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ON THE RADIUS OF CONVERGENCE OF INTERCONNECTED ANALYTIC NONLINEAR INPUT-OUTPUT SYSTEMS*

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Abstract. A complete analysis is presented of the radii of convergence of the parallel, product, cascade and feedback interconnections of analytic nonlinear input-output systems represented as Fliess operators. Such operators are described by convergent functional series, which are indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series. Given growth conditions for the coefficients of the generating series for the subsystems, the radius of convergence of each interconnected system is computed assuming the subsystems are either all locally convergent or all globally convergent. In the process of deriving the radius of convergence for the feedback connection, it is shown definitively that local convergence is preserved under feedback. This had been an open problem in the literature until recently.

Key words. nonlinear systems, formal power series, Chen–Fliess series, real-analytic functionals

AMS subject classifications. 93C10, 47H30, 47F15

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1. Introduction.

Most complex systems found in applications can be viewed as a collection of interconnected subsystems. Generally, an interconnection is said to be *well-posed* when the output signal and every internal signal is uniquely defined on some interval $[t_0, t_0 + T]$, $T > 0$ for a given class of inputs, for example, the set of Lebesgue measurable functions $L_{\mathfrak{p}}[t_0, t_0 + T]$. Sometimes additional properties like causality, continuity, and regularity are also included as part of the definition of well-posedness [4, 34]. If one or more subsystems is nonlinear, a variety of sufficient conditions are available to ensure that an interconnected system is well-posed [1, 2, 31].

This paper focuses on a class of analytic nonlinear input-output systems known as *Fliess operators* [11, 12, 13]. Such operators are described by functional series, which are indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series. It is known that the parallel, product, and cascade connections as shown in Figure 1(a)–(c) are well-posed in the sense that any two locally convergent Fliess operators interconnected in such a manner always yield another locally convergent Fliess operator [17]. Recently, it was shown in [14, 29] that the feedback connection as shown in Figure 1(d) also preserves local convergence. It is also known that a notion of global convergence is preserved under the parallel and product connections but not in general by the cascade or feedback connection [7, 8, 16].

The goal of this paper is to pursue a much finer analysis of the situation by introducing the notion of the *radius of convergence* for a given interconnection. This concept will describe in some sense the largest class of admissible input that an interconnected system of Fliess operators can accept and still remain well-posed (i.e.,

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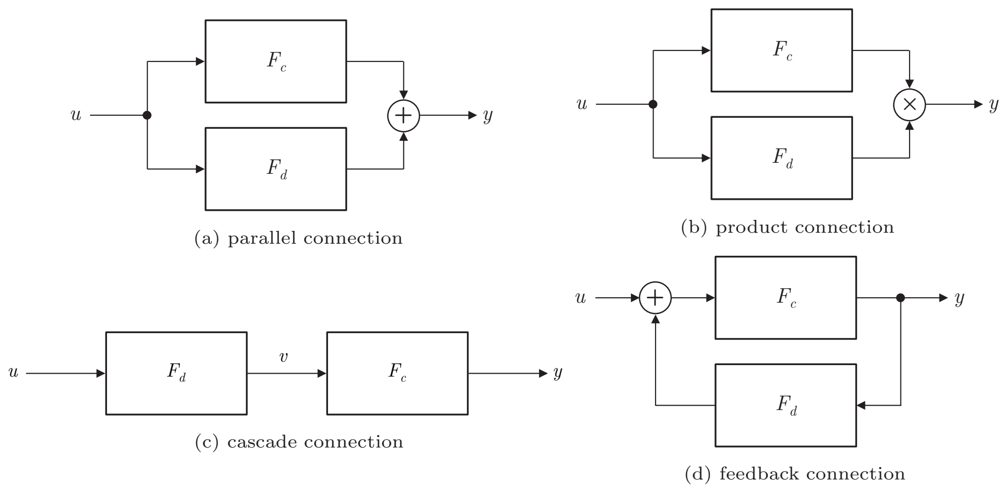

FIG. 1. *Four basic system interconnections.*

convergent). Lower bounds for this radius of convergence are available in [32] for the product connection and in [17] for the cascade and self-excited feedback connections (i.e., when $u = 0$). But it will be shown in this paper, by computing the exact radius of convergence in each instance, that these bounds are in general very conservative. It should be noted that the unity feedback interconnection, which at first inspection does not appear to be of the form shown in Figure 1(d) since the identity map, I, in the feedback path is *not* realizable by a Fliess operator, can in fact be treated as a special case of the analysis to be presented. These results first appeared in [18]. They have special significance when feedback is viewed in the context of Hopf algebras [14, 15]. Two classes of interconnections will be considered individually, one where the subsystems are all locally convergent and the other where they are all globally convergent. In every case, it will be shown precisely what additional benefits one gains by having globally convergent subsystems. For example, in a unity feedback system, this will effectively double the radius of convergence of the closed-loop system when compared to the case where only locally convergent subsystems are present. Finally, specific examples for which the radius of convergence is achieved are provided for each interconnection.

The paper is organized as follows. In the next section, various mathematical preliminaries are introduced to establish the notation, to more precisely frame the problems of interest, and to introduce the main analysis tools. Then sections 3 through 6 present, respectively, the radius of convergence analysis for the parallel, product, cascade, and feedback connections. The conclusions of the paper are summarized in the final section, and some directions for future research are proposed.

2. Preliminaries. A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η, while $|\eta|_{x_i}$ is the number of times the letter x_i appears in η. The set of all words with length k is denoted by X^k. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. Given a subset $L \subseteq X^*$, the *characteristic series* of

L is defined by $\mathrm{char}(L) = \sum_{\eta \in L} \eta$. The notation $c \leq d$ means that the component series satisfy $(c_i, \eta) \leq (d_i, \eta)$ for all $\eta \in X^*$ and $i = 1, 2, \ldots, \ell$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X \rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product and a commutative and associative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}\langle\langle X \rangle\rangle \times \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle$ uniquely specified by the shuffle product of two words

$$(x_i \eta) \sqcup\!\sqcup (x_j \xi) = x_i(\eta \sqcup\!\sqcup (x_j \xi)) + x_j((x_i \eta) \sqcup\!\sqcup \xi)$$

and $\eta \sqcup\!\sqcup \emptyset = \eta$ for all $\eta, \xi \in X^*$ with $(c \sqcup\!\sqcup d)_i := c_i \sqcup\!\sqcup d_i$, $1 \leq i \leq \ell$ when $c, d \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ [11].

2.1. Fliess operators and their convergence. One can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : \ 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L_{\mathfrak{p}}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define iteratively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to \mathcal{C}[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{1}$$

[11, 12, 13]. If there exists real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| := \max_i |(c_i, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{2}$$

then F_c constitutes a well-defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0+T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0+T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [19]. The set of all such *locally convergent* series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$. In particular, when $\mathfrak{p} = 1$ it was shown in [5, 6] that the series (1) converges uniformly and absolutely on $[0, T]$ if

$$\max\{R, T\} < \frac{1}{M_c(m+1)}. \tag{3}$$

A similar conclusion can be drawn when $\mathfrak{p} = \infty$ provided $RT < 1/M_c(m+1)$ [11, 21, 32]. In either case, it is important in applications to identify the *smallest* possible geometric growth constant, M_c, in order to avoid over restricting the domain of F_c. So let $\pi : \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle \to \mathbb{R}^+ \cup \{0\}$ take each series c to the infimum of all M_c satisfying (2). Therefore, $\mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$ can be partitioned into equivalence classes, and the number $1/M_c(m+1)$ will be referred to as the *radius of convergence* for the class $\pi^{-1}(M_c)$. This is in contrast to the usual situation where a radius of convergence is assigned to individual series [24]. In practice, it is not difficult to estimate the minimal M_c for many series, in which case the radius of convergence for $\pi^{-1}(M_c)$ provides an easily computed *lower bound* for the radius of convergence of c in the usual sense. Finally, when c satisfies the more stringent growth condition

$$|(c,\eta)| \le K_c M_c^{|\eta|}, \quad \eta \in X^*, \tag{4}$$

the series (1) defines an operator from the extended space $L_{\mathfrak{p},e}^m(t_0)$ into $\mathcal{C}[t_0,\infty)$, where

$$L_{\mathfrak{p},e}^m(t_0) := \{u : [t_0,\infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L_{\mathfrak{p}}^m[t_0,t_1] \; \forall t_1 \in (t_0,\infty)\}$$

and $u[t_0,t_1]$ denotes the restriction of u to $[t_0,t_1]$ [19]. The set of all such *globally convergent* series is designated by $\mathbb{R}_{GC}^\ell\langle\langle X\rangle\rangle$. Henceforth, given any locally or globally convergent c, M_c will always denote the smallest possible geometric growth constant satisfying (2) or (4), respectively.

2.2. State space realization of a Fliess operator. A Fliess operator F_c defined on $B_{\mathfrak{p}}^m(R)[t_0,t_0+T]$ is said to be *realized* by a state space realization when there exists a system of n differential equations and ℓ output functions

$$\dot{z} = g_0(z) + \sum_{i=1}^m g_i(z)\, u_i, \quad z(t_0) = z_0, \tag{5a}$$

$$y = h(z), \tag{5b}$$

where each g_i is an analytic vector field expressed in local coordinates on some neighborhood $\mathcal{W}$ of z_0 and h is an analytic function on $\mathcal{W}$ such that (5a) has a well-defined solution $z(t)$, $t \in [t_0,t_0+T]$ on $\mathcal{W}$ for any given input $u \in B_{\mathfrak{p}}^m(R)[t_0,t_0+T]$ and $F_c[u](t) = h(z(t))$, $t \in [t_0,t_0+T]$ [11, 12, 13, 19, 21]. Let $G = \{g_0,g_1,\ldots,g_m\}$. It is well known that F_c is realizable if and only if $c \in \mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$ has finite *Lie rank* [12, 13, 21, 23, 28]. In this case, the generating series c is related to the realization (G,h,z_0) by $(c,\eta) = L_{g_\eta}h(z_0)$, $\eta \in X^*$, where the iterated *Lie derivatives* are defined by

$$L_{g_\eta}h = L_{g_{i_1}} \cdots L_{g_{i_k}} h, \quad \eta = x_{i_k}\cdots x_{i_1} \in X^*$$

with $L_{g_i} : h \mapsto \partial h/\partial z \cdot g_i$ and $L_\emptyset h = h$.

2.3. The composition product. The cascade connection of two Fliess operators as depicted in Figure 1(c) was shown by Ferfera in [7, 8] to always yield an input-output system having a Fliess operator representation. To describe its generating series explicitly, let $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ and define the family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup\, e),$$

where $i = 0,1,\ldots,m$ and $d_0 := 1$. Assume $D_\emptyset$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Such maps can be composed in the obvious way so that $D_{x_ix_j} := D_{x_i}D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ under the catenation product. The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$(\underbrace{x_{i_k}x_{i_{k-1}}\cdots x_{i_1}}_{\eta}) \circ d = D_{x_{i_k}}D_{x_{i_{k-1}}}\cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ the definition is extended linearly as

$$c \circ d = \sum_{\eta \in X^*} (c,\eta)\, \eta \circ d.$$

In this case, for any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ the identity $F_c \circ F_d = F_{c\circ d}$ is satisfied. It is known in general that the composition product is associative and distributive to the left over the shuffle product. For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ with the ultrametric dist : $(c, d) \mapsto \sigma^{\mathrm{ord}(c-d)}$, where σ is any real number $0 < \sigma < 1$ [7, 17]. (The *order* of a series c, $\mathrm{ord}(c)$, is taken as the length of the shortest word in the support of c.) The following theorem states that local convergence is preserved under composition.

