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J. J. Swetits  
*Old Dominion University*

B. Wood

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# Local $L_p$ -Saturation of Positive Linear Convolution Operators

J. J. SWETITS

*Department of Mathematical Sciences, Old Dominion University,  
Norfolk, Virginia 23508*

AND

B. WOOD

*Department of Mathematics, University of Arizona,  
Tucson, Arizona 85721*

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Local  $L_p$ -saturation of positive linear convolution operators is investigated. Results are obtained for two important classes of operators previously studied by Bojanic, DeVore, Korovkin and the authors.

## 1. INTRODUCTION

Let  $I = [0, r]$ ,  $1 \leq p < \infty$ , and denote by  $L_p(I)$  the space of real valued  $p$ th power Lebesgue integrable functions on  $I$ . Let  $\{H_n(t)\}$  be a sequence of non-negative, even and continuous functions on  $[-r, r]$  such that

$$\int_{-r}^r H_n(t) dt = 1, \quad n = 1, 2, \dots \quad (1.1)$$

$$\int_{-r}^r t^2 H_n(t) dt \equiv \mu_n \rightarrow 0, \quad n \rightarrow \infty, \quad (1.2)$$

and

$$\int_{-r}^r t^4 H_n(t) dt = O(\mu_n^2), \quad n \rightarrow \infty. \quad (1.3)$$

For  $f \in L_p(I)$  and  $0 \leq x \leq r$ , define the convolution operators

$$K_n(f(t), x) = \int_0^r f(t) H_n(t-x) dt, \quad n = 1, 2, \dots \tag{1.4}$$

These linear operators map  $L_p(I)$  into  $L_p(I)$  and are positive on  $I$ . There are two important examples of (1.4).

EXAMPLE 1.1 (Korovkin operators). Let  $\phi$  be a non-negative, even and continuous function on  $[-r, r]$ , decreasing on  $[0, r]$  and such that  $\phi(0) = 1$ . Let

$$K_n(f, x) = \rho_n \int_0^r f(t) (\phi(t-x))^n dt, \tag{1.5}$$

where

$$\rho_n^{-1} = 2 \int_0^r (\phi(t))^n dt. \tag{1.6}$$

This approximating method was introduced by Korovkin [5], who used it for the uniform approximation of continuous functions. Later Bojanic and Shisha [2] showed that

$$\lim_{t \rightarrow 0^+} \frac{1 - \phi(t)}{t^\lambda} = c \tag{1.7}$$

for some positive constants  $\lambda$  and  $c$  implies

$$\mu_n = O(n^{-2/\lambda}), \quad n \rightarrow \infty, \tag{1.8}$$

and

$$\sup_{0 \leq x \leq r} K_n((t-x)^4, x) = O(n^{-4/\lambda}), \quad n \rightarrow \infty. \tag{1.9}$$

Numerous important special cases of (1.5) that satisfy (1.7) are listed in [2]. Conditions (1.1)–(1.3) follow from (1.6), (1.8) and (1.9).

EXAMPLE 1.2. (Bojanic–DeVore operators). Let  $\{P_n\}$  be a sequence of orthogonal polynomials on  $[-1, 1]$  whose weight function  $w$  is even, non-negative and satisfies

$$0 < m \leq w(x) \leq M < \infty, \quad x \in [-r, r], \quad 0 < r \leq 1.$$

Let

$$R_n(x) = c_n \left[ \frac{P_{2n}(x)}{(x^2 - \alpha_{2n}^2)(x^2 - \alpha_{2n-1}^2)} \right]^2,$$

where  $\alpha_{2n}, \alpha_{2n-1}$  are the smallest positive zeros of  $P_{2n}$  and  $c_n > 0$  is chosen so that

$$\int_{-r}^r R_n(x) dx = 1, \quad n = 1, 2, \dots \quad (1.10)$$

Let

$$K_n(f, x) = \int_0^r f(t) R_n(t-x) dt. \quad (1.11)$$

Bojanic [1] and DeVore [3] have shown that

$$\mu_n = O(n^{-2}), \quad (n \rightarrow \infty), \quad (1.12)$$

and

$$\sup_{0 \leq x \leq r} K_n((t-x)^4, x) = O(n^{-4}), \quad n \rightarrow \infty. \quad (1.13)$$

Conditions (1.1)–(1.3) follow from (1.10), (1.12) and (1.13).

