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Original Publication Citation

Kaneko, H., & Xu, Y. (1996). Superconvergence of the iterated Galerkin methods for Hammerstein equations. *SIAM Journal on Numerical Analysis, 33*(3), 1048-1064. doi:10.1137/0733051

SIAM J. NUMER. ANAL.
Vol. 33, No. 3, pp. 1048–1064, June 1996

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SUPERCONVERGENCE OF THE ITERATED GALERKIN METHODS FOR HAMMERSTEIN EQUATIONS*

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Abstract. In this paper, the well-known iterated Galerkin method and iterated Galerkin–Kantorovich regularization method for approximating the solution of Fredholm integral equations of the second kind are generalized to Hammerstein equations with smooth and weakly singular kernels. The order of convergence of the Galerkin method and those of superconvergence of the iterated methods are analyzed. Numerical examples are presented to illustrate the superconvergence of the iterated Galerkin approximation for Hammerstein equations with weakly singular kernels.

Key words. the iterated Galerkin method, the iterated Galerkin–Kantorovich regularization, Hammerstein equations with weakly singular kernels, superconvergence

AMS subject classifications. 65B05, 45L10

1. Introduction.

In this paper, we consider the following Hammerstein equation:

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))ds = f(t), \quad 0 \le t \le 1, \tag{1.1}$$

where k, f, and ψ are known functions and x is the function to be determined. Define $k_t(s) \equiv k(t,s)$ for $t, s \in [0,1]$ to be the t section of k. We assume throughout this paper, unless stated otherwise, the following conditions on k, f, and ψ:

1. $\lim_{t\to\tau} \|k_t - k_\tau\|_\infty = 0$, $\tau \in [0,1]$;
2. $M \equiv \sup_t \int_0^1 |k(t,s)|ds < \infty$;
3. $f \in C[0,1]$;
4. $\psi(s,x)$ is continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which
$$|\psi(s,x_1) - \psi(s,x_2)| \le C_1|x_1 - x_2| \text{ for all } x_1, x_2 \in (-\infty,\infty);$$
5. the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that
$$|\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \le C_2|x_1 - x_2| \text{ for all } x_1, x_2 \in (-\infty,\infty); \tag{1.2}$$
1. for $x \in C[0,1]$, $\psi(.,x(.))$, $\psi^{(0,1)}(.,x(.)) \in C[0,1]$.

Additional assumptions will be given in §§2, 3, and 4 when they are needed.

Numerical methods for approximating the solutions of the Hammerstein equations have been studied extensively in the literature. A variation of Nyström's method was proposed by Lardy [23]. A new collocation type method was presented by Kumar and Sloan [22] and its superconvergence properties were obtained by Kumar [21]. Two different discrete collocation methods were proposed by Kumar [20] and Atkinson and Flores [4]. A degenerate kernel scheme was introduced by Kaneko and Xu [16] for equations (1.1) with smooth kernels. A product integration method and a collocation method were used to solve Hammerstein equations with weakly singular kernels, and certain superconvergence properties of the approximate solutions were discovered by Kaneko, Noren, and Xu [15]. The review paper by Atkinson

*Received by the editors February 3, 1993; accepted for publication (in revised form) August 17, 1994.
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[3] is recommended to the readers who require more information on the numerical treatments of Hammerstein equations. Some theoretical results about Hammerstein equations may be found in a book by Zeidler [34]. The purpose of this paper is to investigate the superconvergence property of the iterated Galerkin method and iterated Galerkin–Kantorovich method for the solution of the Hammerstein equation (1.1). The iterated method may be viewed as a nonlinear transformation (iteration) that accelerates the convergence of the approximate solutions obtained from the Galerkin approximation. The general theory of the acceleration of convergence of a sequence by linear or nonlinear transformations was studied by Wimp [33] and Delahaye [8] and in the references cited there.

For the Fredholm integral equations of the second kind, the Galerkin and the iterated Galerkin methods have been investigated by many authors, e.g., see Graham [12]; Graham, Joe, and Sloan [13]; Sloan [27], [28]; Sloan and Thomee [29]; and Vainikko, Pedas, and Uba [32]. In those papers that deal with the iterated Galerkin method, it has been shown that under some suitable conditions the iterated Galerkin method gives a rate of convergence that is faster than the rate obtainable by the Galerkin method, a phenomenon commonly known as superconvergence.

The order of convergence for Galerkin approximation for the solutions of Hammerstein equations with weakly singular kernels can be obtained by a direct extension of the corresponding result in the Fredholm case. However, it does not seem to be available in the literature. Hence, we include the results in §2 for completeness. A substantial number of proofs of the theorems in §2 will be omitted since they are straightforward and follow from the work of Vainikko [30] and Atkinson and Potra [6]. In the latter paper, the reader can find the general theory of the Galerkin and the iterated Galerkin methods for the equation $x = \mathcal{K}x$, where $\mathcal{K}$ is a completely continuous operator of a domain in a Banach space into itself. Our present approach and results differ from those of Atkinson and Potra [6] in a number of ways. For instance, we establish an estimate of improvement that we can expect when the iterated Galerkin scheme is applied to the weakly singular Hammerstein equations. This will be done in §3. Several related results on superconvergence are also established in §3. In §3, we deal with equations with weakly singular kernels and "nice" forcing terms, while in §4, we tackle equations with both singular kernels and singular forcing terms by employing the classical Kantorovich regularization technique. We extend the results of the iterated Galerkin method to the iterated Galerkin–Kantorovich regularization method. Numerical examples are given in §5 to illustrate the theoretical estimates.

