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A two-population insurgency in Colombia: Quasi-predator-prey models – A trend towards simplicity

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ABSTRACT

A sequence of analytic mathematical models has been developed in the context of the "low-level insurgency" in Colombia, from 1993 to the present. They are based on generalizations of the two-population "predator-prey" model commonly applied in ecological modeling, and interestingly, the less sophisticated models yield more insight into the problem than the more complicated ones, but the formalism is available to adapt the model "upwards" in the event that more data becomes available, or as the situation increases in complexity. Specifically, so-called "forcing terms" were included initially in the coupled differential equations to represent the effects of government policies towards both the narco-terrorist or insurgent (I) and susceptible (S) populations. These terms are in general functions of time, since it is to be expected that changes in policy will occur as the outcomes of previously implemented policies are recognized. Both continuous and discontinuous forcing functions can be appropriate for each population. Although nonhomogeneous systems are discussed for both populations, the majority of the analysis focused on a system with forcing terms for the terrorist population only.

Two categories of models emerged: 1 — in which the time-dependent forcing terms were independent of the two populations, and 2 — in which these terms were directly proportional to the respective populations. Model 2 in fact, can be considered as a generalization of the unforced (or homogeneous) version of model 1, and as such is implemented to describe the data obtained for the insurgent population from 1993–2003. This provides some restrictions on the unknown parameters in the model. Because of new government policy towards the insurgent population as a result of the election of President Uribe in 2002, a slight but significant modification of this model is used to describe the I-population from 2003 to the present time, again resulting in useful relationships between the various model parameters. The data suggest that a further simplification in this model is appropriate beyond 2003, and an analytic solution was found for both populations. A further simplification results from examining the related decoupled system of equations, yielding a straightforward predictive model for the behavior of the S-population in particular. Finally, in the Appendix, detailed analysis of model 2 yields a general analytic solution of the system for a wide range of sub-models, expressible in terms of confluent hypergeometric functions.

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1. Introduction

Like many of the Latin American states of Central and South America, Colombia has experienced the spectrum of ills that come with being a former possession of a global power, despite (or perhaps as a result of) being the oldest democracy in South America. With that discontent, came a war for independence followed by long-term domestic discord over political ideals supported by two dominant parties: the Conservatives and the Liberals. These conflicts precluded Colombia from developing a sense of nationalism. According to sources in the bibliography below, its government (led alternately by both the Conservatives and the Liberals) was never able to unite its multi-ethnic population as Colombians. This "internal commotion" gave way to marginalization for political and economic power by those who saw a way to capitalize on Colombia's most lucrative export: the plant *coca*. The drug cocaine is chemically extracted from its new fresh leaf tips in a similar fashion to tea bush harvesting. The effects of the sale of coca on the state of Colombia are profound. The trafficking of the drug introduced a form of terrorism that brought with it paramilitaries, private armies, corruption, human rights abuses, internally displaced peoples, and a "dirty war" against many innocent civilians. The Colombian counter-insurgency was relatively ineffective until the U.S. recognized the war on drugs must also be waged in Colombia. With U.S support and an ideological shift that 'morphed' the war on drugs into the war against terrorism, the number of guerrillas, insurgents, or traffickers appears to have decreased — yet coca still finds its way to the markets.

In the 1980s crack cocaine opened the floodgates of the drug trade. Crack was cheaper, more addictive, and smoked versus injected. This lethal combination made crack available for casual use to a much larger audience. By 1989 crack had a direct influence on U.S. national security as President George H.W. Bush declared drugs a threat to America. In the 1990s the conflicts between and among Colombian government forces, anti-government insurgent groups, and illegal paramilitary groups (the latter two heavily funded by the drug trade) escalated. However, the insurgents lacked the military or popular support necessary to overthrow the government and take control of the country. From 1993 to the close of the decade two cartels controlled the Colombian drug trade: the Medillin and the Cali Cartels. Both organizations developed a well-financed mafia-like operating system with enough influence and audacity to propose to the Colombian government their willingness to pay off the national debt. The Cartels were the first to hire private militaries to protect property, operations, and individuals. In the late 1990s a third insurgent group, the United Self-Defense Forces of Colombia (AUC), comprised of right-wing paramilitary groups supported by wealthy landowners, drug cartels, and segments of the Colombian military came into existence. These forces are known to have committed numerous human rights abuses including assassinations of politicians, leftist guerrillas, activists, and civilians. The evolution of Colombia's insurgency is directly tied to a confluence of variables, namely these are, what has been described as the government's lack of control over both the illicit and licit (drugs and oil), human rights abuses, the fluctuation of insurgents, a societal-based counter-insurgency, and a weak social contract.

The democratization of the drug trade (the result of the dissolution of the two large cartels) in the mid 1990s spawned many splinter groups. The insurgents have access to weaponry of all sorts and they have established relationships both across the globe and nearer to home. The mindset of the insurgents appears to be more along the lines that this is an intractable contest as opposed to a resistance to resolution [1]. All involved, the insurgents, the counter-insurgency, and the civilian population suffered war fatigue during this period. The U.S. has funded the counter effort to a great degree. In fact, Colombia became the third largest recipient of U.S. aid. Yet, still the Colombians faced the drugs, violence, and the fear of violence [2]. The causes and support of the insurgency are significant and are only slightly affected by U.S. funding a greater effort against drug trafficking in the form of *Plan Colombia*. This is an integrated strategy to meet the most pressing challenges confronting Colombia today – promoting the peace process, combating the narcotics industry, reviving the Colombian economy, and strengthening the democratic pillars of Colombian society [19].