The degree of  $L_p$ -approximation with (1.4) was investigated in [10, 11]. In this work we consider local saturation of (1.4). Global saturation for the operators (1.5) is studied in [10].  $L_p$  saturation for other types of operators has been studied by Ditzian and May [4], Maier [6], Müller and Maier [8] and Reimenschneider [9].

## 2. DIRECT RESULTS

In the sequel, we let  $e_i(x) = x^i$  for  $i = 0, 1, 2, \dots$ .

LEMMA 2.1. *For  $n = 1, 2, \dots$ , we have*

$$\int_0^r H_n(t-x) dt = K_n(e_0, x) \leq 1, \quad 0 \leq x \leq r, \quad (2.1)$$

$$\int_0^r H_n(t-x) dx \leq 1, \quad 0 \leq t \leq r, \quad (2.2)$$

$$|K_n((t-x), x)| = O(\mu_n), \quad n \rightarrow \infty, \quad 0 < \delta \leq x \leq r - \delta < r, \quad (2.3)$$

$$|K_n(e_0, x) - 1| = O(\mu_n), \quad n \rightarrow \infty, \quad 0 \leq \delta \leq x \leq r - \delta < r, \quad (2.4)$$

$$K_n((t-x)^2, x) \leq \mu_n, \quad 0 \leq x \leq r, \quad (2.5)$$

$$K_n(|t-x|, x) \leq \mu_n^{1/2}, \quad 0 \leq x \leq r. \quad (2.6)$$

and

$$\|K_n(e_i) - e_i\|_{L_p[0,r]} = O(\mu_n^{1/2p}), \quad n \rightarrow \infty, \quad i = 0, 1. \quad (2.7)$$

*Proof.* Equations (2.1), (2.2) and (2.5)–(2.7) were established in [11]. Using (1.2), (1.3) and the method of [11, Lemma 1], it is easy to obtain (2.3) and (2.4).

Let  $0 \leq a < a_1 < b_1 < b \leq r$  and define

$$\begin{aligned} \chi(t) &= 0, \quad t \in [a, b] \\ &= 1, \quad t \notin [a, b]. \end{aligned}$$

LEMMA 2.2. *If  $f \in L_p[0, r]$ , then  $\|K_n(\chi f)\|_{L_p[a_1, b_1]} = O(\mu_n)$  ( $n \rightarrow \infty$ ).*

*Proof.* For  $p = 1$  we have

$$\begin{aligned} \|K_n(\chi f)\|_{L_1[a_1, b_1]} &= \int_{a_1}^{b_1} \left| \int_0^r \chi(t) f(t) H_n(t-x) dt \right| dx \\ &\leq \int_0^r \chi(t) |f(t)| \int_{a_1}^{b_1} H_n(t-x) dx dt \\ &\leq \|f\|_{L_1[0,r]} \left( \sup_{t \notin [a,b]} \int_{a_1}^{b_1} H_n(t-x) dx \right). \end{aligned}$$

If  $t \notin [a, b]$  and  $x \in [a_1, b_1]$ , then  $|t-x| \geq \min(a_1 - a, b - b_1) > 0$ . Using (1.2) and (1.3) we obtain

$$\sup_{t \notin [a,b]} \int_{a_1}^{b_1} H_n(t-x) dx = o(\mu_n), \quad n \rightarrow \infty.$$

