4-1990

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Original Publication Citation

Lasseigne, D. G., & Olmstead, W. E. (1990). Stability of a viscoelastic burgers flow. *SIAM Journal on Applied Mathematics, 50*(2), 352-360. doi:10.1137/0150022

SIAM J. APPL. MATH.
Vol. 50, No. 2, pp. 352-360, April 1990

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STABILITY OF A VISCOELASTIC BURGERS FLOW*

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This paper is dedicated to Edward L. Reiss on the occasion of his 60th birthday.

Abstract. The system of equations proposed by Burgers to model turbulent flow in a channel is extended to include viscoelastic affects. The stability and bifurcation properties are examined in the neighborhood of the critical Reynolds number. For highly elastic fluids, the bifurcated state is periodic with a shift in frequency.

Key words. bifurcation, stability, viscoelastic fluids

AMS(MOS) subject classifications. 35K55, 45KO5

1. Introduction. In some recent work of Olmstead et al. [8], the model problem

$$u_t(x, t) = \int_{-\infty}^{t} K(t, s) u_{xx}(x, s)\, ds + \bar{R}u(x, t) - u^3(x, t), \tag{1.1}$$

$$u(0, t) = u(\pi, t) = 0, \tag{1.2}$$

was proposed to study the bifurcation and stability properties associated with a viscoelastic fluid. The interpretation of (1.1), (1.2) is that $u(x, t)$ is the velocity perturbation for a Bénard type flow in a channel ($0 \leqq x \leqq \pi$). The viscoelastic force of a non-Newtonian fluid with memory is modeled by the integral term in (1.1). The constant $\bar{R} \geqq 0$, which corresponds to the Rayleigh number, becomes the bifurcation parameter in the analysis of (1.1), (1.2).

The introduction of the model problem (1.1), (1.2) was motivated by the difficulty of identifying viscoelastic phenomena from the full-fledged Navier-Stokes equations (e.g., Joseph [6] and Eltayeb [3]). The analysis of [8] was carried out for a Jeffreys kernel,

$$K(t, s) = \frac{1-\delta}{\lambda} \exp\left(-\frac{t-s}{\lambda}\right) + 2\delta \underline{\delta}\,(t-s), \quad \lambda \geqq 0, \quad 0 \leqq \delta < 1, \tag{1.3}$$

where λ is the relaxation time, δ is the ratio of a retardation time to the relaxation time, and $\underline{\delta}(t-s)$ is the Dirac delta function. The limiting case in which $\delta \to 0$, $\lambda \to 0$ is interpreted as a Newtonian fluid.

The results of [8] show that, for fixed δ, the relaxation time λ controls the behavior of the bifurcation parameter $\bar{R}$. As $\bar{R}$ is increased, the first bifurcation is to a steady state for λ sufficiently small and to a periodic state for λ sufficiently large. This behavior is consistent with the linear stability analysis of Sokolov and Tanner [9] for a more realistic formulation of the Bénard problem.

Our goal here is to extend the (weakly) nonlinear bifurcation and stability analysis of [8] to a fluid mechanical model which more closely resembles the Navier-Stokes

* Received by the editors June 12, 1989; accepted for publication (in revised form) June 21, 1989. This research was supported in part by the National Science Foundation under grant DMS-8700962.

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problem. In place of (1.1) we consider the system

$$U_t(t) = P - U(t) - \frac{2}{\pi R^*}\int_0^\pi u^2(\xi, t)\, d\xi, \tag{1.4a}$$

$$u_t(x, t) + 2u(x, t)u_x(x, t) = \int_{-\infty}^t K(t, s)u_{xx}(x, s)\, ds + R^* U(t)u(x, t). \tag{1.4b}$$

The Newtonian version of this model, where the integral term in (1.4b) is replaced by $u_{xx}(x, t)$, was proposed by Burgers [1] as a model for turbulent flow in a channel $(0 \leqq x \leqq \pi)$. In that model, $U(t)$ denotes a mean velocity driven by the pressure gradient P, while R^* represents the Reynolds number. The integral term in (1.4a) plays the role of the Reynolds stress which governs energy conversion between the mean flow and the velocity perturbation $u(x, t)$. Various aspects of the stability and bifurcation properties for the Newtonian model have been investigated by Stuart [10], Eckhaus [2], Golia and Abel [4], Horgan and Olmstead [5], and Olmstead and Davis [7].

