2001

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Original Publication Citation

Babel, L., Kloks, T., Kratochvil, J., Kratsch, D., Muller, H., & Olariu, S. (2001). Efficient algorithms for graphs with few P_4's. *Discrete Mathematics, 235*(1-3), 29-51. doi:10.1016/s0012-365x(00)00258-2

DISCRETE MATHEMATICS

Discrete Mathematics 235 (2001) 29–51

www.elsevier.com/locate/disc

Efficient algorithms for graphs with few P_4's

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Abstract

We show that a large variety of NP-complete problems can be solved efficiently for graphs with 'few' P_4's. We consider domination problems (domination, total domination, independent domination, connected domination and dominating clique), the Steiner tree problem, the vertex ranking problem, the pathwidth problem, the path cover number problem, the hamiltonian circuit problem, the list coloring problem and the precoloring extension problem. We show that all these problems can be solved in linear time for the class of $(q, q-4)$-graphs, for every fixed q. These are graphs for which no set of at most q vertices induces more than $q-4$ different P_4's.

1. Introduction

Motivated by a number of practical applications (see, e.g., [17]), the classes of (q, t)-graphs have been introduced in [3] in order to capture local density properties of a graph. Specifically, in a (q, t)-graph no set of at most q vertices induces more than t distinct P_4's. In particular, the $(q, q-4)$-graphs extend and generalize the well known and intensively studied cographs and several classes of graphs with few P_4's, as e.g. P_4-sparse graphs and P_4-extendible graphs for which many NP-complete graph problems can be solved by linear time algorithms [7,10,15,17]. Note that the cographs are precisely the $(4,0)$-graphs, the P_4-sparse graphs are the $(5,1)$-graphs and the C_5-free P_4-extendible graphs coincide with the $(6,2)$-graphs.

It was shown in [1] that the weighted versions of the clique and independent set problem and the chromatic number and clique cover problem, can all be solved in

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[1] Supported in part by Czech Grant GAČR 201/99/0242.
[2] Supported in part by Czech Research Grant GAUK 1999/158.

0012-365X/01/$ - see front matter
PII: S0012-365X(00)00258-2

linear time for $(q, q-4)$-graphs, for fixed q. In a later paper [2], it was shown that also the treewidth and minimum fill-in problem can be solved in linear time for these classes of graphs. These results are due to a unique tree representation of the primeval decomposition, introduced by Jamison and Olariu [16]. It turns out that, in case of $(q, q-4)$-graphs for fixed q, the problems mentioned above can be solved efficiently for the subgraphs corresponding to the leaves of this tree (the p-components), and that the graph parameters for the subgraphs corresponding to internal nodes can be computed from the corresponding parameters of the children of the node.

In this work, we show that a large variety of problems can be solved efficiently for $(q, q-4)$-graphs. We consider various domination problems, the vertex ranking problem, the Steiner tree problem, pathwidth, vertex ranking, path cover and hamiltonicity, and also the listcoloring and precoloring extension problems. We show that all these problems can be solved in linear time for the class of $(q, q-4)$-graphs, for every fixed q, using a dynamic programming technique on the primeval tree decomposition.

In [15] it was shown that the path cover number can be computed and hamiltonicity can be decided in linear time for P_4-sparse and P_4-extendible graphs. We extend these results by showing that path cover and hamiltonicity can be solved in linear time for $(q, q-4)$-graphs for every fixed q, thereby settling an open problem mentioned in [15].

We organized this paper as follows. We start with some preliminaries on the primeval decomposition, the homogeneous decomposition and the class of $(q, q-4)$-graphs. Then we give for each of the problems some preliminaries, we show how the problem can be solved for the two basic operations disjoint union and disjoint sum and, finally, we present efficient algorithms for $(q, q-4)$-graphs. For some of the problems we show how homogeneous substitutions can be handled.

2. Preliminaries

In [16], Jamison and Olariu introduced the *homogeneous decomposition* of a graph, which extends the well-known *modular decomposition* (see, e.g., [22]). We use the *primeval tree decomposition* introduced in [16], and the characterization of the p-components of $(q, q-4)$-graphs given in [3] to solve a variety of problems for $(q, q-4)$-graphs.

As usual we denote by P_k a chordless path on k vertices (i.e. a P_k means an induced path).

Definition 1. A graph $G = (V, E)$ is *p-connected* if for every partition of V into nonempty subsets V_1 and V_2 there is a *crossing* P_4, that is, a P_4 with vertices both in V_1 and in V_2.

Definition 2. A maximal subset C of vertices such that $G[C]$ is p-connected is called a p-component of G.

It is easy to see (see, e.g., [16]) that each graph has a unique partition into p-components. The p-components are connected and closed under complementation, i.e., a p-component of G is also a p-component of $\bar{G}$.

Definition 3. A partition (C_1, C_2) of a p-component C into nonempty subsets C_1 and C_2 is called a *separation* of C if every P_4 with vertices both in C_1 and in C_2 has both midpoints in C_1 and both endpoints in C_2.

A p-component C is called *separable* if there is a separation (C_1, C_2) of C.

Definition 4. A subset M of V with $1 \leqslant |M| \leqslant |V|$ is called a *module* if each vertex outside is either adjacent to all vertices of M or to none of them. A module M is called a *homogeneous set* if $1 < |M| < |V|$.

The graph obtained from a graph G by shrinking every maximal homogeneous set to one single vertex is called the *characteristic graph* of G.

A graph is called *split graph* if its vertex set splits (can be partitioned into) a clique K and an independent set S. One of the results of [16] (see also [1]) is the following.

Lemma 1. *A p-connected graph G is separable if and only if its characteristic graph is a split graph.*

Furthermore, in [16] it is shown that the separation (C_1, C_2) of a separable p-component C is *unique*. Clearly, if (K, S) is the splitting of the characteristic graph of $G[C]$, then every module $M \subseteq C_1$ shrinks to a vertex in the clique K, and every module $M \subseteq C_2$ shrinks to a vertex in the independent set S.

We need the main result of [16], called the *structure theorem*.

Theorem 1. *For an arbitrary graph G exactly one of the following conditions is satisfied*:

- *G is disconnected.*
- *$\bar{G}$ is disconnected.*
- *There is a unique proper separable p-component H of G with separation (H_1, H_2) such that every vertex outside H is adjacent to all vertices in H_1 and to no vertex in H_2.*
- *G is p-connected.*

2.1. The primeval decomposition

In order to define the primeval decomposition tree we introduce three graph operations each acting on two graphs G_1 and G_2, corresponding with the first three cases of the structure theorem.

- For operation 0, both G_1 and G_2 are arbitrary graphs. Operation 0 takes the disjoint union of G_1 and G_2.
- For operation 1, both G_1 and G_2 are arbitrary. Operation 1 takes the disjoint sum of G_1 and G_2, i.e., every vertex of G_1 is made adjacent to every vertex of G_2.
- For operation 2, G_1 is *not* arbitrary: G_1 is a separable p-connected graph with separation (V_1^1, V_1^2). G_2 is an arbitrary graph. Operation 2 makes every vertex of G_2 adjacent to every vertex of V_1^1 and to no vertex of V_1^2.

These operations suggest a tree representation for arbitrary graphs which is unique up to isomorphism. For our purposes it is more convenient to deal with a binary tree which can be constructed from the original tree in a straightforward way. The leaves of this rooted binary tree are exactly the p-components of the graph. The root corresponds with the input graph G. Internal nodes are labeled with integers $i \in \{0, 1, 2\}$ where an i-node means that the subgraph at this node is obtained by an i-operation applied to the two subgraphs corresponding to the two sons of the node.

