2004

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Akhil K. Vaish

N. Rao Chaganty
Old Dominion University, rchagant@odu.edu

Original Publication Citation
Vaish, A. K., & Chaganty, N. R. (2004). Wishartness and independence of matrix quadratic forms for Kronecker product covariance structures. *Linear Algebra and Its Applications, 388*, 379-388. doi:10.1016/j.laa.2003.10.009

Available online at www.sciencedirect.com

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LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 388 (2004) 379–388

www.elsevier.com/locate/laa

Wishartness and independence of matrix quadratic forms for Kronecker product covariance structures

Akhil K. Vaish [a], N. Rao Chaganty [b,*]

[a]Statistics Research Division, Research Triangle Institute, Research Triangle Park, NC 27709, USA

[b]Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529-0077, USA

Received 7 January 2002; accepted 13 October 2003

Submitted by H.J. Werner

Abstract

Let $\mathbf{X}$ be distributed as matrix normal with mean $\mathbf{M}$ and covariance matrix $\mathbf{W} \otimes \mathbf{V}$, where $\mathbf{W}$ and $\mathbf{V}$ are nonnegative definite (nnd) matrices. In this paper we present a simple version of the Cochran's theorem for matrix quadratic forms in $\mathbf{X}$. The theorem is used to characterize the class of nnd matrices $\mathbf{W}$ such that the matrix quadratic forms that occur in multivariate analysis of variance are independent and Wishart except for a scale factor.

AMS classification: Primary 62H10, 62E15; Secondary 15A63

Keywords: Cochran's theorem; Matrix normal; Wishart distribution

1. Introduction

Let $\mathbf{x}_j = (x_{1j}, \ldots, x_{pj})'$ be a column vector consisting of measurements on p variables or at p time points taken on the jth subject, for $j = 1, \ldots, n$. Suppose that $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \sim N_{p,n}(\mathbf{M}, \boldsymbol{\Sigma})$, that is, $\tilde{\mathbf{x}} \sim N_{p,n}(\widetilde{\mathbf{m}}, \boldsymbol{\Sigma})$, where $\tilde{\mathbf{x}} = \mathrm{vec}(\mathbf{X})$, $\widetilde{\mathbf{m}} = \mathrm{vec}(\mathbf{M})$ and $\boldsymbol{\Sigma}$ is a nonnegative definite (nnd) covariance matrix of order np. Here $\mathrm{vec}(\mathbf{X}) = [\mathbf{x}_1', \mathbf{x}_2', \ldots, \mathbf{x}_n']'$ denotes the operator which stacks the columns of the $p \times n$ matrix $\mathbf{X}$ into a single column vector of length np. There are numerous papers in the literature generalizing the classical Craig–Sakamoto and Cochran

* Corresponding author. Tel.: +1-757-683-3897; fax: +1-757-683-3885.

E-mail addresses: avaish@rti.org (A.K. Vaish), rchagant@odu.edu (N.R. Chaganty).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2003.10.009

theorems on the necessary and sufficient conditions for Wishartness (chi-squaredness) and independence of (matrix) quadratic forms in $\mathbf{X}$ when $\boldsymbol{\Sigma}$ is of the form $\mathbf{W} \otimes \mathbf{V}$, where $\otimes$ denotes the Kronecker product. See for example, [7–10,12,13,17,18,20,21] and more recently [2]. Other authors, for example [11,19], have obtained similar results for a general covariance matrix $\boldsymbol{\Sigma}$. See [4] for an extensive bibliography on the Cochran's theorem. The purpose of this paper is to simplify various versions of the Cochran's theorem obtained by the previous authors in the case where $\boldsymbol{\Sigma}$ is the Kronecker product of two nnd matrices. The results are used to characterize the class of nnd matrices $\mathbf{W}$ such that the matrix quadratic forms that occur in multivariate analysis of variance, are independent and Wishart except for a scale factor.

The organization of this paper is as follows. In Section 2, we present two important lemmas and use them to obtain a simple version of the Cochran's theorem. Section 3 contains some applications to multivariate statistics. For example, we characterize the class of covariance matrices such that the one and two sample Hotelling's T^2 statistic and the distribution of quadratic forms that occur in multivariate analysis of variance remain the same except for a scale factor.