THEOREM 1 (see [17]). *Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then $c \circ d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$. Specifically,*

$$|(c \circ d, \nu)| \leq K_c((\phi(mK_d) + 1)M)^{|\nu|}(|\nu| + 1)!, \quad \nu \in X^*,$$

where $\phi(x) := x/2 + \sqrt{x^2/4 + x}$ and $M = \max\{M_c, M_d\}$.

In light of (3) and the theorem above, a lower bound on the radius of convergence for $F_{c \circ d}$ is $1/(\phi(mK_d)+1)M(m+1)$. It should be noted that no example has been presented to date for which the radius of convergence corresponds exactly to this bound. Thus, there is some suspicion that this result is conservative. In addition, if c and d are globally convergent, one would expect this stronger property to produce a correspondingly larger radius of convergence. But no such analysis is available in the literature.

2.4. The feedback product. Consider two Fliess operators interconnected to form a feedback system as shown in Figure 1(d). The output y must satisfy the feedback equation $y = F_c[u + F_d[y]]$ for every admissible input u. It was shown in [17, 20] that there always exist a unique generating series e so that $y = F_e[u]$. In this case, the feedback equation becomes equivalent to $F_e[u] = F_c[u + F_{d\circ e}[u]]$. The *feedback product* of c and d is thus defined as $c@d = e$. Specifically, e is the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i),$$

where $\tilde{\circ}$ denotes the *modified* composition product. That is, the product

$$c\tilde{\circ}d = \sum_{\eta \in X^*} (c, \eta)\, \eta\tilde{\circ}d,$$

where $\eta\tilde{\circ}d = \tilde{D}_\eta(1)$ with

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup\, e)$$

and $d_0 := 0$ [17]. Therefore, $e = c@d$ satisfies the fixed point equation $e = c\tilde{\circ}(d \circ e)$. In the case of a unity feedback system, denoted by $c@\delta$, this equation reduces to $e = c\tilde{\circ}e$. The output of a self-excited feedback loop is described by the fixed point $e \in \mathbb{R}^m[[X_0]]$, $X_0 := \{x_0\}$, of the contractive iterated map

$$S : e_i \mapsto e_{i+1} = (c \circ d) \circ e_i.$$

Therefore, e is the solution of the equation $e = (c \circ d) \circ e$. In the case of a unity feedback system (or equivalently, if $c \circ d$ is redefined as c), this equation reduces to $e = c \circ e$. The next theorem states that local convergence of a self-excited system is preserved.

THEOREM 2 (see [17]). *Let $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c \geq 1$ and $M_c > 0$. If $e \in \mathbb{R}^m[[X_0]]$ satisfies $e = c \circ e$, then*

$$|(e, x_0^n)| \leq K_c\phi_g\,((mK_c(2 + \phi_g) + 1)M_c)^n s_n\, n!, \quad n \geq 0,$$

where ϕ_g is the golden ratio, $s_0 := 1/\phi_g$, and $s_{n+1} = \mathcal{B}(C_n) := \sum_{k=0}^{n} \binom{n}{k} C_k$, that is, s_{n+1}, $n \geq 0$ is the binomial transformation of the Catalan integer sequence, C_n, $n \geq 0$.

TABLE 1
Selected integer sequences from the OEIS.

Sequence	OEIS number	$n = 0, 1, 2, \ldots$
C_n	A000108	$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$
s_{n+1}	A007317	$1, 2, 5, 15, 51, 188, 731, 2950, \ldots$
$\bar{b}_n$ (Ex. 2)	A052820	$1, 2, 9, 62, 572, 6604, 91526, \ldots$
$\bar{b}_n$ (Ex. 3)	A000110	$1, 2, 5, 15, 52, 203, 877, 4140, \ldots$
$\bar{e}_n$ (Ex. 5)	A112487	$1, 2, 10, 82, 938, 13778, 247210, \ldots$
$\bar{e}_n$ (Ex. 9)	A000629	$1, 2, 6, 26, 150, 1082, 9366, 94586, \ldots$

The sequence s_{n+1}, $n \geq 0$ is integer sequence number A007317 in the Online Encyclopedia of Integer Sequences (OEIS) [27]. See Table 1 for its first few entries. The asymptotics of C_n, $n \geq 0$ and s_{n+1}, $n \geq 0$ are known to be, respectively,

$$C_n \sim \frac{1}{\sqrt{\pi n^3}}\, 4^n, \quad s_n \sim \frac{\sqrt{5}}{8\sqrt{\pi n^3}}\, 5^n.$$

Therefore, it follows for the single-input, single-output case that

$$|(e, x_0^n)| \leq (\beta(K_c) M_c)^n \, n!, \quad n \geq 0,$$

where $\beta(K_c) := K_c(10 + 5\phi_g) + 5$ for $K_c \geq 1$. From (3) with $R = m = 0$, $F_e[0]$ is guaranteed to converge on at least the interval $[0, 1/\beta(K_c)M_c)$. But again no example has been presented to date for which this interval corresponds exactly to the interval of convergence. Moreover, a version of Theorem 2 tailored to the case where $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ should intuitively yield a larger interval of convergence for the closed-loop system. No analysis presently exists for such a problem.

3. The parallel connection. The analysis begins with the parallel connection shown in Figure 1(a). It is assumed throughout that $m > 0$ in order that the interconnection be well-defined. The main technical result that is needed to accomplish the analysis is the following theorem from complex analysis.

THEOREM 3 (see [33]). *Let $f(z) = \sum_{n\geq 0} a_n z^n/n!$ be a function which is analytic at the origin of the complex plane. Suppose z_0 is a singularity of f having smallest modulus. Then for any $\epsilon > 0$, there exists an integer $N \geq 0$ such that for all $n > N$, $|a_n| < (1/|z_0| + \epsilon)^n n!$. Furthermore, for infinitely many n, $|a_n| > (1/|z_0| - \epsilon)^n n!$.*

The essence of this theorem is that the real number $1/|z_0|$ is the minimum geometric growth constant for the coefficients of the Taylor series of f at $z = 0$. Specifically, one can always introduce a $K > 0$, if necessary, so that $|a_n| \leq K(1/|z_0|)^n n!$, $n \geq 0$, and no number smaller than $1/|z_0|$ has this property. The next lemma applies this theorem and provides the crucial insight into determining the radius of convergence of the parallel connection. The following definition is also used.

DEFINITION 1. *A series $\bar{c} \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ is said to be* locally maximal *with growth constants $K_c, M_c > 0$ if each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$. A series $\bar{c} \in \mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$ is said to be* globally maximal *with growth constants $K_c, M_c > 0$ if each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$.*

LEMMA 1. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ be locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $\bar{b} = \bar{c} + \bar{d}$, then the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function*

$$f(x_0) := \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k) \frac{x_0^k}{k!} = \frac{K_c}{1 - M_c x_0} + \frac{K_d}{1 - M_d x_0}$$

for any $i = 1, 2, \ldots, \ell$. *Moreover, the smallest possible geometric growth constant for* $\bar{b}$ *is* $M_b = \max\{M_c, M_d\}$.

Proof. There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \geq 0$ that

$$(\bar{b}, \nu) = (\bar{c}, \nu) + (\bar{d}, \nu) = (K_c M_c^n + K_d M_d^n)\, n!.$$

Furthermore, $(\bar{b}, \nu) = (\bar{b}, x_0^n)$, $n \geq 0$. The key observation is that $f(t)$ is the zero-input response of $F_{\bar{b}}$. That is,

(6)

$$f(t) = \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k) \frac{t^k}{k!} = F_{\bar{b}}[0](t) = \sum_{k=0}^{\infty} K_c M_c^k t^k + \sum_{k=0}^{\infty} K_d M_d^k t^k = \frac{K_d}{1 - M_c t} + \frac{K_d}{1 - M_d t}.$$

Since f is analytic at the origin, by Theorem 3 the smallest geometric growth constant for the sequence $(\bar{b}, x_0^n)$, $n \geq 0$ and thus for the entire formal power series $\bar{b}$ is determined by the location of any singularity nearest to the origin in the complex plane, say x_0'. Specifically, $M_b = 1/|x_0'|$, where it is easily verified from (6) that $x_0' = 1/\max\{M_c, M_d\}$. This proves the lemma. ∎

The following theorem describes the radius of convergence of the parallel connection of two locally convergent Fliess operators.

THEOREM 4. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ *with growth constants* $K_c, M_c > 0$ *and* $K_d, M_d > 0$, *respectively. If* $b = c + d$, *then*

$$|(b, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{7}$$

for some $K_b > 0$, *where* $M_b = \max\{M_c, M_d\}$. *Furthermore, no smaller geometric growth constant can satisfy* (7), *and thus the radius of convergence is* $1/M_b(m+1)$.

Proof. First observe that for all $\nu \in X^*$ and $i = 1, 2, \ldots, \ell$

$$|(c + d, \nu)| \leq |(c, \nu)| + |(d, \nu)| \leq (\bar{c}_i, \nu) + (\bar{d}_i, \nu) = (\bar{b}_i, \nu),$$

where $\bar{c}, \bar{d}$, and $\bar{b}$ are defined as in Lemma 1. In light of this lemma $(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|} |\nu|!$, $\nu \in X^*$ for some $K_b > 0$. Furthermore, $(\bar{b}_i, x_0^n)$, $n \geq 0$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ∎

For the parallel connection of two globally convergent series, it is trivial to show that global convergence is preserved. Thus, the radius of convergence in this case is taken to be infinity.

4. The product connection. The radius of convergence of the product connection of two locally convergent Fliess operators is calculated next. The following lemma is essential.

LEMMA 2. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c}, \bar{d} \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ *be locally maximal series with growth constants* $K_c, M_c > 0$ *and* $K_d, M_d > 0$, *respectively. If* $\bar{b} = \bar{c} \sqcup\!\sqcup \bar{d}$, *then the sequence* $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$f(x_0) = \frac{K_c K_d}{(1 - M_c x_0)(1 - M_d x_0)}$$

for any $i = 1, 2, \ldots, \ell$. *Moreover, the smallest possible geometric growth constant for* $\bar{b}$ *is* $M_b = \max\{M_c, M_d\}$.