2.2. The homogeneous decomposition

The *homogeneous decomposition* [16] involves, additionally, the homogeneous sets of the graph. Given the primeval tree, it constructs a new tree representation by — loosely speaking — replacing homogeneous sets by single vertices. This substitution is reflected by a fourth operation. Let $G_0, H_1, \ldots, H_k$ be disjoint graphs and let $\{x_1, \ldots, x_k\}$ be a set of vertices of G_0. The graph G arises from $G_0, H_1, \ldots, H_k$ by means of a 3-operation if every vertex x_i in G_0 is replaced by the graph H_i in the obvious way (i.e. all the edges between H_i and H_j are added if x_i and x_j are adjacent in G_0). In the resulting decomposition tree, the leftmost child of a 3-node represents the characteristic graph, the other children are the subtrees which represent the maximal homogeneous sets as illustrated in Fig. 1.

Baumann [4] developed linear time algorithms which compute the primeval and the homogeneous decomposition tree of an arbitrary graph. These algorithms depend

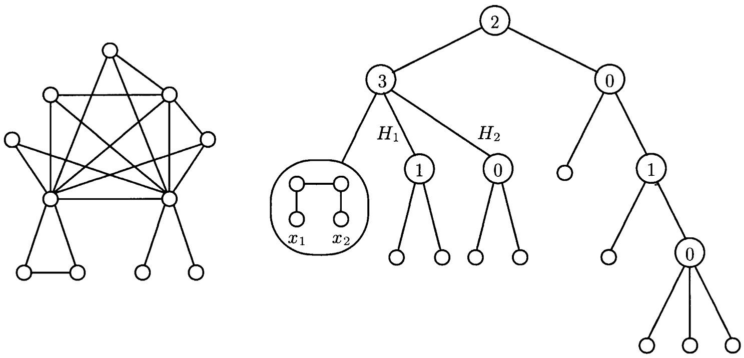

Fig. 1. A graph and the associated homogeneous decomposition tree.

strongly on known methods which compute the modular decomposition of a graph (see, e.g., [8]).

2.3. $(q, q-4)$-graphs

Babel and Olariu introduced the following classes of graphs [3].

Definition 5. A graph is a (q,t)-graph if no set of at most q vertices induces more than t distinct P_4's.

Clearly, the $(4,0)$-graphs are exactly the cographs. A graph is called P_4-*sparse* if it has no induced subgraph isomorphic to one of seven p-connected graphs on five vertices and it turns out that the P_4-sparse graphs are exactly the $(5,1)$-graphs. A graph is called P_4-*extendible* if and only if it has no p-component with more than five vertices. (see [15]). Thus, the C_5-free P_4-extendible graphs are exactly the $(6,2)$-graphs.

The aim of this subsection is to recall the characterization of the p-components of $(q,q-4)$-graphs presented in [3].

Definition 6. A *spider* is a split graph consisting of a clique and an independent set of equal size at least two such that each vertex of the independent set has precisely one neighbor in the clique and each vertex of the clique has precisely one neighbor in the independent set, or it is the complement of such a graph.

Definition 7. A spider is *thin* if every vertex of the independent set has precisely one neighbor in the clique. A spider is *thick* if every vertex of the independent set is nonadjacent to precisely one vertex of the clique.

Notice that a P_4 is both thick and thin and that this is the smallest spider.

The main reason for the linear time solvability of the various problems for $(q,q-4)$-graphs with fixed q is that the p-components are of a very specific type. This is reflected by the following characterization found by Babel and Olariu [1,3].

Theorem 2. *Let G be p-connected.*

1. *If G is a* $(5,1)$-*graph then G is a spider*.
2. *If G is a* $(7,3)$-*graph then* $|V| < 7$ *or G is a spider*.
3. *If G is a* $(q,q-4)$-*graph*, $q=6$ *or* $q \geqslant 8$, *then* $|V| < q$.

3. Vertex ranking

In this section, we describe a linear time algorithm for the ranking problem on $(q,q-4)$-graphs for fixed q. This generalizes the early results of [24] for the ranking problem on cographs.

Definition 8. Let $G=(V,E)$ be a graph and let t be some integer. A (*vertex*) t-ranking is a numbering $c: V \to \{1,\dots,t\}$ such that for every pair of vertices x and y with $c(x)=c(y)$ and for every path between x and y there is a vertex z on the path with $c(z)>c(x)$.

The (*vertex*) *ranking number* of G denoted by $\chi_r(G)$ is the smallest value t for which the graph G admits a t-ranking.

Notice that a ranking is a proper coloring of the graph.

Definition 9. Let $G=(V,E)$ be a graph and let a and b be nonadjacent vertices of G. Then $S\subset V$ is an *a,b-separator* if the removal of S separates a and b in distinct connected components. If no proper subset of the a,b-separator S is itself an a,b-separator then S is a *minimal a,b-separator*. Finally, $S\subset V$ is a *minimal* separator of G if S is a minimal *a,b-separator* for some nonadjacent vertices a and b of G.

The following lemma appeared first in [9].

Lemma 2. *If G is a complete graph on n vertices, $\chi_r(G)=n$. Otherwise,*

$$\chi_r(G)=\min_S\left(|S|+\max_C \chi_r(C)\right),$$

where the minimum is taken over all minimal separators S and the maximum is taken over all components C of $G-S$.

A graph H is a *supergraph* of a graph G if H has the same vertex set as G and the edge set of G is a subset of the edge set of H. A graph G is chordal if each cycle of G of length greater than 3 has a chord, i.e. an edge joining two non consecutive vertices of the cycle. A graph H is a *triangulation* of a graph G if H is a chordal supergraph of G. A triangulation H of G is *P_4-free* if H has no P_4 as induced subgraph, i.e. H is a cograph.

We denote by $\omega(G)$ the *clique number* of the graph G, i.e. the maximum cardinality of a clique of G. For most of our proofs in this section we will make use of the following result presented in [23].

Theorem 3. *For any graph G,*

$$\chi_r(G)=\min\{\omega(H)\,|\,H \text{ is a } P_4\text{-free triangulation of } G\}.$$

First, we consider the disjoint union and the disjoint sum of two graphs.

Lemma 3. *Let G be the disjoint union of G_1 and G_2. Then*

$$\chi_r(G)=\max(\chi_r(G_1),\chi_r(G_2)).$$

Lemma 4. *Let G be the disjoint sum of $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$*

$$\chi_r(G)=\min(\chi_r(G_1)+|V_2|,\chi_r(G_2)+|V_1|).$$

Proof. Consider a P_4-free triangulation H of G. If two vertices of V_1 are not adjacent in H and two vertices of V_2 are not adjacent in H then we obtain a chordless cycle of length 4, which is a contradiction since H is chordal. Hence, either V_1 or V_2 induces a complete subgraph in H. This proves the lemma. □

Given the primeval decomposition tree of a $(q,q-4)$-graph, we start computing the ranking numbers of the leaves of the tree. By Theorem 2 these leaves are either spiders or graphs of bounded size (with less than q vertices). If the graph is a spider, then the ranking number can be determined using the following lemma given in [24].

Lemma 5. *Let G be a split graph with clique K and independent set S such that every vertex of K has at least one neighbor in S. Then $\chi_r(G)=|K|+1$.*

If the leaf corresponds to a graph of bounded size, we can simply list all P_4-free triangulations in constant time, and look for the one that minimizes the clique number.

