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Superconvergence of the iterated collocation methods for Hammerstein equations

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Abstract

In this paper, we analyse the iterated collocation method for Hammerstein equations with smooth and weakly singular kernels. The paper expands the study which began in [16] concerning the superconvergence of the iterated Galerkin method for Hammerstein equations. We obtain in this paper a similar superconvergence result for the iterated collocation method for Hammerstein equations. We also discuss the discrete collocation method for weakly singular Hammerstein equations. Some discrete collocation methods for Hammerstein equations with smooth kernels were given previously in [3, 18].

Keywords: The iterated collocation method; The discrete collocation method; Hammerstein equations with weakly singular kernels; Superconvergence

AMS classification: 65B05; 45L10

1. Introduction

In this paper, we investigate the superconvergence property of the iterated collocation method for Hammerstein equations. In a recent paper [16], the superconvergence of the iterated Galerkin method for Hammerstein equations with smooth as well as weakly singular kernels was established. The paper generalizes the previously reported results on the superconvergence of the iterated Galerkin method for the Fredholm integral equations of the second kind [8, 9, 22]. A more important contribution made in [16] lies in the fact that the superconvergence result was established under weaker assumptions [16, Theorem 3.3]. The approach used in [16] to establish the superconvergence of the iterated Galerkin method can easily be adopted to prove the results of Graham et al. [8], Joe [9] and Sloan [22] under weaker conditions imposed upon the Fredholm equations. This will be demonstrated in Section 3. In Section 2, we review the collocation method for Hammerstein equations as well as

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some necessary known results that will be pertinent to the materials in the ensuing sections. We recall that the collocation method for weakly singular Hammerstein equations was discussed and some superconvergence results of the numerical solutions at the collocation points were discovered by Kaneko, Noren and Xu in [12]. In Section 3, the supereconvergence of the iterated collocation method for Hammerstein equations is established. The results obtained there encompass Hammerstein equations with smooth as well as weakly singular kernels. Finally, in Section 4, we discuss the discrete collocation method for Hammerstein equations with weakly singular kernels. The result obtained in this section extends the results of [3, 18] which deals with the discrete collocation methods for Hammeratein equations with smooth kernels. Some examples are also included in this section.

We note that there have been several other research papers published in recent years that describe various numerical methods for Hammerstein equations. A variant of Nystöm method was proposed by Lardy [19]. The degenerate kernel method was studied by Kaneko and Xu [15]. We point out that a superconvergence of the iterates of the degenerate kernel method was recently observed when a decomposition of the kernel is done properly. This will be reported in a future paper [14]. The reader who is interested in more information on numerical methods for a wider class of nonlinear integral equations may find necessary materials in [2, 5].

2. The collocation method

In this section, the collocation method for Hammerstein equations is presented. Some materials from the approximation theory are also reviewed in this section to make the present paper self-contained. We consider the following Hammerstein equation

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))\,ds = f(t), \quad 0 \leqslant t \leqslant 1, \tag{2.1}$$

where k, f and ψ are known functions and x is the function to be determined. Define $k_t(s) \equiv k(t,s)$ for $t,s \in [0,1]$ to be the t section of k. We assume throughout this paper unless stated otherwise, the following conditions on k, f and ψ:

1. $\lim_{t\to\tau} \| k_t - k_\tau \|_\infty = 0, \quad \tau \in [0,1]$;
2. $M \equiv \sup_t \int_0^1 |k(t,s)|ds < \infty$;
3. $f \in C[0,1]$;
4. $\psi(s,x)$ is continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which

$$|\psi(s,x_1) - \psi(s,x_2)| \leqslant C_1|x_1 - x_2|, \quad \text{for all } x_1,x_2 \in (-\infty,\infty);$$

5. the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

$$|\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \leqslant C_2|x_1 - x_2| \quad \text{for all } x_1,\ x_2 \in (-\infty,\infty); \tag{2.2}$$

6. for $x \in C[0,1]$, $\psi(.,x(.)),\ \psi^{(0,1)}(.,x(.)) \in C[0,1]$.

We let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t,s)\psi(s,x(s))\,\mathrm{d}s.$$

With this notation, Eq. (2.1) takes the following operator form:

$$x - K\Psi x = f. \tag{2.3}$$

For any positive integer n, we let

$$\Pi_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

be a partition of $[0,1]$. Let r and v be nonnegative integers satisfying $0 \leqslant v < r$. Let $S_r^v(\Pi_n)$ denote the space of splines of order r, continuity v, with knots at Π_n, i.e.,

$$S_r^v(\Pi_n) = \{x \in C^v[0,1]: x|_{[t_i,t_{i+1}]} \in \mathcal{P}_{r-1}, \text{ for each } i = 0,1,\ldots,n-1\},$$

where $\mathcal{P}_{r-1}$ denotes the space of polynomials of degree $\leqslant r-1$. For the collocation method, we are interested in the cases $v = 0$ or 1. That is, it is possible to work with the space of piecewise polynomials with no continuity at the knots or with the space of continuous piecewise polynomials with no continuity requirement on the derivatives at the knots. We assume that the sequence of partitions Π_n of $[0,1]$ satisfies the condition that there exists a constant $C > 0$, independent of n, with the property:

$$\frac{\max_{1\leqslant i\leqslant n}(t_i - t_{i-1})}{\min_{1\leqslant i\leqslant n}(t_i - t_{i-1})} \leqslant C, \quad \text{for all } n. \tag{2.4}$$