Proof. There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \geq 0$ that

$$(\bar{b}, \nu) = \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\bar{c}, \eta)(\bar{d}, \xi)(\eta \sqcup\!\sqcup \xi, \nu) = \sum_{j=0}^{n} K_c M_c^j \, j! \, K_d M_d^{n-j} (n-j)! \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\eta \sqcup\!\sqcup \xi, \nu)$$

$$= \sum_{j=0}^{n} K_c M_c^j \, j! \, K_d M_d^{n-j} (n-j)! \binom{n}{j} = K_c K_d \left[\sum_{j=0}^{n} M_c^j M_d^{n-j} \right] n!.$$

Therefore, $\bar{b}$ and the sequence $(\bar{b}, x_0^n)$, $n \geq 0$ will have the same minimal growth constants. Observe that f is the zero-input response of $F_{\bar{b}}$. Specifically,

$$f(t) = \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k) \frac{t^k}{k!} = F_{\bar{b}}[0](t) = \sum_{k=0}^{\infty} K_c M_c^k t^k \sum_{k=0}^{\infty} K_d M_d^k t^k = \frac{K_c K_d}{(1 - M_c t)(1 - M_d t)}.$$

Since f is analytic at the origin, Theorem 3 gives the smallest geometric growth constant $M_b = 1/|x_0'|$, where $x_0' = 1/\max\{M_c, M_d\}$. This proves the lemma. ∎

Now a main result for the product connection is presented below.

THEOREM 5. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, d \in \mathbb{R}_{LC}^{\ell}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \sqcup\!\sqcup d$, then*

$$|(b, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{8}$$

for some $K_b > 0$, where $M_b = \max\{M_c, M_d\}$. Furthermore, no smaller geometric growth constant can satisfy (8), *and thus the radius of convergence is $1/M_b(m+1)$.*

Proof. Assume $\ell = 1$ without loss of generality. Observe that for all $\nu \in X^*$

$$|(c \sqcup\!\sqcup d, \nu)| \leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} |(c, \eta)||(d, \xi)|(\eta \sqcup\!\sqcup \xi, \nu)$$

$$\leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\bar{c}, \eta)(\bar{d}, \xi)(\eta \sqcup\!\sqcup \xi, \nu) = (\bar{b}, \nu),$$

where $\bar{c}, \bar{d}$, and $\bar{b}$ are defined as in Lemma 2. A direct application of this lemma gives $(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|} |\nu|!$, $\nu \in X^*$ for some $K_b > 0$. Furthermore, $(\bar{b}, x_0^n)$, $n \geq 0$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ∎

An interesting observation is that the exponential generating functions in Lemmas 1 and 2 have identical sets of singularities. Therefore, for locally convergent subsystems, the two interconnections have the same radii of convergence. In addition, M_b as defined in Theorem 5 reduces to the geometric growth constant given in [32, p. 23], where it is assumed that $M_c = M_d$. Its minimality, however, was not addressed there. The following theorem states that global convergence is preserved under the product connection. Therefore, its radius of convergence is taken as infinity.

THEOREM 6. *Suppose $c, d \in \mathbb{R}_{GC}^{\ell}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then $c \sqcup\!\sqcup d \in \mathbb{R}_{GC}^{\ell}\langle\langle X \rangle\rangle$. Specifically,*

$$|(c \sqcup\!\sqcup d, \nu)| \leq K_c K_d (M_c + M_d)^{|\nu|}, \quad \nu \in X^*.$$

Proof. Observe for any $\nu \in X^*$ that

$$|(c \sqcup\!\sqcup d, \nu)| = \left| \sum_{j=0}^{|\nu|} \sum_{\substack{\eta \in X^j \\ \xi \in X^{|\nu|-j}}} (c,\eta)(d,\xi)(\eta \sqcup\!\sqcup \xi, \nu) \right| \leq \sum_{i=0}^{|\nu|} \sum_{\substack{\eta \in X^j \\ \xi \in X^{|\nu|-j}}} K_c M_c^j \, K_d M_d^{|\nu|-j} \, (\eta \sqcup\!\sqcup \xi, \nu)$$

$$= K_c K_d \sum_{j=0}^{|\nu|} M_c^j M_d^{|\nu|-j} \binom{\nu}{j} = K_c K_d (M_c + M_d)^{|\nu|}. \qquad \square$$

5. The cascade connection. The analysis of the cascade connection is substantially more complex as compared to that for the parallel and product connections. Therefore, the cases of locally convergent subsystems and globally convergent subsystems will be considered in separate subsections. In several places, state space realizations will be employed to simplify the analysis, but they are not essential for establishing the main claims. See [29, 30] for an alternative approach.

5.1. Locally convergent subsystems. In contrast to the previous sections, the main result is presented first and then the machinery needed for the proof is developed afterwards.

THEOREM 7. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \circ d$, then*

$$|(b,\nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{9}$$

for some $K_b > 0$, where

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d} \exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)}$$

and W denotes the Lambert W-function, namely, the inverse of the function $g(z) = z\exp(z)$ [3]. *Furthermore, no smaller geometric growth constant can satisfy* (9), *and thus the radius of convergence is*

$$\frac{1}{M_d(m+1)} \left[1 - mK_d W\left(\frac{1}{mK_d} \exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)\right].$$

The following three lemmas are prerequisites for the proof of this theorem.

LEMMA 3. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ and $\bar{d} \in \mathbb{R}^{m}_{LC}\langle\langle X \rangle\rangle$ be locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then each component of the series $\bar{b} = \bar{c} \circ \bar{d}$ has coefficients satisfying*

$$0 < (\bar{b}_i, \eta) \leq \left(\bar{b}_i, x_0^{|\eta|}\right), \quad \eta \in X^*, \quad i = 1, 2, \ldots, \ell.$$

Proof. First a state space realization for the input-output map $F_{\bar{c}_i} : u \mapsto y_i$, $i = 1, 2, \ldots, \ell$ will be synthesized using the identity

$$\operatorname{char}(X^k) = \frac{\operatorname{char}(X)^{\sqcup\!\sqcup\, k}}{k!}, \quad k \geq 0.$$

Observe

$$\bar{c}_i = K_c \sum_{k=0}^{\infty} M_c^k k! \operatorname{char}(X^k) = K_c \sum_{k=0}^{\infty} M_c^k \operatorname{char}(X)^{\sqcup\!\sqcup\, k},$$

and thus

$$F_{\bar{c}_i} = K_c \sum_{k=0}^{\infty} M_c^k F_{\mathrm{char}(X)^{\sqcup\!\sqcup\, k}} = K_c \sum_{k=0}^{\infty} \left(M_c F_{\mathrm{char}(X)}\right)^k = \frac{K_c}{1 - M_c F_{\mathrm{char}(X)}}.$$

Defining $z_1 = F_{\bar{c}_i}$, it follows directly that

$$\dot{z}_1 = \frac{M_c}{K_c} z_1^2 \left(1 + \sum_{j=1}^{m} u_j\right), \quad z_1(0) = K_c, \quad y_i = z_1 \tag{10}$$

realizes $y_i = F_{\bar{c}_i}[u]$. A similar realization can be obtained for $F_{\bar{d}_i}$ in coordinate z_2. Since $\bar{c}_i = \bar{c}_j$ and $\bar{d}_i = \bar{d}_j$ for all i, j, setting each input of $F_{\bar{c}}$ equal to the corresponding output of $F_{\bar{d}}$ gives a realization for the cascade system $y_i = F_{(\bar{c}\circ\bar{d})_i}[u]$, namely,

$$\dot{z}_1 = \frac{M_c}{K_c} z_1^2 \left(1 + m z_2\right), \quad z_1(0) = K_c, \tag{11a}$$

$$\dot{z}_2 = \frac{M_d}{K_d} z_2^2 \left(1 + \sum_{j=1}^{m} u_j\right), \quad z_2(0) = K_d, \tag{11b}$$

$$y_i = z_1. \tag{11c}$$

The Lie derivatives of $h(z) = z_1$ with respect to the realization vector fields

$$g_0(z) = \begin{pmatrix} \frac{M_c}{K_c} z_1^2 (1 + m z_2) \\ \frac{M_d}{K_d} z_2^2 \end{pmatrix}, \quad g_j(z) = \begin{pmatrix} 0 \\ \frac{M_d}{K_d} z_2^2 \end{pmatrix},$$

where $j = 1, 2, \ldots, m$, are in terms of polynomials with positive coefficients. Therefore, when evaluated at $z_0 = [K_c \;\; K_d]^T$, it is immediate that $0 < (\bar{b}_i, \eta) = L_{g_\eta} h(z_0)$, $\eta \in X^*$. To prove the remaining inequality in the lemma, let $\eta_k = x_0^{n_0} x_{j_1} x_0^{n_1} \cdots x_{j_k} x_0^{n_k}$, $\eta_{k+1} = \eta_k x_{j_{k+1}} x_0^{n_{k+1}}$, and $\tilde{\eta}_k = \eta_k x_0^{n_{k+1}+1}$, where $1 \leq j_l \leq m$. Noting that $g_0(z) = g_j(z) + \tilde{g}(z)$, where

$$\tilde{g}(z) = \begin{pmatrix} \frac{M_c}{K_c} z_1^2 (1 + m z_2) \\ 0 \end{pmatrix},$$

it follows for any $k > 0$ that

$$\begin{aligned} L_{g_{\tilde{\eta}_k}} h(z_0) &= L_{g_{x_0^{n_{k+1}}}} L_{g_0} L_{g_{\eta_k}} h(z_0) = L_{g_{x_0^{n_{k+1}}}} L_{g_j + \tilde{g}} L_{g_{\eta_k}} h(z_0) \\ &= L_{g_{\eta_{k+1}}} h(z_0) + L_{g_{x_0^{n_{k+1}}}} L_{\tilde{g}} L_{g_{\eta_k}} h(z_0). \end{aligned} \tag{12}$$

Clearly, the second term on the right-hand side of (12) is also a polynomial with positive coefficients, and therefore

$$L_{g_{\eta_{k+1}}} h(z_0) < L_{g_{\tilde{\eta}_k}} h(z_0), \quad k > 0. \tag{13}$$

This inequality is used to complete the proof of the lemma. Specifically, it will be shown by induction on k that

$$(\bar{b}_i, \eta_k) = L_{g_{\eta_k}} h(z_0) \leq L_{g_{x_0^{|\eta_k|}}} h(z_0) = (\bar{b}_i, x_0^{|\eta_k|}), \quad k \geq 0. \tag{14}$$

The claim is trivially true when $k=0$. Now, assume it is true up to some fixed $k \geq 0$. Using (13), it follows that

$$L_{g_{\eta_{k+1}}} h(z_0) < L_{g_{\bar{\eta}_k}} h(z_0) < L_{g_{x_0^{|\bar{\eta}_k|}}} h(z_0) = L_{g_{x_0^{|\eta_{k+1}|}}} h(z_0).$$

Therefore, the claim is verified for all $k \geq 0$, and the lemma is proved. ∎

LEMMA 4. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $\bar{d} \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ be locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $\bar{b} = \bar{c} \circ \bar{d}$, then*

$$|(\bar{b}, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{15}$$

for some $K_b > 0$, where

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d} \exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)}.$$

Furthermore, no smaller geometric growth constant can satisfy (15).