2. Wishartness and independence of matrix quadratic forms

Let $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$. Khatri [7,8] gave necessary and sufficient conditions for the Wishartness and independence of matrix quadratic forms as well as independence of quadratic polynomials, when $\mathbf{W}$ and $\mathbf{V}$ are positive definite matrices. Styan [18] obtained similar results for quadratic forms, when $p = 1$ and $\mathbf{W}$ is nnd, whereas Siotani et al. [17, pp. 95–96], stated the results when $\mathbf{W}$ is nnd and $\mathbf{V}$ is a positive definite matrix. Extensions for the Wishartness and independence of quadratic polynomials in the case where both $\mathbf{W}$ and $\mathbf{V}$ are nnd can be found in [9,19–21]. See also Chapter 7 in Gupta and Nagar [6]. In this section we present a version of the Cochran's theorem in the case where both $\mathbf{W}$ and $\mathbf{V}$ are nnd matrices. We will use this version in Section 3 to study invariance properties of the one and two sample Hotelling's T^2 statistic and the distributions of quadratic forms that occur in multivariate analysis of variance.

The main results of this paper are the following two lemmas. The first lemma is useful to establish Wishartness of matrix quadratic forms in singular normal variables. The second lemma is useful to obtain necessary and sufficient conditions for matrix quadratic forms to be independently distributed. Throughout the paper $\mathbf{O}$ denotes a matrix of zeros.

Lemma 2.1. *Let* $\mathbf{A}$ *and* $\mathbf{W}$ *be symmetric matrices of order* n. *Let* $\mathbf{M}$ *be a matrix of order* $p \times n$. *Consider the following two conditions*:

(a) $\mathbf{W}$ *is an* nnd *matrix such that* $\operatorname{tr}(\mathbf{AW}) = r(\mathbf{A})$,
(b) $r(\mathbf{AW}) = r(\mathbf{A})$.

If the condition (a) *or* (b) *holds then*

$$\begin{aligned}&\text{(i) } \mathbf{WAWAW} = \mathbf{WAW}, \quad \text{(ii) } \mathbf{MAW} = \mathbf{MAWAW} \quad \textit{and}\\ &\text{(iii) } \mathbf{MAM}' = \mathbf{MAWAM}'\end{aligned} \tag{2.1}$$

if and only if

$$\mathbf{AWA} = \mathbf{A}. \tag{2.2}$$

Proof. Sufficiency is easy to check. To prove the necessity, let (a) be given. Then from (2.1)(i), we get

$$\mathbf{T}'\mathbf{ATT}'\mathbf{AT} = \mathbf{T}'\mathbf{AT}, \tag{2.3}$$

where $\mathbf{T}$ is full column rank such that $\mathbf{W} = \mathbf{TT}'$. Therefore $r(\mathbf{T}'\mathbf{AT}) = r(\mathbf{A})$ which implies

$$r(\mathbf{A}) = r(\mathbf{T}'\mathbf{AT}) \leqslant r(\mathbf{AT}) \leqslant r(\mathbf{A})$$

and thus the two column spaces $\mathscr{M}(\mathbf{AT})$ and $\mathscr{M}(\mathbf{A})$ are equal. Hence $\mathbf{A} = \mathbf{ATC} = \mathbf{C}'\mathbf{T}'\mathbf{A}$ for some matrix $\mathbf{C}$. We get (2.2), if we pre- and postmultiply (2.3) by $\mathbf{C}'$ and $\mathbf{C}$, respectively. Suppose now (b) holds, then $\mathscr{M}(\mathbf{AW}) = \mathscr{M}(\mathbf{A})$. Therefore $\mathbf{AWD} = \mathbf{A}$ for some matrix $\mathbf{D}$ such that $\mathbf{AWD} = \mathbf{D}'\mathbf{WA}$. We get (2.2), pre- and postmultiplying (2.1)(i) by $\mathbf{D}'$ and $\mathbf{D}$, respectively. This completes the proof of the lemma. □

Lemma 2.2. *Let* $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{W}$ *be symmetric matrices of order* n. *Let* $\mathbf{M}$ *be a matrix of order* $p \times n$. *Consider the following two conditions*:

(a) $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{W}$ *are* nnd *matrices*,
(b) $r(\mathbf{A}_1\mathbf{W}) = r(\mathbf{A}_1)$ *and* $r(\mathbf{A}_2\mathbf{W}) = r(\mathbf{A}_2)$.