In many cases, Eq. (2.1) possesses multiple solutions (see, e.g., [15]). Hence, it is assumed for the remainder of this paper that we treat an isolated solution x_0 of (2.1). Let $I_i = (t_{i-1}, t_i)$ for each $i = 1,\ldots,n$. Then for $v = 0$, we let $\tau_{i1}, \tau_{i2}, \ldots, \tau_{ir}$ be the Gaussian points (the zeros of the rth degree Legendre polynomial on $[-1,1]$) shifted to the interval I_i. We define

$$G_0 = \{\tau_{ij}: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r\}. \tag{2.5}$$

The points in G_0 give rise to the piecewise collocation method where no continuity between polynomials is assumed. This is the approach taken by Graham, Joe and Sloan [8]. Joe [9], on the other hand, considered the continuous piecewise polynomial collocation method. His method corresponds with taking $v = 1$. Here we define the set G_1 of the collocation points to be the set consisting of the knots along with the Labatto points (the zeros of the first derivative of the $(r-1)$th degree Legendre polynomial) shifted to the interval I_i. Namely, let $\xi_{r-1} = 1$ and for $1 \leqslant l \leqslant r-2$ $(r \geqslant 3)$, let ξ_l denotes the lth Labatto point. If $|I_i|$ denotes the length of I_i, then G_1 contains

$$\tau_{(i-1)(r-1)+l+1} = \tfrac{1}{2}(t_{i-1} + t_i + |I_i|\xi_l), \quad 1 \leqslant i \leqslant n, \ \ 1 \leqslant l \leqslant r-1 \ \text{ and } \ \tau_1 = t_0 = 0. \tag{2.6}$$

The analyses of [8, 9] are very similar. We therefore confine ourselves to developing the collocation method for Hammerstein equations that is analogous to the method of [8]. An obvious extension to the continuous piecewise collocation method will be left to the reader. Define the interpolatory projection P_n from $C[0,1] + S_r^v(\Pi_n)$ to $S_r^v(\Pi_n)$ by requiring that, for $x \in C[0,1] + S_r^v(\Pi_n)$,

$$P_n x(\tau_{ij}) = x(\tau_{ij}), \quad \text{for all } \tau_{ij} \in G_0. \tag{2.7}$$

Then we have, for $x \in C[0,1] + S_r^{\nu}(\Pi_n)$

$$P_n x \to x \quad \text{as } n \to \infty \tag{2.8a}$$

and consequently ,

$$\sup_n \| P_n \| < c. \tag{2.8b}$$

The collocation equation corresponding to (2.3) can be written as

$$x_n - P_n K\Psi x_n = P_n f, \tag{2.9}$$

where $x_n \in S_r^{\nu}(\Pi_n)$. Now we let

$$\hat{T}x \equiv f + K\Psi x \quad \text{and} \quad T_n x_n \equiv P_n f + P_n K\Psi x_n$$

so that Eqs. (2.3) and (2.9) can be written, respectively, as $x = \hat{T}x$ and $x_n = T_n x_n$. We obtain

Theorem 2.1. *Let $x_0 \in C[0,1]$ be an isolated solution of Eq. (2.3). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. Then the collocation approximation equation (2.9) has a unique solution $x_n \in B(x_0, \delta)$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \leqslant \| x_n - x_0 \|_\infty \leqslant \frac{\alpha_n}{1-q}, \tag{2.10}$$

where $\alpha_n \equiv \| (I - T_n'(x_0))^{-1}(T_n(x_0) - \hat{T}(x_0)) \|_\infty$. Finally,

$$E_n(x_0) \leqslant \| x_n - x_0 \|_\infty \leqslant C E_n(x_0), \tag{2.11}$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u \in S_r^{\nu}(\Pi_n)} \| x_0 - u \|_\infty$.

A proof is a straight application of Theorem 2 of [23] and is demonstrated in the proof of Theorem 2.1 [12]. We denote by $W_p^m[0,1]$, $1 \leqslant p \leqslant \infty$, the Sobolev space of functions g whose mth generalized derivative $g^{(m)}$ belongs to $L_p[0,1]$. The space $W_p^m[0,1]$ is equipped with the norm

$$\| g \|_{W_p^m} \equiv \sum_{k=0}^{m} \| g^{(k)} \|_p .$$

It is known from [6, 7] that if $0 \leqslant \nu < r$, $1 \leqslant p \leqslant \infty$, $m \geqslant 0$ and $x \in W_p^m$, then for each $n \geqslant 1$, there exists $u_n \in S_r^{\nu}(\Pi_n)$ such that

$$\| x - u_n \|_p \leqslant C h^{\mu} \| x \|_{W_p^{\mu}}, \tag{2.12}$$

where $\mu = \min\{m, r\}$ and $h = \max_{1 \leqslant i \leqslant n}(t_i - t_{i-1})$. The inequality (2.12) when combined with Theorem 2.1 yields the following theorem.