Proof. A state space realization for the autonomous system $F_{(\bar{c}\circ\bar{d})_i}[0]$, $i = 1, 2, \ldots, \ell$ is clearly given by (11) with $u = 0$. The output equation is

$$y_i^2 \ddot{y}_i - 2y_i \dot{y}_i{}^2 - \frac{M_d (K_c \dot{y}_i - M_c y_i^2)^2}{mK_c K_d M_c} = 0$$

with $y_i(0) = K_c$, $\dot{y}_i(0) = K_c M_c (1 + mK_d)$. It has the unique solution

$$y_i(t) = \frac{K_c}{1 - M_c t + (\frac{mK_d M_c}{M_d}) \ln(1 - M_d t)}, \tag{16}$$

which is analytic at the origin. Clearly, y_i is the exponential generating function of the sequence $(\bar{b}_i, x_0^n)$, $n \geq 0$. Therefore, its smallest geometric growth constant is determined by the location of any singularity nearest to the origin, specifically, $M_b = 1/|t'|$, where it can be verified from (16) that

$$t' = \frac{1}{M_d}\left[1 - mK_d W\left(\frac{1}{mK_d} \exp\left(\frac{M_c - M_d}{mK_d M_c}\right)\right)\right].$$

From Lemma 3, M_b must also be the smallest possible growth constant of the series $\bar{b}$, and the lemma is proved. ∎

The following definition is needed in the proof of the next lemma.

DEFINITION 2. *Given any $\xi \in X^*$, the corresponding* left-shift operator *on X^* is defined as the mapping*

$$\xi^{-1} \,:\, X^* \to \mathbb{R}\langle X\rangle : \xi^{-1}(\eta) = \begin{cases} \eta' & : \ \textit{if } \eta = \xi\eta', \\ 0 & : \ \textit{otherwise.} \end{cases}$$

For any $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, $\xi^{-1}(c) := \sum_{\eta \in X^*} (c, \eta)\xi^{-1}(\eta)$. In addition, $\xi^{-i}(\cdot)$ denotes the left-shift operator $\xi^{-1}(\cdot)$ applied i times.

LEMMA 5. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d, c', d' \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ such that $|c| \leq c'$ and $|d| \leq d'$, where $|c|_i := \sum_{\eta \in X^*} |(c_i, \eta)|\, \eta$, $i = 1, 2, \ldots, \ell$. Then it follows that $|c \circ d| \leq c' \circ d'$.*

Proof. First consider the special case where $c = c' = \xi \in X^*$. The proof is by induction on $k = |\xi| - |\xi|_{x_0}$. Let $\xi_k = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} \cdots x_{i_1} x_0^{n_0}$ for $k \geq 0$, where

$1 \leq i_j \leq m$. For $k = 0$, the claim is trivial since

$$|\xi_0 \circ d| = |x_0^{n_0} \circ d| = x_0^{n_0} = x_0^{n_0} \circ d' = \xi_0 \circ d'.$$

Assume now that $|(\xi_k \circ d, \eta)| \leq (\xi_k \circ d', \eta)$ up to some fixed $k \geq 0$. Observe that

$$\begin{aligned} \xi_{k+1} \circ d &= x_0^{n_{k+1}+1}(d_{i_{k+1}} \sqcup\!\sqcup (\xi_k \circ d)) \\ (\xi_{k+1} \circ d, \eta) &= (d_{i_{k+1}} \sqcup\!\sqcup (\xi_k \circ d), x_0^{-(n_{k+1}+1)}(\eta)) \\ &= \sum_{j=0}^{n} \sum_{\substack{\alpha \in X^j \\ \beta \in X^{n-j}}} (d_{i_{k+1}}, \alpha)(\xi_k \circ d, \beta)(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)), \end{aligned}$$

where $n := |x_0^{-(n_{k+1}+1)}(\eta)| \geq 0$. Therefore,

$$\begin{aligned} |(\xi_{k+1} \circ d, \eta)| &\leq \sum_{j=0}^{n} \sum_{\substack{\alpha \in X^j \\ \beta \in X^{n-j}}} \left|(d_{i_{k+1}}, \alpha)\right| |(\xi_k \circ d, \beta)| \, (\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\ &\leq \sum_{j=0}^{n} \sum_{\substack{\alpha \in X^j \\ \beta \in X^{n-j}}} (d'_{i_{k+1}}, \alpha)(\xi_k \circ d', \beta)(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\ &= (\xi_{k+1} \circ d', \eta). \end{aligned}$$

Thus, the inequality holds for all $k \geq 0$. The general case then follows easily from the left linearity of the composition product. ∎

Proof of Theorem 7. Since $|d| \leq \bar{d}$, it follows from Lemma 5 that for any $\nu \in X^*$

$$|(b, \nu)| \leq \sum_{\eta \in X^*} |(c, \eta)||(\eta \circ d, \nu)| \leq \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|! \, (\eta \circ \bar{d}, \nu) = (\bar{b}_i, \nu),$$

where $\bar{b}_i = (\bar{c} \circ \bar{d})_i$, $i = 1, 2, \ldots, \ell$. In light of Lemma 4, the theorem is proved. ∎

Example 1. Let $X = \{x_0, x_1\}$ and $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $M = M_c = M_d$. Then from Theorem 7,

$$M_b = \frac{M}{1 - K_d W\,(1/K_d)} = \left(\frac{3}{2} + K_d + O\left(\frac{1}{K_d}\right)\right) M \approx K_d M$$

when $K_d \gg 1$. This is consistent with Theorem 1. On the other hand, if $K_d = 1$, then $M_b = (1 - W(1))^{-1}M = 2.3102M$, which is less than the geometric growth constant $(\phi_g + 1)M = 2.6180M$ given by Theorem 1.

It is known that if u is analytic at $t = 0$ with generating series c_u, then $y = F_c[u]$ is also analytic at $t = 0$ [32], and its generating series is given by $c_y = c \circ c_u$ [17, 25, 26]. In this situation, the following corollary is useful for estimating a lower bound on the interval of convergence for y.

COROLLARY 1. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$ and $X_0 = \{x_0\}$. Let $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $c_u \in \mathbb{R}^m_{LC}[[X_0]]$ with growth constants $K_{c_u}, M_{c_u} > 0$. Then, $c_y = c \circ c_u$ satisfies $|(c_y, x_0^k)| \leq K_{c_y} M_{c_y}^k k!$, $k \geq 0$ for some $K_{c_y} > 0$ and*

$$M_{c_y} = \frac{M_{c_u}}{1 - mK_{c_u} W\left(\frac{1}{mK_{c_u}} \exp\left(\frac{M_c - M_{c_u}}{mK_{c_u} M_c}\right)\right)}.$$

Thus, the interval of convergence for the output $y = F_c[u]$ is at least as large as $T = 1/M_{c_y}$.

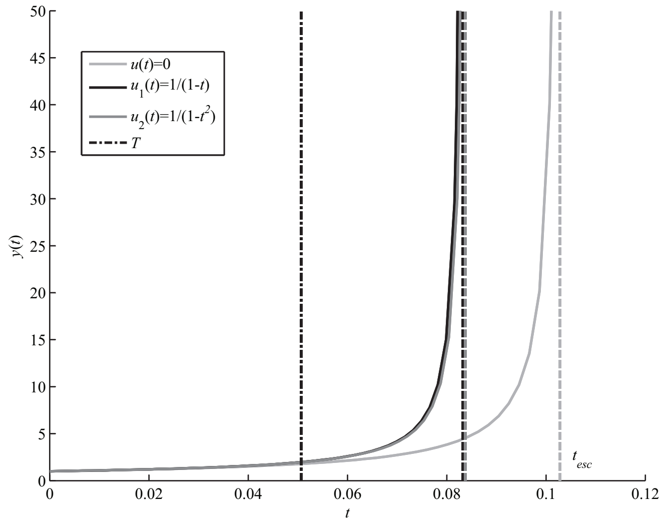

FIG. 2. *Output responses of the cascade system $F_{\bar{c}\circ\bar{d}}$ to various analytic inputs in Example* 2.

Example 2. Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c}\circ\bar{d}$, where $\bar{c}, \bar{d} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ are locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. The output of the cascade system $F_{\bar{c}\circ\bar{d}}$ is described by the state space system (11). A MATLAB generated zero-input response is shown in Figure 2 when $K_c = 1$, $M_c = 2$, $K_d = 3$, and $M_d = 4$. As expected from Lemma 4, the finite escape time of the output is $t_{esc} = 1/M_b = 0.1028$. The output responses corresponding to the analytic inputs $u_1(t) = 1/1-t$ and $u_2(t) = 1/1-t^2$, each having growth constants $K_{c_u} = M_{c_u} = 1$, are also shown in the figure. Their respective finite escape times are 0.08321 and 0.08377. Here u_1 has the shortest escape time since its generating series $c_{u_1} = \sum_{k\geq 0} k!\, x_0^k$ has all its coefficients growing at the maximum rate, whereas $c_{u_2} = \sum_{\geq 0}(2k)!\, x_0^{2k}$ has all its odd coefficients equal to zero. By Corollary 1, any finite escape time for the output corresponding to any analytic input with the given growth constants K_{c_u}, M_{c_u} must be at least as large as $T = 1/M_{c_y} = 0.05073$. Finally, in the case where $K_c = M_c = K_d = M_d = 1$, the exponential generating function of the zero-input response has coefficients $\bar{b}_n$, $n \geq 0$, which are equivalent to the integer sequence A052820 in the OEIS as shown in Table 1.

5.2. Globally convergent subsystems. In this section, the analysis to compute the radius of convergence of the cascade connection of two globally convergent Fliess operators is presented. The following theorem is the first of two main results.

THEOREM 8. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Assume $\bar{c}$ and $\bar{d}$ are globally maximal series with the same growth constants as c and d, respectively. If $b = c\circ d$ and $\bar{b} = \bar{c}\circ\bar{d}$, then*

$$|(b,\nu)| \leq (\bar{b}_i, x_0^{|\nu|}),\ \nu \in X^*,\ i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right).$$

In which case, the radius of convergence of b is infinity.

The following lemma is essential for proving this theorem.

LEMMA 6. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c} \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ *and* $\bar{d} \in \mathbb{R}^{m}_{GC}\langle\langle X\rangle\rangle$ *be globally maximal series with growth constants* $K_c, M_c > 0$ *and* $K_d, M_d > 0$, *respectively. If* $\bar{b} = \bar{c} \circ \bar{d}$, *then*

$$0 < (\bar{b}_i, \nu) \leq (\bar{b}_i, x_0^{|\nu|}), \quad \nu \in X^*,$$

and the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right)$$

for any $i = 1, 2, \ldots, \ell$.