The measure of democracy for the state of Colombia is a seemingly perfect one: the country is the oldest democracy in South America, it has a representative government with free elections, multiple political parties, non-military rule, and it suffers no tyranny [1,3]. In 1991 Colombia introduced a new constitution and it is able to boast having 42 recognized parties [2,4]. Those who support President Alvaro Uribe view his societal approach to the counter-insurgency as legitimate and his "Law and Order" rhetoric as necessary [5–7]. Yet the country suffered as a failed state during this period when the war on drugs was being waged for the obvious reason that the government was unable to perform basic functions and retain its legitimacy. The reasons for this perpetual failing are many: among these, the unending conflict makes it difficult for the country to mature; and many of Colombia's citizens perceive an absence of state and no rule of law as they experience the government's inability to provide for domestic security. This internal disruption can be readily measured by the civilian death toll: between 1980 and 2004 over 100,000 people were killed due to political conflict [8]. This is reflected in the demographics that place Colombia as the third largest nation of internally displaced people with 2.9 million in 2003 [9]. Poverty levels range from "*poverty*" at 64% to "*extreme poverty*" at 23% [8].

The war on drugs was declared by U.S. President Richard Nixon in the mid 1970s as the casual drug use in America became pronounced. With the establishment of the Drug Enforcement Administration came increased funding to wage this war. From Presidents Nixon to Clinton, the U.S. has supported Colombia's counter-insurgent effort as a part of the war on drugs. By the mid-1990s the major cartels were dismantled, but in their wake came the democratization of the drug trade, and the war on drugs continued, albeit in a different form. The military focused solely on this war, as opposed to what militaries traditionally do: secure the nation from external threats. Citizens supported the war against drugs, although many farmers served the drug traffickers either as willing partners or coerced participants. The National Police were also called to serve in this effort as well. The Uribe administration stepped into the executive office with a force of 40,000 troops, 55,000 soldiers, and a combined military 125,000 strong. Uribe's platform during the 1999 election spoke to his commitment to establish new counter-insurgency initiatives [10]. The external funding continued to grow as President Clinton supported the Pan Colombia plan and President Bush supported the Andean Counter Drug Initiative [9]. The long war against drugs, the evolving nature of insurgent activity that resulted in a 'dirty war' left the country to suffer stasis and war fatigue. Colombia had little choice but to change the philosophy and the rules of engagement of its counter-insurgency. These changes would be realized after the events of September 11, 2001.

By the late 1990s the two major cartels had seen their end. What filled that vacuum proved to be equally as deadly for Colombia, as the "democratization" of the drug trade began, and numerous splinter groups sought the control once held by the cartels [2]. The tremendous income from illicit drug sales continued to fund the insurgency. Labor was in steady supply as there was much land to grow the coca and many poor farmers willing to work for the drug traffickers [7]. To measure the strength of the insurgency during the period when it was construed to be a war against drugs and a struggle for the government to regain control, required evaluating qualitative and quantitative data. Measuring the composition of violence such as the number of kidnappings, executions, massacres, and extortions gives considerable insight to the state of terror most of the civilian population endured [5,9,11]. The nature of the insurgency—land, political power, drug trafficking, what was motivating the insurgents, showed the events in Colombia to now be out of the range of traditional insurgency, (political autonomy, nationalism) and centered on the lucrative drug trade and extortion via threats of damage in the oil regions.

Internal support was also atypical. Although some younger Colombians sought the benefits of joining the insurgency (clothes, food, security), a large portion of the peasants were coerced or bribed into participating [12]. The strength of the individual cartels varied as did their strategic approach as some engaged in military like operations while others behaved like hardened criminals. This created a fearful environment with many of the peasants fleeing to protect themselves and their children from impressment. The intent of the insurgency—financial gain—meant the spoils of the insurgency benefited a few [2]. The negative effects of the intent and the disorderly structure of the cartels resulted in a multiplicity and fragmentation of the groups [2]. With the Uribe Administration the drug traffickers were labeled as terrorists and the state engaged a new philosophy of counter-insurgency. In effect, the Insurgency Index was high, but the detractions to the strength of the insurgency were equally high. Most significant is the fact that the majority of the detractions in the ability to conduct the insurgency were self-inflicted.

2. Post September 11: From war on drugs to war against terrorism

It was during this period that the Colombian Government stepped up its efforts to reassert government control throughout the country. The events during the second half of the 1990s with the death of the cartel leaders (and the disintegration of the cartels) brought the democratization of the drug trade. An increased number of insurgent splinter groups fostered a shift in strategy as to how the war on drugs would have to be conducted. The U.S. attacks of September 11 paved the way for Colombia's President Uribe's to introduce his state-based, hard line approach to insurgency. U.S. Attorney General John Ashcroft supported the position that drug trafficking and terrorism are one and the same. This now meant that the counter insurgency once accepted as the war on drugs would be a part of the war against terrorism.

President Uribe had been elected in 2002 and he and U.S. President George W. Bush were of the same mind in their approach to conducting the counter-insurgency. There were two things that both Presidents wanted to focus on: drugs and oil. U.S. support to secure the oil fields and the commitment of the Uribe administration to budget $ 100 million to equip army battalions to protect the pipelines were realized during this period [2]. The changes that occurred from 2002–2006 are the result of lag-time of the financial input and government policy. More and better trained and equipped counter-insurgency forces made things like expanding land ownership and acquisition of arms understandably more difficult for the insurgents.

