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An Invariance Property of Common Statistical Tests*

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ABSTRACT

Let **A** be a symmetric matrix and **B** be a nonnegative definite (nnd) matrix. We obtain a characterization of the class of nnd solutions $\mathbf{\Sigma}$ for the matrix equation $\mathbf{A\Sigma A} = \mathbf{B}$. We then use the characterization to obtain all possible covariance structures under which the distributions of many common test statistics remain invariant, that is, the distributions remain the same except for a scale factor. Applications include a complete characterization of covariance structures such that the chi-squaredness and independence of quadratic forms in ANOVA problems is preserved. The basic matrix theoretic theorem itself is useful in other characterizing problems in linear algebra. © 1997 Elsevier Science Inc.

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1. INTRODUCTION

Many common statistical tests make the assumption that samples are taken independent from one or more normal populations. Much research has been done on the effect of nonnormality on these tests; see [14] for an exposition. Typically, this part of the robustness literature makes the assumption of mutual independence of the observations. While the independence assumption may be approximately valid, due to the choice of experimental designs, clearly the case of dependence between the observations is of practical as well as aesthetic interest. One can even argue that in practical applications, observations are frequently not independent and the physical systems responsible for generation of the observations automatically introduce some dependence among the observations. It is then clear that the statistical interest should be to determine if procedures valid under the independence assumption continue to remain valid with only a simple adjustment when independence assumption is violated. It is our goal to consider this issue in the context of several common statistical problems.

Much prior literature exists in this general area; in particular, see [1, 2, 5], [7–11], [18–20], [23], and [25]. The emphasis in these articles is on examining the effect of specific dependence structures, whereas our goal is to obtain a simpler characterization of the covariance structures under which the usual procedures remain valid with possibly a scale factor adjustment.

The organization of this paper is as follows. In Section 2 we obtain a characterization of the class of nnd solutions $\mathbf{\Sigma}$ for the consistent matrix equation $\mathbf{A\Sigma A} = \mathbf{B}$ where $\mathbf{A}$ is a symmetric matrix and $\mathbf{B}$ is a nnd matrix. This result is of independent interest; in addition, we apply it to some characterization problems in the theory of generalized inverses of matrices. The characterization problem (Theorem 2.1) treated in Section 2 has in fact been considered by several authors; see [6] and [13]. However, we take a different approach and obtain a simpler and minimal representation of the class of all nnd solutions to the aforementioned matrix equation.

In Section 3 we use Theorem 2.1 to characterize the class of covariance matrices such that the distributions of common test statistics remain invariant. For example, let $\mathbf{y} \sim N_n(\mu\mathbf{e}, \mathbf{\Sigma})$, that is, the distribution of the vector of observations $\mathbf{y}$ is multivariate normal of dimension n with mean vector $\mu\mathbf{e}$ and covariance matrix $\mathbf{\Sigma}$, where $\mathbf{e}' = (1, \dots, 1)$. Let

$$\bar{y} = \frac{\mathbf{e}'\mathbf{y}}{n} \quad \text{and} \quad s^2 = \frac{\mathbf{y}'[\mathbf{I} - (1/n)\mathbf{e}\mathbf{e}']\mathbf{y}}{n-1}$$

be the sample mean and sample variance of the vector $\mathbf{y}$, where $\mathbf{I}$ is the identity matrix. In this paper we show that $(n-1)s^2 \sim d\chi^2(n-1)$ and $\bar{y}$ is independent of s^2 if and only if

$$\boldsymbol{\Sigma} = d\left(\mathbf{I} - \frac{1-c}{n}\mathbf{e}\mathbf{e}'\right) \qquad \text{for some} \quad d > 0 \text{ and } c \geqslant 0,$$

that is, the observations are equicorrelated. Also we show that the usual two-sample t-statistic has a t-distribution if the observations in one sample are positively equicorrelated and the observations in the other sample are negatively equicorrelated with the same correlation in absolute value. Other results in the paper include characterization of covariance matrices such that the independence and chi-squaredness of the quadratic forms occurring in ANOVA problems is preserved.

The following general notation and conventions will be used throughout the paper. A random variable having a noncentral chi-square distribution with m degrees of freedom and noncentrality parameter δ is denoted by $\chi^2(m; \delta)$, a random variable having a t-distribution with m degrees of freedom is denoted by $t(m)$, and a random variable having an F-distribution with (m_1, m_2) degrees of freedom is denoted by $F(m_1, m_2)$. We write $\mathbf{x} \sim \mathbf{y}$ to mean that both $\mathbf{x}$ and $\mathbf{y}$ have the same probability distribution. The column space, null space, rank, trace, and transpose of the matrix $\mathbf{A}$ are denoted by $\mathscr{M}(\mathbf{A})$, $\mathscr{N}(\mathbf{A})$, $r(\mathbf{A})$, $\operatorname{tr}(\mathbf{A})$, and $\mathbf{A}'$, respectively. Also, $\mathbf{A}^+$ and $\mathbf{A}^-$ denote the Moore-Penrose inverse and ordinary g-inverse of $\mathbf{A}$, respectively. We will use the definition contained in [16, Table 1, p. 67] concerning nonnegative definiteness of matrices. All nnd matrices are assumed to be symmetric. The vector $\mathbf{e}$ represents a vector of ones of order $n \times 1$, whereas $\mathbf{e}_m$ denotes a vector of ones of order $m \times 1$. Similarly, $\mathbf{I}$ represents the identity matrix of order $n \times n$ whereas $\mathbf{I}_m$ denotes the identity matrix of order $m \times m$. We will denote a vector and a matrix of zeros of appropriate orders by $\mathbf{0}$ and $\mathbf{O}$ respectively. Also, $\operatorname{diag}(d_1, \ldots, d_n)$ denotes a digital matrix of order $n \times n$ with d_i as the ith diagonal element.