Theorem 2.2. *Let x_0 be an isolated solution of Eq.* (2.3) *and let x_n be the solution of Eq.* (2.9) *in a neighborhood of x_0. Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$. If $x_0 \in W^l_\infty$, then*

$$\| x_0 - x_n \|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W^l_p$ $(1 \leqslant p < \infty)$, then

$$\| x_0 - x_n \|_\infty = O(h^\nu),$$

where $\nu = \min\{l - 1, r\}$.

When the kernel k is of weakly singular type, namely, if

$$k(t,s) = m(t,s) g_\alpha(|t - s|), \tag{2.13}$$

where $m \in C^{\mu+1}([0,1] \times [0,1])$ and

$$g_\alpha(s) = \begin{cases} s^{\alpha-1}, & 0 < \alpha < 1, \\ \log s, & \alpha = 1. \end{cases} \tag{2.14}$$

then the solution x_0 of Eq. (2.3) does not, in general, belong to W^m_p. To better characterize the regularity of the solution of (2.3) with weakly singular kernel, consider a finite set S in $[0,1]$ and define the function $\omega_S(t) = \inf\{|t - s| : s \in S\}$. A function x is said to be of *Type*(α, k, S), for $-1 < \alpha < 0$, if

$$|x^{(k)}(t)| \leqslant C[\omega_S(t)]^{\alpha-k} \quad t \notin S,$$

and for $\alpha > 0$, if the above condition holds and $x \in Lip(\alpha)$. Here $Lip(\alpha)=\{x : |x(t)-x(s)| \leqslant C|t-s|^\alpha\}$. It was proved by Kaneko et al. [11] that if f is of *Type*$(\beta, \mu, \{0,1\})$, then a solution of Eq. (2.1) with the kernel defined by (2.13) is of *Type*$(\gamma, \mu, \{0,1\})$, where $\gamma = \min\{\alpha, \beta\}$. The optimal rate of convergence of the collocation solution x_n to x_0 can be recovered by selecting the knots that are defined by

$$t_i = \begin{cases} \frac{1}{2}(2i/n)^q, & 0 \leqslant i \leqslant n/2, \\ 1 - t_{n-i}, & n/2 < i \leqslant n, \end{cases} \tag{2.15}$$

where $q = r/\gamma$ denotes the index of singularity. Details can be found in [12].

3. The iterated collocation method

The faster convergence of the iterated Galerkin method for the Fredholm integral equations of the second kind compared to the Galerkin method was first observed by Sloan in [20, 21]. On the other hand, the superconvergence of the iterated collocation method was studied in [8, 9]. Given the equation of the second kind

$$x - Kx = f, \tag{3.1}$$

where K is a compact operator on $X \equiv C[0,1]$ and $x,\ f \in X$, the collocation approximation x_n is the solution of the following projection equation:

$$x_n - P_n K x_n = P_n f. \tag{3.2}$$

Here P_n is the interpolatory projection of (2.7). The iterated collocation method obtains a solution x_n^{I} by

$$x_n^{\mathrm{I}} = f + K x_n. \tag{3.3}$$

Under the assumption of

$$\| KP_n - K \| \to 0 \quad \text{as } n \to \infty \tag{3.4}$$

it can be shown that

$$\| x - x_n^{\mathrm{I}} \| \leqslant \| (I - KP_n)^{-1} \| \| K(x - P_n x) \| . \tag{3.5}$$

The assumption (3.4) is satisfied if $X = L_2$ and P_n is the orthogonal projection satisfying $\| P_n g - g \| \to 0$ for all g in the closure of the range of the adjoint K^* of K since in this case $\| KP_n - K \| = \| P_n K^* - K^* \|$. The results of Sloan were recently generalized to the iterated Galerkin method for Hammerstein equations by Kaneko and Xu [16]. The main theorem of [16], Theorem 3.3, that guarantees the superconvergence of the iterates was proved by making use of the collectively compact operator theory.

The purpose of this section is to study the superconvergence of the iterated collocation method. For the collocation solution x_n of (2.9), we define

$$x_n^{\mathrm{I}} = f + K\Psi x_n. \tag{3.6}$$

A standard argument shows that x_n^{I} satisfies

$$x_n^{\mathrm{I}} = f + K\Psi P_n x_n^{\mathrm{I}}. \tag{3.7a}$$

We denote the right side of (3.7a) by $S_n x_n^{\mathrm{I}}$, namely,

$$S_n x_n^{\mathrm{I}} \equiv f + K\Psi P_n x_n^{\mathrm{I}}. \tag{3.7b}$$

We recall the following two lemmas from [16].

