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Fixed Points of Generalized Contractive Multi-valued Mappings

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In a recent paper N. Mizoguchi and W. Takahashi gave a positive answer to the conjecture of S. Reich concerning the existence of fixed points of multi-valued mappings that satisfy a certain contractive condition. In this paper, we provide an alternative and somewhat more straightforward proof for the theorem of Mizoguchi and Takahashi. Also the problems associated with fixed points of weakly contractive multi-valued mappings are studied. Finally, we make a few comments that improve other results from their paper (*J. Math. Anal. Appl.* **141** (1989), 177–188).

1. INTRODUCTION

Let (X, d) be a metric space. A subset K of X is called *proximinal* if, for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf\{d(x, y) : y \in K\}$. The family of all bounded proximinal subsets of X is denoted by $P(X)$. We denote the family of all nonempty closed and bounded subsets of X by $\mathrm{CB}(X)$. A mapping $\phi: X \times X \to [0, \infty)$ is called *compactly positive* if $\inf\{\phi(x, y) : a \leq d(x, y) \leq b\} > 0$ for each finite interval $[a, b] \subseteq (0, \infty)$. A mapping $T: X \to \mathrm{CB}(X)$ is called

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weakly contractive if there exists a compactly positive mapping ϕ such that

$$H(T(x), T(y)) \leq d(x, y) - \phi(x, y)$$

for each x, $y \in X$, where H denotes the Hausdorff metric on $\mathrm{CB}(X)$ induced by d. We state below Lemma 1 of [6] for convenience.

LEMMA 1.1. *The following statements about a mapping $T: X \rightarrow P(X)$ are equivalent:*

(a) *T is weakly contractive.*

(b) *$H(T(x), T(y)) \leq h(x, y)d(x, y)$ for some nonnegative function h that satisfies*

$$\sup\{h(x, y) : a \leq d(x, y) \leq b\} < 1$$

for each finite closed interval $[a, b] \subseteq (0, \infty)$.

(c) *$H(T(x), T(y)) \leq \psi(x, y)$, where ψ is such that $d - \psi$ is compactly positive.*

Dugundji and Granas [4] proved that a single-valued weakly contractive mapping of a complete metric space into itself has a unique fixed point. Using the equivalent characterization (b) in Lemma 1.1 for the weakly contractive mapping, Kaneko [6] gave a partial generalization (Theorem 1.3 below) of the theorem of Dugundji and Granas to the multi-valued mappings. To the best of our knowledge, a complete generalization is not yet available in the literature. The following two theorems were proved in [6].

THEOREM 1.2. *Let (X, d) be a complete metric space and $T: X \rightarrow P(X)$. If α is a monotone increasing function such that $0 \leq \alpha(t) < 1$ for each $t \in (0, \infty)$ and if $H(T(x), T(y)) \leq \alpha(d(x, y))d(x, y)$ for each x, $y \in X$, then T has a fixed point in X.*

THEOREM 1.3. *Let (X, d) be a complete metric space and $T: X \rightarrow P(X)$ be such that*

$$H(T(x), T(y)) \leq h(x, y)d(x, y)$$

for each x, $y \in X$ and for some nonnegative function h that satisfies

$$\sup\{h(x, y) : a \leq d(x, y) \leq b\} < 1$$

for each finite closed interval $[a, b] \subseteq (0, \infty)$. *Assume also that if* $(x_n, y_n) \in X \times X$ *is such that* $\lim_{n\to\infty} d(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} h(x_n, y_n) = k$ *for some* $k \in [0, 1)$. *Then T has a fixed point in X.*

Theorems 1.2 and 1.3 were investigated in response to a problem which was put forth by Reich. Reich [9] proposed the following problem.

Conjecture 1.4. Let (X, d) be a complete metric space. Suppose that $T: X \to \mathrm{CB}(X)$ satisfies

$$H(T(x), T(y)) \leq k(d(x, y))d(x, y)$$

for all $x, y \in X$, $x \neq y$, where $k: (0, \infty) \to [0, 1)$ and $\limsup_{r\to t^+} k(r) < 1$ for all $0 < t < \infty$. Then T has a fixed point in X.