2. SOME RESULTS IN LINEAR ALGEBRA

This section contains a characterization of the class of all nnd solutions to a general matrix equation that occurs in statistical distribution theory. We begin this section with an elementary but an important lemma. Lemma 2.1 below is simply a multivariate analogue of the problem of finding the restrictions on the coefficients of a quadratic equation $f(x) = ax^2 + 2bx + c$ such that $f(x) \geqslant 0$ for all x. It plays a crucial role in the proof of our main Theorem 2.1 of this section. The lemma is also useful in quadratic programming problems where the objective is to minimize the multivariate quadratic loss function.

LEMMA 2.1. *Let* $\mathbf{A}$ *be a symmetric matrix and* $\mathbf{b}$ *be a vector in* $\mathfrak{R}^n$. *Let* c *be a real number. In order that*

$$f(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} + 2\mathbf{x}'\mathbf{b} + c \geqslant 0 \qquad \textit{for all} \quad \mathbf{x} \in \mathfrak{R}^n \tag{2.1}$$

it is necessary and sufficient that

(1) $\mathbf{b} \in \mathscr{M}(\mathbf{A})$,
(2) $\mathbf{A}$ *be nnd, and*
(3) $c - \mathbf{b}'\mathbf{A}^{-}\mathbf{b} \geqslant 0$. *In particular, if* $c = 0$, *then* (2.1) *holds if and only if* $\mathbf{b} = \mathbf{0}$.

Proof. Let $\mathbf{A}$ be a symmetric matrix of rank k. Using the spectral value decomposition, we can write $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}'$, where $\mathbf{T}$ is the orthogonal matrix and $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ with $\lambda_i \neq 0$ for $i = 1, \dots, k$. It is easy to see that (2.1) is equivalent to

$$\mathbf{y}'\boldsymbol{\Lambda}\mathbf{y} + 2\mathbf{y}'\mathbf{m} + c \geqslant 0 \qquad \text{for all} \quad \mathbf{y} \in \mathfrak{R}^n, \tag{2.2}$$

where $\mathbf{y} = \mathbf{T}'\mathbf{x}$ and $\mathbf{m} = \mathbf{T}'\mathbf{b}$. Let $\mathbf{y} = (y_1, \dots, y_n)'$ and $\mathbf{m} = (m_1, \dots, m_n)'$. Then (2.2) can be rewritten as

$$\sum_{i=1}^{k} \lambda_i y_i^2 + 2\sum_{i=1}^{n} y_i m_i + c \geqslant 0 \qquad \text{for all} \quad y_1, \dots, y_n \in \mathfrak{R}. \tag{2.3}$$

It is very simple to show that (2.3) holds if and only if

(a) $m_i = 0$ for $i > k$,
(b) $\lambda_i \geqslant 0$ for all i, and
(c) $c - \sum_{i=1}^{k} m_i^2 \lambda_i^{-1} \geqslant 0$.

Clearly, (a) holds if and only if $\mathbf{m} \in \mathscr{M}(\boldsymbol{\Lambda})$, which is equivalent to (1); (b) is equivalent to (2); and (c) is equivalent to

$$c - \mathbf{m}'\boldsymbol{\Lambda}^{-}\mathbf{m} \geqslant 0, \tag{2.4}$$

which is same as (3), since $\boldsymbol{\Lambda}^{-} = \mathbf{T}'\mathbf{A}^{-}\mathbf{T}$. If $c = 0$, then (2.1) holds if and only if $\mathbf{b} \in \mathscr{M}(\mathbf{A})$ and $\mathbf{b}'\mathbf{A}^{-}\mathbf{b} = 0$, or equivalently $\mathbf{b} = \mathbf{0}$. This completes the proof of the lemma. ∎

We are now in a position to state the main theorem of this section. The matrix equation (2.5) occurs in statistical distribution theory and is well known to statisticians. Since the pioneering work of Professor C. R. Rao [16, 17], several authors have obtained characterizations of the class of all solutions $\mathbf{\Sigma}$ to the matrix equation (2.5). However, in statistical theory, $\mathbf{\Sigma}$, being a covariance matrix, is always nnd. Therefore, it is of natural interest to obtain characterizations of the subclass of nnd solutions to the matrix equation (2.5), as well as other matrix equations of this type; see [6, 13] and references contained therein. In Theorem 2.1, we obtain a characterization of the class of nnd solutions $\mathbf{\Sigma}$ to the matrix equation (2.5). The statistical applications of Theorem 2.1 concerning invariance properties of some univariate test statistics are discussed in Section 3. Other applications of our theorems in this section for multivariate test statistics are in [24].

THEOREM 2.1. *Let* $\mathbf{A}$ *be a symmetric matrix and* $\mathbf{B}$ *be a nnd matrix such that*

$$\mathbf{A\Sigma A} = \mathbf{B} \tag{2.5}$$

is a consistent equation. Let $\mathbf{Q_A} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})$. *Then the class of all nnd* $\mathbf{\Sigma}$*'s satisfying* (2.5) *is given by*

$$\mathbf{\Sigma} = \mathbf{A}^+\mathbf{B}\mathbf{A}^+ + \mathbf{Q_A U} + \mathbf{U Q_A} - \mathbf{Q_A U Q_A} \tag{2.6}$$

where $\mathbf{U}$ *is a symmetric matrix satisfying the following two conditions*:

(a) $\mathscr{M}(\mathbf{AUQ_A}) \subseteq \mathscr{M}(\mathbf{B})$.

(b) $\mathbf{V} \overset{\text{def}}{=} \mathbf{Q_A U Q_A} - \mathbf{Q_A U A B^- A U Q_A}$ *is nnd.*

If $\mathbf{Q_A U Q_A}$ *is null matrix, then* $\mathbf{\Sigma}$ *given by* (2.6) *is a nnd solution for* (2.5) *if and only if* $\mathbf{UQ_A} = \mathbf{O}$.

