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G. Kerr
Old Dominion University

G. Melrose
Old Dominion University, gmelrose@odu.edu

J. Tweed
Old Dominion University

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Some Triple Sine Series

G. KERR, G. MELROSE AND J. TWEED
Old Dominion University
Norfolk, VA 23529, U.S.A.
tweed@math.odu.edu

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Abstract—Two types of triple sine series are investigated. They are reduced to singular integral equations with kernels involving elliptic functions. Closed form solutions are obtained.

Keywords—Triple sine series, Mixed boundary value problems, Singular integral equations, Elliptic functions.

1. INTRODUCTION

Mixed boundary value problems in elasticity and potential theory can often be reduced to the solution of triple trigonometric series. Many such series have been discussed in the literature [1–4], but not those we now wish to consider, which take the form

$$\begin{aligned} G(x) &= \sum_{n=1}^{\infty} nA_n \sin nx = 0, && (0 < x < a), \\ F(x) &= \sum_{n=1}^{\infty} \Omega_n^s A_n \sin nx = f(x), && (a < x < b), \\ G(x) &= \sum_{n=1}^{\infty} nA_n \sin nx = 0, && (b < x < \pi), \end{aligned} \tag{1}$$

where $s = \pm 1$, $\lambda > 0$ and

$$\Omega_n = \coth(2n\pi\lambda) + (-)^n \operatorname{csch}(2n\pi\lambda), \qquad n \geq 1. \tag{2}$$

Let

$$A_n = \frac{2}{\pi n} \int_a^b p(t) \sin(nt)\, dt, \qquad n \geq 1; \tag{3}$$

then

$$G(x) = H\left[(b - x)(x - a)\right] p(x), \tag{4}$$

where $H[x]$ is the Heaviside function, and the first and third of the triple equations (1) are satisfied automatically. Furthermore,

$$F(x) = \frac{1}{\pi} \int_a^b M(x,t)\, p(t)\, dt, \tag{5}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where

$$M(x,t) = 2\sum_{n=1}^{\infty} \Omega_n^s n^{-1} \sin(nx)\, \sin(nt), \tag{6}$$

so that the second triple equation will also be satisfied if $p(t)$ is given by the integral equation

$$\frac{1}{\pi}\int_a^b M(x,t)\,p(t)\,dt = f(x), \qquad (a < x < b). \tag{7}$$

2. THE CASE $s = 1$

In this case, we find it convenient to write

$$\Omega_n = \frac{1+q^{2n}}{1-q^{2n}}, \quad q = e^{i\pi[(1/2)+i\lambda]}. \tag{8}$$

Then [5, p. 221],

$$\frac{\partial}{\partial x} M(x,t) = -2\eta t + 2\zeta(t) + \frac{\wp'(t)}{\wp(t)-\wp(x)}, \tag{9}$$

where

$$\zeta(x) = \zeta\left(x, \pi, \pi\left[\frac{1}{2} + i\lambda\right]\right) \tag{10}$$

is the Weierstrass Zeta Function,

$$\wp(x) = \wp\left(x, \pi, \pi\left[\frac{1}{2} + i\lambda\right]\right) \tag{11}$$

is the Weierstrass p-function, and in terms of the theta function

$$\vartheta_1(x,q) = 2\sum_{n=0}^{\infty} (-)^n\, q^{(n+(1/2))^2}\, \sin(2n+1)\,x, \tag{12}$$

η is a constant given by

$$\eta = -\frac{1}{12}\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)}. \tag{13}$$

The Weierstrass p-function $\wp(x) = \wp(x,\omega_1,\omega_3)$ appearing above has one real parameter $\omega_1 = \pi$ and one complex parameter $\omega_3 = \omega_1[(1/2)+i\lambda]$, $\lambda > 0$, and in this case [5, p. 175] is a real valued function of the real variable x. Furthermore, it decreases monotonically from infinity to $e_1 = \wp(\pi,\pi,\pi[(1/2)+i\lambda])$ as x increases from 0 to π.

In order to solve the integral equation (7), we differentiate both sides with respect to x, thereby obtaining the Cauchy type integral equation

$$\frac{1}{\pi}\int_a^b \frac{\wp'(t)}{\wp(t)-\wp(x)}\,p(t)\,dt = f'(x) + B \tag{14}$$

in which

$$B = \frac{1}{\pi}\int_a^b [2\eta t - 2\zeta(t)]\;p(t)\,dt. \tag{15}$$

It follows [6] that

$$p(t) = \frac{C + B\,[\wp(a)+\wp(b)-2\wp(t)]}{2\Delta(t)} - \frac{1}{\pi\Delta(t)}\int_a^b \frac{\Delta(x)\,f'(x)\,\wp'(x)}{\wp(x)-\wp(t)}\,dx, \tag{16}$$

where C is an undetermined constant and

$$\Delta(t) = \sqrt{\{[\wp(a) - \wp(t)]\,[\wp(t) - \wp(b)]\}}. \tag{17}$$

Define

$$\begin{aligned} I_{mn} &= \frac{1}{\pi}\int_a^b t^m\,[\wp(a) + \wp(b) - 2\wp(t)]^n\,\frac{dt}{\Delta(t)}, \\ J_{mn} &= \frac{1}{\pi}\int_a^b \{\zeta(t)\}^m\,[\wp(a) + \wp(b) - 2\wp(t)]^n\,\frac{dt}{\Delta(t)}, \\ K_0 &= \int_0^a \frac{dt}{\Delta_1(t)}, \end{aligned} \tag{18}$$

where

$$\Delta_1(t) = \sqrt{\{[\wp(a) - \wp(t)]\,[\wp(b) - \wp(t)]\}}. \tag{19}$$

Observe that

$$M(x,t) = 2\,[\zeta(x) - \eta x]\;t + \int_0^t \frac{\wp'(x)}{\wp(x) - \wp(\tau)}\,d\tau \tag{20}$$

and hence, that

$$\frac{1}{\pi}\int_a^b \frac{M(x,t)}{\Delta(x)}\,dx = K_0 + 2[J_{10} - \eta I_{10}]\,t. \tag{21}$$