This conjecture has now been proven valid in an almost complete form by Mizoguchi and Takahashi [7]. They replaced the condition on k by the following stronger condition:

$$\limsup_{r\to t^+} k(r) < 1 \qquad \text{for all } 0 \leq t < \infty. \tag{M-T}$$

In this paper, we reaffirm this positive response by Mizoguchi and Takahashi to the conjecture of Reich by giving an alternative proof. This will be done in Theorem 2.1. The proof is, in our opinion, more straightforward and more succinct than the one used by Mizoguchi and Takahashi. The argument of Mizoguchi and Takahashi is, however, used to prove Theorem 2.2 concerning the fixed point of weakly contractive multi-valued mappings. This theorem generalizes Theorem 1.3 by allowing T to take values in $\mathrm{CB}(X)$. Finally, we make some comments and observations in Section 3 that improve certain results of [7].

2. Generalized Contractive Multi-valued Mappings

The proof of Theorem 2.1 below is inspired by the paper of Nadler [8] in which the classical Banach contraction principle was extended to hold in the setting of multi-valued mappings. Theorem 2.1 is Theorem 5 of Mizoguchi and Takahashi [7]. The proof is quite different.

THEOREM 2.1. *Let* (X, d) *be a complete metric space and* $T: X \to \mathrm{CB}(X)$. *If* α *is a function of* $(0, \infty)$ *to* $[0, 1)$ *such that* $\limsup_{r\to t^+} \alpha(r) < 1$ *for every* $t \in [0, \infty)$ *and if*

$$H(T(x), T(y)) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Select a positive integer n_1 such that

$$\alpha^{n_1}(d(x_0, x_1)) \leq \{1 - \alpha(d(x_0, x_1))\}d(x_0, x_1).$$

We may select $x_2 \in T(x_1)$, using the definition of the Hausdorff metric, so that

$$d(x_2, x_1) \leq H(T(x_1), T(x_0)) + \alpha^{n_1}(d(x_0, x_1)).$$

We then have

$$d(x_2, x_1) \leq \alpha(d(x_1, x_0))d(x_1, x_0) + \alpha^{n_1}(d(x_0, x_1)) < d(x_1, x_0).$$

Now choose a positive integer $n_2 > n_1$ so that $\alpha^{n_2}(d(x_2, x_1)) < \{1 - \alpha(d(x_2, x_1))\}d(x_2, x_1)$. Since $T(x_2) \in \text{CB}(X)$, select $x_3 \in T(x_2)$ so that $d(x_3, x_2) \leq H(T(x_2), T(x_1)) + \alpha^{n_2}(d(x_2, x_1))$. Then we have

$$\begin{aligned} d(x_3, x_2) &\leq H(T(x_2), T(x_1)) + \alpha^{n_2}(d(x_2, x_1)) \\ &\leq \alpha(d(x_2, x_1))d(x_2, x_1) + \alpha^{n_2}(d(x_2, x_1)) \\ &< d(x_2, x_1). \end{aligned}$$

Repeating this process, since $T(x_k) \in \text{CB}(X)$ for each k, we may select a positive integer n_k such that

$$\alpha^{n_k}(d(x_k, x_{k-1})) < \{1 - \alpha(d(x_k, x_{k-1}))\}d(x_k, x_{k-1}).$$

Now select $x_{k+1} \in T(x_k)$ so that

$$d(x_{k+1}, x_k) \leq H(T(x_k), T(x_{k-1})) + \alpha^{n_k}(d(x_k, x_{k-1})).$$

Then $d(x_{k+1}, x_k) < d(x_k, x_{k-1})$ so that $d_k \equiv d(x_k, x_{k-1})$ is a monotone nonincreasing sequence of nonnegative numbers.

We now show that the sequence $\{d_k\}$ so generated is Cauchy.

Let $\lim_{k\to\infty} d_k = c \geq 0$. By assumption, $\limsup_{t\to c^+} \alpha(t) < 1$. Hence there is k_0 such that $k \geq k_0$ implies that $\alpha(d_k) < h$, where $\limsup_{t\to c^+} \alpha(t) < h < 1$.