Proof. It follows from Theorem 2.3.2 of [17] that the equation (2.5) is consistent if and only if $\mathbf{AA^-BA^-A} = \mathbf{B}$ for any g-inverse $\mathbf{A}^-$ of $\mathbf{A}$; in which case, the general solution is given by (2.6) where $\mathbf{U}$ is an arbitrary matrix. Therefore, our problem reduces to characterizing the class of all $\mathbf{U}$'s such that $\mathbf{\Sigma}$ given by (2.6) is a nnd matrix. Without loss of generality we can assume that $\mathbf{U}$ is symmetric; otherwise we can replace $\mathbf{U}$ by $\mathbf{U}^* = (\mathbf{U} + \mathbf{U}')/2$. Note that $\mathbf{U}$ and $\mathbf{U}^*$ generate the same $\mathbf{\Sigma}$. We can rewrite the matrix $\mathbf{\Sigma}$ in (2.6) as

$$\begin{aligned}\mathbf{\Sigma} &= \mathbf{A}^+\mathbf{B}\mathbf{A}^+ + \mathbf{Q_A U} + \mathbf{U Q_A} - \mathbf{Q_A U Q_A} \\ &= \mathbf{A}^+\mathbf{B}\mathbf{A}^+ + \mathbf{Q_A U A^+ A} + \mathbf{A^+ A U Q_A} + \mathbf{Q_A U Q_A}.\end{aligned} \tag{2.7}$$

Since $\mathbf{A}$ is symmetric, we have $\mathbf{A}^+\mathbf{A} = \mathbf{A}\mathbf{A}^+$ and $\mathscr{M}(\mathbf{A}^+\mathbf{A}) = \mathscr{M}(\mathbf{A})$. Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ be the orthogonal decomposition of $\mathbf{x}$ where $\mathbf{x}_1 \in \mathscr{M}(\mathbf{A}^+\mathbf{A})$ and $\mathbf{x}_2 \in \mathscr{M}(\mathbf{Q}_\mathbf{A})$. It is easy to see that

$$\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \geqslant 0 \qquad \forall \mathbf{x} \in \mathfrak{R}^n \tag{2.8}$$

if and only if

$$\mathbf{x}_1'\mathbf{A}^+\mathbf{B}\mathbf{A}^+\mathbf{x}_1 + 2\mathbf{x}_1'\mathbf{U}\mathbf{x}_2 + \mathbf{x}_2'\mathbf{U}\mathbf{x}_2 \geqslant 0 \qquad \forall \mathbf{x}_1 \in \mathscr{M}(\mathbf{A}) \quad \forall \mathbf{x}_2 \in \mathscr{M}(\mathbf{Q}_\mathbf{A}). \tag{2.9}$$

Since (2.5) is a consistent equation, we have $\mathbf{A}\mathbf{A}^+\mathbf{B}\mathbf{A}^+\mathbf{A} = \mathbf{B}$. Therefore, from (2.9) we get that $\boldsymbol{\Sigma}$ is nnd if and only if

$$\mathbf{v}'\mathbf{B}\mathbf{v} + 2\mathbf{v}'\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} + \mathbf{w}'\mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} \geqslant 0 \qquad \forall \mathbf{v} \in \mathfrak{R}^n \quad \forall \mathbf{w} \in \mathfrak{R}^n. \tag{2.10}$$

By Lemma 2.1 it follows that (2.10) holds if and only if the following two conditions are satisfied:

(1) $\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} \in \mathscr{M}(\mathbf{B})$, $\forall \mathbf{w} \in \mathfrak{R}^n$
(2) $\mathbf{w}'\mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} - \mathbf{w}'\mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{A}\mathbf{B}^-\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} \geqslant 0$ $\forall \mathbf{w} \in \mathfrak{R}^n$.

It is easy to see that condition (1) is equivalent to (a) and condition (2) is equivalent to (b). If $\mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}$ is a null matrix, then by Lemma 2.1 and (2.10) we have that $\boldsymbol{\Sigma}$ is nnd if and only if $\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}\mathbf{w} = \mathbf{0}$ for $\mathbf{w} \in \mathfrak{R}^n$ or equivalently $\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}$ is a null matrix. Hence $\mathbf{U}\mathbf{Q}_\mathbf{A} = \mathbf{U}\mathbf{Q}_\mathbf{A} - \mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A} = (\mathbf{I} - \mathbf{Q}_\mathbf{A})\mathbf{U}\mathbf{Q}_\mathbf{A} = \mathbf{A}^+\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A} = \mathbf{O}$, since we have assumed that $\mathbf{Q}_\mathbf{A}\mathbf{U}\mathbf{Q}_\mathbf{A}$ is a null matrix. This proves the last assertion of the theorem. ∎

REMARK 2.1. Khatri and Mitra [13, Lemma 2.1] gave an alternative characterization of the class of nnd solutions $\boldsymbol{\Sigma}$ to (2.5). But our representation of the class of all nnd solutions $\boldsymbol{\Sigma}$ in Theorem 2.1 is minimal in the sense that two symmetric matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ generate the same $\boldsymbol{\Sigma}$ if and only if $\mathbf{Q}_\mathbf{A}\mathbf{U}_1 = \mathbf{Q}_\mathbf{A}\mathbf{U}_2$. Also, from the proof of the above theorem it follows that the class of all positive definite (positive semidefinite) $\boldsymbol{\Sigma}$'s satisfying (2.5) is obtained by choosing $\mathbf{U}$ such that $\mathbf{V}$ is positive definite (positive semidefinite) matrix. In the case where $\mathbf{Q}_\mathbf{A}\mathbf{U} = \mathbf{O}$, the only nnd solution $\boldsymbol{\Sigma} = \mathbf{A}^+\mathbf{B}\mathbf{A}^+$ is positive definite or positive semidefinite according as $\mathbf{B}$ is a positive definite or positive semidefinite matrix.