Lemma 3.1. *Let $x_0 \in C[0,1]$ be an isolated solution of (2.3). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently large n, the operators $I - S_n'(x_0)$ are invertible and there exists a constant $L > 0$ such that*

$$\| (I - S_n'(x_0))^{-1} \|_\infty \leqslant L \quad \textit{for sufficiently large } n.$$

Lemma 3.2. *Let $x_0 \in C[0,1]$ be an isolated solution of Eq. (2.3) and x_n be the unique solution of (2.9) in the sphere $B(x_0, \delta_1)$. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently* *large n, x_n^{I} defined by the iterated scheme (3.6) is the unique solution of (3.7) in the sphere $B(x_0,\delta)$. Moreover, there exists a constant $0<q<1$, independent of n, such that*

$$\frac{\beta_n}{1+q} \leqslant \| x_n^{\mathrm{I}} - x_0 \|_\infty \leqslant \frac{\beta_n}{1-q},$$

where $\beta_n = \| (I - S_n'(x_0))^{-1}[S_n(x_0) - \hat{T}(x_0)] \|_\infty$. Finally,

$$\| x_n^{\mathrm{I}} - x_0 \|_\infty \leqslant CE_n(x_0).$$

The definitions of δ and δ_1 are described in [16]. Following the development made in [16], we let

$$\psi(s,y) = \psi(s,y_0) + \psi^{(0,1)}(s, y_0 + \theta(y-y_0))(y-y_0), \tag{3.8}$$

where $\theta := \theta(s,y_0,y)$ with $0<\theta<1$. Also let

$$g(t,s,y_0,y,\theta) = k(t,s)\psi^{(0,1)}(s, y_0+\theta(y-y_0)),$$

$$(G_n x)(t) = \int_0^1 g(t,s,P_n x_0(s), P_n x_n^{\mathrm{I}}(s),\theta)x(s)\,\mathrm{d}s,$$

and $(Gx)(t) = \int_0^1 g_t(s)x(s)\,\mathrm{d}s$, where $g_t(s)=k(t,s)\psi^{(0,1)}(s,x_0(s))$. Now we are ready to state and prove the main theorem of this paper. The proof is a combination of the idea used in [Theorem 3.3; 8, Theorem 4.2].

Theorem 3.3. *Let $x_0 \in C[0,1]$ be an isolated solution of Eq. (2.3) and x_n be the unique solution of (2.9) in the sphere $B(x_0,\delta_1)$. Let x_n^{I} be defined by the iterated scheme (3.7). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Assume that $x_0 \in W_1^l$ $(0<l\leqslant 2r)$ and $g_t \in W_1^m$ $(0<m\leqslant r)$ with $\| g_t \|_{W_1^m}$ bounded independently of t. Then*

$$\| x_0 - x_n^{\mathrm{I}} \|_\infty = \mathrm{O}(h^\gamma), \quad \text{where } \gamma = \min\{l, r+m\}.$$

Proof. From Eqs. (2.3) and (3.7), we obtain

$$x_0 - x_n^{\mathrm{I}} = K(\Psi x_0 - \Psi P_n x_n^{\mathrm{I}}) = K(\Psi x_0 - \Psi P_n x_0) + K(\Psi P_n x_0 - \Psi P_n x_n^{\mathrm{I}}). \tag{3.9}$$

Using (3.8), the last term of (3.9) can be written as

$$K(\Psi P_n x_0 - \Psi P_n x_n^{\mathrm{I}})(t) = (G_n P_n(x_0 - x_n^{\mathrm{I}}))(t).$$

Eq. (3.9) then becomes

$$x_0 - x_n^{\mathrm{I}} = K(\Psi x_0 - \Psi P_n x_0) + G_n P_n (x_0 - x_n^{\mathrm{I}}). \tag{3.10}$$

Using the Lipschitz condition (2.2) imposed on $\psi^{(0,1)}$, for $x \in C[0,1]$,

$$\| (G_n x) - (Gx) \|_\infty \leqslant C_2 \sup_{0\leqslant t\leqslant 1} \int_0^1 |k(t,s)|\,\mathrm{d}s \; \| x \|_\infty (\| P_n x_0 - x_0 \|_\infty + \| P_n \|_\infty \| x_n^{\mathrm{I}} - x_0 \|_\infty).$$

This shows that

$$\| G_n - G \|_\infty \leqslant MC_2(\| P_n x_0 - x_0 \|_\infty + c \| x_n^{\mathrm{I}} - x_0 \|_\infty) \to 0 \quad \text{as} \quad n \to \infty.$$

Also, for each $x \in C[0,1]$,

$$\sup_{0 \leqslant t \leqslant 1} |(GP_n x)(t) - (Gx)(t)| = \sup_{0 \leqslant t \leqslant 1} \left| \int_0^1 g_t(s)[P_n x(s) - x(s)]\, ds \right| \leqslant MM_1 \| P_n x - x \|_\infty,$$

where

$$M_1 = \sup_{0 \leqslant t \leqslant 1} |\psi^{(0,1)}(t, x_0(t))| < +\infty.$$

It follows that $GP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. Again since P_n is uniformly bounded, we have for each $x \in C[0,1]$,

$$\| G_n P_n x - Gx \|_\infty \leqslant \| G_n - G \|_\infty \| P_n \|_\infty \| x \|_\infty + \| GP_n x - Gx \|_\infty .$$

Thus, $G_n P_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. By Assumptions 2, 5, and 6, we see that there exists a constant $C > 0$ such that for all n

$$|\psi^{(0,1)}(s, P_n x_0(s) + \theta(P_n x_n^{\mathrm{I}}(s) - P_n x_0(s)))| \leqslant C_2 \| P_n x_0 - x_0 \|_\infty + \theta C_2 P \| x_n^{\mathrm{I}} - x_0 \|_\infty + M_1 \leqslant C.$$