Now, we consider an internal vertex of the primeval tree. If the label of this vertex is a 0-operation, the ranking number of the subgraph can be determined from the ranking number of the two sons using Lemma 3. If the label is a 1-operation, we can use Lemma 4.

In the rest of this section we concentrate on the type 2-operation. Hence, we assume that the graph G is obtained from a separable p-connected graph $G_1=(V_1,E_1)$ with separation (V_1^1,V_1^2) and a graph $G_2=(V_2,E_2)$ by making every vertex of G_2 adjacent to every vertex of V_1^1. Since G_1 is a p-component, G_1 is either a spider or a graph with less than q vertices of which the characteristic is a split graph by Theorem 2 and Lemma 1.

3.1. G_1 is a spider

First, consider the case where G_1 is a spider. Let K be the clique and S be the independent set of G_1. Then the separation of G_1 is (K,S).

Lemma 6. *If $V_2\neq\emptyset$ then $\chi_r(G)=\chi_r(G_2)+|K|$.*

Proof. Label the vertices of G_2 by an optimal ranking with $1,\dots,\chi_r(G_2)$. Label the vertices of K with $\chi_r(G_2)+1,\dots,\chi_r(G_2)+|K|$. Finally, label all vertices of S with 1. This is a ranking of G implying $\chi_r(G)\leqslant\chi_r(G_2)+|K|$.

Now we show $\chi_r(G)\geqslant\chi_r(G_2)+|K|$. Consider a P_4-free triangulation H of G. Then the subgraph H_2 of H induced by V_2 is a P_4-free triangulation of G_2. Hence, $\omega(H)\geqslant\omega(H_2)+|K|\geqslant\chi_r(G_2)+|K|$. This proves the lemma. □

3.2. $|V_1| < q$ and the characteristic of G_1 is a split graph

We consider two types of triangulations of G. First, we consider those P_4-free triangulations H_1 of G_1 for which V_1^1 is a clique. Let G^* be the graph obtained from G by making a clique of V_1^1. Let H_2 be a P_4-free triangulation of G_2. Notice that the 2-operation of H_1 and H_2 in this case is always a P_4-free triangulation of G. Hence

$$\chi_r(G^*) = \min_{H_1} \max(\omega(H_1), \chi_r(G_2) + |V_1^1|),$$

where the minimum is taken over all P_4-free triangulations H_1 of G_1 for which V_1^1 is a clique.

Now, consider the case of a P_4-free triangulation H where V_1^1 is not a clique. Let x and y be the two nonadjacent vertices of $H[V_1^1]$. Then x and y belong to a maximal homogeneous set $X \subseteq V_1^1$ of G, since the characteristic graph of $G_1[V_1^1]$ is complete. Let $S_X = V_1^2 \cap N(X)$.

Notice that every minimal x, y-separator of H contains at least the common neighbours of x and y in H. Since in a chordal graph every minimal separator must be a clique (see, e.g., [11]) we see that $(V_1^1 \setminus X) \cup S_X \cup V_2$ must be a clique in H. If we make $(V_1^1 \setminus X) \cup S_X \cup V_2$ into a clique, and X into a chordal cograph with minimum clique number, it follows that the only P_4's left in the graph must have vertices in $V_1^2 \setminus S_X$ and $V_1^1 \setminus X$, where $V_1^1 \setminus X$ is a clique. Hence, to obtain the optimal clique number of this type we have to determine the ranking number of every graph $G_1[(V_1^1 \setminus X) \cup (V_1^2 \setminus S_X)]$ with $V_1^1 \setminus X$ turned into a clique, for every maximal homogeneous set $X \subseteq V_1^1$ of G_1. Since G_1 has at most q vertices this can be done in constant time.

Hence we obtain the following result.

Theorem 4. *For every integer $q \geqslant 4$, there is a linear time algorithm to determine the vertex ranking number of a $(q, q-4)$-graph.*

4. Pathwidth

We show that the pathwidth problem can be solved in polynomial time for $(q, q-4)$-graphs for every fixed q. Note that the pathwidth problem remains NP-complete for starlike graphs [12] and that the characteristic of each starlike graph is a split graph.

Definition 10. A *path-decomposition* of G is a sequence $[X_1, X_2, \ldots, X_\ell]$ of subsets of vertices, such that

- every vertex appears in some subset,
- the endvertices of every edge appear in some common subset, and
- for every vertex x, the subsets containing x appear consecutively in the sequence.

The *pathwidth* of a graph equals the minimum width of a path-decomposition, where the width of a path-decomposition is the maximum size of a subset minus one.

A graph G is an *interval graph* if it is an intersection graph of a collection of intervals of the real line. Interval graphs form a subclass of the class of chordal graphs (see also [11]). We make use of the well known result (see, e.g., [5]) that the pathwidth of a graph G is equal to the smallest clique number of any interval supergraph of G decreased by one.

We consider the disjoint union and disjoint sum.

Lemma 7. *Let G be the disjoint union of G_1 and G_2. Then*

$$\mathrm{pw}(G) = \max(\mathrm{pw}(G_1), \mathrm{pw}(G_2))$$

Lemma 8. *Let G be the disjoint sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then*

$$\mathrm{pw}(G) = \min(\mathrm{pw}(G_1) + |V_2|, \mathrm{pw}(G_2) + |V_1|)$$

Proof. Let H be an interval supergraph of G. Hence, either V_1 or V_2 is a clique in H. Suppose V_2 is a clique in H. Then $\omega(H) = \omega(H[V_1]) + |V_2|$. Since $H[V_1]$ is an interval supergraph of G_1 we obtain $\omega(H) \geqslant \mathrm{pw}(G_1) + 1 + |V_2|$. Taking an interval supergraph H_1 of G_1 with $\omega(H_1) = \mathrm{pw}(G_1) + 1$ and making V_2 a clique we obtain an interval supergraph H of G with $\omega(H) = \mathrm{pw}(G_1) + 1 + |V_2|$.

Analogously $\omega(H) \geqslant \mathrm{pw}(G_2) + |V_1|$ for each interval supergraph H of G, where V_1 is a clique of H, and there is an interval supergraph H for which equality holds. □

For $(q, q-4)$-graphs we start again with the primeval tree. First, we compute the pathwidth of the leaves. These leaves are either spiders or graphs of bounded size. If the graph is of bounded size, we can simply try all interval supergraphs to find the pathwidth. The following lemma leads to formulas for the pathwidth of spiders. For a proof of this lemma see [19].

Lemma 9. *Let G be a split graph with maximum size clique K. Then the pathwidth is either $|K|$ or $|K| - 1$. The pathwidth is $|K| - 1$ if and only if there are vertices x and y in K with $N(x) \cap N(y) \subseteq K$.*

The following corollary takes care of those leaves that are spiders.

Corollary 1. *If G is a thin spider with $|K| > 1$ then* $\mathrm{pw}(G) = |K| - 1$. *If G is a thick spider with $|K| > 2$ then* $\mathrm{pw}(G) = |K|$.

Now, we can concentrate on the type 2-operation. We assume that the graph G is obtained from a separable p-connected graph $G_1 = (V_1, E_1)$ with separation (V_1^1, V_1^2) and a graph $G_2 = (V_2, E_2)$ by making every vertex of G_2 adjacent to every vertex of V_1^1.