Proof. As in the local case, a state space realization for the input-output map $F_{\bar{c}_i} : u \mapsto y_i$, $i = 1, 2, \ldots, \ell$, is useful. Observe that

$$\bar{c}_i = K_c \sum_{k=0}^{\infty} M_c^k \operatorname{char}(X^k) = K_c \sum_{k=0}^{\infty} \frac{M_c^k}{k!} \operatorname{char}(X)^{\sqcup\!\sqcup\, k},$$

and thus

$$F_{\bar{c}_i} = K_c \sum_{k=0}^{\infty} \frac{M_c^k}{k!} F_{\operatorname{char}(X)^{\sqcup\!\sqcup\, k}} = K_c \sum_{k=0}^{\infty} \frac{\left(M_c F_{\operatorname{char}(X)}\right)^k}{k!} = K_c \exp(M_c F_{\operatorname{char}(X)}).$$

In which case, defining $z_1 = F_{\bar{c}_i}$, it follows directly that

$$\dot{z}_1 = M_c z_1 \left(1 + \sum_{j=1}^{m} u_j\right), \quad z_1(0) = K_c, \quad y_i = z_1. \tag{17}$$

A similar realization can be obtained for $F_{\bar{d}_i}$ in coordinate z_2. Setting each input of $F_{\bar{c}}$ equal to a corresponding output of $F_{\bar{d}}$ gives the realization for the cascade system $y_i = F_{(\bar{c}\circ\bar{d})_i}[u]$ as follows:

$$\dot{z}_1 = M_c z_1 \left(1 + m z_2\right), \quad z_1(0) = K_c, \tag{18a}$$

$$\dot{z}_2 = M_d z_2 \left(1 + \sum_{j=1}^{m} u_j\right), \quad z_2(0) = K_d, \tag{18b}$$

$$y_i = z_1. \tag{18c}$$

The Lie derivatives of $h(z) = z_1$ with respect to the vector fields

$$g_0(z) = \begin{pmatrix} M_c z_1(1 + m z_2) \\ M_d z_2 \end{pmatrix}, \quad g_j(z) = \begin{pmatrix} 0 \\ M_d z_2 \end{pmatrix},$$

where $j = 1, 2, \ldots, m$, are also in terms of polynomials with positive coefficients. Therefore, when evaluated at $z_0 = [K_c \quad K_d]^T$, it is immediate that $0 < (\bar{b}_i, \eta) = L_{g_\eta} h(z_0)$, $\eta \in X^*$. To prove the remaining inequality in the lemma, let $\eta_k = x_0^{n_0} x_{j_1} x_0^{n_1}$

$\cdots x_{j_k} x_0^{n_k}$, $\eta_{k+1} = \eta_k x_{j_{k+1}} x_0^{n_{k+1}}$, and $\tilde{\eta}_k = \eta_k x_0^{n_{k+1}+1}$, where $1 \leq j_l \leq m$. Noting that $g_0(z) = g_j(z) + \tilde{g}(z)$, where

$$\tilde{g} = \begin{pmatrix} M_c z_1(1 + m z_2) \\ 0 \end{pmatrix},$$

the Lie derivative of h corresponding to the word $\tilde{\eta}_k$ for any $k > 0$ evaluated at $z = z_0$ is given by (12). Clearly, the second term on the right-hand side of (12) is a polynomial with positive coefficients, and therefore (13) and (14) hold in the present context as well. To determine the exponential generating function of $(\bar{b}_i, x_0^k)$, $k \geq 0$, the zero-input response $F_{\bar{b}}[0]$ is computed from (18). The output equation is

$$y_i \ddot{y}_i - \dot{y}_i^2 - M_d y_i(\dot{y}_i - M_c y_i) = 0 \tag{19}$$

with $y_i(0) = K_c$ and $\dot{y}_i(0) = M_c K_c(1 + m K_d)$. Equation (19) has the unique solution

$$y_i(t) = K_c \exp\left(\frac{m K_d \exp(M_d t) + M_d t - m K_d}{M_d / M_c}\right),$$

which is entire. This proves the lemma. □

Proof of Theorem 8. From Lemma 5, it follows for any $\nu \in X^*$ that

$$|(b, \nu)| \leq \sum_{\eta \in X^*} |(c, \eta)||(\eta \circ d, \nu)| \leq \sum_{\eta \in X^*} K_c M_c^{|\eta|} \, (\eta \circ \bar{d}, \nu) = (\bar{b}_i, \nu).$$

By Lemma 6, $(\bar{b}_i, \nu)$ is bounded by $(\bar{b}_i, x_0^{|\nu|})$, which has the exponential generating function f. Thus, the theorem is proved. □

It is worth noting that the Bell numbers B_n have the exponential generating function $e^{e^x - 1}$. Their asymptotic behavior is $B_n \sim n^{-\frac{1}{2}}(\lambda(n))^{n+\frac{1}{2}} e^{\lambda(n)-n-1}$, where $\lambda(n) = n/W(n)$. Thus, the Lambert W-function appears to also play a role in the global problem. It is known that the Bell numbers play a central role in the convergence analysis of function composition [9]. Most importantly, since the double exponential appearing in Lemma 6 has no finite singularities, as appeared in the local case, the following main result is immediate.

THEOREM 9. *The output of the cascade connection of two globally convergent Fliess operators is always well-defined over any finite interval of time when* $u \in L_{1,e}^m(t_0)$.

It is important to understand that this theorem is *not* saying that the cascade system has a globally convergent generating series in the sense of (4). If this were the case, then it would be possible to bound $y(t) = F_{c \circ d}[0]$ by a single exponential function rather than a double exponential function (see [19, Theorem 3.1]). Thus, the fastest possible growth rate for the coefficients of a cascade connection involving subsystems with globally convergent generating series falls somewhere strictly *in between* the local growth condition (2) and the global growth condition (4). For example, the Bell numbers, whose asymptotics behave similarly, satisfy the following three limits: $\lim_{n\to\infty} B_n/M^n = \infty$ for $M > 0$, $\lim_{n\to\infty} B_n/(n!)^s = \infty$ for $0 < s < 1$, and $\lim_{n\to\infty} B_n/n! = 0$ [22, Theorem 9].

Example 3. Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c} \circ \bar{d}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|} \eta$ and $\bar{d} = \sum_{\eta \in X^*} K_d M_d^{|\eta|} \eta$. The output of the cascade system is described by (18). A MATLAB generated zero-input response of this system is shown on a double logarithmic scale in Figure 3 when $K_c = M_c = K_d = M_d = 1$. As expected from Lemma 6, this plot asymptotically approaches that of $\tilde{y}(t) = t$ as $t \to \infty$. Also in this case, the coefficients $\bar{b}_n$, $n \geq 0$ form the integer sequence A000110 in the OEIS as shown in Table 1.

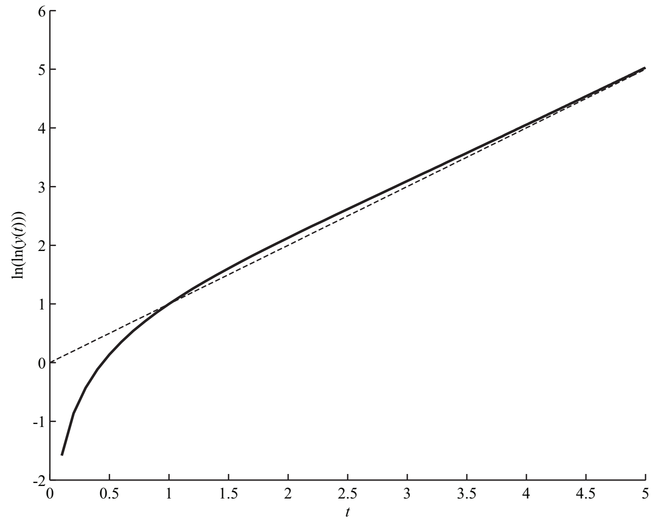

FIG. 3. *Zero-input response of the cascade system* $F_{\bar{c}\circ\bar{d}}$ *in Example* 3 *on a double logarithmic scale and the function* $\tilde{y}(t) = t$ *(dashed line).*

6. The feedback connection. In this section, the radius of convergence is computed for the feedback connection. It is by far the most difficult case. Again, the local and global cases will be treated individually.

6.1. Locally convergent subsystems. The following theorem, which describes the radius of convergence of the feedback connection of two locally convergent subsystems, is the main result of this section.

THEOREM 10. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $c, d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *with growth constants* $K_c, M_c > 0$ *and* $K_d, M_d > 0$, *respectively. If* $e = c@d$, *then*

$$|(e,\eta)| \leq K_e M_e^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{20}$$

for some $K_e > 0$, *where*

$$M_e = \frac{1}{\int_0^{1/M_c} \frac{W(\exp(f(z)))}{1+W(\exp(f(z)))}\, dz} \tag{21}$$

and

$$f(z) = \frac{1 - M_d z}{m K_d} + \ln\left(\frac{(1 - M_c z)^{\frac{K_c M_d}{K_d M_c}}}{m K_d}\right). \tag{22}$$

Furthermore, no geometric growth constant smaller than M_e *can satisfy* (20), *and thus the radius of convergence is* $1/M_e(m+1)$.

The following four lemmas and one theorem are needed for the proof.

LEMMA 7. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c}, \bar{d} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *be locally maximal series with growth constants* $K_c, M_c > 0$ *and* $K_d, M_d > 0$, *respectively. Then each component of the series* $\bar{e} = \bar{c}@\bar{d}$ *has coefficients satisfying*

$$0 < (\bar{e}_i, \eta) \leq \left(\bar{e}_i, x_0^{|\eta|}\right), \quad \eta \in X^*, \quad i = 1, 2, \ldots, m.$$

Proof. Using (10) to realize $F_{\bar{c}_i}$ and $F_{\bar{d}_i}$, the closed-loop system $y_i = F_{\bar{e}_i}[u]$ is realized by

$$\dot{z}_1 = \frac{M_c}{K_c} z_1^2 \left(1 + m z_2 + \sum_{i=1}^{m} u_i\right), \quad z_1(0) = K_c, \tag{23a}$$

$$\dot{z}_2 = \frac{M_d}{K_d} z_2^2 \left(1 + m z_1\right), \quad z_2(0) = K_d, \tag{23b}$$

$$y_i = z_1. \tag{23c}$$

The Lie derivatives of $h(z) = z_1$ with respect to the realization vector fields

$$g_0(z) = \begin{pmatrix} \frac{M_c}{K_c} z_1^2(1 + m z_2) \\ \frac{M_d}{K_d} z_2^2(1 + m z_1) \end{pmatrix}, \quad g_i(z) = \begin{pmatrix} \frac{M_c}{K_c} z_1^2 \\ 0 \end{pmatrix},$$

where $i = 1, 2, \ldots, m$, consist of polynomials with positive coefficients. Noting that $g_0(z) = g_i(z) + \tilde{g}(z)$, where

$$\tilde{g}(z) = \begin{pmatrix} \frac{m M_c}{K_c} z_1^2 z_2 \\ \frac{M_d}{K_d} z_2^2(1 + m z_1) \end{pmatrix},$$

the rest of the proof follows exactly as in Lemma 3. □

The following well known result from complex analysis is used in the proof of the next lemma.

THEOREM 11 (Pringsheim theorem [10]). *Let $f(z) = \sum_{n\geq 0} a_n z^n/n!$ be a function which is analytic at the origin of the complex plane with radius of convergence R. If each $a_n \geq 0$, then the point $z = R$ is a singularity of f.*

LEMMA 8. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ be locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then the zero-input response of the closed-loop system $F_{\bar{c}@\bar{d}}$ has a finite escape time given by*

$$t_{esc} = \int_0^{1/M_c} \frac{W(\exp(f(z)))}{1 + W(\exp(f(z)))}\, dz, \tag{24}$$

where f is defined in (22). *Furthermore, the coefficients of the generating series $\bar{e} = \bar{c}@\bar{d}$ satisfy*

$$|(\bar{e}, \eta)| \leq K_e M_e^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{25}$$

for some $K_e > 0$, where M_e is defined in (21), *and no geometric growth constant smaller than M_e can satisfy* (25).

