$

with subsidiary conditions

$$f(-1) = 0 \quad \text{and} \quad \int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}}\, ds = 0 \tag{48}$$

and right-hand side

$$g(x) = \frac{16\pi}{(5-4x)^2} + \pi \ln\left(\frac{5-4x}{8}\right) + \pi\left(\ln 2 + \frac{1}{4}\right). \tag{49}$$

In this case the exact solution is

$$f(s) = \frac{3}{5-4s} - \frac{3+2s}{3}. \tag{50}$$

Table 2 exhibits the results obtained for Problem 2. It shows the exact and computed values of $f(s)$ for several values of s when 25 terms of the expansion are taken and 25 quadrature points used. This table also shows the time taken for each of the numerical schemes used.

Table 2. Test results for Problem 2 with 25 series terms and 25 quadrature points.

s	Values of $f(s)$			
	Exact	Galerkin	Collocation with x_k^0	Collocation with x_k^1
−1.00	0.0000000	0.0000000	0.0000000	0.0000000
−0.80	−0.1008130	−0.1008130	−0.1008130	−0.1008130
−0.60	−0.1945946	−0.1945946	−0.1945946	−0.1945946
−0.40	−0.2787879	−0.2787879	−0.2787880	−0.2787879
−0.20	−0.3494253	−0.3494253	−0.3494253	−0.3494253
0.00	−0.4000000	−0.4000000	−0.3999999	−0.4000000
0.20	−0.4190476	−0.4190476	−0.4190476	−0.4190476
0.40	−0.3843137	−0.3843137	−0.3843139	−0.3843138
0.60	−0.2461538	−0.2461538	−0.2461540	−0.2461539
0.80	0.1333333	0.1333334	0.1333332	0.1333333
1.00	1.3333333	1.3333332	1.3333333	1.3333333
Elapsed Time (Sec)		12.74	0.12	0.12

PROBLEM 3. Lastly, we consider the problem of solving the integral equation

$$\int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \left\{ \frac{1}{(s-x)^2} + \ln|s-x| + 2(s-x)^2 \right\} ds = g(x), \qquad x \in (-1,1), \tag{51}$$

with subsidiary conditions

$$\int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}}\, ds = 0 \quad \text{and} \quad \int_{-1}^{1} \frac{s\, f(s)}{\sqrt{1-s^2}}\, ds = 0 \tag{52}$$

and right-hand side

$$g(x) = \frac{16\pi}{(5-4x)^2} + \pi \ln\left(\frac{5-4x}{8}\right) + \pi\left(\ln 2 + \frac{1}{4} + x\right). \tag{53}$$

Table 3. Test results for Problem 3 with 25 series terms and 25 quadrature points.

s	Values of $f(s)$			
	Exact	Galerkin	Collocation with x_k^0	Collocation with x_k^1
−1.00	0.3333333	0.3333333	0.3333333	0.3333333
−0.80	0.1658537	0.1658536	0.1658536	0.1658537
−0.60	0.0054054	0.0054054	0.0054054	0.0054054
−0.40	−0.1454545	−0.1454546	−0.1454546	−0.1454546
−0.20	−0.2827586	−0.2827586	−0.2827586	−0.2827586
0.00	−0.4000000	−0.4000000	−0.3999999	−0.4000000
0.20	−0.4857143	−0.4857143	−0.4857143	−0.4857143
0.40	−0.5176471	−0.5176470	−0.5176472	−0.5176471
0.60	−0.4461538	−0.4461538	−0.4461540	−0.4461539
0.80	−0.1333333	−0.1333333	−0.1333335	−0.1333334
1.00	1.0000000	0.9999998	1.0000000	1.0000000
Elapsed Time (Sec)		12.77	0.13	0.13

In this case the exact solution is

$$f(s) = \frac{3}{5-4s} - 1 - s. \tag{54}$$

Table 3 exhibits the results obtained for Problem 3. It shows the exact and computed values of $f(s)$ for several values of s when 25 terms of the expansion are taken and 25 quadrature points used. This table also shows the time taken for each of the numerical schemes used.

The computations associated with the problems discussed above were performed on a 233 Mhz Pentium II computer with 64 Mb of RAM. Each of the proposed algorithms converged quickly providing high accuracy with relatively few terms in the series approximation. For the problems considered, the collocation algorithms were clearly more efficient than the Galerkin algorithm.

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