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Four Anharmonic Oscillators on a Circle

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ABSTRACT

Four identical, uniformly separated particles interconnected by ideal anharmonic springs are constrained to move on a fixed, frictionless circular track. The Lagrangian for the system is written and then transformed by matrix operations suggested by the symmetry of the arrangement of springs and particles. The equations of motion derived from the transformed Lagrangian yield four natural frequencies of motion.

INTRODUCTION

In this paper, we shall consider an idealized mechanical system of four identical particles constrained to move on a fixed, horizontal circular track. Each particle is connected to its two neighbors by identical massless springs whose motions are also confined to the circle of the track. All motions of the particles and springs are taken to proceed without friction so that no energy imparted to the system will be dissipated as heat. The equilibrium positions of the particles are equally spaced on the circle.

In the past, we have exploited the geometries of coupled systems such as that just described to separate their equations of motion either completely or to a significant extent. After the separated equations of motion were written in terms of symmetry coordinates, it was then not a difficult matter to obtain the natural frequencies of vibration corresponding to the various symmetry coordinates (Boyd and Raychowdhury, 2001c; Boyd, Hudepohl, and Raychowdhury, 2001a; Boyd, Hudepohl, and Raychowdhury, 2001b). In each case, the coupling between neighboring particles was provided by harmonic springs.

More recently, we have used matrix and Lagrangian techniques to discover natural frequencies for the transverse vibrations of a linear array of three Hooke's Law springs and two masses with the two endpoints of the array fixed in space. The transverse vibrations are anharmonic with restoring forces on the masses proportional to the cubes of their displacements away from equilibrium (Boyd, Hudepohl, and Raychowdhury, 2002b).

It has been our ambition for quite some time to apply the techniques which have been successful for coupled harmonic oscillators to systems of coupled anharmonic oscillators. The linearity of the harmonic equations of motion accounts for the relative ease with which we have been able to separate those equations. The nonlinearity of the equations of the anharmonic oscillators challenge the essentially linear techniques which we have been using. This paper represents our first attack upon a fairly complicated anharmonic system.

Harmonic springs provide tensions which are proportional to the amount by which they are stretched or compressed away from their natural lengths. Thus the equations of motion for harmonic systems are linear. The elastic potential energy of an harmonic spring is proportional to the square of the change in its length by either stretch or compression.

The coupling in the system under study is provided by anharmonic springs. The tensions in such springs are proportional to the cubes of their changes in length, and the elastic potential energy stored in each of these springs is proportional to the fourth power of its change in length.

We have been able to combine the matrix and Lagrangian techniques which were successful for the simple system of two coupled transverse anharmonic oscillators with a transformation suggested by the symmetry operations used in the investigation of the larger systems of harmonically coupled oscillators. The result is that we have obtained natural frequencies of vibration for the four particles on a circle as first described in the case that the springs are anharmonic. We shall describe that work in this paper. Our emphasis will be upon the use rather than the development of the transformation matrices which simplify our computations. The group representation theory underlying the construction of transformation matrices can be found in numerous places (Duffey, 1973; Hammermesh, 1962; Nussbaum, 1968). It was the proper formulation of the potential energy matrices that enabled us to complete our calculations.

LAGRANGIAN FOR FOUR PARTICLES COUPLED WITH ANHARMONIC SPRINGS

We represent in Figure 1, the system of four particles and springs constrained to move on their fixed circle. Thus the vibrations of the system will be longitudinal. We denote the mass of each particle by m and an anharmonic force constant for each spring by β. The spring constant will be defined by the way in which we write the elastic potential energies for the springs. The counter-clockwise displacement of the particles from their equilibrium positions are denoted by x_1, x_2, x_3, and x_4. The corresponding velocities of the particles are denoted by $\dot{x}_1, \dot{x}_2, \dot{x}_3$, and $\dot{x}_4$ and the total kinetic energy of the vibrating masses as they move is $KE = (m/2)\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2\right)$.

This total kinetic energy may be written in matrix notation as

$$KE = (m/2)\dot{X}I\dot{X}^T$$

where I represents the 4-by-4 identity matrix, $\dot{X} = \left(\dot{x}_1\dot{x}_2\dot{x}_3\dot{x}_4\right)$ represents the row velocity matrix, and $\dot{X}^T$ represents the transpose of the row velocity matrix.