Several things occurred with Colombia's domestic policy between 2002 and 2006. The Uribe landside victory served as a mandate to expand war and to adhere to a "Law and Order" rhetoric that incorporated a Democratic Defense and Security Policy [7,13]. In 2002 the U.S. authorized President Uribe to use $ 1.7 billion to change the military balance and contain violence. Partnering with the U.S in the war against terrorism expanded the role, mission, and authority of the Colombian military [2]. The U.S. endorsed a strategy of overwhelming military power and the use of the military for a quicker, effective result versus a diplomatic approach, and President Uribe proceeded with unilateral authority [14]. At the same time the media emphasized images of Bin Laden and the Taliban, serving to introduce the populous to what would now be a war against terrorism [7]. The 1.2% war tax that President Uribe introduced in 2002, coupled with his declaration of emergency powers, yielded over $ 800 million with 70% on defense spending for 2003 (according to the U.S. State Department). He armed his 20,000 peasant soldiers in support of his state-based solution as a way to retain societal support of his policy [6]. By 2004, U.S support reached $ 3.3 billion with special training for 23,000 soldiers [14]. The counter-insurgency strategy included a policy of zero tolerance, i.e. no dialogue with narco-terrorists, and the forfeiture of civil rights for those concerned [2,7]. According to the U.S. State Department, between 2002 and 2006 Colombia saw decreases in homicides by 37%, kidnappings by 78%, terrorist attacks by 63%, and attacks on the country's infrastructure by 60%. President Uribe's approval rating rose with these successes. Colombia's statistics show that in 2004 over 10,000 insurgents were either captured or killed and 3,000 were disarmed [6]. This left the insurgents in a disjointed state with 15,000–20,000 fighters on 105 fronts affecting 40 percent of the country [7]. While President Uribe's constituents may have mixed feelings of support, a 2005 survey indicated

Latin Americans prefer strong, authoritarian governments. In May 2006 he was re-elected with 62 percent of the vote [15]. Also significant to waging this counter-insurgency is the fact that in 2002 he sought to increase the military from 125,000 to 225,000. As of 2004 his military numbered 350,000 [9].

The purpose of the mathematical study below is two-fold. One is to investigate simple population dynamics models to see if their properties are readily adaptable to insurgency modeling. The second is to develop possible analytical solutions that would serve for future models, both in this context and others. Certainly it appears that general mathematical representations of insurgency models using population dynamics are feasible. If data on insurgency strength as a function of time is known but a detailed 'factor analysis' is not possible, such models may still be able to represent the insurgent behavior by means of these mathematical representations.

2.1. Insurgency models 1 and 2: "Forced" predator-prey systems

Given the above background information, what follows is a description of the analytic models (to some extent at least) *as they evolved* during the course of this investigation. As such, (and the models being quite elementary) it may be valuable in a pedagogic context. However, this paper is intended to be of much more than "historical" or educational interest, for it may be that different aspects of these models can be of use in subsequent studies of insurgencies in this and other contexts. Circumstances will determine which sets of equations are the most relevant to a particular situation. The paper should be seen therefore as representing a novel application of elementary systems of ordinary differential equations, each of which may be appropriate for one cultural/economic/political context or another, or at least form a basis for the development of new models. Furthermore, several predictions are made here for broad qualitative behavior of both insurgent and "susceptible" populations in this two-population study.

3. Model 1

In this model, the equations below are formulated as a "forced" version of the classical Lotka–Volterra predator-prey equations, namely

$$\frac{\mathrm{d}S}{\mathrm{d}t} = aS - bI + f_1(t); \tag{1a}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -cI + dS - f_2(t). \tag{1b}$$

where $S(t)$ and $I(t)$ are the "susceptible" (or indigenous) and narco-terrorist sub-populations respectively. Thus here S is considered to be the "prey" population and I is the "predator" population. The constants a, b, c and d are all considered positive for the moment (later, c will be negative), and the functions $f_1(t)$ and $f_2(t)$ are both non-negative [but note the sign of f_2 in Eq. (1b)]. These functions represent the government/military (G/M) policy towards each sub-population, and in an ecological context would correspond to "harvesting" at a rate independent of the sub-populations.

More succinctly, we write this system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} a & -b \\ d & -c \end{pmatrix}\begin{pmatrix} S \\ I \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{1c}$$

or

$$\frac{\mathrm{d}\mathbf{X}_1}{\mathrm{d}t} = \mathbf{A}_1\mathbf{X}_1 + \mathbf{F}_1. \tag{1d}$$

4. Model 2

In this model the forcing terms are proportional to the respective populations, corresponding to a proportional harvesting effort in an ecological context:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = aS - bI + f_1(t)S; \tag{2a}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -cI + dS - f_2(t)I. \tag{2b}$$

Again, this system is equivalently expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} a + f_1 & -b \\ d & -c - f_2 \end{pmatrix}\begin{pmatrix} S \\ I \end{pmatrix}. \tag{2c}$$

or

$$\frac{\mathrm{d}\mathbf{X}_2}{\mathrm{d}t} = \mathbf{A}_2\mathbf{X}_2. \tag{2d}$$

4.1. Choices for $f_1(t)$ and $f_2(t)$

One possible choice for these terms is just a piecewise constant function for each, representing a change of policy (one way or the other) from a prior one, the change occurring at time $t = T$. Thus one might choose two possible "step functions" for each of $f_1(t)$ and $f_2(t)$; either a step *up* or *down* from $f_1 = F_{11}$ to $f_1 = F_{12}$, say, at time $t = T$. The former represents an increased (supportive or positive) policy by the government (and hence military) towards the sub-population S; the latter a reduced or less supportive policy. Similarly, for f_2 one might posit either a step *up* or *down* from $f_2 = F_{21}$ to $f_2 = F_{22}$ at $t = T$. In this case, since the term f_2 in Eq. (1b) is subtractive, the former represents a stronger, more forceful policy towards the terrorist population I, while the latter corresponds to a weaker policy towards the same, both being initiated at time $t = T$. Mathematically, in terms of the Heaviside function,

$$f_1(t) = F_{11}\left[1 - H(t-T)\right] + F_{12}H(t-T), \quad \text{and} \tag{3a}$$

$$f_2(t) = F_{21}\left[1 - H(t-T)\right] + F_{22}H(t-T). \tag{3b}$$

More realistically perhaps for Colombia, in view of the situation described in the introduction, one might choose $f_1(t) \equiv 0$ (i.e. no effective government policy to support the populace S) but

$$f_2(t) = D\left(1 - \mathrm{e}^{-wt}\right), \tag{4}$$

representing, over time, an increasingly aggressive governmental policy towards the population I, increasing from zero at $t = 0$ to a maximum of "D" as $t \to \infty$; clearly w is a measure of how rapidly this limiting maximum is approached.