The analysis here of the viscoelastic Burgers system (1.4) will also be for the Jeffreys kernel (1.3). It is convenient to define

$$S(x, t) = \frac{1-\delta}{\lambda}\int_{-\infty}^t \exp\left(-\frac{t-s}{\lambda}\right) u(x, s)\, ds, \tag{1.5}$$

$$V(t) = R^*[P - U(t)] \tag{1.6}$$

so that (1.4) can be replaced by the system

$$u_t = S_{xx} + \delta u_{xx} + Ru - uV - 2uu_x, \tag{1.7a}$$

$$\lambda S_t = -S + (1-\delta)u, \tag{1.7b}$$

$$V_t = -V + \frac{2}{\pi}\int_0^\pi u^2(\xi, t)\, d\xi, \tag{1.7c}$$

where $R = PR^*$ will be the bifurcation parameter.

The corresponding system for (1.1), treated in [8], involves the same linear terms that appear in (1.7a) and (1.7b). The two nonlinear terms in (1.7a) are replaced by $-u^3$, and (1.7c) is absent.

For the analysis that follows, it will be convenient to define the system operator

$$L_R = \begin{bmatrix} \frac{\partial}{\partial t} - \delta\frac{\partial^2}{\partial x^2} - R & -\frac{\partial^2}{\partial x^2} & 0 \\ -(1-\delta) & \lambda\frac{\partial}{\partial t} & 0 \\ 0 & 0 & \frac{\partial}{\partial t} + 1 \end{bmatrix}. \tag{1.8}$$

Then (1.7) can be expressed in the form

$$L_R \begin{bmatrix} u \\ S \\ V \end{bmatrix} = \begin{bmatrix} -uV - 2uu_x \\ 0 \\ 2/\pi \int_0^\pi u^2(\xi, t)\, d\xi \end{bmatrix}. \tag{1.9}$$

We will investigate the solution of (1.9), subject to the boundary conditions

$$u(0, t) = u(\pi, t) = S(0, t) = S(\pi, t) = 0, \tag{1.10}$$

and initial conditions

(1.11) $$u(x,0)=\varepsilon h(x),\quad S(x,0)=0,\quad V(0)=0,$$

where $\varepsilon h(x)$, $0<\varepsilon\ll 1$, is the given initial velocity perturbation. The other conditions in (1.11) correspond to a null history; that is, for all $t<0$ the mean velocity in the channel is constant ($U\equiv P$) with no perturbation ($u\equiv 0$).

2. Linear stability analysis. By linearizing the system (1.7), and introducing normal modes, we can identify the branch points were bifurcation can occur for the nonlinear problem. We consider

(2.1) $$L_R\begin{bmatrix}\bar{u}_n\, e^{(\sigma_n+i\omega_n)t}\sin nx\\ \bar{S}_n\, e^{(\sigma_n+i\omega_n)t}\sin nx\\ \bar{V}_n\, e^{(\sigma_n+i\omega_n)t}\end{bmatrix}=0,\qquad n=1,2,\cdots,$$

where the σ_n and ω_n are real. It follows that $\bar{V}_n=0$, $n=1,2,\cdots$, while for arbitrary $\bar{u}_n$ and $\bar{S}_n$ there is a characteristic equation which separates into real and imaginary parts as

(2.2a) $$\lambda(\sigma_n^2-\omega_n^2)+\sigma_n(1-\lambda R+n^2\lambda\delta)+n^2-R=0,$$

(2.2b) $$\omega_n(1+2\lambda\sigma_n+\lambda n^2\delta-\lambda R)=0,\qquad n=1,2\cdots.$$

Solving (2.2) yields two distinct cases. For branch points corresponding to *steady bifurcations*, we set $\omega_n=0$, $\sigma_n=0$ and find

(2.3) $$R_n^{(S)}=n^2,\qquad n=1,2,\cdots.$$

For branch points corresponding to *periodic bifurcations*, we set $\omega_n\neq 0$, $\sigma_n=0$ and find