Assume  $1 < p < \infty$ ,  $a_1 \leq x \leq b_1$ , and  $p + q = pq$ . By (2.1) and Hölder's inequality for positive linear operators,

$$|K_n(\chi f, x)| \leq K_n(\chi, x)^{1/q} (K_n(\chi |f|^p, x)^{1/p}).$$

Hence

$$\begin{aligned} \|K_n(\chi f, x)\|_{L_p[a_1, b_1]} &\leq \left( \sup_{a_1 \leq x \leq b_1} (K_n(\chi, x))^{1/q} \right. \\ &\quad \cdot \left. \left( \int_0^r \int_{a_1}^{b_1} \chi(t) |f(t)|^p H_n(t-x) dx dt \right)^{1/p} \right) \\ &\leq \left( \sup_{a_1 \leq x \leq b_1} (K_n(\chi, x))^{1/q} \|f\|_{L_p[0, r]} \right. \\ &\quad \times \left. \left( \sup_{t \notin [a, b]} \int_{a_1}^{b_1} H_n(t-x) dx \right)^{1/p} \right). \end{aligned}$$

For  $a_1 \leq x \leq b_1$  and  $\delta = \min(a_1 - a, b - b_1)$  we have

$$\begin{aligned} K_n(\chi, x) &= \int_0^r \chi(t) H_n(t-x) dt \\ &\leq \frac{1}{\delta^4} \int_0^r (t-x)^4 H_n(t-x) dt \\ &= O(\mu_n^2), \quad n \rightarrow \infty. \end{aligned}$$

For  $t \notin [a, b]$ ,

$$\begin{aligned} \int_{a_1}^{b_1} H_n(t-x) dx &\leq \frac{1}{\delta^4} \int_0^r (t-x)^4 H_n(t-x) dx \\ &= O(\mu_n^2), \quad n \rightarrow \infty. \end{aligned}$$

Thus

$$\begin{aligned} \|K_n(\chi f, x)\|_{L_p[a_1, b_1]} &\leq \|f\|_{L_p[0, r]} \cdot O(\mu_n^{2/q}) \cdot O(\mu_n^{2/p}) \\ &= \|f\|_{L_p[0, r]} \cdot O(\mu_n^2) \\ &= o(\mu_n), \quad n \rightarrow \infty. \end{aligned}$$

This completes the proof of Lemma 2.2.

For  $1 < p < \infty$  and  $0 \leq a < b \leq r$ , we denote by  $L_p^2[a, b]$  the space of those functions  $f$  such that  $f \in L_p[0, r]$ ,  $f'$  is absolutely continuous on  $[a, b]$  and  $f'' \in L_p[a, b]$ .  $BV[a, b]$  denotes the space of functions of bounded variation on  $[a, b]$ .

**THEOREM 2.3.** *Let  $f \in L_p[0, r]$ ,  $1 \leq p < \infty$  and  $0 < a < a_1 < b_1 < b < r$ . Then*

(i) if  $1 < p < \infty$ , then  $f \in L_p^2[a, b]$  implies  $\|K_n(f) - f\|_{L_p[a_1, b_1]} = O(\mu_n)$ ,  $n \rightarrow \infty$ ;

(ii) if  $p = 1$ , then  $f' \in BV[a, b]$  implies  $\|K_n(f) - f\|_{L_p[a_1, b_1]} = O(\mu_n)$ ,  $n \rightarrow \infty$ ;

(iii) if  $1 \leq p < \infty$ , then  $f$  linear on  $[a, b]$  implies  $\|K_n(f) - f\|_{L_p[a_1, b_1]} = o(\mu_n)$ ,  $n \rightarrow \infty$ .