This implies that $\{G_n P_n\}$ is a family of collectively compact operators [1]. Since $G = (K\Psi)'(x_0)$ is compact and $(I - G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I - G_n P_n)^{-1}$ exists and is uniformly bounded for sufficiently large n. Now using (3.10), we see that

$$\| x_0 - x_n^{\mathrm{I}} \|_\infty \leqslant C \| K(\Psi x_0 - \Psi P_n x_0) \| .$$

Hence we need to estimate $\| K(\Psi x_0 - \Psi P_n x_0) \|$. The following four inequalities are known [8, Theorem 4.2]. Let $\psi_n \in S_l^0(\Pi_n)$ be such that

$$\sum_{i=1}^{n} \| (x_0 - \psi_n)^{(j)} \|_{W_1^m(I_i)} \leqslant ch^{l-j} \| x_0 \|_{W_1^l}, \quad 0 \leqslant j \leqslant l, \tag{3.11}$$

$$\max_{1 \leqslant i \leqslant n} \| \psi_n^{(j)} \|_{W_\infty^m(I_i)} \leqslant c \| x_0 \|_{W_1^l}, \quad j \geqslant 0. \tag{3.12}$$

Also for each $t \in [0,1]$, there exists $\varphi_{n,t} \in S_m^0(\Pi_n)$ such that

$$\sum_{i=1}^{n} \| (g_t - \varphi_{n,t})^{(j)} \|_{W_1^m(I_i)} \leqslant ch^{m-j} K_m, \quad 0 \leqslant j \leqslant m, \tag{3.13}$$

$$\max_{1 \leqslant i \leqslant n} \| \varphi_{n,t}^{(j)} \|_{W_\infty^m(I_i)} \leqslant cK_m, \quad j \geqslant 0, \tag{3.14}$$

where $K_m = \sup_{0 \leqslant t \leqslant 1} \| k_t \|_{W_1^m} < \infty$. Now for $t \in [0,1]$ we have

$$\begin{aligned} K(\Psi x_0 - \Psi P_n x_0)(t) &= (g_t - \varphi_{n,t}, x_0 - P_n x_0) + (\varphi_{n,t}, (I - P_n)(x_0 - \psi_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n)\psi_n). \end{aligned} \tag{3.15}$$

Using Eqs. (3.11)–(3.14) along with the arguments from [8, p. 362] we can show that each of the three terms is bounded by ch^γ uniformly in t. This completes our proof. □

One way to establish the superconvergence of the iterated collocation method for the Fredholm equation is to assume (3.4). In the context of the present discussion, (3.4) is equivalent to assuming

$$\| (K\Psi)'(x_0)(I-P_n)|_{C[a,b]} \|_\infty \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Theorem 3.3 was thus proved under weaker assumptions. The idea used to prove Theorem 3.3 originates from [4, Section 6] in which the superconvergence of the iterated collocation method for the Fredholm equations was established by showing that $\{KP_n\}$ is a family of collectively compact operators.

Finally in this section, we investigate the superconvergence of the iterated collocation method for weakly singular Hammerstein equation. Specifically, we consider Eq. (2.3) with kernel given by (2.13) and (2.14). An enhancement in the rate of convergence is given in the following theorem.

Theorem 3.4. *Let $x_0 \in C[0,1]$ be an isolated solution of Eq.* (2.3) *and x_n be the unique solution of* (2.9) *in the sphere $B(x_0, \delta_1)$ with kernel defined by* (2.13) *and* (2.14) *and knots defined by* (2.15). *Let x_n^{I} be defined by the iterated scheme* (3.7). *Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$ and that $\psi^{(0,1)}(\cdot, x_0(\cdot))$ is of Type$(\alpha, r, \{0,1\})$ for $\alpha > 0$ whenever x_0 is of the same type. Then*

$$\| x_0 - x_n^{\mathrm{I}} \|_\infty = \mathrm{O}(h^{r+\alpha}).$$

Proof. We follow the proof of Theorem 3.3 exactly the same way to (3.15), which is

$$\begin{aligned} K(\Psi x_0 - \Psi P_n x_0)(t) &= (g_t - \varphi_{n,t}, x_0 - P_n x_0) + (\varphi_{n,t}, (I-P_n)(x_0 - \psi_n)) \\ &\quad + (\varphi_{n,t}, (I-P_n)\psi_n). \end{aligned}$$

The difference in superconvergence arises from the degree to which we may bound the first term. As in [16, Theorem 3.6], using an argument similar to [17], it can be proved that there exists $u \in S_r^\nu(\Pi_n)$ with knots Π_n given by (2.15) such that $\| g_t - u \|_1 = \mathrm{O}(h^\alpha)$. Here $h = \max_{1 \leqslant i \leqslant n}\{x_i - x_{i-1}\}$. Then

$$\begin{aligned} |(g_t - \varphi_{n,t}, x_0 - P_n x_0)| &\leqslant \| g_t - \varphi_{n,t} \|_1 \| x_0 - P_n x_0 \|_\infty \\ &= \mathrm{O}(h^{\alpha+r}). \end{aligned}$$