Proof. It was shown in the proof of Lemma 3 that $F_{\bar{c}_i} = K_c/1 - M_c F_{\text{char}(X)}$, $i = 1, 2, \ldots, m$. The proof is simpler, however, when a state space realization distinct from (10) is used. Specifically, defining the state $z_1 = F_{\text{char}(X)}$, $F_{\bar{c}_i}$ has the realization

$$\dot{z}_1 = 1 + \sum_{j=1}^{m} u_j, \quad z_1(0) = 0, \quad y_i = \frac{K_c}{(1 - M_c z_1)}.$$

In which case, the closed-loop system is realized by

$$\begin{aligned}\dot{z}_1 &= 1 + \frac{mK_d}{1 - M_d z_2} + \sum_{j=1}^{m} u_j, \quad z_1(0) = 0,\\ \dot{z}_2 &= 1 + \frac{mK_c}{1 - M_c z_1}, \quad z_2(0) = 0,\\ y_i &= \frac{K_c}{1 - M_c z_1}.\end{aligned}$$

To characterize the zero-input state response set $u_j = 0$, $j = 1, 2, \ldots, m$, and observe that

$$\ddot{z}_1 = \frac{M_d}{mK_d}(\dot{z}_1 - 1)^2 \left(1 + \frac{mK_c}{1 - M_c z_1}\right). \tag{26}$$

Letting $v := \dot{z}_1$, (26) becomes

$$\frac{v}{(v-1)^2}\, dv = \frac{M_d}{mK_d}\left(1 + \frac{mK_c}{1 - M_c z_1}\right) dz_1.$$

Integrating both sides of this expression gives

$$\ln(v - 1) - \frac{1}{(v-1)} = -f(z_1),$$

where f is given in (22) and $z_1 < 1/M_c$. Thus,

$$\frac{dz_1}{dt} = v = \frac{W(\exp(f(z_1))) + 1}{W(\exp(f(z_1)))},$$

or in integral form

$$\int_0^{z_1(t)} \frac{W(\exp(f(\zeta)))}{1 + W(\exp(f(\zeta)))}\, d\zeta = t$$

with $z_1(t) < 1/M_c$. Since $\dot{z}_1(t) > 0$ for all $z_1(t) \in [0, 1/M_c]$, z_1 is invertible on that interval. Therefore, $z^{-1}(1/M_c)$ is the only singularity of $y_i(t) = K_c/[1 - M_c z_1(t)]$ on the real axis. However, by Lemma 7, $y_i(t)$ has only positive Taylor series coefficients. Therefore, using Theorem 11, the singularity $z^{-1}(1/M_c)$ is a singularity nearest to the origin, and thus y_i has a finite escape time, namely,

$$t_{esc} = \int_0^{1/M_c} \frac{W(\exp(f(\zeta)))}{1 + W(\exp(f(\zeta)))}\, d\zeta.$$

Since $y(t)$ is the exponential generating function of the sequence $(\bar{e}, x_0^n)$, $n \geq 0$, its smallest geometric growth constant is given by $M_e = 1/t_{esc}$. Finally, from Lemma 7, M_e is the smallest possible geometric growth constant of the complete series $\bar{e}$, and the lemma is proved. ∎

LEMMA 9. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d, c', d' \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ such that $|c| \leq c'$ and $|d| \leq d'$. Then it follows that $|c \tilde{\circ} d| \leq c' \tilde{\circ} d'$.*

Proof. The proof is similar to that for Lemma 5. ∎

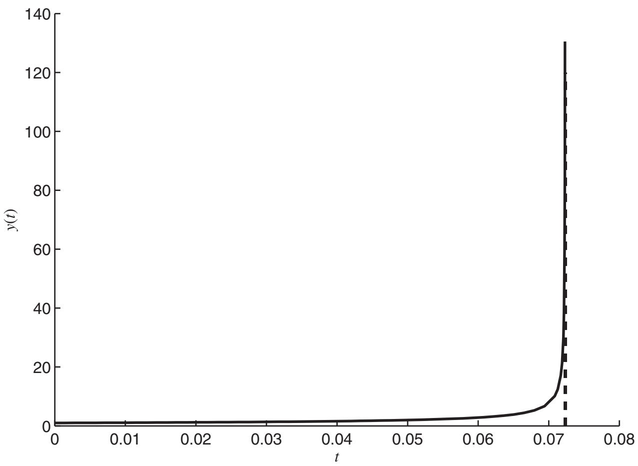

Fig. 4. *Output of the self-excited feedback system in Example* 4.

Lemma 10. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, \bar{c}, d, \bar{d} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively, and assume $\bar{c}, \bar{d}$ are locally maximal series. If $e = c@d$ and $\bar{e} = \bar{c}@\bar{d}$, then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

Proof. Since the mapping $e \mapsto c\tilde{\circ}(d \circ e)$ is a contraction, it follows that if $e_i(k) := c\tilde{\circ}(d\circ e_i(k-1))$, $k \geq 2$ with $e_i(1) = 0$, then $e_i = \lim_{k\to\infty} e_i(k)$. Likewise, one can define a sequence $\bar{e}_i(k)$ using $\bar{c}$ and $\bar{d}$. It will first be shown by induction that $|e_i(k)| \leq \bar{e}_i(k)$, $k \geq 1$. The $k = 1$ case is trivial. Assume the claim holds up to some fixed $k \geq 1$. Then, using Lemma 5, Lemma 9, and the induction hypothesis, for any $\eta \in X^*$

$$
\begin{aligned}
|(e_i(k+1), \eta)| = |((c\tilde{\circ}(d \circ e(k)))_i, \eta)| &\leq \left|\left(\sum_{\xi\in X^*} (c_i, \xi)(\xi\tilde{\circ}(d \circ e(k)))_i, \eta\right)\right| \\
&\leq \sum_{\xi\in X^*} |(c_i, \xi)|\, |((\xi\tilde{\circ}(d \circ e(k)))_i, \eta)| \leq \sum_{\xi\in X^*} K_c M_c^{|\xi|} |\xi|!\, ((\xi\tilde{\circ}(\bar{d} \circ \bar{e}(k)))_i, \eta) \\
&= ((\bar{c}\tilde{\circ}(\bar{d} \circ \bar{e}(k)))_i, \eta) = (\bar{e}_i(k+1), \eta).
\end{aligned}
$$

Thus, the initial claim is established. Next, by a property of the limit supremum,

$$
\limsup_{k\to\infty} |(e_i(k), \eta)| \leq \limsup_{k\to\infty} (\bar{e}_i(k), \eta).
$$

Since each sequence converges, it follows that $|e_i| \leq \bar{e}_i$. ∎

Proof of Theorem 10. In Lemma 8, it was shown that $|(\bar{e}, \eta)| \leq K_e M_e^{|\eta|} |\eta|!$, $\eta \in X^*$. Therefore, using Lemma 10, $|(e, \eta)| \leq |(\bar{e}, \eta)| \leq K_e M_e^{|\eta|} |\eta|!$, $\eta \in X^*$. As demonstrated in the proof of Lemma 8, $\bar{c}$ and $\bar{d}$ are the series for which each component of the corresponding feedback generating series $\bar{e}$ achieves exactly the growth rate $K_e M_e^{|\eta|} |\eta|!$. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ∎

Example 4. Suppose $X = \{x_0, x_1\}$. Let $\bar{e} = \bar{c}@\bar{d}$, where $\bar{c}, \bar{d} \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ are locally maximal series with growth constants $K_c = 1, M_c = 2, K_d = 3$, and $M_d = 4$. Numerical integration of (24) for this case gives $t_{esc} = 0.0723$. A MATLAB generated solution of the corresponding system (23) with $u = 0$ is shown in Figure 4. As expected, the finite escape time is $t_{esc} \approx 0.0723$, which is the radius of convergence.

It will be shown next that the radius of convergence of a unity feedback system can be obtained directly from Theorem 10. The following two lemmas simply establish that the radius of convergence of a unity feedback system is determined by its zero-input response. The proofs are very similar to those which have already been presented. A pleasant surprise is that for this special case, (21) has an *explicit* form.

LEMMA 11. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *be a locally maximal series with growth constants* $K_c, M_c > 0$. *Then each component of the series* $\bar{e} = \bar{c}@\delta$ *satisfies*

$$0 < (\bar{e}_i, \eta) \le \left(\bar{e}_i, x_0^{|\eta|}\right), \ \ \eta \in X^*, \ \ i = 1, 2, \ldots, m.$$

Proof. This proof is a variation of that for Lemma 7. □

LEMMA 12. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $c, \bar{c} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *with growth constants* $K_c, M_c > 0$, *and assume* $\bar{c}$ *is a locally maximal series. If* $e = c@\delta$ *and* $\bar{e} = \bar{c}@\delta$, *then* $|e_i| \le \bar{e}_i$, $i = 1, 2, \ldots, m$.

Proof. The proof is similar to that of Lemma 10. □

COROLLARY 2. *Suppose* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *with growth constants* $K_c, M_c > 0$. *If* $e = c@\delta$, *then*

$$|(e, \eta)| \le K_e(\alpha(K_c)M_c)^{|\eta|}|\eta|!, \ \ \eta \in X^*,$$

for some $K_e > 0$, *where*

$$\alpha(K_c) = \frac{1}{1 - mK_c \ln\left(1 + \frac{1}{mK_c}\right)}.$$

Furthermore, no geometric growth constant smaller than $\alpha(K_c)M_c$ *is possible, and thus the radius of convergence is* $1/\alpha(K_c)M_c(m+1)$.

Proof. It has been established in all feedback connections considered that the zero-input response determines the radius of convergence. Furthermore, it is easy to show that the self-excited feedback equation $e = c \circ (d \circ e)$ and the self-excited unity feedback equation $e = c \circ e$ have the same solution when $c = d$. Thus, the smallest possible geometric growth constant for $c@\delta$ can be computed by setting $K_c = K_d$ and $M_c = M_d$ in (21) and (22). In this case, it follows directly that

$$M_e = \frac{1}{\int_0^{1/M_c} \frac{1 - M_d z}{mK_d + 1 - M_d z}\, dz} = \frac{M_c}{1 - mK_c \ln\left(1 + \frac{1}{mK_c}\right)}. \qquad \square$$

It is worth noting that Corollary 2 first appeared in [18] and was proved by entirely different means, specifically without the use of any state space models. In addition, it is easy to show that $\alpha(K_c) < \beta(K_c)$ for all $K_c \ge 1$ and $\beta(K_c)/\alpha(K_c) \approx 9$ for $K_c \gg 1$, where $\beta(K_c)$ is defined in Theorem 2. Thus, Corollary 2 constitutes an order of magnitude improvement over the lower bound on the radius of convergence given in Theorem 2.

The following corollary is useful for the convergence analysis of unity feedback systems having analytic inputs.