If the condition (a) *or* (b) *holds then*

$$\begin{aligned}&\text{(i) } \mathbf{WA}_1\mathbf{WA}_2\mathbf{W} = \mathbf{O}, \quad \text{(ii) } \mathbf{WA}_1\mathbf{WA}_2\mathbf{M}' = \mathbf{O} = \mathbf{WA}_2\mathbf{WA}_1\mathbf{M}' \quad \textit{and}\\ &\text{(iii) } \mathbf{MA}_1\mathbf{WA}_2\mathbf{M}' = \mathbf{O}\end{aligned} \tag{2.4}$$

if and only if

$$\mathbf{A}_1\mathbf{WA}_2 = \mathbf{O}. \tag{2.5}$$

Proof. The sufficiency is easy, so we outline only the proof of the necessary part. Let (a) be given, then it follows from [15,16] that (2.4) implies (2.5) (see also Theorem 4s in [14, p. 71]). Suppose now (b) is given. Then as shown in Lemma 2.1, we have $\mathbf{A}_1\mathbf{WC} = \mathbf{A}_1$ and $\mathbf{A}_2\mathbf{WD} = \mathbf{A}_2$ for some matrices $\mathbf{C}$ and $\mathbf{D}$. We get (2.5) by pre- and postmultiplying (2.4)(i) by $\mathbf{C}'$ and $\mathbf{D}$, respectively. □

Note that if $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{I} \otimes \mathbf{V})$, where $\mathbf{V}$ is nnd, then $\mathbf{XX}'$ is distributed as noncentral Wishart distribution $W_p(n, \mathbf{V}; \Omega)$, where $\Omega = \mathbf{MM}'$ is the noncentrality parameter. We now present theorems concerning the distribution of matrix quadratic forms.

Theorem 2.1. *Let* $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ *where* $\mathbf{W}$, $\mathbf{V}$ *are* nnd *matrices. Let* $\mathbf{A}$ *be a symmetric matrix of order* n. *Then* $\mathbf{Q}(\mathbf{X}) = \mathbf{XAX}' \sim W_p(r(\mathbf{A}), \mathbf{V}; \mathbf{Q}(\mathbf{M}))$ *if and only if* $\mathbf{AWA} = \mathbf{A}$.

Proof. Follows from Lemma 2.1 and Corollary 3.2 in [19]. □

Corollary 2.2. *The distribution of* $\mathbf{Q}(\mathbf{X})$ *in Theorem* 2.1 *is a central Wishart distribution* $W_p(r(\mathbf{A}), \mathbf{V})$ *if and only if* $\mathbf{AM}' = \mathbf{O}$.

Proof. It is easy to see that if $\mathbf{AM}' = \mathbf{O}$ then $\mathbf{MAM}' = \mathbf{O}$. Hence $\mathbf{Q}(\mathbf{X})$ has a central Wishart distribution. For the converse, note that if $\mathbf{W} = \mathbf{TT}'$ is the rank factorization then $\mathbf{MAM}' = \mathbf{O}$ implies $\mathbf{MAT} = \mathbf{O}$, which in turn implies that $\mathbf{MATT}'\mathbf{A} = \mathbf{O}$. Hence $\mathbf{AM}' = \mathbf{O}$ since $\mathbf{AWA} = \mathbf{A}$. □

Corollary 2.3. *Under the assumptions of Theorem* 2.1, *we have* $\mathbf{Q}(\mathbf{X}) \sim dW_p(r(\mathbf{A}), \mathbf{V}; \Omega)$ *if and only if* $\mathbf{AWA} = d\mathbf{A}$ *where* $d > 0$ *and* $\Omega = \frac{1}{d}\mathbf{Q}(\mathbf{M})$.

The next theorem gives necessary and sufficient conditions for the independence of two quadratic forms. The theorem is an extension of the results contained in [15,18] for $p = 1$.

Theorem 2.4. *Let* $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ *where* $\mathbf{W}$ *and* $\mathbf{V}$ *are* nnd *matrices. Let* $\mathbf{Q}_1(\mathbf{X}) = \mathbf{XA}_1\mathbf{X}'$ *and* $\mathbf{Q}_2(\mathbf{X}) = \mathbf{XA}_2\mathbf{X}'$, *where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are symmetric matrices of order* n. *Consider the conditions*: (a) $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are* nnd *matrices and* (b) $r(\mathbf{A}_1\mathbf{W}) = r(\mathbf{A}_1)$ *and* $r(\mathbf{A}_2\mathbf{W}) = r(\mathbf{A}_2)$. *If the condition* (a) *or* (b) *holds then* $\mathbf{Q}_1(\mathbf{X})$ *and* $\mathbf{Q}_2(\mathbf{X})$ *are independently distributed if and only if* $\mathbf{A}_1\mathbf{WA}_2 = \mathbf{O}$.