The rest of proof follows in the same way as described in [8, p. 362]. □

4. The discrete collocation method for weakly singular Hammerstein equations

Several papers have been written on the subject of the discrete collocation method. Joe [10] gave an analysis of discrete collocation method for second kind Fredholm integral equations. A discrete collocation-type method for Hammerstein equations was described by Kumar in [18]. Most recently Atkinson and Flores [3] put together the general analysis of the discrete collocation methods for nonlinear integral equations. In this section, we describe a discrete collocation method for weakly singular Hammerstein equations. In the aforementioned papers [10, 18, 3], their discussions are primarily concerned with integral equations with smooth kernels. Even though, in principle, an analysis for the discrete collocation method for weakly singular Hammerstein equations is similar to the one given in [3], we feel that a detailed discussion on some specific points pertinent to weakly singular equations, e.g., a selection of a particular quadrature scheme, a convergence analysis, etc., will be of great interest to practioners. Our convergence analysis of the discrete collocation method presented in this section is different from the one given in [3] in that it is based upon Theorem 2 of Vainikko [23]. The idea of the quadrature used here was recently developed by Kaneko and Xu [17] and a complete Fortran program based on the idea is being developed by Kaneko and Padilla [13]. A particular case of the quadrature schemes developed in [16] is concerned with an approximation of the integral

$$I(f)=\int_0^1 f(s)\,\mathrm{d}s, \tag{4.1}$$

where $f\in Type(\alpha,2r,S)$ with $\alpha>-1$. For simplicity of demonstration, we assume $S=\{0\}$. We define $q=(2r+1)/(\alpha+1)$ and a partition

$$\pi_\alpha\colon s_0=0,\quad s_1=n^{-q},\quad s_j=j^q s_1,\quad j=2,3,\ldots,n. \tag{4.2}$$

Now we construct a piecewise polynomial S_r of degree $r-1$ by the following rule; $S_r(s)=0$, $s\in[s_0,s_1)$ and $S_r(s)$ is the Lagrange polynomial of degree $r-1$ interpolating f at $\{u_j^{(i)}\}_{j=1}^r$ for $s\in[s_i,s_{i+1})$, $i=1,2,\ldots,n-2$ and for $x\in[x_{n-1},x_n]$. Here $\{u_j^{(i)}\}_{j=1}^r$ denote the zeros of the rth degree Legendre polynomial transformed into $[s_i,s_{i+1})$. Our approximation process consists of two stages. First, $I(f)$ is approximated by

$$\hat{I}(f)=\int_{x_1}^1 f(s)\,\mathrm{d}s=\sum_{i=1}^{n-1}\int_{s_i}^{s_{i+1}} f(s)\,\mathrm{d}s. \tag{4.3}$$

Second, $\hat{I}(f)$ is approximated by $\hat{I}(S_r)=\int_{s_1}^1 S_r(s)\,\mathrm{d}s$. A computation of $\hat{I}(S_r)$ can be accomplished as follows; let $\theta:[s_i,s_{i+1}]\to[-1,1]$ be defined by $\theta=[2s-(s_{i+1}+s_i)]/(s_{i+1}-s_i)$ so that

$$\hat{I}(f)=\int_{-1}^1 F_f(\theta)\,\mathrm{d}\theta, \tag{4.4}$$

where

$$F_f(\theta)=\sum_{i=1}^{n-1}\frac{1}{2}(s_{i+1}-s_i)f\left(\tfrac{1}{2}(s_{i+1}-s_i)\theta+\tfrac{1}{2}(s_{i+1}+s_i)\right).$$

If $\{u_i\colon i=1,2\ldots,r\}$ denotes the zeros of the Legendre polynomial of degree r, then

$$S_r(s)=\sum_{i=1}^r F_f(u_i)l_i(s)$$

with $l_i(s)$ the fundamental Lagrange polynomial of degree $r-1$ so that

$$\hat{I}(S_r)=\sum_{i=1}^r w_iF_f(u_i),\quad \text{where } w_i=\int_{-1}^1 l_i(s)\,\mathrm{d}s. \tag{4.5}$$

It was proved in [17] that

$$|I(f)-\hat{I}(S_r)|=\mathrm{O}(n^{-2r}). \tag{4.6}$$

In this section, we examine Eq. (2.1) with the kernel k defined by (2.13) and (2.14). When the knots are selected according to (2.15), as stated earlier, it was shown in [12] that the solution x_n of the collocation Eq. (2.9) converges to the solution x of (2.1) in the rate that is optimal to the degree of polynomials used. Specifically, x_n must be found by solving

$$x_n(u_j^{(i)}) - \int_0^1 g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)},s)\psi(s,x_n(s))\,\mathrm{d}s = f(u_j^{(i)}) \tag{4.7}$$

where $i = 0,1,\ldots,n-1$ and $j = 1,2,\ldots,r$.