Now

$$\begin{aligned} d_{k+1} &= d(x_{k+1}, x_k) \\ &\le H(T(x_k), T(x_{k-1})) + \alpha^{n_k}(d_k) \\ &\le \alpha(d_k)d_k + \alpha^{n_k}(d_k) \\ &\le \alpha(d_k)\alpha(d_{k-1})d_{k-1} + \alpha(d_k)\alpha^{n_{k-1}}(d_{k-1}) + \alpha^{n_k}(d_k) \\ &\dots \\ &\le \prod_{i=1}^{k} \alpha(d_i)d_1 + \sum_{m=1}^{k-1} \prod_{i=m+1}^{k} \alpha(d_i)\alpha^{n_m}(d_m) + \alpha^{n_k}(d_k) \\ &\le \prod_{i=1}^{k} \alpha(d_i)d_1 + \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \alpha(d_i)\alpha^{n_m}(d_m) + \alpha^{n_k}(d_k) \equiv A. \end{aligned}$$

In the last inequality, we have taken advantage of the fact that $\alpha < 1$ to delete some α factors from the product. We now focus on the sum of products,

$$\begin{aligned} \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \alpha(d_i)\alpha^{n_m}(d_m) &\le (k_0 - 1)h^{k-k_0+1} \sum_{m=1}^{k_0-1} \alpha^{n_m}(d_m) \\ &\quad + \sum_{m=k_0}^{k-1} h^{k-m}\alpha^{n_m}(d_m) \\ &\le (k_0 - 1)h^{k-k_0+1} \sum_{m=1}^{k_0-1} \alpha^{n_m}(d_m) + \sum_{m=k_0}^{k-1} h^{k-m+n_m} \\ &\le Ch^k + \sum_{m=k_0}^{k-1} h^{k-m_{n_m}} \\ &\le Ch^k + h^{k+n_{k_0}-k_0} + h^{k+n_{k_0}-1-(k_0-1)} \\ &\quad + \cdots + h^{k+n_{k-1}-(k-1)} \\ &\le Ch^k + \sum_{m=k+n_{k_0}-k_0}^{k+n_{k-1}-(k-1)} h^m \\ &= Ch^k + \frac{h^{k+n_{k_0}-k_0+1} - h^{k+n_{k-1}-k+2}}{1-h} \\ &= Ch^k + h^k \frac{h^{n_{k_0}-k_0+1}}{1-h} \\ &= Ch^k, \end{aligned}$$

where C is a generic positive constant. Now we can continue our inequalities,

$$\begin{aligned} A &\leq \prod_{i=1}^{k} \alpha(d_i)d_1 + Ch^k + \alpha^{n_k}(d_k) \\ &< h^{k-k_0+1} \prod_{i=1}^{k_0-1} \alpha(d_i)d_1 + Ch^k + h^{n_k} \\ &< Ch^k + Ch^k + h^k \\ &= Ch^k, \end{aligned}$$

C again being a generic constant. Now it is easy to show in the usual way that $\{x_k\}$ is Cauchy. For $k \geq k_0$, $m \in N$,

$$\begin{aligned} d(x_k, x_{k+m}) &\leq d(x_k, x_{k+1}) + \cdots + d(x_{k+m-1}, x_{k+m}) \\ &= \sum_{i=k+1}^{k+m} d_i \\ &< \sum_{i=k+1}^{k+m} Ch^{i-1} \\ &= C\frac{h^{k+1} - h^{k+m}}{1-h} \\ &\leq h^k, \end{aligned}$$

which tends to zero as $k \to \infty$. Let $x_k \to x \in X$; then

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_k) + d(x_k, T(x)) \\ &\leq d(x, x_k) + H(T(x_{k-1}), T(x)) \\ &\leq d(x, x_k) + \alpha(d(x_{k-1}, x))d(x_{k-1}, x). \end{aligned}$$

Since both terms in the last expression tend to zero as $k \to \infty$, we obtain $x \in T(x)$. ▮

COROLLARY 2.2. *Let (X, d) be a complete metric space and T: $X \to$ CB(X). If α is a monotone increasing function such that $0 \leq \alpha(t) < 1$ for each $t \in (0, \infty)$ and if*

$$H(T(x), T(y)) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, then T has a fixed point in X.