Since G_1 is a p-component, G_1 is either a spider or a graph with less than q vertices of which the characteristic is a split graph (Theorem 2 and Lemma 1).

4.1. G_1 is a spider

First, we consider the case where G_1 is a spider. Let K be the clique and S be the independent set of G_1. Thus, (K,S) is the separation of G_1.

Lemma 10. *If* $V_2 \neq \emptyset$ *then* $\mathrm{pw}(G) = \mathrm{pw}(G_2) + |K|$.

Proof. Let H be an interval supergraph of G. Then $\omega(H) \geqslant \omega(H[K \cup V_2]) \geqslant \mathrm{pw}(G_2) + 1 + |K|$. Thus, $\mathrm{pw}(G) \geqslant \mathrm{pw}(G_2) + |K|$.

We show that there is a path-decomposition of G of width $\mathrm{pw}(G_2) + |K|$. Take a path-decomposition $[X_1, X_2, \ldots, X_\ell]$ of G_2 with width $\mathrm{pw}(G_2)$. Let $S = \{s_1, s_2, \ldots, s_q\}$. Then $[X_1 \cup K, X_2 \cup K, \ldots, X_\ell \cup K, \{s_1\} \cup K, \{s_2\} \cup K, \ldots, \{s_q\} \cup K]$ is path-decomposition of G and its width is $\mathrm{pw}(G_2) + |K|$. □

4.2. $|V_1| < q$ and the characteristic of G_1 is a split graph

Let (V_1^1, V_1^2) be the separation of G_1. We consider two cases. First, we consider triangulations of G into an interval graph where V_1^1 is a clique.

4.2.1. Triangulations with V_1^1 a clique

Let G^* be the graph obtained from G by making a clique of V_1^1. By Lemma 8, an optimal path-decomposition of $G^*[V_2 \cup V_1^1]$ can be made by adding V_1^1 to every subset of an optimal path-decomposition for G_2.

We claim that G^* has an optimal path-decomposition in which all subsets containing a vertex of G_2 occur consecutively. Let $\mathscr{P} = [X_1, X_2, \ldots, X_\ell]$ be an optimal path-decomposition of G^* without this property. Let X_i be the leftmost and X_k be the rightmost subset containing a vertex of G_2. For $j = i, \ldots, k$ let $Y_j = X_j \setminus V_2$ and $Z_j = X_j \setminus V_1^2$. Then $\mathscr{P}' = [X_1, \ldots, X_{i-1}, Y_i, \ldots, Y_k, Z_i, \ldots, Z_k, X_{k+1}, \ldots, X_\ell]$ is a path decomposition such that $\mathrm{width}(\mathscr{P}') \leqslant \mathrm{width}(\mathscr{P})$ since $V_1^1 \subseteq Z_j$ for $j = i, \ldots, k$. This proves the claim.

Therefore, the algorithm to compute the pathwidth of G^* works as follows. Compute all possible path-decompositions for $G^*[V_1]$. We can of course restrict to those path-decompositions with subsequent subsets not equal; hence, the set of all possible path-decompositions of $G^*[V_1]$ is bounded by a constant depending only on q.

Let $\mathscr{P} = [X_1, \ldots, X_\ell]$ be such a path-decomposition. Add $X_0 = X_{\ell+1} = \emptyset$. Consider the subsequence $X_i, \ldots, X_j$ of subsets that contain the clique V_1^1 and determine the minimum of $|(X_s \cap X_{s+1}) \setminus V_1^1|$ for $s = i-1, \ldots, j$. Let the minimum cardinality be attained for $(X_t \cap X_{t+1}) \setminus V_1^1$. Then a path-decomposition for G^* can be obtained by making an optimal path-decomposition for G_2, adding $V_1^1 \cup (X_t \cap X_{t+1})$ to every subset and putting this sequence of subsets between X_t and X_{t+1} in the path-decomposition $\mathscr{P}$. Then the width of this path-decomposition is $\max(\mathrm{width}(\mathscr{P}), \mathrm{pw}(G_2) + |V_1^1 \cup (X_t \cap X_{t+1})|)$.

Consequently, the pathwidth of G^* can be computed in linear time.

4.2.2. Triangulations with V_1^1 not a clique

Now, consider the case where V_1^1 is not a clique in an optimal interval supergraph H of G. Then there is a maximal homogeneous set X of $G_1[V_1^1]$ with vertices x and y that are nonadjacent in H. The common neighbors of x and y must form a clique in H. Hence, in H, $(V_1^1 - X) \cup S_X \cup V_2$ is a clique, where S_X is the set of all those vertices of V_1^2 which are adjacent to a vertex in X (and thus to all vertices of X) in G.

Hence, an optimal path-decomposition for $G[V_1^1 \cup S_X \cup V_2]$ with clique $(V_1^1 - X) \cup S_X \cup V_2$ consists of an optimal path-decomposition of $G[X]$ with $S_X \cup V_2 \cup (V_1^1 - X)$ added to every subset. Since $|X| < q$, an optimal path-decomposition of $G[X]$ can be found in constant time.

If $|V_2| < q$ then the graph G has at most $2q-2$ vertices, thus its pathwidth can be determined in constant time. Otherwise, the optimal width of a path-decomposition of G for which the maximal module X of $G[V_1^1]$ is not a clique is equal to the optimal width of a path-decomposition of $G[V_1^1 \cup S_X \cup V_2]$ with clique $(V_1^1 - X) \cup S_X \cup V_2$. This is a consequence of the fact that a subset $(V_1^1 - X) \cup (S - S_X)$ can be added to the latter path-decomposition without increasing its width, since $|V_2| - 1 \geqslant q - 1$.

Varying over all possible maximal modules X of $G_1[V_1^1]$ we obtain the smallest width of a path-decomposition of G in this case.

Theorem 5. *For every integer $q \geqslant 4$, there exists a linear time algorithm to compute the pathwidth for $(q, q-4)$-graphs.*

5. Path cover and hamiltonicity

In this section, we show how to decide hamiltonicity and how to compute the path cover number of a $(q, q-4)$-graph for fixed q in linear time. This extends the corresponding results of [15] on P_4-sparse and P_4-extendible graphs.

Definition 11. A family $P_1, \ldots, P_k$ of paths in G is called a *path cover* of G if every vertex of G is contained in exactly one of these paths. The *path cover number* $\pi(G)$ of G is the minimum cardinality of a path cover of G.

First, we consider the disjoint union and the disjoint sum of two graphs.

Lemma 11 (Hochstättler [15]). *Let G be the disjoint union of G_1 and G_2. Then $\pi(G) = \pi(G_1) + \pi(G_2)$.*

Lemma 12. *Let G be the disjoint sum of G_1 and G_2. Let $m = \max(\pi(G_1) - |V_2|, \pi(G_2) - |V_1|)$. Then G is hamiltonian if and only if $m \leqslant 0$. The path cover number satisfies $\pi(G) = \max(1, m)$.*

Proof. Assume G is hamiltonian. Consider a hamiltonian cycle of G. It induces a path cover $P_1,\dots,P_\ell$ for G_1 and a path cover $Q_1,\dots,Q_\ell$ for G_2. Then clearly $\pi(G_1)\leqslant\ell\leqslant|V_2|$ and $\pi(G_2)\leqslant\ell\leqslant|V_1|$. Hence $m\leqslant 0$.

For the remaining parts of the proof we refer to Lemma 3 in [15]. □

Notice that the hamiltonian circuit problem is NP-complete for split graphs [11, p. 155].