Therefore, on multiplying (7) by $1/(\pi\Delta(x))$ and integrating with respect to x from $x = a$ to $x = b$, we obtain the relationship

$$\frac{1}{\pi}\int_a^b \{K_0 + 2[J_{10} - \eta I_{10}]\,t\}\,p(t)\,dt = \frac{1}{\pi}\int_a^b \frac{f(x)}{\Delta(x)}\,dx. \tag{22}$$

On substituting from (16) into (15) and (22), we now obtain two simultaneous linear algebraic equations for the constants B and C and thereby determine $p(t)$ completely.

$$\begin{aligned} [\eta I_{11} - J_{11} - 1]\;B + [\eta I_{10} - J_{10}]\;C &= R_1, \\ [K_0 I_{01} + 2\{J_{10} - \eta I_{10}\}\,I_{11}]\;B + [K_0 I_{00} + 2\{J_{10} - \eta I_{10}\}\,I_{10}]\;C &= R_2, \end{aligned} \tag{23}$$

where

$$R_1 = \frac{2}{\pi}\int_a^b [\eta t - \zeta(t)]\;\frac{1}{\pi\Delta(t)}\int_a^b \frac{\Delta(x)\,f'(x)\,\wp'(x)}{\wp(x) - \wp(t)}\,dx\,dt \tag{24}$$

and

$$\begin{aligned} R_2 = &\frac{2}{\pi}\int_a^b [K_0 + 2\{J_{10} - \eta I_{10}\}\,t]\;\frac{1}{\pi\Delta(t)}\int_a^b \frac{\Delta(x)\,f'(x)\,\wp'(x)}{\wp(x) - \wp(t)}\,dx\,dt. \\ &+ \frac{2}{\pi}\int_a^b \frac{f(x)}{\Delta(x)}\,dx \end{aligned} \tag{25}$$

Last, we note that

$$F(x) = \int_0^x F'(\xi)\,d\xi, \tag{26}$$

where

$$F'(x) = H\,[(b-x)(x-a)]\;f'(x) + \operatorname{sgn}(a+b-2x)\,H\,[(x-a)(x-b)]\;\frac{F_1(x)}{\Delta_1(x)} \tag{27}$$

and

$$F_1(x) = \frac{C + B\,[\wp(a) + \wp(b) - 2\wp(x)]}{2} - \frac{1}{\pi}\int_a^b \frac{\Delta(t)\,f'(t)\,\wp'(t)}{\wp(t) - \wp(x)}\,dt. \tag{28}$$

3. THE CASE $s = -1$

In this case, we have

$$\Omega_n^{-1} = \frac{1-(-)^n q^n}{1+(-)^n q^n}, \quad q = e^{-2\pi\lambda}, \tag{29}$$

and hence [7, 2.18; 8, 123.05],

$$M(x,t) = \log\left|\frac{h(x)+h(t)}{h(x)-h(t)}\right|, \tag{30}$$

where

$$h(x) = \frac{\operatorname{sn}(Kx/\pi,k)\,\operatorname{dn}(Kx/\pi,k)}{\operatorname{cn}(Kx/\pi,k)}, \tag{31}$$

$$k = 4\left[\frac{\sum_{n=0}^{\infty} q^{(n+(1/2))^2}}{1+2\sum_{n=1}^{\infty} q^{n^2}}\right]^2, \tag{32}$$

and $K = K(k)$ is a complete elliptic integral of the first kind. Integral equation (7) thus takes the form

$$\frac{1}{\pi}\int_a^b \log\left|\frac{h(x)+h(t)}{h(x)-h(t)}\right| dt = f(x), \qquad (a < x < b). \tag{33}$$

It follows at once [4] that

$$p(t) = \frac{h'(t)}{\delta(t)} F_1(t), \tag{34}$$

where

$$\delta(t) = \sqrt{\{[h^2(b)-h^2(t)]\,[h^2(t)-h^2(a)]\}}, \tag{35}$$

$$F_1(t) = C - \frac{2}{\pi}\int_a^b \frac{\delta(y)\,f'(y)\,h(y)}{h^2(y)-h^2(t)}\,dy, \tag{36}$$

$$C = \frac{h^2(b)}{K_1K_1'}\int_a^b \frac{h'(x)\,f(x)}{\delta(x)}\,dx + \frac{2h(b)}{\pi K_1'}\int_a^b \frac{h'(t)}{\delta(t)}\int_a^b \frac{\delta(x)\,h(x)\,f'(x)}{h^2(x)-h^2(t)}\,dx\,dt, \tag{37}$$

and $K_1 = K(k_1)$, $K_1' = K(\sqrt{1-k_1^2})$ are complete elliptic integrals of the first kind with parameter $k_1 = h(a)/h(b)$. We now see that

$$F(x) = \int_0^x F'(\xi)\,d\xi, \tag{38}$$

where

$$F'(x) = H\left[(b-x)(x-a)\right]\,f'(x) + \operatorname{sgn}(a+b-2x)\,H\left[(x-a)(x-b)\right]\,h'(x)\,\frac{F_1(x)}{\delta_1(x)}, \tag{39}$$

and

$$\delta_1(x) = \sqrt{\{[h^2(b)-h^2(x)]\,[h^2(a)-h^2(x)]\}}. \tag{40}$$

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