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The Strong Perfect Graph Conjecture for Pan-Free Graphs

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A graph G is perfect if for every induced subgraph F of G, the chromatic number $\chi(F)$ equals the largest number $\omega(F)$ of pairwise adjacent vertices in F. Berge's famous Strong Perfect Graph Conjecture asserts that a graph G is perfect if and only if neither G nor its complement $\bar{G}$ contains an odd chordless cycle of length at least five. Its resolution has eluded researchers for more than twenty years. We prove that the conjecture is true for a class of graphs which strictly contains the claw-free graphs.

1. INTRODUCTION

In the early 1960s, Claude Berge [1] proposed the study of *perfect* graphs: these are graphs G such that for every induced subgraph F of G the chromatic number $\chi(F)$ of F equals the largest number $\omega(F)$ of pairwise adjacent vertices in F. He conjectured that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. This conjecture was proved by Lovász [4] and is known as the Perfect Graph Theorem.

A graph G is called *minimal imperfect* if G itself is imperfect but every proper induced subgraph of G is perfect.

The only known minimal imperfect graphs are the odd chordless cycles of length at least five (also called *odd holes*) and their complements (termed *odd anti-holes*). Berge [2] conjectured that these are the only minimal imperfect graphs. This conjecture is the celebrated Strong Perfect Graph Conjecture (SPGC, for short) and it is still open.

We define a *k-pan* to be the graph obtained from a chordless cycle C_k $(k \geqslant 4)$ and a vertex x outside the cycle, by joining x by an edge to precisely one vertex of the cycle (see Fig. 1).

Call a graph *pan-free* if it contains no induced subgraph isomorphic to

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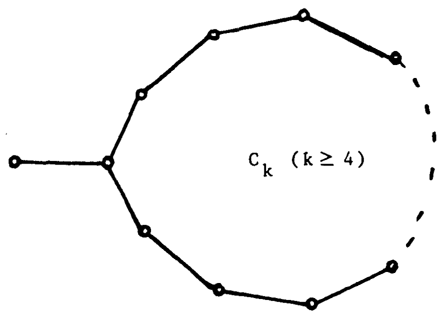

FIGURE 1

a k-pan $(k \geqslant 4)$. It is customary to refer to the graph with vertices a, b, c, d and edges ab, bc, bd as the *claw*.

Trivially, claw-free graphs are also pan-free, but not conversely. Thus, the class of pan-free graphs strictly contains the class of claw-free graphs.

Parthasarathy and Ravindra [6] proved the SPGC for claw-free graphs. The purpose of this work is to prove that the SPGC holds true for pan-free graphs.

2. THE RESULTS

Vašek Chvátal [3] defined the notion of *star-cutset*: this is a non-empty set C of vertices of a graph G such that $G - C$ is disconnected and some vertex in C is adjacent to all the remaining vertices in C. Chvátal [3] also proved the following result. (Actually, similar results were proved by Olaru [5] and Tucker [7].)

THE STAR-CUTSET LEMMA. *No minimal imperfect graph contains a star-cutset.*

As usual, we shall use *minimal* with respect to set inclusion, not size. Furthermore, we let the symbol N stand for neighbourhood: $N(w)$ denotes the set of all vertices of a graph G adjacent to w (we assume that adjacency is not reflexive, and so $w \notin N(w)$); $N'(w)$ stands for the set of all the vertices adjacent to w in the complement $\bar{G}$ of G.

We shall find it convenient to use the following simple properties:

(P1) Let G have at least three vertices. If neither G nor $\bar{G}$ has a star-cutset, then the neighbourhood $N(u)$ of every vertex u is a minimal cutset in G.

(P2) If a graph G contains a proper subset H of at least two vertices such that every vertex outside H is either adjacent to all the vertices in H

or to none of them, then G or $\bar{G}$ has a star-cutset. (A set H with the property described above is often referred to as *homogeneous.*)

[P1) is immediate; (P2) is a restatement of Theorem 1 in Lovász [4]].

We are now ready to state our main result.

THEOREM 1. *The Strong Perfect Graph Conjecture holds true for pan-free graphs.*

Our proof of Theorem 1 relies on the following result which is of independent interest.

THEOREM 2. *Let G be a pan-free graph. At least one of the following statements is true.*

(i) *G or $\bar{G}$ has a star-cutset,*

(ii) *G is claw-free.*

To see that Theorem 2 implies Theorem 1, consider a pan-free minimal imperfect graph. Theorem 2, the Star-Cutset Lemma, and the Perfect Graph Theorem combined guarantee that G must be claw-free. Now the result of Parthasarathy and Ravindra [5] implies that G or $\bar{G}$ is an odd hole.

Proof of Theorem 2. Let $G = (V, E)$ be a graph satisfying the hypothesis of Theorem 2. We only need to prove that if neither G nor its complement $\bar{G}$ has a star-cutset, then G is claw-free.

For this purpose, we shall assume that G has at least three vertices, for otherwise there is nothing to prove. If G is a clique, then we are trivially done.

Now G is not a clique and hence there exists a cutset in G. Let C be a minimal cutset in G, and enumerate the connected components of $G - C$ as $V_1, V_2, ..., V_t$ $(t \geqslant 2)$.

For further reference, we make the following simple observation whose justification is trivial.

Observation 1. For non-adjacent vertices v, w in C and for any choice of the subscript j, $1 \leqslant j \leqslant t$, there exists a chordless path joining v and w and having all the internal vertices in V_j.

In addition, we shall rely on the following intermediate results which we present as facts.

FACT 1. *For every component V_j and for every pair of distinct, non-adjacent vertices u, v in $V - (C \cup V_j)$, $N(u) \cap N(v) \cap C$ is a clique in G.*

Proof of Fact 1. Let V' stand for $N(u) \cap N(v) \cap C$. We only need to derive a contradiction from the assumption that V' is not a clique.