The anharmonic springs provide forces proportional to $\left|x_j - x_l\right|^3$ on the j-th and l-th particles to which they are attached. These forces tend to restore the particles to their equilibrium positions. We take the elastic potential energy stored in the spring connecting the j-th and l-th particles to be $(\beta/4)\left(x_j - x_l\right)^4$ where β is a positive number. The total elastic potential energy for the system becomes

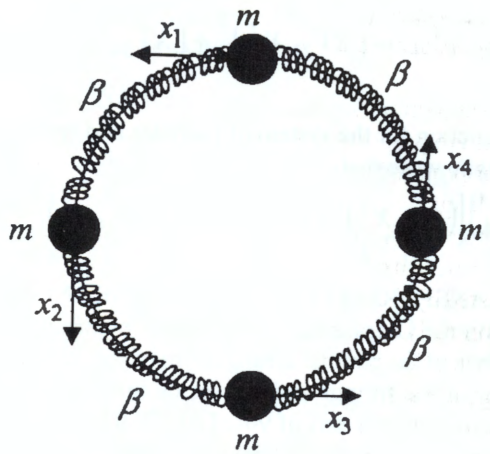

FIGURE 1. The Four Particles on Their Circle.

$$PE = \left(\beta/4\right)\left[(x_1 - x_2)^4 + (x_2 - x_3)^4 (x_3 - x_4)^4 + (x_4 - x_1)^4\right]$$

The potential energy involves the raising of four binomials to the fourth power. Our task is to discover a matrix formulation for the total elastic potential energy of the system which will accomplish the binomial algebra. To continue the notation adopted for expressing the kinetic energy, we let $X = (x_1 x_2 x_3 x_4)$ represent the row displacement matrix and X^T the transpose of X. We then choose the following four potential energy matrices:

$$V_{12} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad V_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \text{ and } \quad V_{41} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Straightforward computation will justify the statement that

$$PE = \left(\beta/4\right)\left[\left(XV_{12}X^T\right)^2 + \left(XV_{23}X^T\right)^2 + \left(XV_{34}X^T\right)^2 + \left(XV_{41}X^T\right)^2\right]$$

The Lagrangian function for the system of particles and springs is given by $L = KE - PE$ and we may write that

$$L = \left(m/2\right)\dot{X}I\dot{X}^T - \left(\beta/4\right)\left[\left(XV_{12}X^T\right)^2 + \left(XV_{23}X^T\right)^2 + \left(XV_{34}X^T\right)^2 + \left(XV_{41}X^T\right)^2\right]. \quad (1)$$

A TRANSFORMATION OF THE LAGRANGIAN

We seek to simplify the Lagrangian of our system by means of a transformation based upon the geometry of the circular arrangement of springs and masses. The rigid eometrical symmetries of the springs and masses on their circle are four reflections and counterclockwise, plane rotations of 90°, 180°, 270°, and 360° about the center of the circle. Taken together, these eight symmetry operations comprise the nonabelian group C_{4v} in which the rotation through 360° serves as the identity element. Familiarity with the matrix group representations of C_{4v} suggested to us that the orthogonal matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} / 2$$

might provide us with a transformation which would simplify the Lagrangian of equation 1 and, hence, the equations of motion which follow from the Lagrangian. The orthogonal transformation with S together with the choice of the potential energy matrices V_{12}, V_{23}, V_{34}, and V_{41} does lead to our goal of simplifying the equations of motion of the system.

Let us transform the coordinate and velocity vectors by

$$SX = S\left(x_1 x_2 x_3 x_4\right) = \left(z_1 z_2 z_3 z_4\right) = Z, x^T S^{-1} = Z^T,$$

$$S\dot{X} = S\left(\dot{x}_1 \dot{x}_2 \dot{x}_3 \dot{x}_4\right) = \left(\dot{z}_1 \dot{z}_2 \dot{z}_3 \dot{z}_4\right) = \dot{Z},$$

$$\text{and } \dot{X}^T S^{-1} = \dot{Z}^T.$$

We shall refer to the coordinates $z_j, j = 1,2,3,4$, as symmetry coordinates. Their corresponding velocities are $\dot{z}_j$.