(2.4) $$R_n^{(p)}=\frac{1}{\lambda}+n^2\delta,\qquad \omega_n=\frac{1}{\lambda}[(1-\delta)n^2\lambda-1]^{1/2},$$

with the stipulation that

(2.5) $$\lambda>\lambda_n=\frac{1}{(1-\delta)n^2}.$$

These results suggest that the critical value of R where the null solution loses stability is given by

(2.6) $$R_c=\begin{cases}R_1^{(S)}=1, & \lambda<\lambda_1,\\ R_1^{(p)}=1-\left(\dfrac{1}{\lambda_1}-\dfrac{1}{\lambda}\right), & \lambda>\lambda_1,\end{cases}$$

where $\lambda_1=1/(1-\delta)$. The physical implication of (2.6) is that λ_1 represents a measure of how much elasticity the fluid must have for the loss of stability to have an oscillatory nature.

Since the linearized versions of (1.4) and (1.1) are identical, it is to be expected that the bifurcation sites given by (2.3) and (2.4) are the same as those found in [8]. However, we can anticipate some differences in the evolutionary nature of the nonlinear problem.

3. Bifurcation analysis for $\lambda<\lambda_1$. In view of (2.6), we anticipate the nonlinear problem (1.7) to exhibit its first bifurcation, as R increases from zero, at $R=R_1^{(S)}=1$ when $\lambda<\lambda_1$. Moreover, we expect that an initial perturbation of the null solution will evolve slowly to a steady birucated state for $R > R_1^{(S)}$. To analyze that bifurcation, we set

$$R = R_1^{(S)} + \varepsilon^2 = 1 + \varepsilon^2, \tag{3.1}$$

where ε is the same small parameter that scales the initial data in (1.11). We also introduce the slow time

$$\tau = \varepsilon^2 t, \tag{3.2}$$

and seek an asymptotic solution of (1.9)-(1.11) through the expansion

$$\begin{aligned} u &= \sum_{j=0}^{\infty} \varepsilon^{j+1} u_j(x, t, \tau), \\ S &= \sum_{j=0}^{\infty} \varepsilon^{j+1} S_j(x, t, \tau), \\ V &= \sum_{j=0}^{\infty} \varepsilon^{j+1} V_j(t, \tau). \end{aligned} \tag{3.3}$$

By introducing (3.1)-(3.3), our problem (1.9)-(1.11) takes the form

$$L_1 \begin{bmatrix} u \\ S \\ V \end{bmatrix} = \begin{bmatrix} -uV - 2uu_x \\ 0 \\ 2/\pi \int_0^\pi u^2 \, d\xi \end{bmatrix} - \varepsilon^2 \begin{bmatrix} u_\tau - u \\ \lambda S_\tau \\ V_\tau \end{bmatrix}, \tag{3.4}$$

$$u_j = S_j = 0 \quad \text{for } j \geqq 0 \quad \text{at } x = 0, \pi, \tag{3.5}$$

$$u_0 = h, \quad S_0 = V_0 = 0, \quad u_j = S_j = V_j = 0 \quad \text{for } j \geqq 1 \quad \text{at } t = \tau = 0. \tag{3.6}$$

At $O(\varepsilon)$ it follows from (3.4) that

$$L_1 \begin{bmatrix} u_0 \\ S_0 \\ V_0 \end{bmatrix} = 0. \tag{3.7}$$

Upon solving (3.7) subject to (3.5) and (3.6), while retaining only the terms that survive as $t \to \infty$, we find

$$\begin{bmatrix} u_0 \\ S_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} \lambda_1 C_0(\tau) \sin x \\ C_0(\tau) \sin x \\ 0 \end{bmatrix} + O(e^{-\gamma t}) \tag{3.8}$$

for some $\gamma > 0$. Here

$$C_0(\tau) = \frac{A_0(\tau) - \lambda B_0(\tau)}{\lambda_1 - \lambda}, \tag{3.9}$$

where

$$\begin{aligned} A_0(\tau) &= \frac{2}{\pi} \int_0^\pi u_0(x, 0, \tau) \sin x \, dx, \\ A_0(0) &= \frac{2}{\pi} \int_0^\pi h(x) \sin x \, dx = h_0 \end{aligned} \tag{3.10a}$$

$$B_0(\tau) = \frac{2}{\pi} \int_0^\pi S_0(x, 0, \tau) \sin x \, dx, \qquad B_0(0) = 0. \tag{3.10b}$$