*Proof.* (i) Let  $\chi$  be defined as in Lemma 2.2 and let  $\chi_1(t) = 1 - \chi(t)$ . Then

$$\|K_n(f) - f\|_{L_p[a_1, b_1]} \leq \|K_n(\chi f)\|_{L_p[a_1, b_1]} + \|K_n(\chi_1 f) - (\chi_1 f)\|_{L_p[a_1, b_1]}.$$

By Lemma 2.2,

$$\|K_n(\chi f)\|_{L_p[a_1, b_1]} = O(\mu_n), \quad n \rightarrow \infty.$$

Assume  $1 < p < \infty$  and  $f \in L_p^2[a, b]$ . Then

$$\begin{aligned} & \|K_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p[a_1, b_1]} \\ &= \left( \int_{a_1}^{b_1} \left| \int_0^r (\chi_1(t)f(t) - f(x)) H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\leq \left( \int_{a_1}^{b_1} \left| \int_a^b (f(t) - f(x)) H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\quad + \left( \int_{a_1}^{b_1} |f(x)|^p \left( \int_0^a H_n(t-x) dt \right)^p dx \right)^{1/p} \\ &\quad + \left( \int_{a_1}^{b_1} |f(x)|^p \left( \int_b^r H_n(t-x) dt \right)^p dx \right)^{1/p}. \end{aligned}$$

By (1.2) and (1.3), the last two terms are  $o(\mu_n)$ ,  $n \rightarrow \infty$ , since  $|t-x| \geq \delta = \min(a_1 - a, b - b_1) > 0$ . The first term is dominated by

$$\begin{aligned} & \left( \int_{a_1}^{b_1} \left| \int_a^b f'(x)(t-x) H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &+ \left( \int_{a_1}^{b_1} \left| \int_a^b \left( \int_x^t (t-u) f''(u) du \right) H_n(t-x) dt \right|^p dx \right)^{1/p}, \end{aligned}$$

since  $f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u)f''(u) du$ . Using (1.2) and (1.3), it can be shown that the first term is dominated by

$$\sup_{a_1 < x \leq b_1} |f'(x)| \cdot \|K_n(t-x, x)\|_{L_p[a_1, b_1]} + o(\mu_n) = o(\mu_n), \quad n \rightarrow \infty.$$

Now let

$$\theta(f'', x) = \sup_{\substack{a \leq t \leq b \\ t \neq x}} \frac{1}{(t-x)} \int_x^t |(f''(u))| du, \quad a_1 \leq x \leq b_1,$$

denote the Hardy–Littlewood majorant of  $f''$  at  $x$ . Since  $f'' \in L_p[a, b]$  and  $p > 1$ ,

$$\|\theta(f'', x)\|_{L_p[a, b]} \leq A_p \|f''\|_{L_p[a, b]},$$

where  $A_p > 0$  depends only on  $p$  [12, Theorem 13.15]. Hence

$$\begin{aligned} & \left( \int_{a_1}^{b_1} \left| \int_a^b \left( \int_x^t (t-u)f''(u) du \right) H_n(t-x) dt \right|^p dx \right)^{1/p} \\ & \leq \left( \int_{a_1}^{b_1} |\theta(f'', x)|^p \left( \int_a^b (t-x)^2 H_n(t-x) dt \right)^p dx \right)^{1/p} \\ & = O(\mu_n), \quad n \rightarrow \infty. \end{aligned}$$

Finally,

$$\begin{aligned} \|K_n(\chi_1 f) - (\chi_1 f)\|_{L_p[a_1, b_1]} & \leq \|K_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p[a_1, b_1]} \\ & \quad + \|(\chi_1 f)(x)(K_n(e_0, x) - 1)\|_{L_p[a_1, b_1]} \end{aligned}$$

and part (i) follows from (2.4) and the above calculations.