5. Model 1 revisited

Eliminating $S(t)$ from Eqs. (1a) and (1b), the resulting inhomogeneous differential equation for $I(t)$ is

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} - (\mathrm{Trace}\,\mathbf{A})\frac{\mathrm{d}I}{\mathrm{d}t} + (\mathrm{Det}\,\mathbf{A})I = af_2 + df_1 - f_2' \equiv g_1(t), \tag{5a}$$

or

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + (c-a)\frac{\mathrm{d}I}{\mathrm{d}t} + (bd - ac)I = g_1(t). \tag{5b}$$

Note that the presence of the derivative of $f_2(t)$ in Eq. (5a) for $g_1(t)$ implies that

$$f_2'(t) = (F_{22} - F_{21})\,\delta(t-T), \tag{6}$$

where $\delta(t-T)$ is the Dirac delta "function". [Strictly, both $H(t-T)$ and $\delta(t-T)$ should be regarded as *distributions* in a technical sense, but this need not concern us unduly here.]

An equation similar to (5b) holds for $S(t)$ of course, and details are provided in the Appendix. *Note that Model 2 can be considered a generalization of the unforced (or homogeneous) version of Model 1*. We have already noted that by matching the solutions of two piecewise-constant versions of Model 1 at some time $t = T$ we can effectively incorporate changes of policy towards the I-population in a relatively simple fashion. In light of this the unforced system will be examined first. It is perhaps also worth noting that Model 2 bears very close resemblence to a model developed for *cancer chemotherapy* [17]; this connection should not be too surprising upon reflection: the latter two-compartment model dealt with the effects of chemotherapeutic modalities ("external forcing") on both *normal* and *cancer* cells, so the models are in one sense isomorphic, (apart, that is from the periodic nature of the forcing term in chemotherapy).

6. The unforced equation

The unforced equation for the terrorist population $I(t)$ is

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} - \beta\frac{\mathrm{d}I}{\mathrm{d}t} + \gamma I = 0, \tag{7}$$

where $\beta = \mathrm{Trace}\,\mathbf{A} = a - c$ and $\gamma = \mathrm{Det}\,\mathbf{A} = bd - ac$. The solution, for initial conditions $I(0) = I_0$, $I'(0) = I_0'$ is, for $\lambda > 0$,

$$I(t) = \mathrm{e}^{\beta t/2}(A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t), \tag{8}$$

where

$$\lambda = \gamma - \frac{\beta^2}{4}, \qquad A = I_0 \quad \text{and} \quad B = \lambda^{-1/2}\left(I_0' - \frac{\beta}{2}I_0\right). \tag{9}$$

For $\lambda < 0$ the corresponding solution is

$$I(t) = \mathrm{e}^{\beta t/2}\left(A\cosh\sqrt{|\lambda|}t + B\sinh\sqrt{|\lambda|t}\right), \tag{10}$$

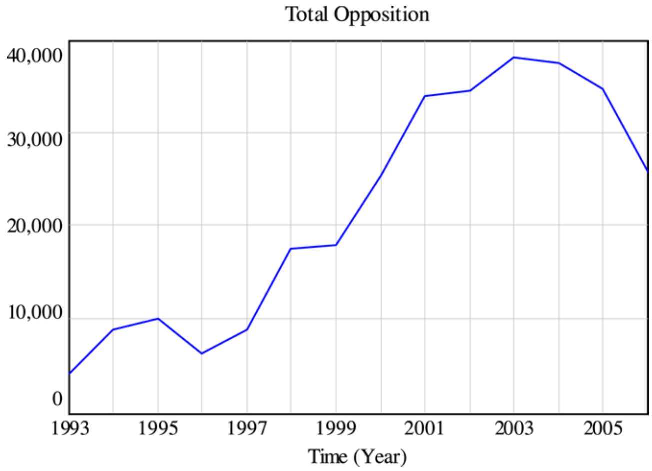

Fig. 1. (Chart provided by the Center for Army Analysis Counterinsurgency Database).

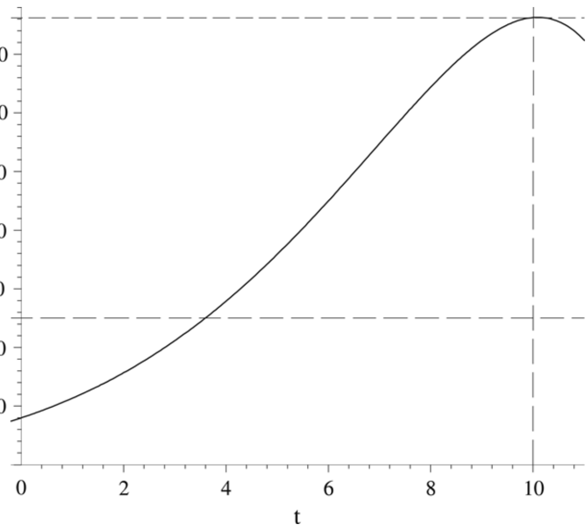

Fig. 2. The graph of $I(t)$ according to Eq. (13).

where now

$$B = |\lambda|^{-1/2}\left(I_0' - \frac{\beta}{2}I_0\right). \tag{11}$$

In the (unlikely) event that λ is *identically* zero,

$$I(t) = \mathrm{e}^{\beta t/2}(A + Bt), \tag{12}$$

where $A = I_0$ and $B = (I_0' - \beta I_0/2)$.

Now we are in a position to examine the relatively limited data on the insurgency in Colombia. Fig. 1 shows a piecewise linear graph of the estimated number of insurgents $I(t)$ from 1993 to 2006, according to the Center for Army Analysis Counterinsurgency Database. This forms the basis for the numerical work that follows.