At $O(\varepsilon^2)$ it follows from (3.4) that

$$L_1\begin{bmatrix} u_1 \\ S_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} -\lambda_1^2 C_0^2(\tau)\sin 2x \\ 0 \\ \lambda_1^2 C_0^2(\tau) \end{bmatrix} + O(e^{-\gamma t}). \tag{3.11}$$

Upon solving (3.11) subject to (3.5) and (3.6), while retaining only the terms that survive as $t\to\infty$, we find

$$\begin{bmatrix} u_1 \\ S_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} \lambda_1[C_1(\tau)\sin x - \frac{1}{3}\lambda_1 C_0^2(\tau)\sin 2x] \\ C_1(\tau)\sin x - \frac{1}{3}\lambda_1 C_0^2(\tau)\sin 2x \\ \lambda_1^2 C_0^2(\tau) \end{bmatrix} + O(\varepsilon^{-\gamma t}). \tag{3.12}$$

At $O(\varepsilon^3)$ it follows from (3.4) that

$$L_1\begin{bmatrix} u_2 \\ S_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} G \\ Q \\ 0 \end{bmatrix} + O(e^{-\gamma t}), \tag{3.13}$$

where

$$G(t,\tau,x) = \{-\lambda_1 C_0'(\tau) + C_0(\tau) - \lambda_1^3 C_0^3(\tau)[1-\tfrac{4}{3}(\cos 2x + \cos^2 x)] - 4\lambda_1 C_1(\tau) C_0(\tau)\cos x\}\sin x, \tag{3.14}$$

$$Q(t,\tau,x) = -\lambda C_0'(\tau)\sin x. \tag{3.15}$$

For the solution of (3.13), subject to (3.5) and (3.6), to remain bounded in t, we require the solvability condition,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^\pi [G(s,\tau,x) - Q(s,\tau,x)]\sin x\,dx\,ds = 0. \tag{3.16}$$

This yields the nonlinear differential equation

$$(\lambda_1-\lambda)C_0'(\tau) - \lambda_1 C_0(\tau) + \tfrac{4}{3}\lambda_1^3 C_0^3(\tau) = 0, \tag{3.17}$$

subject to the initial condition $C_0(0) = h_0/(\lambda_1-\lambda)$ from (3.9). It follows that

$$C_0(\tau) = \sqrt{3}\,h_0\{[3(\lambda_1-\lambda) - 4h_0^2\lambda_1^2]\,e^{-2\lambda_1\tau/(\lambda_1-\lambda)} + 4\lambda_1^2 h_0^2\}^{-1/2}. \tag{3.18}$$

This, together with (3.8), determines the slow evolution toward the steady bifurcated state. It follows that as $\tau\to\infty$,

$$\begin{bmatrix} u \\ S \\ V \end{bmatrix} \sim \begin{bmatrix} \lambda_1 \\ 1 \\ 0 \end{bmatrix}\frac{\sqrt{3}}{2\lambda_1}\varepsilon\sin x + O(\varepsilon^2). \tag{3.19}$$

4. Bifurcation analysis for $\lambda > \lambda_1$. As suggested by (2.6), when the fluid elasticity is sufficiently large, i.e., $\lambda > \lambda_1$, the nonlinear problem (1.7) will exhibit its first bifurcation at $R = R_1^{(p)} < 1$. Moreover, we expect that an initial perturbation of the null solution will evolve slowly to a periodic bifurcated state for $R > R_1^{(p)}$. To analyze that bifurcation, we set

$$R = R_1^{(p)} + \varepsilon^2 = 1 - \left(\frac{1}{\lambda_1} - \frac{1}{\lambda}\right) + \varepsilon^2. \tag{4.1}$$

As in § 3, we introduce the slow time $\tau = \varepsilon^2 t$ and seek an asymptotic solution of (1.9)-(1.11) in the form of the expansions (3.3).

For this case, our problem (1.9)-(1.11) takes the form (3.4) with L_1 replaced by $L_{1+(1/\lambda)-(1/\lambda_1)}$. The boundary conditions (3.5) and initial condition (3.6) again apply.