As an important application of Theorem 2.1 we obtain below an elegant characterization of the class of all nnd g-inverses of the centering matrix, $\mathbf{A}^* = \mathbf{I} - (1/n)\mathbf{ee}'$. Sharpe and Styan [21] studied the problem of characterization of the class of all g-inverses of the centering matrix and obtained an expression similar to (2.11). See Equation (29) in their paper [21]. However, they did not characterize the subclass of symmetric nnd g-inverses of the centering matrix. Corollary 2.1 below shows that the subclass of symmetric nnd g-inverses of the centering matrix can be obtained by choosing the vector $\mathbf{a}$ occurring in the expression (2.11) so that the inequality (2.12) is satisfied.

COROLLARY 2.1. *The class of all nnd g-inverses, $\mathscr{G}_n$, of the centering matrix $\mathbf{A}^*$ is given by*

$$\boldsymbol{\Sigma} = \mathbf{A}^* + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a}}{n}\mathbf{ee}' \tag{2.11}$$

where $\mathbf{a}' = (a_1, \dots, a_n)$ is such that

$$\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2 \leqslant \bar{a} \tag{2.12}$$

and $\bar{a} = (\mathbf{a}'\mathbf{e})/n$ is the mean of the components of the vector $\mathbf{a}$.

Proof. The proof follows easily from Theorem 2.1. ■

REMARK 2.2. The representation of the class $\mathscr{G}_n$ by matrices $\boldsymbol{\Sigma}$ given by (2.11) is minimal in the sense that the ordered n-tuple $\{a_1, \dots, a_n\}$ satisfying (2.12) uniquely determines a nnd g-inverse of $\mathbf{A}^*$.

REMARK 2.3. It is easy to show that $n - 2$ of the eigenvalues of $\boldsymbol{\Sigma}$ given by (2.11) are one and the other two eigenvalues are the solutions of the following quadratic equation:

$$\lambda^2 - \lambda(1 + \bar{a}) + \left(\bar{a} - \frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2\right) = 0. \tag{2.13}$$

We can show that both the roots of the equation (2.13) are nonnegative if and only if the condition (2.12) is satisfied. Thus, we have an alternative proof of Corollary 2.1.

REMARK 2.4. A different characterization of the class $\mathscr{G}_n$ is in [11]. However, our characterization is useful in that it provides an easy method of generating a $\boldsymbol{\Sigma}$ in the class $\mathscr{G}_n$. All we need to do is to choose a set $\{a_1, \dots, a_n\}$ of n numbers and, if the inequality (2.12) is not satisfied by the a_i's, then translate them by an appropriate constant. Recall that translation of a set of numbers increases the mean but does not change the variance.

REMARK 2.5. Note that for $d > 0$, the class of nnd matrices $\boldsymbol{\Sigma}$ satisfying the equation $\mathbf{A}^* \boldsymbol{\Sigma} \mathbf{A}^* = d\mathbf{A}^*$ is simply given $\mathscr{G}_{d,n} = \{d\boldsymbol{\Sigma} : \boldsymbol{\Sigma} \in \mathscr{G}_n\}$.

We can use Theorem 2.1 to obtain a characterization of the class of nnd g-inverses of a nnd matrix $\mathbf{A}$. Note that in Example 2.1, condition (a) is trivially satisfied, since $\mathbf{B} = \mathbf{A}$.

EXAMPLE 2.1. Let $\mathbf{A}$ be a nnd matrix and let $\mathbf{Q_A} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})$. Then the class of all nnd g-inverses of the matrix $\mathbf{A}$ is given by

$$\boldsymbol{\Sigma} = \mathbf{A}^+ + \mathbf{Q_A U} + \mathbf{U Q_A} - \mathbf{Q_A U Q_A} \tag{2.14}$$

where $\mathbf{U}$ is a symmetric matrix such that

$$\mathbf{V} = \mathbf{Q_A U Q_A} - \mathbf{Q_A U A U Q_A} \tag{2.15}$$

is a nnd matrix. If $\mathbf{Q_A U Q_A} = \mathbf{O}$ then $\mathbf{U Q_A} = \mathbf{Q_A U} = \mathbf{O}$, and in this special case $\boldsymbol{\Sigma} = \mathbf{A}^+$.

Let us now look at the case where $\mathbf{A}$ and $\mathbf{B}$ can be expressed as linear combinations of k orthogonal and idempotent matrices. The following lemma is needed in the proofs of Theorems 2.2 and 3.4, and it is stated without a proof.

LEMMA 2.2. *Let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k$ be symmetric, idempotent matrices such that $\mathbf{A}_i \mathbf{A}_j = \mathbf{O}$ for all $i \neq j$. Let $\mathbf{A} = \sum_{i=1}^k \mathbf{A}_i$ and let $\mathbf{B} = \sum_{i=1}^k c_i \mathbf{A}_i$ where $c_i > 0$ for all $1 \leqslant i \leqslant k$. Then $\mathbf{A \Sigma A} = \mathbf{B}$ if and only if*

$$\mathbf{A}_i \boldsymbol{\Sigma} \mathbf{A}_j = \begin{cases} c_i \mathbf{A}_i & \text{if } i = j, \\ \mathbf{O} & \text{if } i \neq j. \end{cases} \tag{2.16}$$

Furthermore, $\mathscr{M}(\mathbf{A}) = \mathscr{M}(\mathbf{B})$.

The next theorem is a special case of Theorem 2.1, and it is useful to study the invariance properties of common statistical tests for dependent observations. We state the theorem without a proof, since it follows easily from Lemma 2.2 and Theorem 2.1.