Corollary 2.2 above generalizes Theorem 1.2 by extending the range of T from $P(X)$ to $\mathrm{CB}(X)$. Now we make the same generalization to Theorem 1.3 by extending the range of T. It is remarked that in order to prove Theorem 1.3, we have used condition (b) of Lemma 1.1. In Theorem 2.3 below, we shall use the original definition of the weak contractiveness.

THEOREM 2.3. *Let (X, d) be a complete metric space and $T\colon X \to \mathrm{CB}(X)$ weakly contractive. Assume that*

$$\liminf_{\beta \to 0} \frac{\lambda(\alpha, \beta)}{\beta} > 0 \qquad (0 < \alpha \le \beta),$$

where $\lambda(\alpha, \beta) \equiv \inf\{\phi(x, y) \mid x, y \in X,\ \alpha \le d(x, y) \le \beta\}$ for each finite interval $[\alpha, \beta] \subset (0, \infty)$. Then T has a fixed point in X.

Proof. Select $x_1 \in X$ and let $t_1 \equiv d(x_1, T(x_1))$. If $t_1 > 0$, let $T_1 \equiv \lambda(t_1, 2t_1)/2t_1$. Note that $T_1 \le 1$, and since ϕ is compactly positive, $T_1 > 0$. Now select ε_1 such that $0 < \varepsilon_1 < \min\{T_1/(1 - T_1), 1\}$. Select $x_2 \in T(x_1)$ such that $d(x_1, x_2) < (1 + \varepsilon_1)d(x_1, T(x_1))$. Note that $d(x_2, T(x_2)) \le H(T(x_1), T(x_2)) \le d(x_1, x_2) - \phi(x_1, x_2)$, and

$$\begin{aligned}
&d(x_1, T(x_1)) - d(x_2, T(x_2)) \\
&\quad \ge \frac{1}{1+\varepsilon_1} d(x_1, x_2) - \{d(x_1, x_2) - \phi(x_1, x_2)\} \\
&\quad = \frac{1}{1+\varepsilon_1} d(x_1, x_2) - \left\{1 - \frac{\phi(x_1, x_2)}{d(x_1, x_2)}\right\} d(x_1, x_2) \qquad (*) \\
&\quad \ge \frac{1}{1+\varepsilon_1} d(x_1, x_2) - \left\{1 - \frac{\lambda(t_1, 2t_1)}{2t_1}\right\} d(x_1, x_2) \\
&\quad = \left\{\frac{1}{1+\varepsilon_1} - (1 - T_1)\right\} d(x_1, x_2).
\end{aligned}$$

By the assumption on ε_1 we see that $1/(1 + \varepsilon_1) - (1 - T_1) > 0$. Now let $t_2 \equiv d(x_2, T(x_2))$. If $t_2 > 0$, put $T_2 \equiv \lambda(t_2, \frac{3}{2} t_2)/\frac{3}{2} t_2$. From (*), $t_1 - t_2 > 0$, so that $t_1/t_2 > 1$. Select ε_2 such that $0 < \varepsilon_2 < \min\{T_2/(1 - T_2), t_1/t_2 - 1, \frac{1}{2}\}$. Next, find $x_3 \in T(x_2)$ such that $d(x_2, x_3) \le (1 + \varepsilon_2)d(x_2, T(x_2))$. As before, $d(x_2, T(x_2)) - d(x_3, T(x_3)) \ge \{1/(1 + \varepsilon_2) - (1 - T_2)\}d(x_2, x_3)$. In general, if $t_i > 0$ for $i = 1, \ldots, n$, let $t_n \equiv d(t_n, T(x_n))$ and $T_n \equiv \lambda(t_n, ((n + 1)/n)\, t_n)/((n + 1)/n)\, t_n$. Select ε_n such that $0 < \varepsilon_n < \min\{T_n/(1 - T_n), (t_{n-1}/t_n) - 1, 1/n\}$ and pick $x_{n+1} \in T(x_n)$ so that $d(x_n, x_{n+1}) \le (1 + \varepsilon_n)d(x_n, T(x_n))$. We then have

$$d(x_n, T(x_n)) - d(x_{n+1}, T(x_{n+1})) \geq \left\{ \frac{1}{1+\varepsilon_n} - (1 - T_n) \right\} d(x_n, x_{n+1}).$$