From this data fit provided by the Center for Army Analysis, we take 1993 as the starting point, i.e. $t = 0$, and $t = 10$ (2003) as the terminal point, so that $I(0) = I_0 = 4000$ and $I(10) = 38,100$. An estimate of I_0' is also readily determined from this graph. For comparison, the solution (8) for $I(t)$ (referred to as the Quasi-predator prey model, or QPPM) is shown in Fig. 2. Specifically, the resulting unforced QPPM function plotted below is

$$I(t) = \mathrm{e}^{0.287t}\left[4000\cos(0.153t) + 2000\sin(0.153t)\right]. \tag{13}$$

In Fig. 2 the horizontal dashed line represents the data for $I = 12,500$, which is the approximate value for $t = 4$ (1997) indicated by the data. It can be seen that the QPPM is a very good qualitative fit. In principle then, we can determine *some* information about the parameters of the QPP model based on the seemingly close fit of this model with the data on the interval [0, 10]. In particular, from the fit $\beta/2 = (a - c)/2 \approx 0.287$, and $\sqrt{\lambda} \approx 0.153$, then

$$\gamma = bd - ac = \lambda + \frac{\beta^2}{4} \approx 0.106,$$

which provides some restrictions on the choices for the parameter set $\{a, b, c, d\}$, but not enough to determine them uniquely.

7. Modification of Model 1

Recall that for the *unforced* Model 1 we were using the following equations, noting that they were in fact a special case of Model 2:

$$\frac{dS}{dt} = aS - bI; \tag{14a}$$

$$\frac{dI}{dt} = -cI + dS. \tag{14b}$$

The rationale for this form of system was that in *ecological* predator-prey models (this linear model being contained in the fully nonlinear version) the predator population (I) in particular decayed in the absence of the prey (S). It was therefore posited that $a > 0, b > 0. c > 0$ and $d > 0$. There is a significant disparity between the two population levels ($S \gtrsim 10^3 I$), and the (rather sparse) available data for the narco-terrorist population $I(t)$ indicates that $c < 0$ at least until 2003 (when $t = T = 10$). Therefore a slight variation of the above set for 1993–2003 is proposed, to reflect this, by using a piecewise unforced predator-prey system. Specifically, for $0 \leq t \leq 10$ (the period 1993–2003) we utilize

$$\frac{dS}{dt} = aS - bI; \tag{15a}$$

$$\frac{dI}{dt} = cI + dS, \tag{15b}$$

where $a > 0, b > 0, c > 0$ and $d > 0$. Note that the negative sign is missing for the term cI in the second equation. Now $\beta = a + c$ and $\gamma = bd + ac$. For the period $t > 10$ (i.e. from 2003 onward) the original form is used, i.e.

$$\frac{dS}{dt} = \tilde{a}S - \tilde{b}I; \tag{16a}$$

$$\frac{dI}{dt} = -\tilde{c}I + \tilde{d}S. \tag{16b}$$

where the new coefficients (denoted by tildes) are in general different from the set in (15a) and (15b). In particular, the per-capita population growth-rate a has decreased from a value of about 1.9% in 1995 to 1.4% in 2007, and is projected to be about 1.2% in 2015. Nevertheless, one average value ($a = 0.019$) will be used for the interval [0, 10] and a lower one ($\tilde{a} = 0.014$) for $t > 10$.Note that the negative sign is re-introduced in the second equation, and $\tilde{a} > 0, \tilde{b} > 0, \tilde{c} > 0$ and $\tilde{d} > 0$, and $\tilde{\beta} = \tilde{a} - \tilde{c}$ and $\tilde{\gamma} = \tilde{b}\tilde{d} - \tilde{a}\tilde{c}$.

As noted above, one of the disadvantages of the earlier QPP models was that there was not enough information to determine the parameter set (a, b, c, d) uniquely. While this is still the case, it is less of a problem than before, because the estimated value of $\lambda \approx 0.02$ indicates that the oscillatory part of the solution in the model is *far less significant* than the exponential part. This follows because the period of oscillation in Eq. (13) is $(2\pi)/0.153 \approx 41$ years, whereas the e-folding time $(0.287)^{-1} \approx 3.5$ years. In particular, an exponential growth model is consistent with the fit for $0 \leq t \leq 10$ and an exponential decay model is consistent with the fit for $t > 10$. As we will show below, this enables us to calculate the *insurgent* per capita growth ($0 \leq t \leq 10$) and decay ($t > 10$) rates c and $\tilde{c}$, with some qualitative information on the relationship between the parameters $b, d, \tilde{b}$ and $\tilde{d}$. In Fig. 3 below, the following functions are depicted:

$$\text{(i) } I(t) = 4000e^{0.225t}, \quad (0 \leq t \leq 10) \quad \text{and} \tag{17a}$$

$$\text{(ii) } I(t) = 38100e^{-0.075(t-10)} \quad (t \geq 10). \tag{17b}$$

It can be seen that is some respects this is a better qualitative fit to the date in Fig. 1 than that provided by Fig. 2 (which is shown for comparison). Since these latest graphs are essentially pure exponential functions, the question can be asked: what advantages does the QPPM have over simple exponential growth and decay? The answer is that in more general situations it is unlikely that the I-population is completely independent of the S-population, although as here, aspects of the model can be "turned off" when the data suggests it. This depends very much of course on the context of the problem.

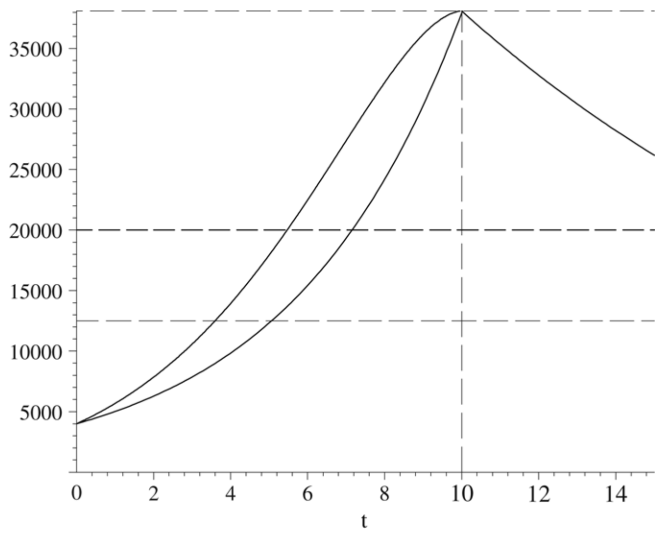

Fig. 3. The piecewise exponential solution (17a) and (17b).