THEOREM 2.2. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ *and* $\mathbf{B}$ *be as in Lemma 2.2. Suppose that* $\sum_{i=1}^{k} \mathbf{A}_i = \mathbf{A}^*$, *where* $\mathbf{A}^*$ *is the centering matrix. Then the class of all nnd matrix solutions for the equation*

$$\mathbf{A}^* \boldsymbol{\Sigma} \mathbf{A}^* = \mathbf{B} \tag{2.17}$$

is given by

$$\boldsymbol{\Sigma} = \mathbf{B} + \frac{1}{n}(\mathbf{e}\mathbf{a}' + \mathbf{a}\mathbf{e}') - \frac{\bar{a}}{n}\mathbf{e}\mathbf{e}' \tag{2.18}$$

where $\mathbf{a}$ *is an arbitrary vector satisfying*

$$\frac{1}{n} \sum_{i=1}^{k} \frac{\mathbf{a}'\mathbf{A}_i\mathbf{a}}{c_i} \leqslant \bar{a}. \tag{2.19}$$

3. STATISTICAL APPLICATIONS

In this section we present some statistical applications of the theorems of Section 2. The applications are concerned with the problem of characterizing the class of all covariance matrices such that the distributions of common test statistics remain invariant, that is, the distributions are preserved except for a scale factor. We begin with the following lemma concerning the distributions of quadratic forms in normal variables with a nnd covariance matrix.

LEMMA 3.1. *Let* $\mathbf{z} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *where* $\boldsymbol{\Sigma}$ *is a nnd matrix. Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be nnd matrices of order* $n \times n$, *and* $\mathbf{a}$ *be a vector in* $\mathfrak{R}^n$. *Then*:

(1) $\mathbf{z}'\mathbf{A}\mathbf{z} \sim \chi^2(r(\mathbf{A}); \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})$ *if and only if* $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{A}$.
(2) $\mathbf{z}'\mathbf{A}\mathbf{z}$ *and* $\mathbf{z}'\mathbf{B}\mathbf{z}$ *are independent if and only if* $\mathbf{A}\boldsymbol{\Sigma}\mathbf{B} = \mathbf{O}$.
(3) $\mathbf{z}'\mathbf{a}$ *and* $\mathbf{z}'\mathbf{B}\mathbf{z}$ *are independent if and only if* $\mathbf{B}\boldsymbol{\Sigma}\mathbf{a} = \mathbf{0}$.

Proof. From Styan [22, Theorem 4] we have $\mathbf{z}'\mathbf{A}\mathbf{z} \sim \chi^2(r(\mathbf{A}); \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})$ if and only if the following conditions are satisfied: (i) $r(\mathbf{A}) = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})$, (ii) $\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}$, and (iii) $\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}$. It follows from the Lemma in Khatri [12] that these three conditions are equivalent to $\mathbf{A\Sigma A} = \mathbf{A}$. This completes the proof of (1). We can deduce (2) and (3) from Theorem 3 in [15] as special cases, after noting that when both $\mathbf{A}$ and $\mathbf{B}$ are nnd we have $\mathbf{\Sigma A \Sigma B \Sigma} = \mathbf{O}$ iff $\mathbf{A\Sigma B} = \mathbf{O}$ and $\mathbf{\Sigma B \Sigma a} = \mathbf{0}$ iff $\mathbf{B\Sigma a} = \mathbf{0}$. ■

The next three examples and Theorem 3.1 are simple consequences of the results in Section 2 and Lemma 3.1.

EXAMPLE 3.1. Let $\mathbf{z} \sim N_n(\boldsymbol{\mu}, \mathbf{\Sigma})$ where $\mathbf{\Sigma}$ is a nnd matrix. Let $\mathbf{A}$ be a nnd matrix of order $n \times n$. By Lemma 3.1(1), we have

$$\mathbf{z}'\mathbf{A}\mathbf{z} \sim \chi^2(r(\mathbf{A}); \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}) \tag{3.1}$$

if and only if $\mathbf{A\Sigma A} = \mathbf{A}$. Therefore, for a given nnd matrix $\mathbf{A}$, the class of all nnd $\mathbf{\Sigma}$'s for which (3.1) holds is given by Example 2.1.

EXAMPLE 3.2. In Example 3.1, $\mathbf{z}'\mathbf{A}\mathbf{z} \sim d\chi^2(r(\mathbf{A}); \delta)$ if and only if $\mathbf{A\Sigma A} = d\mathbf{A}$ where $d > 0$ and $\delta = (1/d)\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$. Thus, we can use Theorem 2.1 to obtain a complete characterization of the covariance matrices $\mathbf{\Sigma}$ such that $\mathbf{z}'\mathbf{A}\mathbf{z} \sim d\chi^2(r(\mathbf{A}); \delta)$.

EXAMPLE 3.3. Let $\mathbf{y} \sim N_n(\mu\mathbf{e}, \mathbf{\Sigma})$ where μ is a constant and $\mathbf{\Sigma}$ is a nnd matrix. Let s^2 be the sample variance of the vector $\mathbf{y}$. Then for any $d > 0$, we have $(n-1)s^2 \sim d\chi^2(n-1)$ if and only if $\mathbf{\Sigma} \in \mathscr{G}_{d,n}$.

THEOREM 3.1. *Let* $\mathbf{y}_1 \sim N_{n_1}(\mu_1\mathbf{e}_{n_1}, \mathbf{\Sigma}_1)$ *and* $\mathbf{y}_2 \sim N_{n_2}(\mu_2\mathbf{e}_{n_2}, \mathbf{\Sigma}_2)$ *where* μ_1, μ_2 *are constants and* $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2$ *are nnd matrices. Assume that* $\mathbf{y}_1$ *is independent of* $\mathbf{y}_2$. *Let* s_1^2 *and* s_2^2 *be the sample variances of the vectors* $\mathbf{y}_1$ *and* $\mathbf{y}_2$, *respectively. Then for* $c > 0$, *we have that* s_1^2/s_2^2 *is distributed as* $cF(n_1 - 1, n_2 - 1)$ *if and only if* $\mathbf{\Sigma}_1 \in \mathscr{G}_{cd, n_1}$ *and* $\mathbf{\Sigma}_2 \in \mathscr{G}_{d, n_2}$ *for some constant* $d > 0$.