Proof. Follows from Lemma 2.2 and Corollary 3.5 in [19]. □

The following is an easy consequence of the above theorem.

Corollary 2.5. *Let* $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ *where* $\mathbf{W}$ *and* $\mathbf{V}$ *are* nnd *matrices. Let* $\mathbf{A}$ *be a* nnd *matrix of order* n *and* $\mathbf{L}$ *be a matrix of order* $p \times n$. *Then* $\mathbf{Q}(\mathbf{X}) = \mathbf{XAX}'$ *and* $\mathbf{XL}'$ *are independently distributed if and only if* $\mathbf{AWL}' = \mathbf{O}$.

It is interesting to note that the necessary and sufficient conditions in Theorems 2.1, 2.4 and the above corollary do not depend on $\mathbf{M}$. We now present a version of the Cochran's theorem.

Theorem 2.6. *Let* $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ *where* $\mathbf{W}$ *and* $\mathbf{V}$ *are* nnd *matrices. Let* $\mathbf{A}_i$ $(i = 1, \dots, k)$ *and* $\mathbf{A}$ *be symmetric matrices of order n such that* $\mathbf{A} = \sum_{i=1}^{k} \mathbf{A}_i$. *Consider the following conditions*:

(a_1) $\mathbf{X}\mathbf{A}_i\mathbf{X}' \sim W_p(r(\mathbf{A}_i), \mathbf{V}; \Omega_i)$ *where* $\Omega_i = \mathbf{M}\mathbf{A}_i\mathbf{M}'$ *for* $i = 1, \dots, k$,
(a_2) $\mathbf{X}\mathbf{A}_i\mathbf{X}'$ *and* $\mathbf{X}\mathbf{A}_j\mathbf{X}'$ *are mutually independent for* $i \neq j = 1, \dots, k$,
(a_3) $\mathbf{X}\mathbf{A}\mathbf{X}' \sim W_p(r(\mathbf{A}), \mathbf{V}; \Omega)$ *where* $\Omega = \mathbf{M}\mathbf{A}\mathbf{M}'$,
(b_1) $\mathbf{A}_i\mathbf{W}\mathbf{A}_i = \mathbf{A}_i$ *for* $i = 1, \dots, k$,
(b_2) $\mathbf{A}_i\mathbf{W}\mathbf{A}_j = \mathbf{O}$ *for* $i \neq j = 1, \dots, k$,
(b_3) $\mathbf{AWA} = \mathbf{A}$,
(b_4) $\sum_{i=1}^{k} r(\mathbf{A}_i) = r(\mathbf{A})$.

Then

(1) *any two of the three conditions* (a_1), (a_2), (a_3) *or*
(2) *any two of the three conditions* (b_1), (b_2), (b_3) *or*
(3) *any two conditions* (a_i) *and* (b_j) *for* $i \neq j = 1, 2, 3$ *or*
(4) (b_3) *and* (b_4) *or*
(5) (a_3) *and* (b_4)

are necessary and sufficient for all the remaining conditions: (a_1)–(b_4).

Proof. The proof is based on Theorems 2.1 and 2.4. We will only prove that (a_3) and (b_4) imply all the remaining conditions. Suppose that (a_3) and (b_4) hold, then from Theorem 2.1, we get (b_3). Let $\mathbf{B} = \mathbf{T}'\mathbf{A}\mathbf{T}$ and $\mathbf{B}_i = \mathbf{T}'\mathbf{A}_i\mathbf{T}$ for $i = 1, \dots, k$ where $\mathbf{W} = \mathbf{T}\mathbf{T}'$ is the rank factorization of $\mathbf{W}$. Then $\mathbf{B} = \sum_{i=1}^{k} \mathbf{B}_i$. Also from ($b_3$), we get $\mathbf{B}^2 = \mathbf{B}$. Using condition ($b_4$), we have

$$r(\mathbf{A}) = \operatorname{tr}(\mathbf{AW}) = \operatorname{tr}(\mathbf{B}) = r(\mathbf{B}) \leqslant \sum_{i=1}^{k} r(\mathbf{B}_i) \leqslant \sum_{i=1}^{k} r(\mathbf{A}_i) = r(\mathbf{A}),$$

since $\operatorname{tr}(\mathbf{AW}) = r(\mathbf{AW}) = r(\mathbf{A})$ from (b_3). Hence $r(\mathbf{B}) = \sum_{i=1}^{k} r(\mathbf{B}_i)$. From Theorem 1 in [5], we get $\mathbf{B}_i^2 = \mathbf{B}_i$ and $\mathbf{B}_i\mathbf{B}_j = \mathbf{O}$ for $i \neq j = 1, \dots, k$. It follows from Lemma 2.3 that $r(\mathbf{B}_i) = r(\mathbf{A}_i)$ for $i = 1, \dots, k$. Applying Lemmas 2.1 and 2.2 for $\mathbf{A}_i$ we can see that (b_1) and (b_2) hold. Now (a_1) and (a_2) follow from Theorems 2.1 and 2.4. □