8. Some approximate bounds

To illustrate this point, if the oscillatory part of the solution *over the short term* (about 5–10 years) is negligible, i.e. in the solutions of (15a), (15b) and (16a), (16b) for $I(t)$ we may treat both the parameters λ and $\tilde{\lambda}$ as *approximately* zero (but not identically so, or solution (12) would apply), then (i) $\beta^2 \approx 4\gamma$, i.e.

$$(a+c)^2 \approx 4(bd+ac), \tag{18}$$

and (ii) $\tilde{\beta}^2 = 4\tilde{\gamma}$, i.e.

$$(\tilde{a}-\tilde{c})^2 \approx 4\left(\tilde{b}\tilde{d}-\tilde{a}\tilde{c}\right). \tag{19}$$

Eq. (17) implies that

$$(a-c)^2 \approx 4bd, \quad \text{or} \quad |a-c| \approx 2\sqrt{bd}, \tag{20}$$

and (18) implies that

$$(\tilde{a}+\tilde{c})^2 \approx 4\tilde{b}\tilde{d}, \qquad \text{i.e. } |\tilde{a}+\tilde{c}| \approx 2\sqrt{\tilde{b}\tilde{d}}. \tag{21}$$

We can find the value of a from various sources (such as the online CIA Factbook); the estimated per capita birth rate for 2007 is 20.16 births per 1000 population, the estimated death rate is 5.54 per thousand, and the net migration rate is (−)0.29 per thousand, leading to a net percentage growth rate of 1.43%, or 0.014(3). Now from the exponential growth term on [0, 10], the exponent

$$\beta/2 = (a+c)/2 = 0.225,$$

and for $t > 10$,

$$\tilde{\beta}/2 = (\tilde{a}-\tilde{c})/2 = -0.075.$$

Therefore $a+c = 0.450$ and $\tilde{a}-\tilde{c} = -0.150$. Now $c = 0.450 - a = 0.450 - 0.019 = 0.431$ and $\tilde{c} = \tilde{a} + 0.150 = 0.014 + 0.150 = 0.164$. Then using (20) and (21), the obvious inequality $\tilde{c} < c$ can be written as

$$\tilde{c} \approx 2\sqrt{\tilde{b}\tilde{d}} - \tilde{a} < c \approx 2\sqrt{bd} + a.$$

This implies that

$$\sqrt{\tilde{b}\tilde{d}} \lessapprox \sqrt{bd} + \frac{a+\tilde{a}}{2}. \tag{22}$$

The values for c and $\tilde{c}$ correspond to a 43% growth rate and a 16% decay rate in the insurgent population respectively. To put this in perhaps more stark terms, this means that, in the interval from 1993 to 2003, according to this model, for every 100 insurgents at a given time, one year later there were, on the average, about 143; beyond 2003, this model predicts that for every hundred, one year later there would be 86. These figures, of course, should not be taken too literally at this stage, pending further validation.

9. A decoupled system

The parameter b is a *relative* per capita population decay rate due to the insurgency, i.e. if we write Eq. (15a) as

$$\frac{1}{S}\frac{\mathrm{d}S}{\mathrm{d}t} = a - b\frac{I}{S},$$

and typically, as mentioned earlier, the ratio $I/S \approx 0.001$. Suppose that the parameters d and $\tilde{d}$ are both zero; that is, the relative per capita growth rate of the insurgent population from the susceptible population is zero. Then our generic system becomes

$$\frac{\mathrm{d}S}{\mathrm{d}t} = aS - bI; \tag{23a}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = cI, \tag{23b}$$

for $0 \le t \le 10$ and

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \tilde{a}S - bI; \tag{24a}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\tilde{c}I, \tag{24b}$$

for $t > 10$, where we have assumed that $b = \tilde{b}$. The advantage, mathematically at least, of these equations is that there is a readily available analytic solution because the I-population is decoupled from the S-population. Consider (23a), (23b) first: clearly $I(t) = I_0 \mathrm{e}^{ct}$, and therefore

$$\frac{\mathrm{d}S}{\mathrm{d}t} - aS = -bI_0 \mathrm{e}^{ct}. \tag{25}$$

This nonhomogeneous first-order equation has the integrating factor e^{-at}, so the general solution is

$$S(t) = K\mathrm{e}^{at} - \frac{bI_0}{c-a}\mathrm{e}^{ct}, \tag{26}$$

K being a constant of integration with value

$$K = S_0 + \frac{bI_0}{c-a}.$$

Thus

$$S(t) = S_0\mathrm{e}^{at} + \frac{bI_0}{c-a}(\mathrm{e}^{at} - \mathrm{e}^{ct}), \quad 0 \le t \le 10. \tag{27}$$

Since $S(t) \approx 4 \times 10^7$, we write the solution in terms of this quantity by setting $S_0 = 1$. For $t > 10$, since $a \to \tilde{a}$ and $c \to -\tilde{c}$, the solution, continuous at $t = 10$, is

$$S(t) = S(10)\left(\mathrm{e}^{\tilde{a}(t-10)} - \frac{bI_0}{\tilde{c}+\tilde{a}}(\mathrm{e}^{\tilde{a}(t-10)} - \mathrm{e}^{-\tilde{c}(t-10)})\right), \quad t > 10. \tag{28}$$

The graph of this function is shown in Fig. 4 for arbitrarily chosen values of b.