Proof. Since s_1^2 and s_2^2 are independent, it follows from the results contained in [4] that s_1^2/s_2^2 is distributed as $cF(n_1 - 1, n_2 - 1)$ if and only if $(n_1 - 1)s_1^2 \sim cd\chi^2(n_1 - 1)$ and $(n_2 - 1)s_2^2 \sim d\chi^2(n_2 - 1)$. The theorem now follows from Example 3.3. ■

We now characterize the class of covariance matrices such that the mean and variance are independent for a normal sample of dependent observations. Theorem 3.2 below shows that the sample variance is distributed as chi-square except for a constant and is independent of the sample mean if and only if the observations are equicorrelated, that is, the correlation is the same between each pair of observations.

THEOREM 3.2. *Let* $\mathbf{y} \sim N_n(\mu\mathbf{e}, \boldsymbol{\Sigma})$ *where* μ *is a constant and* $\boldsymbol{\Sigma}$ *is a nnd matrix. Let* $\bar{y}$ *and* s^2 *be the mean and variance of the vector* $\mathbf{y}$. *Then* $(n-1)s^2 \sim d\chi^2(n-1)$, *and* $\bar{y}$ *is independent of* s^2 *if and only if*

$$\boldsymbol{\Sigma} = d\left(\mathbf{I} - \frac{1-c}{n}\mathbf{ee}'\right) \qquad \textit{for some} \quad c \geqslant 0 \ \textit{and} \ d > 0.$$

Proof. For any $d > 0$, we have $(n-1)s^2 \sim d\chi^2(n-1)$ if and only if $\boldsymbol{\Sigma} \in \mathscr{G}_{d,n}$ (see Example 3.3). From Lemma 3.1 (3), $\bar{y}$ and s^2 are independent if and only if

$$\left(\mathbf{I} - \frac{1}{n}\mathbf{ee}'\right)\boldsymbol{\Sigma}\mathbf{e} = \mathbf{0}. \tag{3.2}$$

It is easy to check that $\boldsymbol{\Sigma} \in \mathscr{G}_{d,n}$ and satisfies (3.2) if and only if

$$\left(\mathbf{I} - \frac{1}{n}\mathbf{ee}'\right)\mathbf{a} = \mathbf{0}. \tag{3.3}$$

Now, (3.3) holds if and only if $\mathbf{a} = c\mathbf{e}$ where $c = \bar{a} \geqslant 0$. Therefore $(n-1)s^2 \sim d\chi^2(n-1)$ and s^2 is independent of $\bar{y}$ if and only if

$$\begin{aligned}\boldsymbol{\Sigma} &= d\left[\left(\mathbf{I} - \frac{1}{n}\mathbf{ee}'\right) + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a}}{n}\mathbf{ee}'\right] \\ &= d\left[\left(\mathbf{I} - \frac{1}{n}\mathbf{ee}'\right) + \frac{2c}{n}\mathbf{ee}' - \frac{c}{n}\mathbf{ee}'\right] \\ &= d\left(\mathbf{I} - \frac{1-c}{n}\mathbf{ee}'\right) \qquad (3.4)\end{aligned}$$

where $c \geqslant 0$. This completes the proof of the theorem. ∎

Our next result is concerned with an invariance property of the two sample t-test. Theorem 3.3 below shows that the commonly used two sample

t-statistic has a t-distribution if one of the samples is positively equicorrelated and the other is negatively equicorrelated so that the correlation is the same in absolute value in both the samples.

THEOREM 3.3. *Let* $\mathbf{y}_1 \sim N_{n_1}(\mu_1 \mathbf{e}_{n_1}, \mathbf{\Sigma}_1)$ *and* $\mathbf{y}_2 \sim N_{n_2}(\mu_2 \mathbf{e}_{n_2}, \mathbf{\Sigma}_2)$ *where* μ_1, μ_2 *are constants and* $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2$ *are nnd matrices. Suppose that* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *are independently distributed. Let* $\bar{y}_1, s_1^2$ *and* $\bar{y}_2, s_2^2$ *be the mean and variance of the two vectors* $\mathbf{y}_1$ *and* $\mathbf{y}_2$, *respectively. Let* $s_p^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/(n_1 + n_2 - 2)$ *be the pooled sample variance. Then*

$$\frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1 + n_2 - 2) \tag{3.5}$$

if

$$\mathbf{\Sigma}_1 = d(\mathbf{I}_{n_1} + \beta \mathbf{e}_{n_1}\mathbf{e}'_{n_1}) \quad \text{and} \quad \mathbf{\Sigma}_2 = d(\mathbf{I}_{n_2} - \beta \mathbf{e}_{n_2}\mathbf{e}'_{n_2}) \tag{3.6}$$

for some constants d *and* β *such that* $d > 0$ *and* $-1/n_1 < \beta < 1/n_2$.