Lemma 2.3. *Let* $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{A}$ *and* $\mathbf{B}$ *be as defined in Theorem* 2.6. *Let* $r(\mathbf{A}) = \sum_{i=1}^{k} r(\mathbf{A}_i) = \sum_{i=1}^{k} r(\mathbf{B}_i) = r(\mathbf{B})$. *Then* $r(\mathbf{B}_i) = r(\mathbf{A}_i)$ *for* $i = 1, \dots, k$.

Proof. It is obvious from the definition of $\mathbf{B}_i$'s that $r(\mathbf{B}_i) \leqslant r(\mathbf{A}_i)$ for $i = 1, \dots, k$. Also, $r(\mathbf{B}_i) = r(\mathbf{B}) - \sum_{j \neq i=1}^{k} r(\mathbf{B}_j) = r(\mathbf{A}) - \sum_{j \neq i=1}^{k} r(\mathbf{B}_j) \geqslant r(\mathbf{A}) -$

$\sum_{j\neq i=1}^{k} r(\mathbf{A}_j) = r(\mathbf{A}_i)$. Hence, repeating the same argument we can show that $r(\mathbf{B}_i) \geqslant r(\mathbf{A}_i)$ for $i = 1, \ldots, k$. $\square$

3. Applications to multivariate statistics

Here we present applications of the theorems in Section 2 to multivariate statistical theory. Suppose that $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{W}$ and $\mathbf{V}$ are nnd matrices. Let $\bar{\mathbf{x}} = \frac{1}{n}\mathbf{Xe}$ be the sample mean and $\mathbf{S} = \mathbf{XRX}'/(n-1)$ be the sample covariance matrix, where $\mathbf{R} = \mathbf{I} - \frac{1}{n}\mathbf{ee}'$ is the centering matrix, and $\mathbf{e}$ is a vector of ones. Basu et al. [1] considered the equicorrelated structure for $\mathbf{W}$ and showed that $\bar{\mathbf{x}}$ and $\mathbf{S}$ are independently distributed and their distributions are preserved except for a constant. We prove the converse in Theorem 3.2.

Theorem 3.1. *Let $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{W}$ and $\mathbf{V}$ are* nnd *matrices. Then for any $d > 0$, we have $(n-1)\mathbf{S} \sim dW_p(n-1, \mathbf{V}; \Omega)$ where $\Omega = \frac{1}{d}\mathbf{MRM}'$, if and only if*

$$\mathbf{W} = d\left[\mathbf{R} + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a}}{n}\mathbf{ee}'\right], \tag{3.1}$$

where $\bar{a} = (\mathbf{a}'\mathbf{e})/n$, and $\mathbf{a} = (a_1, \ldots, a_n)'$ is an arbitrary vector satisfying $\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2 \leqslant \bar{a}$.

Proof. Using Corollary 2.3 we can see that $\mathbf{W}$ has to satisfy $\mathbf{RWR} = d\mathbf{R}$. The theorem follows from Remark 2.5 in [3]. $\square$

Theorem 3.2. *Let $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{W}$ and $\mathbf{V}$ are* nnd *matrices. Then $(n-1)\mathbf{S} \sim dW_p(n-1, \mathbf{V}; \Omega)$ and is independent of $\bar{\mathbf{x}}$ if and only if $\mathbf{W} = d(\mathbf{I} - \frac{(1-c)}{n}\mathbf{ee}')$ for some $c \geqslant 0$, $d > 0$ and Ω is given in Theorem 3.1.*

Proof. From Corollary 2.5, we have $\bar{\mathbf{x}}$ is independent of $\mathbf{S}$ if and only if

$$\mathbf{RWe} = \mathbf{0}. \tag{3.2}$$

Now $\mathbf{W}$ satisfies (3.1) and (3.2) if and only if $\mathbf{Ra} = \mathbf{0}$ which is true iff $\mathbf{a} = c\mathbf{e}$ where $c = \bar{a} \geqslant 0$. The proof is completed substituting $\mathbf{a} = c\mathbf{e}$ in (3.1). $\square$

As a consequence of the preceding theorem we have the following property of the one sample Hotelling's T^2 statistic.