If $t_n = 0$ for any n, then we are through. If $t_n > 0$ for all n, then by hypothesis, $\liminf_n T_n > 0$ and we have $\limsup_n (1 - T_n) < 1$ so that for some $b > 0$,

$$d(x_n, T(x_n)) - d(x_{n+1}, T(x_{n+1})) \geq b d(x_n, x_{n+1})$$

for all sufficiently large n. Since $\{t_n\}$ is a monotone decreasing sequence of positive numbers, it converges. Now

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\leq \frac{1}{b} \sum_{j=n}^{m-1} \{d(x_j, T(x_j)) - d(x_{j+1}, T(x_{j+1}))\} \\ &= \frac{1}{b} \{d(x_n, T(x_n)) - d(x_m, T(x_m))\} \\ &= \frac{1}{b}(t_n - t_m) \to 0 \qquad \text{as } n, m \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x^* \in X$. Then

$$H(T(x_n), T(x^*)) \leq d(x_n, x^*) - \phi(x_n, x^*) \leq d(x_n, x^*).$$

By Lemma 2 of Assad and Kirk [1], we conclude that $x^* \in T(x^*)$; thus T has a fixed point in X. ∎

We note that the condition on λ taken in Theorem 2.3 above, namely

$$\liminf_{\beta \to 0} \frac{\lambda(\alpha, \beta)}{\beta} > 0$$

is weaker than the following condition assumed on the function h:

$$\lim_{n \to \infty} h(x_n, y_n) = k \text{ for some } k \in [0, 1) \text{ whenever } \lim_{n \to \infty} d(x_n, y_n) = 0.$$

This can be verified easily or the reader can find an argument in Corollary 2 of [6].

3. Comments on the Paper of Mizoguchi and Takahashi

As was stated in Section 2, Mizoguchi and Takahashi [7] gave a positive answer to the conjecture of Reich [9]. In addition to this result, the paper of Mizoguchi and Takahashi contains numerous other interesting theorems. The purpose of this section is twofold. First Theorem 1 of [7] will be used to establish a fixed point theorem for generalized contractions. This will be done in Theorem 3.3 below. Theorem 3.3 generalizes the fixed point theorem (Corollary 1.1) of Kaneko [5]. Second, we shall improve Theorem 3 of [7] by relaxing its hypotheses. This will be done in Theorem 3.5. We now state Theorems 1 and 3 of [7] for convenience.

THEOREM 3.1. (Mizoguchi and Takahashi). *Let (X, d) be a complete metric space and let T be a mapping of X into the family of nonempty subsets of X such that for each $x \in X$ there exists $y \in T(x)$ satisfying*

$$\psi(y) + d(x, y) \leq \psi(x),$$

where ψ is a proper (i.e., not identically equal to $+\infty$) bounded below and lower semicontinuous function of X into $(-\infty, +\infty]$. Then T has a fixed point in X.

As the authors note in [7], this theorem is a direct consequence of Caristi's fixed point theorem [3]. Let $g: X \to X$ and let $F(g)$ denote the set of fixed points of g. We recall that g is called *quasi-nonexpansive* if $d(g(x), y) \leq d(x, y)$ for all $x \in X$ and $y \in F(g)$. Also for a multi-valued mapping T defined on X and for a single-valued mapping g, g and T are said to *commute* if $g(T(x)) \subseteq T(g(x))$ for all $x \in X$. Finally, T is called *k-contractive* if there exists $k \in [0, 1)$ such that $H(T(x), T(y)) \leq kd(x, y)$ for each $x, y \in X$.

THEOREM 3.2 (Mizoguchi and Takahashi). *Let K be a closed convex subset of a uniformly convex Banach space, let g be a quasi-nonexpansive mapping of K into itself and let T be a k-contractive mapping of K into $\mathrm{CB}(K)$ such that $T(x)$ is convex for each $x \in K$. If g and T commute, then there exists $z \in K$ with $g(z) = z \in T(z)$.*

We are now in a position to prove our first theorem in this section.