(ii) Now assume  $p = 1$  and  $f' \in BV[a, b]$ . Then we have, for  $x, t \in [a, b]$ ,

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u) df'(u).$$

From the proof of (i), we need only show that

$$\left\| K_n \left( \chi_1(t) \int_x^t (t-u) df'(u), x \right) \right\|_{L_1[a_1, b_1]} = O(\mu_n), \quad n \rightarrow \infty.$$

Fix  $\delta > 0$ . We have

$$\begin{aligned} & \left| \int_a^b \int_x^t (t-u) df'(u) H_n(t-x) dt \right| \\ & \leq \int_a^b |t-x| H_n(t-x) \left| \int_x^t df'(u) \right| dt \\ & = \int_a^b |t-x| H_n(t-x) \left| \int_0^{t-x} |df'(x+y)| \right| dt \\ & \leq \sum_{j=0}^{[(b-a)/\delta]} I_{nj}(x), \end{aligned}$$



where

$$I_{nj}(x) = \int_{j\delta \leq |t-x| \leq (j+1)\delta} |t-x| H_n(t-x) \int_0^{|t-x|} |df'(x+y)| dt.$$

Clearly,

$$I_{nj}(x) \leq S_{nj}(\delta, x) \int_0^{(j+1)\delta} |df'(x+y)|,$$

where

$$S_{nj}(\delta, x) = \int_{j\delta \leq |t-x| \leq (j+1)\delta} |t-x| H_n(t-x) dt.$$

Next, we estimate  $S_{nj}(\delta, x)$  for  $j=0$  and  $1 \leq j \leq [(b-a)/\delta]$  separately. We have

$$\begin{aligned} S_{n0}(\delta, x) &= \int_{0 \leq |t-x| < \delta} |t-x| H_n(t-x) dt \\ &< \delta \int_{-r}^r H_n(t) dt \\ &= \delta. \end{aligned}$$

For  $1 \leq j$ ,

$$\begin{aligned} S_{nj}(\delta, x) &\leq \frac{1}{(j\delta)^3} \int_{j\delta \leq |t-x| \leq (j+1)\delta} (t-x)^4 H_n(t-x) dt \\ &\leq \frac{1}{(j\delta)^3} \int_{-r}^r t^4 H_n(t) dt \\ &= \frac{1}{(j\delta)^3} \cdot O(\mu_n^2) \end{aligned}$$

by (1.3).

It follows that

$$\begin{aligned} \sum_{j=0}^{[(b-a)/\delta]} I_{nj}(x) &\leq \delta \int_0^\delta |df'(x+y)| \\ &+ \left( \sum_{j=1}^{[(b-a)/\delta]} \frac{1}{(j\delta)^3} \int_0^{(j+1)\delta} |df'(x+y)| \right) \cdot O(\mu_n^2). \end{aligned}$$

Integrating this last inequality with respect to  $x$ , and taking into account that

$$\int_{a_1}^{b_1} \int_0^{(j+1)\delta} |df'(x+y)| dx \leq (j+1)\delta \|f'\|_{BV[a,b]}$$

we obtain

$$\begin{aligned} & \left\| K_n \left( \chi_1(t) \int_x^t (t-u) df'(u) \right), x \right\|_{L_1[a_1,b_1]} \\ & \leq \delta^2 \|f'\|_{BV[a,b]} + \frac{\|f'\|_{BV[a,b]}}{\delta^2} \sum_{j=1}^{\infty} \frac{j+1}{j^3} \cdot O(\mu_n^2). \end{aligned}$$

If we choose  $\delta = \mu_n^{1/2}$ , the last term is  $O(\mu_n)$ ,  $n \rightarrow \infty$ .

(iii) If  $f$  is linear on  $[a, b]$ , the above calculations, along with (2.3), (2.4) and Lemma 2.2, imply the conclusion of (iii).

### 3. A LOCAL SATURATION THEOREM

The following theorem provides a local inverse to Theorem 2.3 and shows that the convolution operators (1.4) are locally saturated with order  $O(\mu_n)$ .