We must also transform the potential energy matrices in a manner consistent with the transformations of coordinates and velocities. Those transformations may be accomplished by the following computations:

$$SV_{12}S^{-1}, SV_{23}S^{-1} SV_{34}S^{-1}, \text{ and } SV_{41}S^{-1}.$$

The Lagrangian as given by equation 1 may now be rewritten in terms of the symmetry coordinates and their velocities as

$$L = \left(\frac{m}{2}\right)\dot{X}S^{-1}SIS^{-1}S\dot{X}^{T}$$
$$-\left(\frac{\beta}{4}\right)\left(XS^{-1}SV_{12}S^{-1}SX^{T}\right)^{2}+\left(XS^{-1}SV_{23}S^{-1}SX^{T}\right)^{2}$$
$$+\left(XS^{-1}SV_{34}S^{-1}SX^{T}\right)^{2}+\left(XS^{-1}SV_{41}S^{-1}SX^{T}\right)^{2} \qquad (2)$$
$$=\left(\frac{m}{2}\right)\left(\dot{z}_1^2+\dot{z}_2^2+\dot{z}_3^2+\dot{z}_4^2\right)-\beta\left(z_2^4+3z_2^2z_3^2+\frac{z_3^4}{2}+3z_2^2z_4^2+\frac{z_4^4}{2}\right).$$

We note that we have resorted to the computer algebra system *Mathematica* to perform the matrix manipulations leading to this expression for L in terms of z_j and $\dot{z}_j$.

EQUATIONS OF MOTION AND NATURAL FREQUENCIES

Equations of motion in terms of the new symmetry coordinates and their accelerations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_j}\right)-\left(\frac{\partial L}{\partial z_j}\right)=0 \qquad (3)$$

Thus, we may write

$$m\ddot{z}_1 = 0, \qquad (3.1)$$
$$m\ddot{z}_2+\beta\left(4z_2^3+6z_2z_3^2+6z_2z_4^2\right)=0, \qquad (3.2)$$
$$m\ddot{z}_3+\beta\left(6z_2^2z_3+2z_3^3\right)=0, \text{ and} \qquad (3.3)$$
$$m\ddot{z}_4+\beta\left(6z_2^2z_4+2z_4^3\right)=0. \qquad (3.4)$$

Although equations 3.1, 3.2, 3.3, and 3.4 are not completely separated, we observe that, if we set as initial conditions that all $z_j = 0$, and $\dot{z}_j = 0$ at $t = 0$ except for $j = k$, the equation governing the variation in time of z_k involves no other symmetry coordinate. Thus the vibrations associated with each of z_1, z_2, and z_4 may be stimulated and sustained while the other symmetry coordinates remain suppressed as time progresses.

Such would not be the case if we had expanded the original Lagrangian of equation 2 in the coordinates x_1, x_2, x_3, x_4 and velocities $\dot{x}_1\,\dot{x}_2\,\dot{x}_3\,\dot{x}_4$. In that expansion, the appearance of terms of the form $4x_j x_l^3$ with $j \neq l$ contribute a coupling in the equations of motion which make it impossible for one coordinate to change without affecting other coordinates.

Returning to equation 3.2, we choose initial conditions $z_3 = z_4 = 0$ and $\dot{z}_3 = \dot{z}_4 = 0$. For equation 3.3, we take $z_2 = 0$ and $\dot{z}_2 = 0$. For equation 3.4, we take $z_2 = 0$ and $\dot{z}_2$ as well. Thus we are led to the next four equations of motion from which we can compute the frequencies of the system vibrating in natural or symmetrical modes:

$$m\ddot{z}_1 = 0 \tag{4.1}$$

$$m\ddot{z}_2 + 4\beta z_2^3 = 0 \tag{4.2}$$

$$m\ddot{z}_3 + 2\beta z_3^3 + 0, \text{ and} \tag{4.3}$$

$$m\ddot{z}_4 + 2\beta z_4^3 = 0. \tag{4.4}$$

The symmetry coordinate z_1 corresponds to a rotation at constant angular velocity with frequency $f_1 = 0$ since there is no vibration. Solutions in closed form for z_2, z_3, and z_4 may be written with the Jacobi cosine and sine amplitude functions (Dixon, 1984). These functions are denoted by $cn(t,\alpha)$ and $sn(t,\alpha)$, respectively. They are doubly periodic, analytic functions of the complex variable *t*. The parameter α is known as the modulus of its function. When the independent variable *t* is taken to be real valued (as is time in our problem) the functions have only a single period and resemble the trigonometric functions. These functions appear in the exact solutions of the equations of motion for the simple pendulum with large displacements and for a uniform sphere of specific gravity 0.5 bobbing in water (Boyd, 1991). Those readers who wish to investigate those elliptic functions will be interested to know that the functions have been written into the *Mathematica* software(Wolfram, 1999).