Proof. It follows from the proof of Theorem 3.2 that for any $d > 0$, $(n_i - 1)s_i^2$ is distributed as $d\chi^2(n_i - 1)$ and s_i^2 is independent of $\bar{y}_i$ if and only if

$$\mathbf{\Sigma}_i = d\left(\mathbf{I}_{n_i} - \frac{1 - c_i}{n_i}\mathbf{e}_{n_i}\mathbf{e}'_{n_i}\right) \tag{3.7}$$

where $c_i \geqslant 0$ for $i = 1, 2$. Thus for $c_1, c_2 > 0$, we have

$$\frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\dfrac{c_1}{n_1} + \dfrac{c_2}{n_2}}} \sim t(n_1 + n_2 - 2) \tag{3.8}$$

if the $\mathbf{\Sigma}_i$'s are given by (3.7) for $i = 1, 2$. Now for (3.5) to hold we require

$$\frac{c_1}{n_1} + \frac{c_2}{n_2} = \frac{1}{n_1} + \frac{1}{n_2} \tag{3.9}$$

or equivalently,

$$\frac{-(1-c_1)}{n_1} = \frac{1-c_2}{n_2} = \beta \qquad (\text{say}). \tag{3.10}$$

Since $c_1, c_2 > 0$, we have $-1/n_1 < \beta < 1/n_2$. The theorem now follows from (3.7), (3.8), and (3.10). ∎

We need the following version of Cochran's theorem for the distribution of quadratic forms in normal variables with nnd covariance matrix. A slightly different version and new version of Cochran's theorem using partial ordering among symmetric matrices instead of the usual rank additivity condition are available in [22] and [3]. Theorem 3.4 is useful for deriving the invariance properties of the quadratic forms in ANOVA models.

THEOREM 3.4. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ *by symmetric and idempotent matrices of order* $n \times n$ *such that* $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ *for all* $i \neq j$. *Let* $\mathbf{A} = \sum_{i=1}^k \mathbf{A}_i$ *and* $\mathbf{B} = \sum_{i=1}^k c_i\mathbf{A}_i$ *where* $c_i > 0$ *for* $1 \leqslant i \leqslant k$. *Let* $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *where* $\boldsymbol{\Sigma}$ *is a nnd matrix. Let* $Q_i = \mathbf{y}'\mathbf{A}_i\mathbf{y}$ *for* $1 \leqslant i \leqslant k$. *Then the quadratic forms* Q_i *are pairwise independent and distributed as* $c_i\chi^2(r(\mathbf{A}_i); \delta_i)$, $\delta_i = (1/c_i)\boldsymbol{\mu}'\mathbf{A}_i\boldsymbol{\mu}$, *for* $1 \leqslant i \leqslant k$ *if and only if* $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{B}$.

Proof. From Example 3.2 and Lemma 3.1(2), the Q_i's are pairwise independent and distributed as $c_i\chi^2(r(\mathbf{A}_i); \delta_i)$ for $1 \leqslant i \leqslant k$ if and only if

$$\mathbf{A}_i\boldsymbol{\Sigma}\mathbf{A}_j = \begin{cases} c_i\mathbf{A}_i & \text{if} \quad i=j, \\ \mathbf{O} & \text{if} \quad i \neq j. \end{cases} \tag{3.11}$$

The theorem now follows from Lemma 2.2. ∎

In the next theorem we obtain an invariance property of the distribution of quadratic forms in an ANOVA table.

THEOREM 3.5. *Let* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$ *be symmetric and idempotent matrices of order* $n \times n$ *such that* $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ *for all* $i \neq j$. *Let* $\sum_{i=1}^k \mathbf{A}_i = \mathbf{A}^*$ *and* $B = \sum_{i=1}^k c_i\mathbf{A}_i$, *where* $\mathbf{A}^*$ *is the centering matrix and* $c_i > 0$ *for* $1 \leqslant i \leqslant k$. *Let* $\mathbf{y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *where* $\boldsymbol{\Sigma}$ *is a nnd matrix. Let* $Q_i = \mathbf{y}'\mathbf{A}_i\mathbf{y}$ *for* $1 \leqslant i \leqslant k$. *Then the quadratic forms* Q_i *are pairwise independent and distributed as* $c_i\chi^2(r(\mathbf{A}_i); \delta_i)$, $\delta_i = (1/c_i)\boldsymbol{\mu}'\mathbf{A}_i\boldsymbol{\mu}$, *for* $1 \leqslant i \leqslant k$ *if and only if* $\boldsymbol{\Sigma}$ *is of the form* (2.18) *where* $\mathbf{a}$ *is an arbitrary vector satisfying* (2.19).

Proof. From Theorem 3.4, we get that Q_i's are pairwise independent and distributed as $c_i\,\chi^2(r(\mathbf{A}_i);\delta_i)$ for $1 \leqslant i \leqslant k$ if and only if $\mathbf{A}^*\boldsymbol{\Sigma}\mathbf{A}^* = \mathbf{B}$. Theorem 3.5 now follows from Theorem 2.2. ∎

REMARK 3.1. Note that in the case where $\boldsymbol{\mu} = \mu\mathbf{e}$ where μ is some constant, the Q_i's defined in Theorem 3.5 are pairwise independent and distributed as $c_i\,\chi^2(r(\mathbf{A}_i))$ for $1 \leqslant i \leqslant k$ if and only if $\mathbf{A}^*\boldsymbol{\Sigma}\mathbf{A}^* = \mathbf{B}$.

As another simple application of the results of Section 2, we get the following characterization of the covariance matrices such that the null distribution of the quadratic forms in one way ANOVA remains invariant.

THEOREM 3.6. *Consider the one way ANOVA model*

$$y_{ij} = \mu_i + \varepsilon_{ij}, \qquad j = 1,\dots,n_i \text{ and } i = 1,\dots,g. \tag{3.12}$$

Let $\boldsymbol{\varepsilon}' = (\varepsilon_{11},\dots,\varepsilon_{1n_1},\dots,\varepsilon_{g1},\dots,\varepsilon_{gn_g})$ *and* $n = \Sigma_{i=1}^{g} n_i$. *Assume that* $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0},\boldsymbol{\Sigma})$ *where* $\boldsymbol{\Sigma}$ *is a nnd matrix. Let* $\bar{y}_{i\cdot} = \Sigma_{j=1}^{n_i} y_{ij}/n_i$ *and* $\bar{y}_{\cdot\cdot} = \Sigma_{i=1}^{g}\Sigma_{j=1}^{n_i} y_{ij}/n$. *Let* SSR $= \Sigma_{i=1}^{g} n_i(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$ *and* SSE $= \Sigma_{i=1}^{g}\Sigma_{j=1}^{n_i}(y_{ij} - \bar{y}_{i\cdot})^2$ *be the treatment and the error sum of squares, respectively. Then, under the hypothesis* $\mu_i = \mu$ *for* $1 \leqslant i \leqslant g$, *one has*

(a) SSR $\sim d\chi^2(g-1)$,
(b) SSE $\sim d\chi^2(n-g)$,
(c) SSR *is independent of SSE*

if and only if $\boldsymbol{\Sigma} \in \mathscr{G}_{d,n}$ *for some constant* $d > 0$.