**THEOREM 3.1.** *If  $0 < a < b < r$  and  $f \in L_p[0, r]$ , we have*

(i) *if  $1 < p < \infty$ , then  $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$ ,  $n \rightarrow \infty$ , implies  $f \in L_p^2[a, b]$ ;*

(ii) *if  $p = 1$ , then  $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$ ,  $n \rightarrow \infty$ , implies  $f' \in BV[a, b]$ ;*

(iii) *if  $1 \leq p < \infty$ , then  $\|K_n(f) - f\|_{L_p[a,b]} = o(\mu_n)$ ,  $n \rightarrow \infty$ , implies  $f$  is linear on  $[a, b]$ .*

*Proof.* To prove (i) and (ii), let  $f \in L_p[0, r]$  and choose  $\psi \in C^2[a, b]$  with  $\psi(a) = \psi'(a) = \psi''(a) = \psi(b) = \psi'(b) = \psi''(b) = 0$ . Let  $\psi(x) = 0$  for  $x \notin [a, b]$ , so that  $\psi \in C^2[0, r]$  and  $\psi(x) = \psi'(x) = 0$  for  $x \notin [a, b]$ . We utilize the bilinear functional (compare [6, 8])

$$A_n(f, \psi) = \frac{1}{\mu_n} \int_a^b K_n(f(t), x) - f(x) \psi(x) dx.$$

Fix  $\psi$  as above. We first demonstrate that  $\{A_n(\cdot, \psi)\}$  is uniformly bounded on  $L_p[0, r]$ . We have

$$\begin{aligned} \int_a^b K_n(f(t), x) \psi(x) dx &= \int_0^r \psi(x) \int_0^r f(t) H_n(t-x) dt dx \\ &= \int_0^r f(t) \int_0^r \psi(x) H_n(t-x) dx dt. \end{aligned}$$

For  $x, t \in [0, r]$  we can write

$$\psi(x) = \psi(t) + \psi'(t)(x-t) + \frac{\psi''(\eta)}{2}(x-t)^2$$

for some  $\eta$  between  $x$  and  $t$ . Hence

$$\begin{aligned} &\int_0^r \int_0^r f(t) \psi(x) H_n(t-x) dx dt \\ &= \int_0^r f(t) \psi(t) \int_0^r H_n(t-x) dx dt \\ &\quad + \int_0^r f(t) \psi'(t) \int_0^r (x-t) H_n(t-x) dx dt \\ &\quad + \frac{1}{2} \int_0^r f(t) \int_0^r \psi''(\eta)(x-t)^2 H_n(t-x) dx dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using (1.1) we have

$$\begin{aligned} I_1 &= \int_0^r f(t) \psi(t) \int_{t-r}^t H_n(u) du dt \\ &= \int_0^r f(t) \psi(t) dt - \int_0^r f(t) \psi(t) \int_{-r}^{t-r} H_n(u) du dt \\ &\quad - \int_0^r f(t) \psi(t) \int_t^r H_n(u) du dt. \end{aligned}$$

Since  $\psi(t) = 0$  for  $t \notin [a, b]$ , (1.2) and (1.3) yield

$$\begin{aligned} &\left| \int_0^r f(t) \psi(t) \int_t^r H_n(u) du dt \right| \\ &\leq \frac{1}{a^4} \int_0^r |f(t)| |\psi(t)| \int_t^r u^4 H_n(u) du dt \\ &\leq \sup_{0 \leq t \leq r} |\psi(t)| \left( \frac{1}{a^4} \int_{-r}^r u^4 H_n(u) du \right) \|f\|_{L_1[0,r]} \\ &= o(\mu_n) \|f\|_{L_p[0,r]}, \quad n \rightarrow \infty. \end{aligned}$$

Similarly

$$\left| \int_0^r f(t) \psi(t) \int_{-r}^{t-r} H_n(u) du dt \right| = o(\mu_n) \|f\|_{L_p[0,r]}, \quad n \rightarrow \infty.$$

Hence

$$I_1 = \int_0^r f(t) \psi(t) dt + o(\mu_n) \|f\|_{L_p[0,r]}, \quad n \rightarrow \infty.$$

Similar calculations and the fact that  $\psi'(t) = 0$  for  $t \notin [a, b]$  yield  $I_2 = o(\mu_n) \|f\|_{L_p[0,r]}$ ,  $n \rightarrow \infty$ . Finally

$$|I_3| \leq \frac{1}{2} \sup_{0 < x < r} |\psi''(x)| \mu_n \|f\|_{L_1[0,r]},$$

and it follows that  $\{A_n(\cdot, \psi)\}$  is uniformly bounded on  $L_p[0, r]$ .