For this purpose, consider a component H with at least two vertices of the subgraph of $\bar{G}$ induced by V'. Since neither G nor $\bar{G}$ has a star-cutset, H cannot be a homogeneous set. We find, therefore, a vertex w outside H, adjacent to some, but not all the vertices in H. By the connectedness of H in $\bar{G}$, we find vertices h, h' in H that are non-adjacent in G, and such that $wh \in E$, $wh' \notin E$. The desired contradiction will be achieved as soon as we prove that the vertex w cannot exist.

First, we note that w is distinct from both u and v and, by the definition of H, w is not in V'.

Next, w is not in V_j, for otherwise $\{u, v, h, h', w\}$ would induce a k-pan with $k=4$.

Further, w is not in $V-(C \cup V_j)$. To see this, note that by Observation 1, there exists a chordless path P joining h and h' and having all the internal vertices in V_j. If w were in $V-(C \cup V_j)$, then w would be adjacent to both u and v, for if not, then $P \cup \{w, z\}$ would induce a k-pan ($k \geqslant 4$), with $z=u$ or $z=v$. However, now $\{h', h'', u, v, w\}$ induces a k-pan with $k=4$, for any neighbour h'' of h' in V_j.

Finally, w is not in $C - V'$. To see that this is the case, note that if w is in $C - V'$, then w cannot be adjacent to both u and v (else w would be in V'). If w is adjacent to neither u nor v, then $\{u, v, h, h', w\}$ induces a k-pan with $k = 4$. Hence, w is adjacent to precisely one of the vertices u and v. We shall assume, without loss of generality, that w is adjacent to v. Observation 1 guarantees the existence of a chordless path P' joining h' and w and having all the internal vertices in V_j. Thus, $P' \cup \{u, v\}$ induces a k-pan ($k \geqslant 4$), a contradiction.

This completes the proof of Fact 1. ∎

FACT 2. *For every component V_j, and for every vertex v in C, $N(v) \cap (V-(C \cup V_j))$ is a clique.*

Proof of Fact 2. Let V'' stand for $N(v) \cap (V-(C \cup V_j))$. We only need derive a contradiction from the assumption that V'' contains non-adjacent vertices.

For this purpose, let x and y be non-adjacent vertices in V''. We claim that

the intermediate vertices of all the paths in G joining x or y to a vertex in $C-N(v)$ contain v or a neighbour of v. (1)

Suppose not; there exists a path

$$P, \qquad z=w_0, w_1, ..., w_p \; (p \geqslant 2)$$

joining a vertex z in $\{x, y\}$ to some vertex w_p in C, and such that $w_i \notin \{v\} \cup N(v)$, for $i \geqslant 1$. Let P be the shortest path violating (1), and let r $(1 \leqslant r \leqslant p)$ be the first subscript such that $w_r \in C$.

Now Observation 1 guarantees the existence of a chordless path Q joining v and w_r, with all the internal vertices in V_j.

We note that Q together with $\{z, w_1, ..., w_{r-1}\}$ determines a chordless cycle Γ in G of length at least 4.

Let z' stand for the vertex in $\{x, y\}$ distinct from z. If $r = 1$, then $z'w_r \in E$, for otherwise $Q \cup \{z, z'\}$ induces a k-pan $(k \geqslant 4)$. But now, the vertices z, z' contradict Fact 1.

We may, therefore, assume $r \geqslant 2$. Clearly, $z'w_r \notin E$, for if not, then since $zw_r \notin E$, $Q \cup \{z, z'\}$ induces a k-pan $(k \geqslant 4)$, a contradiction.

Let s $(1 \leqslant s \leqslant r-1)$ be the first subscript for which $z'w_s \in E$. Trivially, $\{v, v', z, w_1, ..., w_s, z'\}$ induces a k-pan $(k \geqslant 4)$, for any neighbour v' of v in V_j. Therefore, z' is adjacent to no vertex w_i with $0 \leqslant i \leqslant r$. However, now $\Gamma \cup \{z'\}$ induces a k-pan $(k \geqslant 4)$, a contradiction.

Hence, (1) must hold, and so G has a star-cutset. This is the desired contradiction. ∎

To complete the proof of Theorem 2, assume that G contains an induced claw with vertices a, b, c, d and edges ab, bc, bd. Since, by assumption, neither G nor $\bar{G}$ has a star-cutset, property (P1) guarantees that the neighbourhood $N(a)$ of a is a minimal cutset in G. Now Fact 2, with $C = N(a)$, $V_j = \{a\}$ implies that $N(b) \cap N'(a)$ is a clique, a contradiction.

Thus G is claw-free, as claimed. ∎

References

1. C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* (1961), 114–115.
2. C. Berge, Sur une conjecture relative au problème des codes optimaux, Comm. 13e Assemblée générale de l'URSI, Tokyo, 1962.
3. V. Chvátal, Star-cutsets and perfect graphs, *J. Combin. Theory Ser. B* **39** (1985) 189–199.
4. L. Lovász, Normal hypergrahs and the perfect graph conjecture, *Discrete Math.* **2** (1972), 253–267.
5. E. Olaru, Uber die Uberdeckung von Graphen mit Cliquen, *Wiss. Z. Tech. Hochsch. Ilmenau* **15** (1969), 115–120.
6. K. R. Parthasarathy and G. Ravindra, The strong perfect graph conjecture is true for $K_{1,3}$-free graphs, *J. Combin. Theory Ser. B* **21** (1976), 212–223.
7. A. Tucker, Critical perfect graphs and perfect 3-chromtic graphs, *J. Combin. Theory Ser. B* **23** (1977), 143–149.