We shall simply discover the periods of vibration corresponding to equations 4 by integration. Let us turn to equation 4.3 and suppose that at $t = 0$, $z_3 = A_3 > 0$ and $z_3 = 0$. Equation 4.3 may be rewritten as

$$\frac{d\dot{z}_3}{dt} = \dot{z}_3 \frac{d\dot{z}_3}{dz_3} = -\frac{2\beta}{m} z_3^3.$$

The first integration

$$\int_0^{\dot{z}_3} u\,du = -\frac{2\beta}{m} \int_{A_3}^{z_3} v^3\,dv$$

yields

$$\frac{\dot{z}_3^2}{2} = -\frac{2\beta}{m}\left(\frac{\dot{z}_3^4}{4} - \frac{A_3^4}{4}\right) = \frac{\beta}{2m}\left(A_3^4 - z_3^4\right).$$

It follows that

$$\frac{dz_3}{dt} = \sqrt{\frac{\beta}{m}}\sqrt{A_3^4 - z_3^4}$$

or

$$dt = -\frac{dz_3}{\sqrt{\frac{\beta}{m}}\sqrt{A_3^4 - z_3^4}}$$

where the negative square root is taken since, as time increases during the first quarter period of motion after $t = 0$, z_3 decreases from A_3 to 0.

Let us denote the period of vibration for symmetry coordinate z_3 by T_3. Integrating the last equation from 0 to $\frac{T_3}{4}$ on the left-hand side and from A_3 to 0 on the right-hand side yields

$$T_3 = 4\int_0^{A_3} \frac{dz_3}{\sqrt{\frac{\beta}{m}}\sqrt{A_3^4 - z_3^4}} = 4\sqrt{\frac{m}{\beta}}\int_0^1 \frac{du}{A_3\sqrt{1-u^4}} = \frac{1}{A_3}\sqrt{\frac{m}{\beta}}(5.24412).$$

Thus the frequency $f_3 = \frac{1}{T_3}$ depends upon the amplitude A_3 of the motion as is known to be the case for the elliptic functions. The integral is an elliptic integral which has been evaluated by *Mathematica.* Inspection of equation 4.4 indicates that

$$T_4 = 4\sqrt{\frac{m}{\beta}}\int_0^1 \frac{du}{A_4\sqrt{1-u^4}} = \frac{1}{A_4}\sqrt{\frac{m}{\beta}}(5.24412)$$

where A_4 is the amplitude of the variation in z_4. Then $f_4 = \frac{1}{T_4}$.

A similar pair of integrations that begin with equation 4.2 leads to the conclusion that

$$T_2 = 4\sqrt{\frac{m}{2\beta}}\int_0^1 \frac{du}{A_2\sqrt{1-u^4}} = \sqrt{\frac{m}{2\beta}}(5.24412)$$

and $f_2 = \dfrac{1}{T_2}$.

CONCLUSION

Since the elliptic functions govern the motions of the system of anharmonic oscillators, the natural frequencies will always depend on the amplitudes of the corresponding vibrations. As previously noted, the Jacobi elliptic functions can be handled in closed form with *Mathematica.* In addition, *Mathematica* permits us to experiment with various matrix forms to develop useful transformations of coordinates. We have taken advantage of this computational power to give exact solutions of equations of motion for a simpler, anharmonic system than that considered in this work (Boyd, Hudepohl, and Raychowdhury 2002a).

We hope to look at other systems, but so far each problem that we have considered has required a solution tailored to the particular problem. It seems clear that no computational program for natural anharmonic frequencies will ever match in elegance and simplicity the symmetry-based calculations for natural harmonic frequencies.

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