Proof. Let $\mathbf{y}' = (y_{11},\dots,y_{1n_1},\dots,y_{g1},\dots,y_{gn_g})$ and $\mu_i = \mu$ for $1 \leqslant i \leqslant g$; then we have $\mathbf{y} \sim N_n(\mu\mathbf{e},\boldsymbol{\Sigma})$. Note that $\mathbf{y}'\mathbf{A}^*\mathbf{y} = \text{SSR} + \text{SSE}$. It follows from Remark 3.1 and Theorem 3.5 with $k = 2$ and $c_1 = c_2 = d > 0$ that (a), (b), and (c) hold if and only if $\mathbf{A}^*\boldsymbol{\Sigma}\mathbf{A}^* = d\mathbf{A}^*$, which is true if and only if $\boldsymbol{\Sigma} \in \mathscr{G}_{d,n}$. This completes the proof of Theorem 3.6. ∎

The next theorem is concerned with the invariance property of the null distribution of the quadratic forms in a one way ANOVA model. Here we assume that observations within each treatment are correlated but observations between different treatments are uncorrelated. Theorem 3.7 shows that the quadratic forms for testing the equality of g means are independent and have chi-square distributions if and only if all the observations are uncorrelated when g is greater than or equal to 3. The case $g = 2$ was already considered in Theorem 3.3.

THEOREM 3.7. *Consider the one way ANOVA model as in Theorem* 3.6. *Let* $\boldsymbol{\varepsilon}_i' = (\varepsilon_{i1}, \ldots, \varepsilon_{in_i})$ *and* $n = \sum_{i=1}^{g} n_i$. *Assume that* $\boldsymbol{\varepsilon}_i$*'s are independent and* $\boldsymbol{\varepsilon}_i \sim N_{n_i}(\mathbf{0}, \boldsymbol{\Sigma}_i)$ *where the* $\boldsymbol{\Sigma}_i$*'s are nnd matrices for* $1 \leqslant i \leqslant g$. *If* $g \geqslant 3$, *under the hypothesis* $\mu_i = \mu$ *for* $1 \leqslant i \leqslant g$, *we have* (a), (b), *and* (c) *of Theorem* 3.6 *if and only if* $\boldsymbol{\Sigma}_i = d\mathbf{I}_{n_i}$ *for* $1 \leqslant i \leqslant g$ *and for some* $d > 0$.

Proof. Let $\mathbf{y}' = (y_{11}, \ldots, y_{1n_1}, \ldots, y_{g1}, \ldots, y_{gn_g})$. Then, under the hypothesis $\mu_i = \mu$ for $1 \leqslant i \leqslant g$, we have $\mathbf{y} \sim N_n(\mu\mathbf{e}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma} = \oplus_{i=1}^{g} \boldsymbol{\Sigma}_i$ with $\oplus$ denoting the direct sum of $\boldsymbol{\Sigma}_i$'s. By Theorem 3.6, we have (a), (b), and (c) if and only if $\boldsymbol{\Sigma} \in \mathscr{S}_{d,n}$ for some $d > 0$, that is,

$$\boldsymbol{\Sigma} = d\left[\left(\mathbf{I} - \frac{1}{n}\mathbf{ee}'\right) + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a}}{n}\mathbf{ee}'\right] \tag{3.13}$$

for some vector $\mathbf{a} \in \mathfrak{R}^n$ satisfying the inequality (2.12). Let σ_{jk} denote the (j,k)th element of $\boldsymbol{\Sigma}$. Since the $\boldsymbol{\varepsilon}_i$'s are uncorrelated, we require

$$\sigma_{jk} = d\left[a_j + a_k - (\bar{a} + 1)\right] = 0 \qquad \text{for} \quad 1 \leqslant j \leqslant n_1, \quad n_1 + 1 \leqslant k \leqslant n, \tag{3.14}$$

for

$$j = n_1 + 1, \ldots, n_1 + n_2, \quad k = 1, \ldots, n_1, n_1 + n_2 + 1, \ldots, n.$$

Since $g \geqslant 3$, we have $n > n_1 + n_2$, and it is easy to check that (3.14) holds if and only if $\mathbf{a} = \mathbf{e}$. Therefore, from (3.13), we get $\boldsymbol{\Sigma} = d\mathbf{I}$ and hence $\boldsymbol{\Sigma}_i = d\mathbf{I}_{n_i}$ for $1 \leqslant i \leqslant g$. This completes the proof of the theorem. ∎

4. SUMMARY AND CONCLUSIONS

The most widely used statistical methods are concerned with drawing inference for the parameters of the normal populations. In these problems the distribution of the test statistics are derived under the assumption that the observations are independent and identically distributed. While the independence assumption may be approximately valid, due to the choice of the experimental designs, exploring the problem of dependence between the observations is of practical as well as aesthetic interest. In this paper, we have characterized the class of covariance structures such that the distributions of the common test statistics remain invariant, that is, the distributions remain the same except for a scale factor. We have shown that in most cases the covariance structure must be equicorrelated for the distribution of the test statistics to remain invariant.

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