Corollary 3.3. *Let $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{W}$ is a* nnd *matrix and $\mathbf{V}$ is a positive definite matrix. Let $\mathbf{M} = \boldsymbol{\mu}\mathbf{e}'$ where $\boldsymbol{\mu}$ is a vector of order $p \times 1$. Assume that $n - 1 \geqslant p$. Then*

$$T^2 = \frac{n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})}{n-1}\frac{n-p}{p} \sim cF(p, n-p) \tag{3.3}$$

if $\mathbf{W} = d(\mathbf{I} - \frac{(1-c)}{n}\mathbf{e}\mathbf{e}')$ *for some* $c \geqslant 0$ *and* $d > 0$.

For the two sample Hotelling's T^2 we have

Theorem 3.4. *For* $k = 1, 2$, *let* $\mathbf{X}_k \sim N_{p,n_k}(\boldsymbol{\mu}_k\mathbf{e}'_{n_k}, \mathbf{W}_k \otimes \mathbf{V})$, *where* $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *be* nnd *matrices, and* $\mathbf{V}$ *be a positive definite matrix. Assume that* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent and* $n_1 + n_2 - 2 \geqslant p$. *Let* $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ *be the sample mean vectors and* $\mathbf{S}_1$, $\mathbf{S}_2$ *be the sample covariance matrices of* $\mathbf{X}_1$ *and* $\mathbf{X}_2$, *respectively. Also, let* $\mathbf{S}_p = [(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2]/(n_1 + n_2 - 2)$ *be the pooled sample covariance matrix and* $\boldsymbol{\theta} = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$. *Then*

$$\begin{aligned} T^2 = {} & \frac{n_1 n_2}{n_1 + n_2}\frac{((\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) - \boldsymbol{\theta})'\mathbf{S}_p^{-1}((\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) - \boldsymbol{\theta})}{n_1 + n_2 - 2}\frac{(n_1 + n_2 - p - 1)}{p} \\ & \sim F(p, n_1 + n_2 - p - 1) \end{aligned} \tag{3.4}$$

if

$$\mathbf{W}_1 = d(\mathbf{I} + \beta\mathbf{e}_{n_1}\mathbf{e}'_{n_1}) \quad \text{and} \quad \mathbf{W}_2 = d(\mathbf{I} - \beta\mathbf{e}_{n_2}\mathbf{e}'_{n_2})$$

for some constants β *and* d *such that* $d > 0$ *and* $-1/n_1 < \beta < 1/n_2$. *Here* $\mathbf{e}_n$ *is a column vector of ones of order* n.

Proof. From Theorem 3.2 we have $(n_k - 1)\mathbf{S}_k \sim dW_p(n_k - 1, \mathbf{V})$ and is independent of $\bar{\mathbf{x}}_k$ if and only if $\mathbf{W}_k = d(\mathbf{I} - \frac{(1-c_k)}{n_k}\mathbf{e}_{n_k}\mathbf{e}'_{n_k})$, where $c_k \geqslant 0$ for $k = 1, 2$ and $d > 0$. For these $\mathbf{W}_k$'s we have

$$\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - \boldsymbol{\theta} \sim N_p\left(\mathbf{0}, d\left(\frac{c_1}{n_1} + \frac{c_2}{n_2}\right)\mathbf{V}\right) \tag{3.5}$$

and

$$\frac{n_1 + n_2 - 2}{d}\mathbf{S}_p \sim W_p(n_1 + n_2 - 2, \mathbf{V}), \tag{3.6}$$

where $\boldsymbol{\theta} = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$. Thus for $c_1 > 0$ and $c_2 > 0$ from (3.5) and (3.6) we can see that

$$\begin{aligned} & \frac{n_1 n_2}{n_1 + n_2}\frac{((\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) - \boldsymbol{\theta})'\mathbf{S}_p^{-1}((\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) - \boldsymbol{\theta})}{n_1 + n_2 - 2}\frac{(n_1 + n_2 - p - 1)}{p} \\ & \quad \sim \frac{c_1 n_2 + c_2 n_1}{n_1 + n_2}F(p, n_1 + n_2 - p - 1). \end{aligned} \tag{3.7}$$

The proof is completed choosing $c_1 = n_1\beta + 1$ and $c_2 = 1 - n_2\beta$ where $-1/n_1 < \beta < 1/n_2$. $\square$

Remark 3.1. Selecting $c_1 = c_2 = c$ in the proof of Theorem 3.4 we can see that the two sample Hotelling's $T^2 \sim cF(p, n_1 + n_2 - p - 1)$ if $\mathbf{W}_k = d(\mathbf{I} - \frac{(1-c)}{n_k}\mathbf{e}_{n_k}\mathbf{e}'_{n_k})$, for $k = 1, 2$, where $d > 0$.