Now we consider spiders.

Lemma 13 (Hochstättler [15]). *Let G be a spider with clique K. If G is a thin spider then $\pi(G)=\lceil\frac{1}{2}|K|\rceil$. If G is a thick spider with $|K|>2$ then G is hamiltonian.*

We concentrate on the type 2-operation. We assume that the graph G is obtained from a separable p-connected graph $G_1=(V_1,E_1)$ with separation (V_1^1,V_1^2) and a graph $G_2=(V_2,E_2)$ by making every vertex of G_2 adjacent to every vertex of V_1^1. Since G_1 is a p-component, G_1 is either a spider or a graph with less than q vertices of which the characteristic is a split graph (Theorem 2 and Lemma 1).

5.1. G_1 is a spider

We can refer one more time to [15]. Let the clique of G_1 be K.

Lemma 14. *If G_1 is a thin spider then*

$$\pi(G)=\pi(G_2)+\max(0,\lceil\tfrac{1}{2}|K|\rceil-\pi(G_2))$$

and if G_1 is a thick spider with $|K|>2$ then $\pi(G)=\pi(G_2)$.

5.2. $|V_1|<q$ and the characteristic of G_1 is a split graph

Any path cover $P_1,\dots,P_\ell$ of G contains a path cover of G_2 of size j with $\pi(G_2)\leqslant j\leqslant|V_2|$, which can be obtained by removing all vertices of G_1. Consider the paths containing vertices of G_1 and G_2 in any path cover of G. These are at most q paths since $|V_1|<q$. Removing all vertices in V_1^1 from these paths, we obtain at most $2q$ subpaths in G_2.

By the above observations we may shrink all subpaths of G_2 in a path of a path cover of G into a single vertex. Therefore, we consider graphs H_j which are obtained by a 2-operation of G_1 with separation (V_1^1,V_1^2) and an empty graph on j vertices.

Clearly, $\pi(G)$ can be computed in constant time if $|V_2|\leqslant 2q$. Thus, we may assume $2q<|V_2|$. Then $\pi(G)$ can be obtained by computing the path cover number of all graphs H_j for which either $\pi(G_2)\leqslant j\leqslant 2q$, if $\pi(G_2)\leqslant 2q$, or only for H_{2q}, if $2q>\pi(G_2)$. These values can be computed in constant time since there are at most $2q$ graphs H_j and each of these graphs has at most $3q$ vertices.

Now, $\pi(G) = \min\{\pi(H_j)\colon \pi(G_2) \leqslant j \leqslant 2q\}$ if $\pi(G_2) \leqslant 2q$ and $\pi(G) = \pi(H_{2q}) + \pi(G_2) - 2q$ if $2q > \pi(G_2)$.

Hamiltonicity can be checked in a similar manner. Thus, we obtain the following result.

Theorem 6. *For every integer $q \geqslant 4$, there exists a linear time algorithm to decide whether a $(q, q-4)$-graph is hamiltonian and to determine its path cover number.*

6. Independent domination

In the following sections, we demonstrate that various domination problems can be solved very efficiently on $(q, q-4)$-graphs.

Definition 12. A set D of vertices is a *dominating set* if every vertex in $V \setminus D$ has a neighbor in D.

The minimum dominating set problem asks to determine a dominating set of smallest cardinality. In the weighted version, each vertex v of the graph is assigned a nonnegative weight $w(v)$ and the problem is to find a dominating set of smallest total weight.

In many applications (see, e.g., [13,14]) dominating sets are subject to additional constraints. In particular, one is frequently interested in dominating sets which are either independent or cliques or induce connected subgraphs. The parameters $\gamma_{\mathrm{i}}(G)$, $\gamma_{\mathrm{cl}}(G)$, and $\gamma_{\mathrm{c}}(G)$ denote, respectively, the *independent domination number*, the *dominating clique number*, and the *connected domination number*, that is, the smallest weight of an independent, complete, and connected dominating set in G.

In this section, we show how the independent domination problem can be solved using the homogeneous decomposition of a graph. The following lemma is obvious.

Lemma 15. *If G is the disjoint union of graphs G_1 and G_2 then*

$$\gamma_{\mathrm{i}}(G) = \gamma_{\mathrm{i}}(G_1) + \gamma_{\mathrm{i}}(G_2).$$

Lemma 16. *Let G be the disjoint sum of graphs G_1 and G_2. Then*

$$\gamma_{\mathrm{i}}(G) = \min(\gamma_{\mathrm{i}}(G_1), \gamma_{\mathrm{i}}(G_2)).$$

Proof. An independent dominating set in G contains vertices from exactly one of the subgraphs G_i. □

Now, we consider a 3-operation. Let H be a homogeneous set in G. We denote by $N(H)$ the set of all neighbors of vertices from H. If an independent dominating set contains no vertices from $N(H)$ then it must consist of an independent dominating set in H and an independent dominating set in $G \backslash (H \cup N(H))$. On the other hand, if it contains a vertex from $N(H)$ then it does not contain a vertex from H. This observation implies that $\gamma_{\mathrm{i}}(G)=\gamma_{\mathrm{i}}(G_H)$, where G_H denotes the graph obtained from G by replacing H by a single vertex of weight $\gamma_{\mathrm{i}}(H)$.

Hence, if G arises by a 3-operation involving homogeneous sets $H_1,\ldots,H_k$, then it suffices to solve the problem on the characteristic graph of G. Naturally, the weight of the vertex representing the homogeneous set H is $\gamma_{\mathrm{i}}(H)$.

Assume that G arises by a 2-operation. Then G consists of a separable p-component G_1 and of a subgraph G_2 outside G_1 which is adjacent to G_1 as stipulated in Theorem 1. By Lemma 1, the characteristic graph of G is a split graph. If the weights of all vertices representing homogeneous sets are already known, then we can easily solve the problem for G. For that purpose, denote the vertices of the clique by $y_1,\ldots,y_r$ and the vertices of the independent set by $z_0,z_1,\ldots,z_s$. The vertex z_0 represents the homogeneous set G_2 and, by convention, belongs to the independent set. If an independent dominating set in the split graph contains a vertex from the clique, say y_i, then it must contain all vertices from the independent set that are nonadjacent to y_i. If an independent dominating set contains no vertex from the clique then it must contain all vertices from the independent set. This shows the following statement.

Lemma 17. *Let G be a split graph with clique $K=\{y_1,\ldots,y_r\}$ and independent set $S=\{z_0,\ldots,z_s\}$. Then*

$$\gamma_{\mathrm{i}}(G)=\min\left(\sum_{j=0}^{s} w(z_j);\ \min_{1\leqslant i\leqslant r}\left(w(y_i)+\sum_{z_j\notin N(y_i)} w(z_j)\right)\right).$$

We now restrict to weighted $(q,q-4)$-graphs. If G contains fewer than q vertices, for some fixed q, then $\gamma_{\mathrm{i}}(G)$ can be determined in constant time. If G is a spider then we can apply Lemma 17. This immediately implies the following result.

Theorem 7. *For every integer $q\geqslant 4$, the independent domination number of a $(q,q-4)$-graph can be computed in linear time.*

7. Connected and clique domination

We now consider the problem of finding a smallest weight connected dominating set. If G is the disjoint union of G_1 and G_2 then no connected dominating set exists. This is indicated by writing $\gamma_{\mathrm{c}}(G)=\infty$.