COROLLARY 3. *Let* $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *with growth constants* $K_c, M_c > 0$, *and assume* $e = c@\delta$. *If* $c_u \in \mathbb{R}^m_{LC}[[X_0]]$ *with growth constants* $K_{c_u}, M_{c_u} > 0$, *then* $c_y = e \circ c_u$ *satisfies*

$$|(c_y, x_0^k)| \le K_{c_y} M_{c_y}^k k!, \ k \ge 0$$

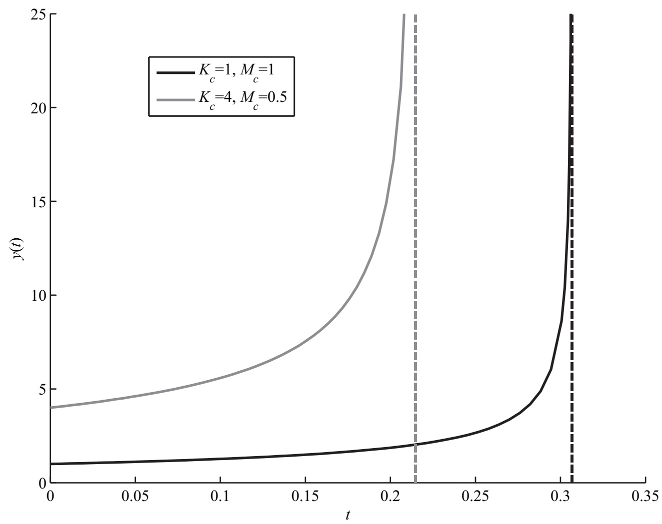

FIG. 5. *Outputs of the self-excited feedback system in Example* 5.

for some $K_{c_y} > 0$ *and*

$$M_{c_y} = \frac{M_{c_u}}{1 - mK_{c_u} W\left(\frac{1}{mK_{c_u}} \exp\left(\frac{\alpha(K_c)M_c - M_{c_u}}{m\alpha(K_c)M_c K_{c_u}}\right)\right)}.$$

Thus, the interval of convergence for the output $y = F_{c_y}[u]$ *is at least as large as* $T = 1/M_{c_y}$.

Proof. The proof is an immediate consequence of Corollaries 1 and 2. ∎

Example 5. Let $X = \{x_0, x_1\}$. Suppose $\bar{e} = \bar{c}@\delta$, where $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ is a locally maximal series with growth constants $K_c, M_c > 0$. The output of the self-excited unity feedback system is described by the solution of the state space system

$$\dot{z} = \frac{M_c}{K_c}(z^2 + z^3), \;\; z(0) = K_c, \;\; y = z.$$

MATLAB generated solutions of this system are shown in Figure 5 when $K_c = M_c = 1$ and when $K_c = 4$, $M_c = 0.5$. As expected from Corollary 2, the respective finite escape times are $t_{esc} = 1/\alpha(1) = 1 - \ln(2) \approx 0.3069$ and $t_{esc} = 2/\alpha(4) \approx 0.2149$. Also, when $K_c = M_c = 1$, $(\bar{e}, x_0^n)$, $n \geq 0$ has the exponential generating function

$$y(t) = \sum_{n=0}^{\infty} (\bar{e}, x_0^n)\frac{t^n}{n!} = \frac{-1}{1 + W(-2\exp(t-2))}.$$

The coefficients $(\bar{e}, x_0^n)$, $n \geq 0$ correspond to OEIS integer sequence A112487 as shown in Table 1.

Example 6. Let $X = \{x_0, x_1\}$ and consider the case where $e = c@\delta$ with $c = \sum_{n\geq 0} n!\, x_1^n$. In comparison to the previous example, c has most of its coefficients equal to zero. Therefore, it is likely that the output will be finite over a longer interval. The output of the corresponding self-excited unity feedback system is described by the solution of

$$\dot{z} = z^3, \;\; z(0) = 1, \;\; y = z.$$

Therefore, $y(t) = 1/\sqrt{1-2t}$ is finite up to $t = 0.5$, which is longer than the finite escape time of $t_{esc} = 0.3069$ obtained in the previous example.

Example 7. Consider the feedback system $c@d$ with $c = d = \sum_{\eta \in X^*} |\eta|!\, \eta$. The output y of the feedback system with $u = 0$ is described by (23) with $K_c = M_c = K_d = M_d = 1$. The output y, as computed by MATLAB, is numerically indistinguishable from the $K_c = M_c = 1$ case shown in Figure 5. This demonstrates that the self-excited feedback connection of two identical Fliess operators reduces to the self-excited unity feedback connection. Thus, the same radius of convergence and finite escape time are obtained as in Example 5.

6.2. Globally convergent subsystems. The following theorem, which describes the radius of convergence of the feedback connection of two globally convergent subsystems, is the main result of this section.

THEOREM 12. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $e = c@d$, then*

$$|(e, \eta)| \leq K_e M_e^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{27}$$

for some $K_e > 0$, where

$$M_e = \frac{1}{\int_0^\infty \frac{1}{1+W(\exp(f(z)))}\, dz} \tag{28}$$

and

$$f(z) = \frac{mK_cM_d}{M_c}(\exp(M_c z) - 1) + M_d z + mK_d + \ln(mK_d). \tag{29}$$

Furthermore, no geometric growth constant smaller than M_e can satisfy (27), *and thus the radius of convergence is $1/M_e(m+1)$.*

The following three lemmas are needed for the proof.

LEMMA 13. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$ be globally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then each component of the series $\bar{e} = \bar{c}@\bar{d}$ satisfies*

$$0 < (\bar{e}_i, \eta) \leq \left(\bar{e}_i, x_0^{|\eta|}\right), \quad \eta \in X^*, \quad i = 1, 2, \ldots, m.$$

Proof. The proof is similar to the proof of Lemma 7, except here the state space realizations of $F_{\bar{c}_i}$ and $F_{\bar{d}_i}$ given in (18) are employed so that the closed-loop system $y_i = F_{\bar{e}_i}[u]$ is realized by

$$\begin{aligned} \dot{z}_1 &= M_c z_1 \left(1 + m z_2 + \sum_{i=1}^m u_i\right), \quad z_1(0) = K_c, \\ \dot{z}_2 &= M_d z_2 \left(1 + m z_1\right), \quad z_2(0) = K_d, \\ y_i &= z_1. \quad \square \end{aligned}$$

LEMMA 14. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$ be globally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then the zero-input response of the closed-loop system $F_{\bar{c}@\bar{d}}$ has a finite escape time given by*

$$t_{esc} = \int_0^\infty \frac{1}{1+W(\exp(f(z)))}\, dz,$$

where f is defined in (29). *Furthermore, the coefficients of the generating series* $\bar{e} = \bar{c}@\bar{d}$ *satisfy*

$$|(\bar{e}, \eta)| \leq K_e M_e^{|\eta|} |\eta|!, \quad \eta \in X^*, \tag{30}$$

for some $K_e > 0$, where M_e is defined in (28)*, and no geometric growth constant smaller than M_e can satisfy* (30).

Proof. It was shown in the proof of Lemma 6 that $F_{\bar{c}_i} = K_c \exp(M_c F_{\mathrm{char}(X)})$, $i = 1, 2, \ldots, m$. But again the proof is easier if a state space realization distinct from (17) is used. Defining the state $z_1 = F_{\mathrm{char}(X)}$, $F_{\bar{c}_i}$ is realized by

$$\dot{z}_1 = 1 + \sum_{i=1}^{m} u_i, \quad z_1(0) = 0, \quad y_i = K_c \exp(M_c z_1).$$

In this case, the closed-loop system has the realization

$$\dot{z}_1 = 1 + mK_d \exp(M_d z_2) + \sum_{i=1}^{m} u_i, \quad z_1(0) = 0, \tag{31a}$$

$$\dot{z}_2 = 1 + mK_c \exp(M_c z_1), \quad z_2(0) = 0, \tag{31b}$$

$$y_i = K_c \exp(M_c z_1). \tag{31c}$$

To characterize the zero-input response, set $u_i = 0$, $i = 1, 2, \ldots, m$, and observe that

$$\ddot{z}_1 = (\dot{z}_1 - 1) M_d (1 + mK_c \exp(M_c z_1)). \tag{32}$$

Letting $v := \dot{z}_1$, (32) becomes

$$\frac{v}{(v-1)}\, dv = M_d \left(1 + mK_c \exp(M_c z_1)\right) dz_1.$$

Integrating both sides of this expression gives

$$v + \ln(v - 1) = f(z_1) + 1,$$

where f is given in (29). Thus,

$$\frac{dz_1}{dt} = v = W(\exp(f(z_1))) + 1,$$

or in integral form

$$\int_0^{z_1(t)} \frac{1}{1 + W(\exp(f(\zeta)))}\, d\zeta = t.$$

Since $\dot{z}_1(t) > 0$ for all $z_1(t) \in [0, \infty)$, z_1 is invertible on that interval. Thus, z_1 has only one singularity on the real axis, namely,

$$t' = \lim_{z_1 \to \infty} \int_0^{z_1} \frac{1}{1 + W(\exp(f(\zeta)))}\, d\zeta, \tag{33}$$

which is also the only real singularity of $y_i(t) = K_c \exp(M_c z_1(t))$. It is necessary to show, however, that the limit in (33) exists. Let $M = \min\{M_c, M_d\}$, $K = \min\{K_c, K_d\}$, and $\tilde{f}(z_1) = mK(\exp(Mz_1) - 1) + Mz_1 + mK + \ln(mK)$. If $M_d > M_c$, it is easy to see that $f(z_1) > \tilde{f}(z_1)$ for all $z_1 \in [0, \infty]$. If $M_d < M_c$, one can show for all $z_1 \in \big[\frac{\ln(M/M_c)}{M-M_c}, \infty\big)$ that $\exp(Mz_1)/M < \exp(M_c z_1)/M_c$, which yields $f(z_1) > \tilde{f}(z_1)$. Consequently,

$$0 < \frac{1}{1+W(\exp(f(z_1)))} < \frac{1}{1+W(\exp(\tilde{f}(z_1)))}.$$

Now observe that in either case

$$t' = \int_0^{\frac{\ln(M/M_c)}{M-M_c}} \frac{1}{1+W(\exp(f(\zeta)))}\, d\zeta + \int_{\frac{\ln(M/M_c)}{M-M_c}}^{\infty} \frac{1}{1+W(\exp(f(\zeta)))}\, d\zeta. \tag{34}$$

However,

$$\begin{aligned} \int_{\frac{\ln(M/M_c)}{M-M_c}}^{\infty} \frac{1}{1+W(\exp(\tilde{f}(\zeta)))}\, d\zeta &\leq \int_0^{\infty} \frac{1}{1+W(\exp(\tilde{f}(\zeta)))}\, d\zeta \\ &= \lim_{z_1\to\infty} \frac{-\ln\big(1+mKe^{Mz_1}\big) + \ln\big(e^{Mz_1}\big) + \ln(1+mK)}{M} \\ &= \frac{1}{M}\ln\left(1+\frac{1}{mK}\right) < \infty. \end{aligned} \tag{35}$$