At $O(\varepsilon)$ it follows that

$$L_{1+(1/\lambda)-(1/\lambda_1)}\begin{bmatrix} u_0 \\ S_0 \\ V_0 \end{bmatrix}=0. \tag{4.2}$$

Upon solving (4.2) subject to (3.5) and (3.6), while retaining only the terms that survive as $t\to\infty$, we find

$$\begin{bmatrix} u_0 \\ S_0 \\ V_0 \end{bmatrix}=\begin{bmatrix} \cos\omega_1 t+\dfrac{1}{\lambda\omega_1}\sin\omega_1 t & -\dfrac{1}{\omega_1}\sin\omega_1 t & 0 \\ \dfrac{1}{\lambda_1\lambda\omega_1}\sin\omega_1 t & \cos\omega_1 t-\dfrac{1}{\lambda\omega_1}\sin\omega_1 t & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} A_0(\tau) \\ B_0(\tau) \\ 0 \end{bmatrix}\sin x + O(e^{-\gamma t}). \tag{4.3}$$

Considerable simplification of (4.3), as well as other results to follow is obtained by introducing the polar representation

$$A_0(\tau)=R(\tau)\sin\Psi(\tau) \tag{4.4a}$$

$$B_0(\tau)=\omega_1 R(\tau)\left[\frac{1}{\lambda\omega_1}\sin\Psi(\tau)-\cos\Psi(\tau)\right]. \tag{4.4b}$$

Then (4.3) becomes

$$\begin{bmatrix} u_0 \\ S_0 \\ V_0 \end{bmatrix}=\begin{bmatrix} \sin[\omega_1 t+\Psi(\tau)] \\ \dfrac{1}{\lambda}\sin[\omega_1 t+\Psi(\tau)]-\omega_1\cos[\omega_1 t+\Psi(\tau)] \\ 0 \end{bmatrix} R(\tau)\sin x+O(e^{-\gamma t}). \tag{4.5}$$

At $O(\varepsilon^2)$ it follows that

$$L_{1+(1/\lambda)-(1/\lambda_1)}\begin{bmatrix} u_1 \\ S_1 \\ V_1 \end{bmatrix}=\begin{bmatrix} -\sin 2x \\ 0 \\ 1 \end{bmatrix} R^2(\tau)\sin^2[\Phi(t,\tau)]+O(e^{-\gamma t}), \tag{4.6}$$

where

$$\Phi(t,\tau)=\omega_1 t+\Psi(\tau). \tag{4.7}$$

Upon solving (4.6) subject to (3.5) and (3.6), while retaining only the terms that survive as $t\to\infty$, we find

$$\begin{bmatrix} u_1 \\ S_1 \\ V_1 \end{bmatrix}=\begin{bmatrix} -\{a+b\cos 2\Phi(t,\tau)+c\sin 2\Phi(t,\tau)\}\sin 2x \\ -\{\bar{a}+\bar{b}\cos 2\Phi(t,\tau)+\bar{c}\sin 2\Phi(t,\tau)\}\sin 2x \\ 1-(1+4\omega_1^2)^{-1}\cos 2\Phi(t,\tau)-2\omega_1(1+4\omega_1^2)^{-1}\sin 2\Phi(t,\tau) \end{bmatrix} R^2(\tau)/2 + O(e^{-\gamma t}), \tag{4.8}$$

where

$$a=\frac{1}{3+\lambda\omega_1^2},\quad b=\frac{-\alpha(1+4\lambda^2\omega_1^2)}{\alpha^2+36\omega_1^2},\quad c=\frac{6\omega_1(1+4\lambda^2\omega_1^2)}{\alpha^2+36\omega_1^2}$$

$$\bar{a}=\frac{1-\delta}{3+\lambda\omega_1^2},\quad \bar{b}=\frac{b-2\lambda\omega_1 c}{\lambda_1(1+4\lambda^2\omega_1^2)},\quad \bar{c}=\frac{2\lambda\omega_1 b+c}{\lambda_1(1+4\lambda^2\omega_1^2)},\tag{4.9}$$

$$\alpha=(1+4\lambda^2\omega_1^2)\left(3-\frac{3}{\lambda_1}-\frac{1}{\lambda}\right)+\frac{4}{\lambda_1}.$$