Lemma 18. *Let G be the disjoint sum of G_1 and G_2. Then*

$$\gamma_{\mathrm{c}}(G)=\min_{i=1,2}(\gamma_{\mathrm{c}}(G_i);\ w_{\min}(G_1)+w_{\min}(G_2)),$$

where $w_{\min}(G_i)$ denotes the smallest weight of a vertex from G_i.

Proof. A smallest weight connected dominating set in G consists either of a smallest weight connected dominating set in one of the subgraphs G_i or it consists of precisely two vertices from different subgraphs G_1 and G_2. $\square$

Consider the problem of finding a dominating clique. If G is the disjoint union of two graphs G_1 and G_2 then no dominating clique exists. Hence, $\gamma_{\mathrm{cl}}(G) = \infty$.

If G is the disjoint sum of G_1 and G_2 then either a dominating clique is a clique in one of the two subgraphs, or two vertices, one from each subgraph.

Assume that G arises by a 3-operation involving the homogeneous sets $H_1, \ldots, H_k$. Consider an arbitrary set H_i. Clearly, any connected dominating set in G must contain at least one vertex from $N(H_i)$. Therefore, a smallest weight connected dominating set contains at most one vertex from H_i and, if this is the case, this vertex has smallest weight in H_i. Hence, we can restrict the problem to the characteristic graph of G. Naturally, the vertex which represents H_i has weight $w_{\min}(H_i)$.

The same idea applies if G arises by a 2-operation. We can restrict our attention to the characteristic graph which now is a split graph.

It is easy to verify that the minimum dominating clique problem can be treated in a quite analogous manner. In case of a 2- or a 3-operation we have to find a dominating clique in the characteristic graph. Hence, for the computation of $\gamma_{\mathrm{cl}}(G)$ we can adopt the ideas used to determine $\gamma_{\mathrm{c}}(G)$.

We have shown that the weighted independent dominating set problem can be solved very easily when restricted to split graphs. However, it is known that the connected dominating set problem is NP-complete for split graphs [20] which also implies the NP-completeness of the minimum dominating clique problem on split graphs (see also [13]).

Hence, in the case of a 2-node, we cannot obtain $\gamma_{\mathrm{c}}(G)$ and $\gamma_{\mathrm{cl}}(G)$ as easily as this was possible for $\gamma_{\mathrm{i}}(G)$. Here, it is still an open problem to compute these parameters for the characteristic graph of G. Nevertheless, the problems are easy to solve for $(q, q-4)$-graphs.

We consider the leaves of the primeval decomposition of a $(q, q-4)$-graph. If G contains fewer than q nodes, the parameters $\gamma_{\mathrm{cl}}(G)$ and $\gamma_{\mathrm{c}}(G)$ can be determined in constant time.

Assume that G is a spider. Denote the vertices of the clique by $y_1, \ldots, y_r$ and the vertices of the independent set by $z_1, \ldots, z_r$, such that y_i is adjacent (resp. nonadjacent) to z_i if G is a thin (resp. thick) spider.

Lemma 19. *If G is a thin spider then*

$$\gamma_{\mathrm{cl}}(G) = \gamma_{\mathrm{c}}(G) = \sum_{i=1}^{r} w(y_i).$$

If G is a thick spider then

$$\gamma_{\mathrm{cl}}(G) = \gamma_{\mathrm{c}}(G) = \min_{1 \leqslant i < j \leqslant r} (w(y_i) + w(y_j)).$$

Proof. If G is a thin spider then the clique is the only minimal connected dominating set. If G is a thick spider then a minimal connected dominating set must consist of two vertices from the clique. □

Now, we consider a 2-operation. If G_1 is a spider then the characteristic of G is a spider plus one additional vertex which is adjacent precisely to the vertices of the clique. In this case, $\gamma_{\mathrm{cl}}(G)$ and $\gamma_{\mathrm{c}}(G)$ are computed in the same way as in the previous lemma. If G_1 has less than q vertices then the characteristic of G has at most q vertices. Hence, we can compute $\gamma_{\mathrm{cl}}(G)$ and $\gamma_{\mathrm{c}}(G)$ in constant time.

This provides the next result.

Theorem 8. *For every integer $q \geqslant 4$, the connected domination number and the dominating clique number of a $(q, q-4)$-graph can be computed in linear time.*

8. Domination and total domination

A further problem which has attracted considerable attention in recent years is the *minimum total dominating set problem* (see, e.g., [6,21]). The task involves finding a dominating set which contains no isolated vertices. We denote by $\gamma(G)$ and $\gamma_{\mathrm{t}}(G)$, respectively, the *domination number* and the *total domination number*, that is, the smallest weight of a dominating set and of a total dominating set in G.

Lemma 20. *Let G be the disjoint union of G_1 and G_2. Then*

$$\gamma(G) = \gamma(G_1) + \gamma(G_2).$$

Lemma 21. *Let G be the disjoint sum of G_1 and G_2. Then*

$$\gamma(G) = \min_{i=1,2}(\gamma(G_i), w_{\min}(G_1) + w_{\min}(G_2)).$$

For $\gamma_{\mathrm{t}}(G)$ we obtain analogous results.

Let now H be a homogeneous set in G. If a minimum dominating set D in G contains no vertices from $N(H)$ then it consists of dominating sets in H and in $G \setminus (H \cup N(H))$. If D contains a vertex from $N(H)$ then at most one vertex from H belongs to D and this vertex has smallest weight. Hence, we have

$$\gamma(G) = \min(\gamma(H) + \gamma(G \setminus (H \cup N(H))); \gamma(G_H)),$$

where G_H is the graph which arises from G by replacing H by an independent set of size two with one vertex having smallest weight in H and the other vertex having weight ∞ (the second vertex guarantees that a vertex from $N(H)$ must belong to a minimum dominating set). An analogous equality holds for $\gamma_{\mathrm{t}}(G)$, however, in G_H it suffices to replace H by one vertex of smallest weight. Since a total dominating set D contains no isolated vertices, it is clear that at least one vertex from $N(H)$ must belong to D.

Let G arise by a 2-operation with a separable p-component G_1 with separation (V_1^1, V_1^2) and a subgraph G_2 outside G_1 which is adjacent to all vertices of V_1^1 and to no vertex of V_1^2. By the previous arguments, we obtain

$$\gamma(G) = \min\left(\gamma(G_2) + \gamma(G[V_1^2]);\ \gamma(G_{G_2})\right).$$

Note that both the total dominating set problem and the dominating set problem are NP-complete for split graphs [20].

We assume that a $(q, q-4)$-graph G is given along with its primeval decomposition tree. If G corresponds to a leaf and if G has fewer than q vertices then both $\gamma(G)$ and $\gamma_t(G)$ can be determined in constant time (we write $\gamma_t(G) = \infty$ in case G contains no total dominating set).