THEOREM 3.3. *Let (X, d) be a complete metric space and $T: X \to \mathrm{CB}(X)$ be such that*

$$H(T(x), T(y)) \leq k \max\{d(x, y), d(x, T(x)), d(y, T(y)), \tfrac{1}{2}[d(x, T(y)) + d(y, T(x))]\}$$

for $0 \le k < 1$, *for all* $x, y \in X$. *If* $x \to d(x, T(x))$ *is lower semicontinuous, then there exists* $z \in X$ *such that* $z \in T(z)$.

Proof. Let $x \in X$ and $\varepsilon > 0$. Here we choose ε so small that $1/(1 + \varepsilon) > k$. Let $y \in T(x)$ so that $d(x, y) \le (1 + \varepsilon)d(x, T(x))$. Then

$$\begin{aligned} d(y, T(y)) &\le H(T(x), T(y)) \\ &\le k \max\{d(x, y), d(x, T(x)), d(y, T(y)), \\ &\quad \tfrac{1}{2}[d(x, T(y)) + d(y, T(x))]\} \\ &= k \max\{d(x, y), d(x, T(x)), \tfrac{1}{2} d(x, T(y))\}. \end{aligned}$$

We need to examine the following three cases. First, suppose that

$$d(x, y) = \max\{d(x, y), d(x, T(x)), \tfrac{1}{2} d(x, T(y))\}.$$

Then

$$\begin{aligned} d(x, T(x)) - d(y, T(y)) &\ge d(x, T(x)) - kd(x, y) \\ &\ge \left(\frac{1}{1 + \varepsilon} - k\right) d(x, y). \end{aligned}$$

Second, suppose that

$$d(x, T(x)) = \max\{d(x, y), d(x, T(x)), \tfrac{1}{2} d(x, T(y))\}.$$

Then

$$\begin{aligned} d(x, T(x)) - d(y, T(y)) &\ge d(x, T(x)) - kd(x, T(x)) \\ &= (1 - k)d(x, T(x)) \\ &\ge \frac{1 - k}{1 + \varepsilon} d(x, y). \end{aligned}$$

Third, for the final case, suppose that

$$\tfrac{1}{2} d(x, T(y)) = \max\{d(x, y), d(x, T(x)), \tfrac{1}{2} d(x, T(y))\}.$$

Then, since $d(y, T(y)) \le (k/2)\{d(x, y) + d(y, T(y))\}$,

$$\begin{aligned} d(x, T(x)) - d(y, T(y)) &\ge d(x, T(x)) - \frac{k}{2 - k} d(x, y) \\ &\ge \left(\frac{1}{1 + \varepsilon} - \frac{k}{2 - k}\right) d(x, y). \end{aligned}$$

Considering these three cases, we conclude that

$$d(x, T(x)) - d(y, T(y)) \geq \left(\frac{1}{1+\varepsilon} - k\right) d(x, y).$$

Now define $\psi(x) \equiv (1/(1 + \varepsilon) - k)^{-1}d(x, T(x))$. Then using Theorem 3.1 above, we are guaranteed of an element $z \in X$ such that $z \in T(z)$. ∎

A slight generalization of Theorem 3.3 can be made in the following way. Here we denote the family of non-empty bounded subsets of X by $\mathrm{B}(X)$. A proof is left to the reader.

THEOREM 3.4. *Let (X, d) be a complete metric space and let $T: X \to \mathrm{B}(X)$ be such that*

$$H(T(x), T(y)) \leq k \max\{d(x, y), d(x, T(x)), d(y, T(y)), \alpha[d(x, T(y)) + d(y, T(x))]\}$$

for all $x, y \in X$, where $0 \leq k < 1$ and $0 < \alpha < 1/(2k + \delta)$ for some $\delta > 0$. If $x \to d(x, T(x))$ is lower semicontinuous on X, then T has a fixed point in X.