At $O(\varepsilon^3)$ it follows that

$$L_{1+(1/\lambda)-(1/\lambda_1)}\begin{bmatrix}u_2\\ S_2\\ V_2\end{bmatrix}=\begin{bmatrix}G\\ Q\\ 0\end{bmatrix}+O(e^{-\gamma t}),\tag{4.10}$$

where

$$\begin{aligned}G=\{&-R'\sin\Phi+R(\sin\Phi-\Psi'\cos\Phi)\\ &-R^3[\tfrac{1}{2}-2(a+b\cos 2\Phi+c\sin 2\Phi)(\cos 2x+\cos^2 x)]\sin\Phi\\ &+\frac{R^3}{2}(1+4\omega_1^2)^{-1}(\cos 2\Phi+2\omega_1\sin 2\Phi)]\}\sin x\end{aligned}\tag{4.11}$$

$$Q=\{-R'(\sin\Phi-\lambda\omega_1\cos\Phi)-R\Psi'(\cos\Phi+\lambda\omega_1\sin\Phi)\}\sin x.$$

For the solution of (4.10), subject to (3.5) and (3.6), to remain bounded in t, we require the solvability conditions (see [8])

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^\pi\left\{G(s,\tau,x)\cos\Phi(s,\tau)+\frac{1}{\lambda\omega_1}[Q(s,\tau,x)-G(s,\tau,x)]\right.\\ &\qquad\left.\sin\Phi(s,\tau)\right\}\sin x\,dx\,ds=0,\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^\pi\left\{G(s,\tau,x)\sin\Phi(s,\tau)+\frac{1}{\lambda\omega_1}[Q(s,\tau,x)-G(s,\tau,x)]\right.\\ &\qquad\left.\cos\Phi(s,\tau)\right\}\sin x\,ds\,ds=0.\end{aligned}\tag{4.12}$$

These solvability conditions yield the nonlinear differential equations

$$2R'(\tau)=R(\tau)-\mu R^3(\tau),\tag{4.13a}$$

$$\Psi'(\tau)=-\left(\frac{\beta}{\mu}-\frac{1}{\lambda\omega_1}\right)\frac{R'(\tau)}{R(\tau)}+\frac{\beta}{\mu},\tag{4.13b}$$

where

$$\begin{aligned}\mu&=\frac{1+a}{2}-\frac{b}{4}+\frac{c}{4\lambda\omega_1}+\frac{(\lambda-2)}{4\lambda(1+4\omega_1^2)}>0\\ \beta&=\frac{1}{4\lambda\lambda_1\omega_1^2}\left(c-\frac{2\omega_1}{1+4\omega_1^2}\right).\end{aligned}\tag{4.14}$$

The initial conditions for (4.13) follow from (4.4) and (3.10) as

$$R(0)=\frac{h_0}{\omega_1\sqrt{\lambda\lambda_1}},\qquad \Psi(0)=\tan^{-1}(\lambda\omega_1).\tag{4.15}$$

Solving (4.13) subject to (4.15) yields

$$R(\tau)=R(0)\{\mu R^2(0)+[1-\mu R^2(0)]\,e^{-\tau}\}^{-1/2} \tag{4.16a}$$

$$\Psi(\tau)=\Psi(0)+\left(\frac{\beta}{\mu}-\frac{1}{\lambda\omega_1}\right)\log\frac{R(\tau)}{R(0)}+\frac{\beta}{\mu}\tau. \tag{4.16b}$$

This determines the slow evolution towards the bifurcated state which is periodic in t. It follows that as $\tau\to\infty$,

$$\begin{bmatrix} u \\ S \\ V \end{bmatrix} \sim \begin{bmatrix} \sin(\tilde{\omega}t+\Psi_\infty) \\ 1/\lambda_1 \sin(\tilde{\omega}t+\Psi_\infty)-\omega_1\cos(\tilde{\omega}t+\Psi_\infty) \\ 0 \end{bmatrix} \frac{\varepsilon}{\sqrt{\mu}}\sin x+O(\varepsilon^2), \tag{4.17}$$

where

$$\tilde{\omega}=\omega_1+\frac{\beta}{\mu}\varepsilon^2, \qquad \Psi_\infty=\Psi(0)-\left(\frac{\beta}{\mu}-\frac{1}{\lambda\omega_1}\right)\log(R(0)\sqrt{\mu}). \tag{4.18}$$

We note that the bifurcated state is periodic with a frequency $\tilde{\omega}$ that contains a shift away from the frequency ω_1 given by the linear analysis.