Note that, depending on the value of b, the model predicts a gradual slow-down in the growth rate of the S-population, followed by anything from a small to a severe drop (resulting from the indirect effects of terrorism on this population), followed by an increase in the S-population, resulting from the change in policy in 2003 ($t = 10$), described by the equation set (24a), (24b). It will be interesting to see if this trend is in fact realized.

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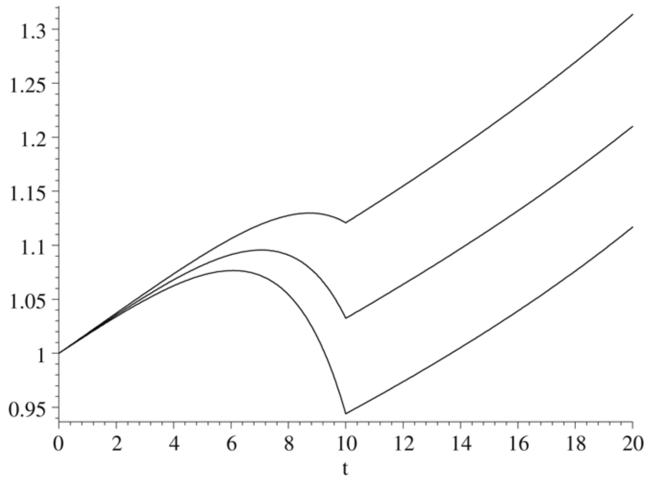

Fig. 4. The solution (28) for $b = 0.5$, $b = 1$, and $b = 1.5$ (upper to lower graph).

Appendix

A.1. Model 1: Analytic solutions

For $f_1(t) = 0$, the solution to the non-homogeneous differential equation

$$I''(t) - \beta I'(t) + \gamma I(t) = af_2(t) - f_2'(t) = D\left[a - (a+w)\,\mathrm{e}^{-wt}\right] \tag{A.1}$$

with initial conditions $I(0) = I_0$, $I'(0) = I_0'$ is, for $\lambda > 0$

$$I(t) = \mathrm{e}^{\beta t/2}(A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t) + \frac{aD}{\gamma} - \frac{D(a+w)\mathrm{e}^{-wt}}{w^2+\beta w+\gamma}, \tag{A.2}$$

where $w^2 + \beta w + \gamma \neq 0$ (i.e. $w \neq -\mu$), and

$$\lambda = \gamma - \frac{\beta^2}{4}, \qquad A = I_0 - \frac{aD}{\gamma} + \frac{D(a+w)}{w^2+\beta w+\gamma} \quad \text{and}$$

$$B = \lambda^{-1/2}\left(I_0' - \frac{\beta}{2}I_0 + \frac{a\beta D}{2\gamma} - \frac{D(a+w)\,(w+\beta/2)}{w^2+\beta w+\gamma}\right). \tag{A.3}$$

For $\lambda < 0$ the corresponding solution is

$$\mathrm{e}^{\beta t/2}(A\cosh\sqrt{|\lambda|}t + B\sinh\sqrt{|\lambda|}t) + \frac{aD}{\gamma} - \frac{D(a+w)\mathrm{e}^{-wt}}{w^2+\beta w+\gamma}, \tag{A.4}$$

where in the coefficient B, the factor $\lambda^{-1/2}$ is replaced by $|\lambda|^{-1/2}$.

The corresponding equation for the susceptible population $S(t)$ for $f_1(t) = 0$ is

$$S''(t) - \beta S'(t) + \gamma S(t) = bf_2(t) = bD\left(1 - \mathrm{e}^{-wt}\right), \tag{A.5}$$

with solution

$$S(t) = \mathrm{e}^{\beta t/2}(\tilde{A}\cos\sqrt{\lambda}t + \tilde{B}\sin\sqrt{\lambda}t) + \frac{bD}{\gamma} - \frac{bD\mathrm{e}^{-wt}}{w^2+\beta w+\gamma}, \tag{A.6}$$

where

$$\tilde{A} = S_0 - \frac{bD}{\gamma} + \frac{bD}{w^2+\beta w+\gamma} \quad \text{and}$$

$$\tilde{B} = \lambda^{-1/2}\left(S_0' - \frac{\beta}{2}S_0 + \frac{b\beta D}{2\gamma} - \frac{bD\,(w+\beta/2)}{w^2+\beta w+\gamma}\right). \tag{A.7}$$

As above, for $\lambda < 0$ the corresponding solution is

$$\mathrm{e}^{\beta t/2}(\tilde{A}\cosh\sqrt{|\lambda|}t + \tilde{B}\sinh\sqrt{|\lambda|}t) + \frac{bD}{\gamma} - \frac{bD\mathrm{e}^{-wt}}{w^2+\beta w+\gamma}, \tag{A.8}$$

where in the coefficient $\tilde{B}$, the factor $\lambda^{-1/2}$ is replaced by $|\lambda|^{-1/2}$. For $\lambda = 0$,

$$I(t) = e^{\beta t/2}(A + Bt) + \frac{aD}{\gamma} - \frac{D(a+w)e^{-wt}}{w^2 + \beta w + \gamma}, \tag{A.9}$$

where

$$A = I_0 - \frac{aD}{\gamma} + \frac{D(a+w)}{w^2 + \beta w + \gamma} \quad \text{and}$$
$$B = \left(I_0' - \frac{\beta}{2} I_0 + \frac{a\beta D}{2\gamma} - \frac{D(a+w)\,(w+\beta/2)}{w^2+\beta w+\gamma}\right), \tag{A.10}$$

and similarly

$$S(t) = e^{\beta t/2}(\tilde{A} + \tilde{B}t) + \frac{bD}{\gamma} - \frac{bDe^{-wt}}{w^2 + \beta w + \gamma}, \tag{A.11}$$

where

$$\tilde{A} = S_0 - \frac{bD}{\gamma} + \frac{bD}{w^2 + \beta w + \gamma} \quad \text{and}$$
$$\tilde{B} = \left(S_0' - \frac{\beta}{2} S_0 + \frac{b\beta D}{2\gamma} - \frac{bD\,(w+\beta/2)}{w^2+\beta w+\gamma}\right). \tag{A.12}$$