Lemma 22. *If G is a thin spider then*

$$\gamma(G) = \sum_{i=1}^{r} \min(w(y_i); w(z_i)) \quad \textit{and} \quad \gamma_t(G) = \sum_{i=1}^{r} w(y_i).$$

If G is a thick spider then

$$\gamma(G) = \min_{1 \leqslant j < k \leqslant r} \left(\sum_{i=1}^{r} w(z_i);\ w(y_j) + w(y_k);\ w(y_j) + w(z_j) \right)$$

and

$$\gamma_t(G) = \min_{1 \leqslant j < k \leqslant r} (w(y_j) + w(y_k)).$$

Consider a 2-operation. Since G is a $(q, q-4)$-graph it follows that G_1 has fewer than q vertices or is a spider. In the first case, both $\gamma(G[V_1^2])$ and $\gamma(G_{G_2})$ can be computed in constant time since $G[V_1^2]$ and G_{G_2} have at most $q+1$ vertices. If G_1 is a spider then

$$\gamma(G[V_1^2]) = \sum_{i=1}^{r} w(z_i).$$

If G_1 is a thin spider then

$$\gamma(G_{G_2}) = \min_{1 \leqslant i \leqslant r} \left(w(y_i) + \sum_{j \neq i} \min(w(y_j); w(z_j)) \right).$$

If G_1 is a thick spider then

$$\gamma(G_{G_2}) = \min_{1 \leqslant i < j \leqslant r} (w(y_i) + w(y_j); w(y_i) + w(z_i)).$$

Similarly, if G_1 is a thin spider then we obtain

$$\gamma_t(G) = \sum_{i=1}^{r} w(y_i)$$

and, if G_1 is a thick spider

$$\gamma_t(G) = \min_{1 \leqslant i < j \leqslant r} (w(y_i) + w(y_j)).$$

The previous considerations immediately imply the following statement.

Theorem 9. *For every integer $q \geqslant 4$, the domination number and the total domination number of a $(q, q-4)$-graph can be computed in linear time.*

9. Steiner tree

The *Steiner tree problem* bears some similarity to the minimum connected dominating set problem.

Definition 13. Given a set T of *target vertices* in a graph $G=(V,E)$, a set S is called a set of *Steiner vertices* if $G[S \cup T]$ is connected.

We are interested in finding a *smallest* set S of Steiner vertices. Naturally, the weighted version of the problem asks for a set S of smallest total weight. Such a set S is usually called a *Steiner set*.

Lemma 23. *Let G be the disjoint union of G_1 and G_2. If G_1 and G_2 both contain vertices of the target set T then, obviously, no Steiner set S exists. If all vertices of T belong to one of the subgraphs, say G_i, then S is completely contained in G_i.*

Hence, in case of a disjoint union, we can restrict the problem to one of the subgraphs.

Lemma 24. *Let G be the disjoint sum of G_1 and G_2. We distinguish two cases:*

1. *If G_1 and G_2 both contain vertices of T, then T induces a connected graph, and so $S = \emptyset$.*
2. *If all vertices from T belong to some subgraph G_i then either S is completely contained in G_i or S contains precisely one vertex, namely a vertex of smallest weight, outside of G_i.*

Hence, in case G is the disjoint sum of two graphs, we solve the problem restricted to G_i and determine a vertex with smallest weight outside G_i. The Steiner set S is the one of the two resulting sets which has minimum weight.

Let H be a homogeneous set in G. We consider three cases. First, assume that $T \subseteq H$ holds. In this case, S is completely contained in H or consists of one vertex of smallest weight in $N(H)$.

Next, assume that $T \cap H = \emptyset$. Now, S contains at most one vertex from H and, if this is the case, this vertex has smallest weight. Hence, we can restrict the problem to the graph obtained from G by replacing H by some vertex of smallest weight in H. Clearly, a Steiner set in the new graph is also a Steiner set in the original graph.

Finally, assume that $T \cap H \neq \emptyset$ and $T \nsubseteq H$. It is an easy observation that S contains no vertices from H (if no vertex from T belongs to $N(H)$ then at least one vertex from $N(H)$ must belong to S). This shows that, as before, it suffices to study the graph where H is replaced by a single vertex which represents the set $T \cap H$. A Steiner set in the latter graph is again a Steiner set in the original graph.

Hence, if G arises by a 2- or 3-operation with homogeneous sets $H_1, \ldots, H_k$ then we have to check whether $T \subseteq H_i$ holds for some $i \in \{1, \ldots, k\}$. In this case, we compute a Steiner set in the subgraph H_i and determine a set containing a single vertex of smallest weight from $N(H_i)$. By the previous arguments, S is the set with smaller weight. Otherwise, we have to compute a Steiner set in the characteristic graph of G where each homogeneous set H_i with $T \cap H_i = \emptyset$ is represented by one of its vertices of smallest weight. The vertices representing sets H_j with $T \cap H_j \neq \emptyset$ belong to the new set T' of target vertices, together with the vertices from T which belong to none of the homogeneous sets.

Note that the Steiner tree problem remains NP-complete when restricted to split graphs [25].

In order to solve the problem for $(q, q-4)$-graphs we first have to show that the problem can be solved efficiently when restricted to the graphs corresponding to the leaves of the primeval decomposition tree.

Let G be such a graph. If G has less than q vertices then the problem can be solved in constant time. Assume that G is a thin spider. If $|T| = 1$ then, clearly, $S = \emptyset$. Therefore, assume that $|T| \geqslant 2$. If a vertex z_i from the independent set of the spider belongs to T and if y_i is not in T then y_i must belong to S. This fact suffices to construct the Steiner set S.

Now, let G be a thick spider and $|T| \geqslant 2$. If at least two vertices y_i and y_j of the clique belong to T then T induces a connected graph and $S = \emptyset$. If only one vertex y_i of the clique belongs to T then we have to consider two cases. If z_i is not in T then T is connected and $S = \emptyset$. If z_i belongs to T then S must contain one vertex from the clique with smallest weight. Finally, if no vertex from the clique belongs to T then again we have two cases. If all vertices from the independent set are in T then S consists of two vertices from the clique having smallest weights. Otherwise, write $T = \{z_1, \ldots, z_k\}$. Then S either contains only one vertex, namely a vertex of smallest weight from $\{y_{k+1}, \ldots, y_r\}$, or it contains two vertices of smallest weight from $\{y_1, \ldots, y_k\}$.

If G is the result of a 2-operation then the characteristic of G either has at most q vertices or is isomorphic to a spider with one additional vertex adjacent precisely to the clique. In the first case, we can solve the problem in constant time, in the second case we proceed analogously as above.

These observations imply that a Steiner set can be found efficiently.

Theorem 10. *For every integer $q \geqslant 4$, the Steiner tree problem for a $(q, q-4)$-graph can be solved in linear time.*

10. List coloring

The problem is NP-complete for cographs [18]. However, if we restrict the number of colors in the union of the lists by a constant then it can be seen that the problem is linear time solvable for $(q, q-4)$-graphs for fixed q.

Definition 14. Given a graph $G=(V,E)$ and for every vertex u, a list $L(u)$ of admissible colors for this vertex, G is called *L-list colorable* if vertices of G can be assigned colors from their lists so that adjacent vertices receive different colors.

Let G be a graph and let L be a list assignment such that $|\bigcup_{u\in V} L(u)| = k$ is a constant. Let $\mathscr{S}(G)$ be the set of subsets of $\bigcup_{u\in V} L(u)$ such that $S \in \mathscr{S}(G)$ iff G has an L-list coloring which uses exactly the colors of S. As $\bigcup_{u\in V} L(u)$ has 2^k subsets, $\mathscr{S}(G)$ can attain at most 2^{2^k} values, which is still a constant number with respect to the input size (of G). Obviously, G is L-list colorable iff $\mathscr{S}(G) \neq \emptyset$.