We turn our attention now to the multivariate analysis of variance. Here the total corrected sum of squares $\mathbf{XRX}'$ is decomposed as $\mathbf{Q}_1 + \cdots + \mathbf{Q}_m$, where $\mathbf{Q}_i = \mathbf{XA}_i\mathbf{X}'$, $1 \leqslant i \leqslant (m-1)$, are matrix quadratic forms useful to test $(m-1)$ orthogonal hypothesis, and $\mathbf{Q}_m = \mathbf{XA}_m\mathbf{X}'$ is the matrix quadratic form representing the residuals. The matrices $\mathbf{A}_i$'s are idempotent and $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ for $i \neq j$. The invariance property of these quadratic forms is given below.

Theorem 3.5. *Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ be symmetric and idempotent matrices of order n such that $\mathbf{A}_i\mathbf{A}_j = \mathbf{O}$ for all $i \neq j$. Let $\sum_{i=1}^m \mathbf{A}_i = \mathbf{R}$ and $\mathbf{B} = \sum_{i=1}^m c_i\mathbf{A}_i$ where $\mathbf{R}$ is the centering matrix and $c_i > 0$ for $1 \leqslant i \leqslant m$. Let $\mathbf{X} \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{W}$ and $\mathbf{V}$ are* nnd *matrices. Let $\mathbf{Q}_i(\mathbf{X}) = \mathbf{XA}_i\mathbf{X}'$ for $1 \leqslant i \leqslant m$. Then $\mathbf{Q}_i(\mathbf{X}) \sim c_iW_p(r(\mathbf{A}_i), \mathbf{V}; \mathbf{Q}_i(\mathbf{M})/c_i)$ for all i and pairwise independent if and only if*

$$\mathbf{W} = \mathbf{B} + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a}}{n}\mathbf{ee}', \tag{3.8}$$

where $\mathbf{a}$ is an arbitrary vector satisfying

$$\frac{1}{n}\sum_{i=1}^{m}\frac{\mathbf{a}'\mathbf{A}_i\mathbf{a}}{c_i} \leqslant \bar{a}. \tag{3.9}$$

Proof. Follows from Theorem 2.6, Corollary 2.3 and Theorem 2.2 in [3]. □

Theorem 3.5 yields some interesting characterizations in the one way model. Suppose that we have from the kth population a matrix of observations $\mathbf{X}_k$ of order $p \times n_k$ such that $E(\mathbf{X}_k) = \boldsymbol{\mu}_k\mathbf{e}'_{n_k}$ for $1 \leqslant k \leqslant g$. Assume that $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_g] \sim N_{p,n}(\mathbf{M}, \mathbf{W} \otimes \mathbf{V})$, where $\mathbf{V}$ is a nnd matrix of order p and $\mathbf{W}$ is a nnd matrix of order $n = \sum_{k=1}^g n_k$. The standard test for testing $\boldsymbol{\mu}_k = \boldsymbol{\mu}$ for all k, that is, $\mathbf{M} = \boldsymbol{\mu}\mathbf{e}'$, partitions the total corrected sum of squares $\mathbf{XRX}'$ as $\mathbf{Q}_t(\mathbf{X}) + \mathbf{Q}_e(\mathbf{X})$ where $\mathbf{Q}_t(\mathbf{X}) = \mathbf{X}(\mathbf{J} - \frac{1}{n}\mathbf{ee}')\mathbf{X}'$, $\mathbf{Q}_e(\mathbf{X}) = \mathbf{X}(\mathbf{I} - \mathbf{J})\mathbf{X}'$. Here $\mathbf{J} = \bigoplus_{k=1}^g \frac{1}{n_k}\mathbf{e}_{n_k}\mathbf{e}'_{n_k}$ and $\bigoplus$ denotes the direct sum. It is well that known that if $\mathbf{W} = \mathbf{I}$ and $\boldsymbol{\mu}_k = \boldsymbol{\mu}$ for all k, then $\mathbf{Q}_t(\mathbf{X}) \sim W_p(g-1, \mathbf{V})$, $\mathbf{Q}_e(\mathbf{X}) \sim W_p(n-g, \mathbf{V})$ and both are independent. The theorem below gives a complete characterization of the class of $\mathbf{W}$'s such that these properties hold.