5. Discussion. We have examined the (weakly) nonlinear stability properties of the null state for a viscoelastic version of the Burgers system. This analysis represents an extension of the work in [8] to a model which more closely resembles the Navier-Stokes problem.

We find that for a non-Newtonian fluid of relatively small elasticity ($\lambda<\lambda_1$) the first instability is encountered at $R=R_1^{(s)}=1$, where there is a bifurcation to a steady state. This is in accord with the results for the Burgers model with a Newtonian fluid ($\lambda\to 0$, $\delta\to 0$), as seen in [7]. On the other hand, for a fluid of sufficiently large elasticity ($\lambda>\lambda_1$), the first instability is encountered at $R=R_1^{(p)}<1$ where there is a bifurcation to a periodic state. These results are qualitatively consistent with the linear stability analysis of [9] for a Navier-Stokes problem and with the (weakly) nonlinear analysis of [8] for the model equation (1.1).

Our results do reveal an interesting feature not suggested by previous investigations. For a fluid of sufficiently large elasticity ($\lambda>\lambda_1$), the evolution to the periodic bifurcated state is accompanied by a frequency shift. As seen in (4.17), (4.18), this shift $\Delta\omega$ is given by

$$\Delta\omega=\frac{\beta}{\mu}\varepsilon^2, \tag{5.1}$$

where β and μ are given by (4.14).

The ratio β/μ can be expressed in terms of only λ and δ, which are the two viscoelastic parameters of the Jeffreys kernel (1.3). Figure 1 provides a graphical illustration of the dependence of β/μ on these parameters. The fact that $\beta/\mu\to-\infty$ as $\lambda\to\lambda_1$ should be ignored since the analysis leading to (5.1) is not valid in this limit. It is significant that for δ sufficiently small, the frequency shift is (essentially) always positive; whereas for δ sufficiently large, the shift is always negative.

Since these results were obtained from the Burgers model, the quantitative value of the frequency shift cannot be regarded as significant. Nevertheless, the fact that there is a frequency shift may provide some insight into future investigations of the full-fledged Navier-Stokes model.

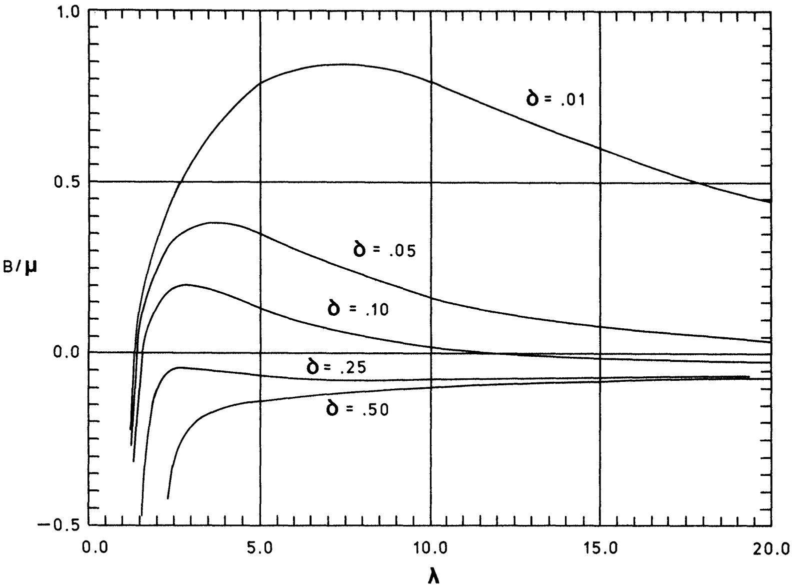

FIG. 1

Acknowledgement. On this occasion the authors are reminded how much Ed Reiss has enlightened us by his scientific expertise and encouraged us by his friendship.

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