Now we shift our attention to Theorem 3.2. In the next theorem, we improve the result of Theorem 3.2 by dropping the convexity of the sets K and $T(x)$ and the uniform convexity of X from the hypotheses. We let $F(g)$ denote the fixed point set of g and $P_A(x) \equiv \{y \in A \mid \|x - y\| = \inf_{z\in A} \|x - z\|\}$ with $A \subseteq X$.

THEOREM 3.5. *Let X be a Banach space and $K \subset X$ a nonempty closed subset. Let $g: K \to K$ be quasi-nonexpansive and let $T: K \to \mathrm{CB}(K)$ be a k-contractive mapping. Suppose that g and T commute on $F(g)$. Then g maps $P_{T(x)}(x)$ into itself. Suppose, further, that for each $x \in X$, $P_{T(x)}(x)$ is nonempty and that $g: P_{T(x)}(x) \to P_{T(x)}(x)$ possesses a fixed point. Then there exists $z \in K$ with $g(z) = z \in T(z)$.*

Proof. As in [7], since g is quasi-nonexpansive, $F(g)$ is closed. Since K is closed, $P_{T(x)}(x)$ is closed and by hypothesis it is nonempty. If $y \in P_{T(x)}(x) \subset T(x)$, then $g(y) \in g(T(x)) \subseteq T(g(x)) = T(x)$ for $x \in F(g)$ and, hence, $g: P_{T(x)}(x) \to T(x)$. But for $x \in F(g)$, $\|g(y) - x\| \leq \|y - x\|$, since g is quasi-nonexpansive and since $y \in P_{T(x)}(x)$, we also have $g(y) \in P_{T(x)}(x)$. This proves that $g: P_{T(x)}(x) \to P_{T(x)}(x)$. If g possesses a fixed point, say $y \in P_{T(x)}(x)$, then $F(g) \cap P_{T(x)}(x) \neq \varnothing$. This obviously implies that $T_\varepsilon(x) \cap F(g) \neq \varnothing$ with $T_\varepsilon(x) = \{y \in T(x) \mid d(x, y) \leq (1 + \varepsilon)d(x, T(x))\}$ and, hence, by applying Lemma 2 of [1], we conclude that there exists $z \in K$ with $g(z) = z \in T(z)$. ∎

In Theorem 3.2, since $T(x)$ is convex for each $x \in X$, it is easy to verify that the set $P_{T(x)}(x)$ is also convex. Hence the assumption that $g: P_{T(x)}(x) \to P_{T(x)}(x)$ has a fixed point is guaranteed by Theorem 1 of Browder [2]. Theorem 3.5 is thus a generalization of Theorem 3.2. Some related results will be included in the ensuing corollaries.

COROLLARY 3.6. *Let X, K, g, and T be as in Theorem* 3.5 *and suppose that $T(x)$ is compact and convex for each $x \in K$. Suppose, further, that g is continuous. Then there exists $z \in K$ with $g(z) = z \in T(z)$.*

Proof. In this case, $P_{T(x)}(x)$ is compact and $g: P_{T(x)}(x) \to P_{T(x)}(x)$, being a continuous mapping, has a fixed point by the classical Schauder's theorem. Hence, $P_{T(x)}(x) \cap F(g) \neq \emptyset$. Again by applying Lemma 2 of [1], we conclude that there exists $z \in K$ with $g(z) = z \in T(z)$.

COROLLARY 3.7. *Let X, K, g, and T be as in Theorem* 3.5 *and suppose further that $T(x)$ is proximinal for each $x \in F(g)$ and that $g: K \to K$ is a k-contraction mapping. Then there exists $z \in K$ with $g(z) = z \in T(z)$.*

Proof. Since g is a k-contraction, $F(g) = \{z\}$ is nonempty and a singleton by the Banach contraction principle. Since $T(x)$ is proximinal, it is closed and $P_{T(x)}(x)$ is closed and nonempty, and it is thus a complete metric space. As in the proof of Theorem 3.5, $g: P_{T(x)}(x) \to P_{T(x)}(x)$ and, again, by the Banach contraction principle, $z \in P_{T(x)}(x)$. Hence, $P_{T(x)}(x) \cap F(g) \neq \emptyset$ for all $x \in F(g)$. Thus $g(z) = z \in T(z)$. ∎

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