Thus, the second integral in (34) is finite, while the first integral is also finite since the integrand has no singularity over its finite interval of integration. Thus, the limit in (33) exists when $M_c \neq M_d$. In the special case where $M_c = M_d$, $f(z_1) \geq \tilde{f}(z_1)$ for all $z_1 \in [0, \infty)$. Hence, (33) is bounded by the right-hand side of (35), which again is finite. Returning to the main argument, it follows from Lemma 13 that $y_i(t)$ has only positive Taylor series coefficients. So from Theorem 11, the real number t' is a singularity nearest to the origin. Thus, $y_i(t)$ must have a finite escape time at $t_{esc} = t'$. Since $y_i(t)$ is the exponential generating function of the sequence $(\bar{e}, x_0^n)$, $n \geq 0$, its smallest geometric growth constant is given by $M_e = 1/t_{esc}$. Finally, by Lemma 13, M_e is the smallest possible geometric growth constant of the complete series $\bar{e}$, and the lemma is proved. ∎

LEMMA 15. *Suppose $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, \bar{c}, d, \bar{d} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively, and assume $\bar{c}, \bar{d}$ are globally maximal series. If $e = c@d$ and $\bar{e} = \bar{c}@\bar{d}$, then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

Proof. The proof is perfectly analogous to that given for Lemma 10. ∎

Proof of Theorem 12. By Lemma 14, $|(\bar{e}, \eta)| \leq K_e M_e^{|\eta|} |\eta|!$, $\eta \in X^*$. Using Lemma 15, $|(e, \eta)| \leq K_e M_e^{|\eta|} |\eta|!$, $\eta \in X^*$. As demonstrated in the proof of Lemma 14, $\bar{c}$ and $\bar{d}$ are the series for which each component of the corresponding feedback generating series $\bar{e}$ achieves exactly the growth rate $K_e M_e^{|\eta|} |\eta|!$. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ∎

Example 8. Suppose $X = \{x_0, x_1\}$. Let $\bar{e} = \bar{c}@\bar{d}$, where $\bar{c}, \bar{d} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ are globally maximal series with growth constants $K_c = 1, M_c = 2, K_d = 3$, and $M_d = 4$. Numerical integration of (28) for this case gives $t_{esc} = 0.1570$. A MATLAB generated solution of the corresponding system (31) is shown in Figure 6. As expected, the finite escape time is $t_{esc} \approx 0.1570$, which is the radius of convergence.

For a unity feedback connection involving a globally convergent Fliess operator, the analysis is perfectly analogous to the local case.

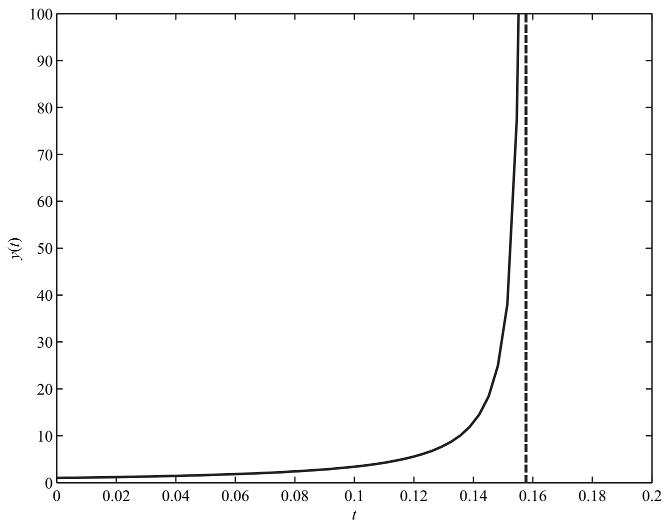

FIG. 6. *Output of the self-excited feedback system in Example* 8.

COROLLARY 4. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e = c@\delta$, then*

$$|(e,\eta)| \leq K_e(\gamma(K_c)M_c)^{|\eta|}|\eta|!, \quad \eta \in X^*, \tag{36}$$

for some $K_e > 0$, where

$$\gamma(K_c) = \frac{1}{\ln\left(1 + \frac{1}{mK_c}\right)}.$$

Furthermore, no geometric growth constant smaller than $\gamma(K_c)M_c$ can satisfy (36), *and thus the radius of convergence is $1/\gamma(K_c)M_c(m+1)$.*

Proof. Set $K_c = K_d$ and $M_c = M_d$ in (28) and then evaluate directly. ∎

Example 9. Suppose $X = \{x_0, x_1\}$. Let $\bar{e} = \bar{c}@\delta$ with $\bar{c} = \sum_{\eta\in X^*} K_c M_c^{|\eta|}\,\eta$. From Corollary 4 it follows that $M_e = \gamma(K_c)M_c$. The output of the self-excited unity feedback system is described by the solution of the state space system

$$\dot{z} = M_c(z + z^2), \quad z(0) = K_c, \quad y = z.$$

MATLAB generated solutions of this system are shown in Figure 7 when $K_c = M_c = 1$ and when $K_c = 4$, $M_c = 0.5$. As expected, the respective finite escape times are $t_{esc} = 1/\gamma(1) = \ln(2) \approx 0.6931$ and $t_{esc} = 2/\gamma(4) \approx 0.4463$. Note that these escape times are in fact about twice that of the respective cases in Example 5. In light of the expansions about $K_c = \infty$,

$$\alpha(K_c) = \frac{4}{3} + 2K_c + O\left(\frac{1}{K_c}\right), \quad \gamma(K_c) = \frac{1}{2} + K_c + O\left(\frac{1}{K_c}\right),$$

the radius of convergence for the global case is *always* about twice that for the local case when $K_c \gg 1$. Also, when $K_c = M_c = 1$, the sequence $(\bar{e}, x_0^n)$, $n > 0$ has the exponential generating function

$$y(t) = \sum_{n=0}^{\infty} (\bar{e}, x_0^n)\,\frac{t^n}{n!} = \frac{\exp(t)}{2 - \exp(t)}.$$

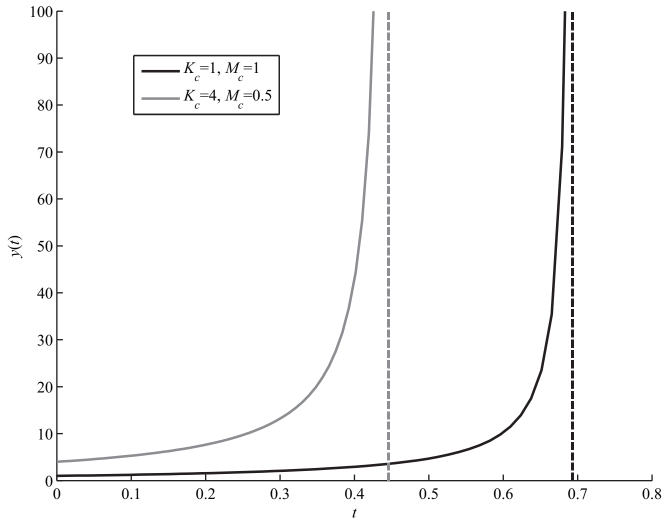

FIG. 7. *Outputs of the self-excited feedback system in Example* 9.

The sequence $(\bar{e}, x_0^n)$, $n \geq 0$ corresponds to OEIS integer sequence A000629 as shown in Table 1.

Example 10. Suppose $X = \{x_0, x_1\}$ and consider the case where $e = c@\delta$ with $c = \sum_{n\geq 0} x_1^n$. The series c has the same growth constants $K_c = M_c = 1$ as in Example 9 except most of its coefficients are zero. Thus, the zero-input response is expected to be finite over a longer interval. The output of the self-excited unity feedback system is described by the solution of

$$\dot{z} = z^2, \;\; z(0) = 1, \;\; y = z.$$

Therefore, $y(t) = 1/(1-t)$ is finite up to $t = 1$, which exceeds the finite escape time of $t_{esc} = 0.6931$ in the previous example.

Example 11. Consider a feedback interconnection involving the globally convergent series $c = x_1$ and $d = \sum_{k\geq 0} x_1^k$. This example first appeared in [16] to demonstrate that global convergence is not preserved under feedback. Here a lower bound is computed for the radius of convergence of the closed-loop system. Observe that $F_{c@d}$ has the state space realization

$$\begin{aligned} \dot{z}_1 &= z_1 z_2, \;\; z_1(0) = 1, \\ \dot{z}_2 &= z_1 + u, \;\; z_2(0) = 0, \\ y &= z_2. \end{aligned}$$

Setting $u = 0$, y satisfies the initial value problem $\ddot{y} - \dot{y}y = 0$, $y(0) = 0$, $\dot{y}(0) = 1$, which has the solution

$$\begin{aligned} y(t) &= \sqrt{2}\tan\left(\frac{t}{\sqrt{2}}\right) = \sum_{k\geq 1} (-1)^{k-1} 2^k (2^{2k-1}) \frac{B_{2k}}{k} \frac{t^{2k-1}}{(2k-1)!} \\ &= t + \frac{t^3}{3!} + 4\,\frac{t^5}{5!} + 34\,\frac{t^7}{7!} + 496\,\frac{t^9}{9!} + \cdots \end{aligned}$$

for $0 \leq t < \pi/\sqrt{2} = t_{esc}$, where B_k denotes the kth Bernoulli number. Setting $K_c = M_c = K_d = M_d = 1$, the interval of convergence should be at least as long as

TABLE 2
Radii of convergence for the four elementary system connections.

Connection	$c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$	$c, d \in \mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$
parallel	$\frac{1}{\max\{M_c, M_d\}(m+1)}$	∞ (GC)
product	$\frac{1}{\max\{M_c, M_d\}(m+1)}$	∞ (GC)
cascade ($\ell = m$ for d)	$\frac{1}{M_d(m+1)}\left[1 - mK_dW\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mK_dM_c}\right)\right)\right]$	∞
feedback ($\ell = m$)	$\frac{1}{(m+1)}\int_0^{1/M_c}\frac{W(\exp(f(z)))}{1+W(\exp(f(z)))}\,dz$ $f(z) = \frac{1-M_dz}{mK_d} + \ln\left(\frac{(1-M_cz)^{\frac{K_cM_d}{K_dM_c}}}{mK_d}\right)$	$\frac{1}{(m+1)}\int_0^{\infty}\frac{1}{1+W(\exp(f(z)))}\,dz$ $f(z) = \frac{mK_cM_d}{M_c}(\exp(M_cz) - 1) + M_dz$ $+ mK_d + \ln(mK_d)$
unity feedback ($\ell = m$)	$\frac{1}{M_c(m+1)}\left[1 - mK_c\ln\left(1 + \frac{1}{mK_c}\right)\right]$	$\frac{1}{M_c(m+1)}\ln\left(1 + \frac{1}{mK_c}\right)$

$[0, 1/\gamma(1)) = [0, 0.6931)$. A MATLAB generated solution of this system has the finite escape time $t_{esc} \approx 2.2214 > 0.6931$.

7. Conclusions and future research. The radii of convergence have been computed for the four fundamental interconnections of two Fliess operators as summarized in Table 2. It was found that the Lambert W-function plays a central role in the analysis of the cascade and feedback connections. This suggests a possible relationship to the combinatorics of rooted nonplanar labeled trees, where the Lambert W-function also plays a prominent role [3, 10]. One could continue to investigate the radius of convergence for other types of system interconnections, for example, interconnections involving subsystems which have a mixture of locally convergent and globally convergent generating series or mixtures of Fliess operators and static nonlinearities.

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