The discrete collocation method for Eq. (2.1) is obtained when the integral in (4.7) is replaced by a numerical quadrature given in (4.5). Let $k_{ij}(s) \equiv g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)},s)$. Then

$$\int_0^1 g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)},s)\psi(s,x_n(s))\,\mathrm{d}s = \int_0^1 k_{ij}(s)\psi(s,x_n(s))\,\mathrm{d}s$$

$$= \int_0^{u_j^{(i)}} + \int_{u_j^{(i)}}^1 k_{ij}(s)\psi(s,x_n(s))\,\mathrm{d}s. \tag{4.8}$$

The integrals in the last expression of (4.8) represent two weakly singular integrals which can be approximated to within $\mathrm{O}(n^{-2r})$ order of accuracy by (4.5) by transforming them to $[-1,1]$ and selecting the points in (4.2) appropriately.

Writing (4.7) as

$$P_n x_n - P_n K \Psi x_n = P_n f, \tag{4.9}$$

we consider the approximation $\tilde{x}_n$ to x_n defined as the solution of

$$\tilde{x}_n = Q_n \tilde{x}_n \equiv P_n K_n \Psi \tilde{x}_n + P_n f, \tag{4.10}$$

where K_n is the discrete collocation approximation to the integrals in (4.8) described above.

We will use [23, Theorem 2] to find a unique solution to (4.10) in some δ neighborhood of x_n, where n is sufficiently large. Clearly, $Q_n'(x) = P_n K_n \Psi'(x)$, where $\Psi'(x)[y](s) = \psi^{(0,1)}(s,x(s))y(s)$. For sufficiently large n, (4.9) has a unique solution in some δ neighborhood of x. To see that $I - Q_n'(x_n)$ is continuously invertible with $\{(I - Q_n'(x_n))^{-1}\}_{n=N}^{\infty}$ uniformly bounded, it is enough to observe that $\{Q_n'(x_n)\}_{n=1}^{\infty}$ is collectively compact, and to do this we will show that

$$|\, Q_n'(x_n)[x](t) - Q_n'(x_n)[x](t') \,| = |\, P_n K_n \Psi'(x_n)x(t) - P_n K_n \Psi'(x_n)x(t') \,| \to 0 \tag{4.11}$$

as $t \to t'$, for each $x \in C[0,1]$, [1]. Here N is some sufficiently large number.

If we show (4.11), then part (a) of Theorem 2 [23] is also verified. In order to verify part (b) of Theorem 2 [23], we only need to establish (because of the uniform boundedness of $\{(I - Q_n(x_n))^{-1}\}_{n=N}^{\infty}$) that

$$\|\, Q_n'(x) - Q_n'(x_n) \,\|_\infty \leqslant L \,\|\, x - x_n \,\|_\infty \leqslant L\delta, \tag{4.12}$$

for some constant L, and

$$\|\, Q_n(x_n) - T_n(x_n) \,\| \to 0 \quad \text{as } n \to \infty. \tag{4.13}$$

Once this is done, Theorem 2 [23] applies yielding a unique solution $\tilde{x}_n$ in some neighborhood of x_n (for sufficiently large n) and

$$\| x_n - \tilde{x}_n \| \leqslant L\tilde{\alpha}_n \leqslant L \| Q_n(x_n) - T_n(x_n) \|_\infty . \tag{4.14}$$

(Here and throughout the remainder of the section, L denotes a generic constant, the exact value of which may differ at each occurrence.) This inequality will be used to obtain the order of convergence.

Considering (4.11), the right-hand side is bounded by $T_1 + T_2 + T_3$, where

$$T_1 = | P_nK_n\Psi'(x_n)x(t) - P_nK\Psi'(x_n)x(t) |,$$

$$T_2 = | P_nK\Psi'(x_n)x(t) - P_nK\Psi'(x_n)x(t') |,$$

$$T_3 = | P_nK_n\Psi'(x_n)x(t') - P_nK_n\Psi'(x_n)x(t') | .$$

Let $\varepsilon > 0$. Since $\{P_n\}_{n=1}^{\infty}$ is uniformly bounded, $T_1 + T_3 < \frac{2}{3}\varepsilon$ by applying (4.6) with $f(s) = \psi^{(0,1)}(s,x_n(s))x(s)$ and letting n be sufficiently large. For T_2 we have $T_2 \leqslant M\int_0^1 | k(t,s) - k(t',s) | \,\mathrm{d}s \leqslant M(S_1 + S_2)$,
where

$$S_1 = \int_0^1 g_\alpha(| s - t |) \, | m(t,s) - m(t',s) | \,\mathrm{d}s$$

and

$$S_2 = \int_0^1 | g_\alpha(| t - s |) - g_\alpha(| t' - s |) | \, | m(t',s) | \,\mathrm{d}s.$$

but

$$S_1 \leqslant \sup_{0\leqslant s\leqslant 1} | m(t,s) - m(t',s) | \int_0^1 g_\alpha(| t - s |)\,\mathrm{d}s$$
$$\leqslant L \sup_{0\leqslant s\leqslant 1} | m(t,s) - m(t',s) | \to 0 \quad \text{as } t \to t',$$

and

$$S_2 \leqslant L \int_0^1 | g_\alpha(| t - s |) - g_\alpha(| t' - s |) | \,\mathrm{d}s$$
$$= (L/\alpha)\{| t^\alpha - (t')^\alpha | + | (1-t)^\alpha - (1-t')^\alpha | + (4/2^\alpha) | t - t' |^\alpha\}$$
$$\to 0 \quad \text{as } t \to t'.$$