A.2. Model 2: An analytic solution

For the choices of forcing function

$$f_1(t) = 0, \quad f_2(t) = D\left(1 - e^{-wt}\right),$$

the governing differential equation from Model II for the insurgent (or narco-terrorist) population $I(t)$ is:

$$I''(t) + \left[A - De^{-wt}\right] I'(t) + \left[B + Ce^{-wt}\right] I(t) = 0, \tag{A.13}$$

where $A = D - \beta$, $B = \gamma - aD$ and $C = D\,(a + w)$. The corresponding differential equation for the susceptible population $S(t)$ is formally the same, i.e.

$$S''(t) + \left[A - De^{-wt}\right] S'(t) + \left[B + \hat{C}e^{-wt}\right] S(t) = 0, \tag{A.14}$$

where $\hat{C} = C + aDe^{-wt}$, so the analysis below carries over to this equation without loss of generality. Returning, then, to Eq. (A.13), we proceed to manipulate it into a standard form via a squence of variable transformations. First, a change of independent variable is in order. If $\xi = e^{-wt}$, then the operators

$$\frac{d}{dt} = \xi'(t)\frac{d}{d\xi} \to -w\xi\frac{d}{d\xi}, \quad \text{and} \quad \frac{d^2}{dt^2} \to w^2\xi\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right) = w^2\left[\xi^2\frac{d^2}{d\xi^2} + \xi\frac{d}{d\xi}\right].$$

After some reduction, Eq. (A.13) becomes

$$\xi^2 I''(\xi) + \xi\left[Dw^{-1}\xi + \left(1 - Aw^{-1}\right)\right] I'(\xi) + w^{-2}\left[B + C\xi\right] I(\xi) = 0, \quad \text{or} \tag{A.15a}$$

$$\xi^2 I''(\xi) + \xi\left[\tilde{D}\xi + \tilde{A}\right] I'(\xi) + \left[\tilde{B} + \tilde{C}\xi\right] I(\xi) = 0 \tag{A.15b}$$

(Strictly speaking, the dependent variable $I(t)$, when referred to a *different* independent variable (ξ), should be itself redefined, i.e. $I(t) \to \tilde{I}(\xi)$, but we shall not require that here.) At this stage, we compare this with the form

$$x^2 y'' + x\left[ax^n + b\right] y' + \left[\alpha x^{2n} + \beta x^n + \gamma\right] y = 0, \tag{A.16}$$

and make the identifications $n = 1$; $a = \tilde{D}$; $b = \tilde{A}$; $\alpha = 0$; $\beta = \tilde{C}$; and $\gamma = \tilde{B}$. Now let k be a root of the quadratic equation

$$k^2 + \left(\tilde{A} - 1\right) k + \tilde{B} = 0, \tag{A.17}$$

and make the change of variable $y = \eta(x)\,x^k$ in Eq. (A.16) (which is now formally identical to (A.15b)), i.e. in

$$x^2 y'' + x\left[\tilde{D}x + \tilde{A}\right] y' + \left[\tilde{C}x + \tilde{B}\right] y = 0. \tag{A.18}$$

After some further algebraic reduction and simplification, we arrive at the differential equation

$$x\eta''(x) + \left[\tilde{D}x + 2k + \tilde{A}\right]\eta'(x) + \left[\tilde{D}k + \tilde{C}\right]\eta(x) = 0. \tag{A.19}$$

This can be compared with the standard form

$$(a_2x + b_2)\eta'' + [a_1x + b_1]\,\eta' + [a_0x + b_0]\,\eta = 0. \tag{A.20}$$

Clearly $a_2 = 1$, $b_2 = 0$, $a_1 = \tilde{D}$, $b_1 = 2k + \tilde{A}$, $a_0 = 0$ and $b_0 = \tilde{D}k + \tilde{C}$. Now some further manipulations are necessary before we can write down an analytic solution in closed form.

1. Let $h = (\tilde{D}^2 - \tilde{D})/2$, $A(h) = 2h + \tilde{D} = \tilde{D}^2$, $B(h) = \left(2k + \tilde{A}\right)h + \tilde{D}k + \tilde{C} = k\tilde{D}^2 + \tilde{A}(\tilde{D}^2 - \tilde{D})/2 + \tilde{C}$.
2. Next let $\lambda = -\tilde{D}^{-2}$, $\nu = x/\lambda = -\tilde{D}^2x$, $\tilde{a} = B(h)/A(h)$.
3. Then the solution of Eq. (A.20) in this nomenclature is

$$\eta(x) = \mathrm{e}^{hx}\mathcal{J}\left(\tilde{a}, 2k + \tilde{A}; -\tilde{D}^2x\right), \tag{A.21}$$

where $\mathcal{J}\left(\tilde{a}, \tilde{b}; \tilde{x}\right)$ is the general solution of the *confluent* (or degenerate) *hypergeometric* equation [18]

$$\tilde{x}z'' + \left(\tilde{b} - \tilde{x}\right)z' - \tilde{a}z = 0. \tag{A.22}$$

In terms of the original variables in Eq. (A.13), we can distribute all the variable changes throughout to get

$$I(\xi) = I\left(\mathrm{e}^{-wt}\right) = \left[\exp\left(h\mathrm{e}^{-wt}\right)\right]\mathcal{J}\left(\tilde{a}, 2k + \tilde{A}; -\tilde{D}^2\mathrm{e}^{-wt}\right). \tag{A.23}$$

Note that this solution is in terms of the "compressed" variable $\xi = \mathrm{e}^{-wt}$, so $I(t)$ must be found by an appropriate degree of "stretching", i.e. by requiring in Eq. (A.23) that $t \to -(\ln t)/w$.

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