Lemma 25. *Let G be the disjoint union of G_1 and G_2. Then $\mathscr{S}(G) = \{A \cup B\colon A \in \mathscr{S}(G_1),\ B \in \mathscr{S}(G_2)\}$.*

Lemma 26. *Let G be the disjoint sum of G_1 and G_2. Then $\mathscr{S}(G) = \{A \cup B\colon A \in \mathscr{S}(G_1),\ B \in \mathscr{S}(G_2),\ A \cap B = \emptyset\}$.*

We further consider the type-2 operation. Let G be obtained from a separable p-connected graph G_1 with separation (V_1^1, V_1^2) and from a graph G_2 by making every vertex of V_1^1 adjacent to every vertex of G_2. We set

$$\mathscr{T}(G_1) = \{(A,B)\colon \exists\ L\text{-coloring } f \text{ s.t. } A = f(V_1^1) \text{ and } B = f(V_1^2)\}.$$

Lemma 27. $\mathscr{S}(G) = \{A \cup B \cup C\colon (A,B) \in \mathscr{T}(G_1),\ C \in \mathscr{S}(G_2),\ A \cap C = \emptyset\}$.

Then $\mathscr{S}(G)$ can be computed in constant time from $\mathscr{T}(G_1)$ and $\mathscr{S}(G_2)$ and it only remains to show how $\mathscr{T}(G_1)$ can be computed in linear time.

$|V_1| < q$: We simply consider all possible L-list colorings of G_1 (there are at most k^q of them) and so $\mathscr{T}(G_1)$ can be computed in constant time.

G_1 *is a spider*: Since in any coloring of a spider the vertices of the clique part have to be colored by mutually distinct colors, G_1 is not list colorable if $|V_1| > 2k$. In such a case we output that G is not L-list colorable. If $|V_1| \leqslant 2k$, G_1 is of constant size and we can again compute $\mathscr{T}(G_1)$ in constant time by brute force.

We summarize our results on list colorings in the following theorem.

Theorem 11. *For fixed integers k and q, k-List-Coloring restricted to $(q, q-4)$-graphs is solvable in linear time.*

11. Precoloring extension

The restricted variant precolor extension is also solvable in polynomial time, even if the total number of colors in each list is unbounded.

Precoloring extension is a special case of list-coloring where lists of admissible colors are either one-element (such vertices are *precolored*) or equal $L(u)$ (being the set of all colors for such vertices u). We may assume that the precoloring is legal, i.e., no two vertices precolored by the same color are adjacent. If k is the total number of colors, the problem is referred to as k-PrExt. Note that not all k colors are necessarily used on the precolored vertices. We will show that k-PrExt can be solved in polynomial time on $(q, q-4)$-graphs even if k is not constant but rather part of the input. Towards this end we denote by $\mathrm{Pr}(G)$ the set of colors used on precolored vertices, $\mathrm{pre}(G) = |\mathrm{Pr}(G)|$ the number of colors used on the precolored vertices, and by $p(G)$ the minimum number of additional colors needed for a precoloring extension.

So G is feasible for k-PrExt iff $p(G) \leqslant k - pre(G)$.

Lemma 28. *Let G be the disjoint union of G_1 and G_2. Then*

$$p(G) = \max\{p(G_1) - |\mathrm{Pr}(G_2) \setminus \mathrm{Pr}(G_1)|, p(G_2) - |\mathrm{Pr}(G_1) \setminus \mathrm{Pr}(G_2)|, 0\}.$$

Lemma 29. *Let G be the disjoint sum of G_1 and G_2. Then $p(G) = p(G_1) + p(G_2)$.*

We further consider the type-2 operation. Let G be obtained from a separable p-connected graph G_1 with separation (V_1^1, V_1^2) and from a graph G_2 by making every vertex of V_1^1 adjacent to every vertex of G_2.

G_1 *is small*: We try all possible precoloring extensions of G_1 using the colors of the precoloring of G_1 and variable colors for the extension. For each such coloring f we determine the minimum number of additional colors needed for extending the coloring to G. We know that G_2 itself needs $p(G_2)$ new colors and that these have to be different from all colors used on V_1^1.

1. Use as many colors of $\mathrm{Pr}(G_2) \setminus \mathrm{Pr}(G_1)$ as possible on the variable colors that are used on V_1^2 only.
2. Unify as many as possible of the remaining variable colors used only on V_1^2 with the new colors used on G_2.
3. The remaining variable colors on G_1 and new colors on G_2 have to be mutually different.

Let the total number of new colors be $p_f(G)$. Then $p(G)=\min_f p_f(G)$ can be computed in time linear in $p(G_2)$.

G_1 *is a spider*: Unlike the case of list colorings with bounded number of colors, in this case the size of G_1 is not bounded. Consider an optimal coloring of G. Since $G[V_1^1]$ is a complete graph, the $p(G_2)$ new colors on G_2 and the $p_{11}=|V_1^1|-\mathrm{pre}(G[V_1^1])$ colors on unprecolored vertices of $G[V_1^1]$ must be all different, and different from colors in $\mathrm{Pr}(G_2)\cup\mathrm{Pr}(G[V_1^1])$. We can color G and determine $p(G)$ by the following procedure:

1. Use as many colors of $\mathrm{Pr}(G[V_1^2])\setminus\mathrm{Pr}(G[V_1^1])$ as possible on the p_{11} unprecolored vertices of $G[V_1^1]$;
2. Use the remaining colors of $\mathrm{Pr}(G[V_1^2]) - \mathrm{Pr}(G[V_1^1])$ (if there are any left) as the new colors on G_2;
3. Use fresh new colors for colors needed on G_2 and for vertices of $G[V_1^1]$ which were not colored in Steps 1 and 2;
4. Color the unprecolored vertices $G[V_1^2]$ using colors used on $G[V_1^1]$.

We clearly get an optimal coloring since each of the Steps 1–4 uses the minimum possible number of new colors.

Claim. *In Step* 4 *we can color all vertices of* $G[V_1^2]$ *since each such vertex is nonadjacent to at least one vertex of* $G[V_1^1]$ *and the color of any such vertex can be used.*

Claim. *Step* 1 *can be performed in polynomial time. Consider the bipartite graph* H *induced in* G_1 *by unprecolored vertices of* V_1^1 *and by precolored vertices of* V_1^2, *vertices precolored by the same color being unified into a single vertex representing that color. Then the maximum number of colors of* $\mathrm{Pr}(G[V_1^2])$ *which can be used on* V_1^1 *is the size of a maximum matching in* $\bar{H}$, *the bipartite complement of* H, *and this can be computed in polynomial time.*

Note that the preceding claim applies to any split graph G_1. However, it uses a bipartite matching algorithm and as such it is not known to be linear. The special structure of spiders can be used to prove linearity.

If G_1 is a *thick spider* then $\bar{H}$ has maximum degree at most 1 and so the size of a maximum matching is simply the number of edges of this graph.

Let G_1 be a *thin spider*. Then H is a bipartite graph whose vertices in $H[V_1^1]$ have degrees $\leqslant 1$. If the other part of H, $\mathrm{Pr}(G[V_1^2])$, contains a vertex (color) adjacent to all vertices of $H[V_1^1]$ then this color cannot be used on V_1^1 and H is a star plus isolated vertices (on the side of colors). In this case, the size of a maximum matching is either p_{11} or $|\mathrm{Pr}(G[V_1^2])|-1$, whichever is smaller. It is easy to see that $\bar{H}$ contains a full matching (of size $|\mathrm{Pr}(G[V_1^2])|$ or p_{11}) otherwise.

We summarize our results on precoloring extensions in the following theorem.

Theorem 12. *For fixed integer* q, k-PrExt *restricted to* $(q,q-4)$-*graphs is solvable in polynomial time.*

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