Hence (4.11) holds. For (4.12),

$$\| Q_n'(x) - Q_n'(x_n) \|_\infty = \| P_nK_n(\Psi'(x) - \Psi'(x_n)) \|$$
$$\leqslant MC \| x - x_n \| \leqslant M\delta = q < 1$$

for δ sufficiently small. Note that we have used the uniform boundedness of $\{P_n\}, \{K_n\}$ and because $\Psi^{(0,1)}(s, y(s))$ is locally Lipschitz, so is the operator $\Psi' : C[0,1] \to B(C[0,1], C[0,1])$ (the space of bounded linear operators from $C[0,1]$ into $C[0,1]$).

For (4.13), we have

$$\| Q_n(x_n) - T_n(x_n) \|_\infty = \| P_n(K_n \Psi x_n - K \Psi x_n) \| \leqslant L \| (K_n - K)\Psi(x_n) \leqslant L(R_1 + R_2 + R_3), \tag{4.15}$$

where

$$R_1 = \| K_n\Psi(x_n) - K_n\Psi(x_0) \|, \qquad R_2 = \| K_n\Psi(x_0) - K\Psi(x_0) \|, \qquad R_3 = \| K\Psi(x_0) - K\Psi(x_n) \| . \tag{4.16}$$

But

$$R_1 \leqslant L \| \Psi(x_n) - \Psi(x_0) \| \leqslant C_1 L \| x_n - x_0 \| \tag{4.17}$$

because Ψ is a Lipschitz operator and $\{K_n\}$ is uniformly bounded, and also

$$R_3 \leqslant M \| \Psi(x_0) - \Psi(x_n) \| \leqslant C_1 M \| x_n - x_0 \| . \tag{4.18}$$

Finally,

$$R_2 = \mathrm{O}(n^{-2r}) \tag{4.19}$$

by (4.6) using $f(s) = \Psi(x, x_0(s))$.

Thus Vainikko's theorem yields a unique solution $\tilde{x}_n$ for n sufficiently large and (4.14) holds. Now (4.14) and (4.15) – (4.19) show that

$$\| x_n - \tilde{x}_n \| = \mathrm{O}(n^{-\beta}), \tag{4.20}$$

where β is the minimum of $2r$ and the order of convergence of $\|x_0 - x_n\|$. We summerize the results obtained above in the following theorem:

Theorem 4.1. *Let x_0 be an isolated solution of Eq.* (2.3) *and let x_n be the solution of Eq.* (2.9) *in a neighborhood of x_0. Moreover, let $\tilde{x}_n$ be the solution of* (4.10). *Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$. If $x_0 \in W^l_\infty$, then*

$$\|x_0 - \tilde{x}_n\|_\infty = \mathrm{O}(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W^l_p$ $(1 \leqslant p < \infty)$, then

$$\|x_0 - \tilde{x}_n\|_\infty = \mathrm{O}(h^\nu),$$

where $\nu = \min\{l - 1, r\}$.

5. Numerical examples

In this section we present two numerical examples. Let $k(s,t) = \mathrm{e}^{s-t}$ and $\Psi(s, x(s)) = \cos(s + x(s))$. The spline coefficients were obtained using a Newton–Raphson algorithm. Also, the Gauss-type

Table 1

n	Errors	
	Non-iterated	Iterated
2	0.153571593748756e − 1	0.286029074365e − 4
3	0.71758714356116e − 2	0.47721991441e − 5
4	0.41291276625525e − 2	0.14180649575e − 5
5	0.26770046422053e − 2	0.5636996160e − 6
Convergence rate	≈ 2	≈ 4

Table 2

n	Errors	
	Non-iterated	Iterated
2	0.157961272540103e − 1	0.24257900549439e − 2
3	0.71150661058771e − 2	0.7663852778203e − 3
4	0.41192622669880e − 2	0.3210258989686e − 3
5	0.25982238843077e − 2	0.1770978040470e − 3
Convergence rate	≈ 2	≈ 3

Table 3

n	Errors	
	Non-iterated	Iterated
2	0.01540556116740788	0.005968844100471715
3	0.00722550448387438	0.002566222099442683
4	0.00416092487581254	0.001371170616411344
5	0.00269785684908008	0.000835161756464808
Convergence rate	≈ 2	≈ 2.2

quadrature algorithm described in [17] is used to calculate all integrations. The computed errors for the solution and the iterated solution are shown in Table 1.

For the second example, let $k(s,t) = \log(|s-t|)$ and $\Psi(s,x(s)) = \cos(s + x(s))$. The computed errors for the solution and iterated solution of the weakly singular integral are shown in Table 2.

For the third example, let $k(s,t) = 1/\sqrt{|s-t|}$, $\Psi(s,x(s)) = \cos(s + x(s))$, and $x(t) = \cos(t)$. The computed errors for the solution and iterated solution of the weakly singular integral are shown in Table 3.

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