Theorem 3.6. *Assume that $\mathbf{X} \sim N_{p,n}(\boldsymbol{\mu}\mathbf{e}', \mathbf{W} \otimes \mathbf{V})$ where $\mathbf{V}$ and $\mathbf{W}$ are* nnd *matrices of order p and n, respectively. Let $c_i > 0$ for $i = 1, 2$. Then*

(1) $\mathbf{Q}_t(\mathbf{X}) \sim c_1W_p(g-1, \mathbf{V})$,
(2) $\mathbf{Q}_e(\mathbf{X}) \sim c_2W_p(n-g, \mathbf{V})$,
(3) $\mathbf{Q}_t(\mathbf{X})$ *is independent of* $\mathbf{Q}_e(\mathbf{X})$

if and only if

$$\mathbf{W} = c_2\mathbf{I} + (c_1 - c_2)\mathbf{J} + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a} + c_1}{n}\mathbf{ee}', \tag{3.10}$$

where **a** *is an arbitrary vector satisfying*

$$\frac{\mathbf{a}'\left(\mathbf{J} - \frac{1}{n}\mathbf{ee}'\right)\mathbf{a}}{nc_1} + \frac{\mathbf{a}'(\mathbf{I} - \mathbf{J})\mathbf{a}}{nc_2} \leqslant \bar{a}. \tag{3.11}$$

Proof. Choose $\mathbf{M} = \boldsymbol{\mu}\mathbf{e}'$ and $\mathbf{A}_1 = \mathbf{J} - \frac{1}{n}\mathbf{ee}'$, $\mathbf{A}_2 = \mathbf{I} - \mathbf{J}$ in Theorem 3.5. □

When observations from the g populations are independent we have

Theorem 3.7. *Assume that* $\mathbf{X}_k \sim N_{p,n_k}(\boldsymbol{\mu}\mathbf{e}'_{n_k}, \mathbf{W}_k \otimes \mathbf{V})$ *where* $\mathbf{W}_k$ *and* $\mathbf{V}$ *are* nnd *matrices of orders* n_k *and* p, *respectively for* $k = 1, \ldots, g$ *and they are independent. Let* $c_i > 0$ *for* $i = 1, 2$. *If* $g \geqslant 3$ *then* (1)–(3) *of Theorem* 3.6 *hold if and only if*

$$\mathbf{W}_k = c_2\mathbf{I} + \frac{(c_1 - c_2)}{n_k}\mathbf{e}_{n_k}\mathbf{e}'_{n_k} \quad \textit{for } 1 \leqslant k \leqslant g. \tag{3.12}$$

Proof. Let $\mathbf{W} = \bigoplus_{k=1}^{g} \mathbf{W}_k$. From Theorem 3.6 we can see that (1)–(3) hold if and only if

$$\mathbf{W} = c_2\mathbf{I} + (c_1 - c_2)\mathbf{J} + \frac{1}{n}(\mathbf{ea}' + \mathbf{ae}') - \frac{\bar{a} + c_1}{n}\mathbf{ee}' \tag{3.13}$$

for some vector **a** satisfying (3.11). Thus w_{ij}, the (i, j)th element of **W** satisfies

$$\begin{aligned} w_{ij} = a_i + a_j - (\bar{a} + c_1) = 0 \quad & \text{for } 1 \leqslant i \leqslant n_1, \quad n_1 + 1 \leqslant j \leqslant n \\ \text{and} \quad & \text{for } \begin{array}{l} i = n_1 + 1, \ldots, n_1 + n_2, \\ j = 1, \ldots, n_1, n_1 + n_2 + 1, \ldots, n. \end{array} \end{aligned} \tag{3.14}$$

Since $g \geqslant 3$ we have $n > n_1 + n_2$ and it is easy to check that (3.14) hold if and only if $\mathbf{a} = c_1\mathbf{e}$. For this choice of **a**, (3.13) simplifies to $\mathbf{W} = c_2\mathbf{I} + (c_1 - c_2)\mathbf{J}$. This completes the proof. □

Acknowledgements

This research was partially supported by the USARO grant DAAD19-01-1-0684. We thank the Associate Editor and a referee for reviewing our paper.

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