Next we observe that, for  $f \in C^2[0, r]$ ,

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = \frac{1}{2} \int_a^b f(x) \psi''(x) dx. \tag{3.1}$$

This follows from the definition of  $\psi$  and

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_n} (K_n(f, x) - f(x)) = \frac{1}{2} f''(x), \tag{3.2}$$

uniformly on  $[a, b]$ . Equation (3.2) is a consequence of Mamedov's asymptotic theorem [7], (1.2), (1.3) and (2.3)–(2.5).

Since  $\{A_n(\cdot, \psi)\}$  is uniformly bounded on  $L_p[0, r]$ , and  $C^2[0, r]$  is dense in  $L_p[0, r]$ , (3.1) yields

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = \frac{1}{2} \int_a^b f(x) \psi''(x) dx \tag{3.3}$$

for any  $f \in L_p[0, r]$ .

Fix  $f \in L_p[0, r]$  and consider the sequence of linear functionals  $\{A_n(f, \cdot)\}$ . Since  $\|K_n(f) - f\|_{L_p[a,b]} = O(\mu_n)$ ,  $n \rightarrow \infty$ , there exists  $h \in L_p[a, b]$  ( $p > 1$ ) and  $h \in BV[a, b]$  ( $p = 1$ ) and a subsequence  $\{A_{n_i}(f, \cdot)\}$  such that

$$\begin{aligned} \lim_{i \rightarrow \infty} A_{n_i}(f, \psi) &= \int_a^b h(x) \psi(x) dx, & p > 1 \\ &= \int_a^b \psi(x) dh(x), & p = 1. \end{aligned} \tag{3.4}$$

Equating (3.3) and (3.4) we obtain

$$\begin{aligned} \frac{1}{2} \int_a^b f(x) \psi''(x) dx &= \int_a^b h(x) \psi(x) dx, & p > 1 \\ &= \int_a^b \psi(x) dh(x), & p = 1. \end{aligned} \tag{3.5}$$

A particular solution to (3.5) is

$$\begin{aligned} \frac{1}{2} f(x) &= \int_n^x \int_n^\xi h(\mu) d\mu d\xi, & p > 1 \\ &= \int_n^x \int_n^\xi dh(\mu) d\xi, & p = 1. \end{aligned}$$

The homogeneous problem

$$\int_a^b f(x) \psi''(x) dx = 0$$

has the general solution  $f(x) = C_1 x + C_2$  for  $a \leq x \leq b$ , since  $\psi \in C^2[a, b]$ ,  $\psi(a) = \psi'(a) = \psi(b) = \psi'(b) = 0$  is arbitrary. Hence, if  $1 < p < \infty$ ,  $f \in L_p^{(2)}[a, b]$ , and if  $p = 1$  then  $f' \in BV[a, b]$ .

If  $\|K_n(f) - f\|_{L_p[a, b]} = o(\mu_n)$ ,  $n \rightarrow \infty$ , then

$$\begin{aligned} |A_n(f, \psi)| &\leq \frac{1}{\mu_n} \int_a^b |K_n(f, x) - f(x)| \cdot |\psi(x)| dx \\ &\leq \left( \sup_{a \leq x \leq b} |\psi(x)| \right) \frac{A_p}{\mu_n} \|K_n(f) - f\|_{L_p[a, b]}, \end{aligned}$$

where  $A_p > 0$  is independent of  $n$ . Hence

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = 0. \tag{3.6}$$

Comparing (3.3) and (3.6) we obtain

$$\frac{1}{2} \int_a^b f(x) \psi''(x) dx = 0$$

and, consequently,  $f$  is linear on  $[a, b]$ .

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