2. The Galerkin methods for Hammerstein equations.

In this section, we develop the Galerkin method for Hammerstein equations and establish the order of convergence. Results concerning the Galerkin approximation using spline functions for the solutions of equation (1.1) with smooth and weakly singular kernels are presented.

Let n be a positive integer and $\{X_n\}$ be a sequence of finite-dimensional subspaces of $C[0,1]$ such that for any $x \in C[0,1]$ there exists a sequence $\{x_n\}$, $x_n \in X_n$, for which

$$\|x_n - x\|_\infty \to 0 \text{ as } n \to \infty. \tag{2.1}$$

Let $P_n\colon L_2[0,1] \to X_n$ be an orthogonal projection for each n. We assume that the projection P_n when restricted to $C[0,1]$ is uniformly bounded, i.e.,

$$P := \sup_n \|P_n|_{C[0,1]}\|_\infty < \infty. \tag{2.2}$$

Then from (2.1) and (2.2) it follows that for each $x \in C[0,1]$,

$$\|P_n x - x\|_\infty \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Now let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t,s)\psi(s,x(s))ds.$$

With this notation, equation (1.1) takes the operator form

$$x - K\Psi x = f. \tag{2.4}$$

In many interesting cases, equation (1.1) allows multiple solutions. Hence it is assumed for the remainder of this paper that we are treating a solution x_0 of equation (1.1) that is isolated.

Let $\{\varphi_{nj}\}_{j=1}^n$ be a set of linearly independent functions that spans X_n. The Galerkin method is to find

$$x_n = \sum_{j=1}^n b_{nj}\varphi_{nj}$$

that satisfies

$$x_n - P_n K\Psi x_n = P_n f. \tag{2.5}$$

Equivalently one is required to find b_{nj}'s that satisfy the system of nonlinear equations described by

$$\sum_{j=1}^n b_{nj}\langle\varphi_{nj},\varphi_{ni}\rangle - \left\langle \int_0^1 k(t,s)\psi(s,\sum_{j=1}^n b_{nj}\varphi_{nj}(s))ds, \varphi_{ni}\right\rangle = \langle f,\varphi_{ni}\rangle,\ 1 \le i \le n, \tag{2.6}$$

where $\langle .,.\rangle$ denotes the inner product in L_2.

We next estimate the error of the Galerkin approximate solutions to the exact solutions. For notational convenience, we introduce operators $\hat{T}$ and T_n by letting

$$\hat{T}x \equiv f + K\Psi x \tag{2.7}$$

and

$$T_n x_n \equiv P_n f + P_n K\Psi x_n \tag{2.8}$$

so that equations (2.4) and (2.5) can be written respectively as $x = \hat{T}x$ and $x_n = T_n x_n$. A proof of the following theorem can be made by directly applying Theorem 2 of Vainikko [30]. The paper of Atkinson and Potra [6] is also useful in this connection.

THEOREM 2.1. *Let $x_0 \in C[0,1]$ be an isolated solution of equation* (2.4). *Assume that* 1 *is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. Then the Galerkin approximation equation* (2.5) *has a unique solution $x_n \in B(x_0,\delta) := \{x \in C[0,1] : \|x - x_0\|_\infty \le \delta\}$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\alpha_n}{1-q}, \tag{2.9}$$

where $\alpha_n \equiv \|(I - T_n'(x_0))^{-1}(T_n(x_0) - \hat{T}(x_0))\|_\infty$. Finally,

$$E_n(x_0) \le \|x_n - x_0\|_\infty \le CE_n(x_0), \tag{2.10}$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u\in X_n}\|x_0 - u\|_\infty$.

We denote by $W_p^m[0, 1]$, $1 \le p \le \infty$, the Sobolev space of functions g whose mth generalized derivative $g^{(m)}$ belongs to $L_p[0, 1]$. The space $W_p^m[0, 1]$ is equipped with the norm

$$\|g\|_{W_p^m} \equiv \sum_{k=0}^{m} \|g^{(k)}\|_p.$$

We now specify the finite-dimensional subspace X_n. For any positive integer n, let

$$\Pi_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

be a partition of [0, 1]. Let r and ν be nonnegative integers satisfying $0 \le \nu < r$. Let $S_r^\nu(\Pi_n)$ denote the space of splines of order r, continuity ν, with knots at Π_n, that is,

$$S_r^\nu(\Pi_n) = \{x \in C^\nu[0, 1] : x|_{[t_i, t_{i+1}]} \in \mathcal{P}_{r-1} \text{ for each } i = 0, 1, \ldots, n-1\},$$

where $\mathcal{P}_{r-1}$ denotes the space of polynomials of degree $\le r-1$. We assume that the sequence of partitions Π_n of [0, 1] satisfies the condition that there exists a constant $C > 0$, independent of n, with the property

$$\frac{\max_{1\le i\le n}(t_i - t_{i-1})}{\min_{1\le i\le n}(t_i - t_{i-1})} \le C \text{ for all } n. \tag{2.11}$$

It is known from de Boor [7] and Douglas, Dupont, and Wahlbin [11] that condition (2.11) implies that the Galerkin projections P_n are uniformly bounded. In addition, it is also well known from Demko [9] and De Vore [10] that if $0 \le \nu < r$, $1 \le p \le \infty$, $m \ge 0$, and $x \in W_p^m$, then for each $n \ge 1$, there exists $u_n \in S_r^\nu(\Pi_n)$ such that

$$\|x - u_n\|_p \le Ch^\mu \|x\|_{W_p^\mu}, \tag{2.12}$$

where $\mu = \min\{m, r\}$ and $h = \max_{1\le i\le n}(t_i - t_{i-1})$. Using Theorem 2.1 and the inequalities (2.10) and (2.12), we obtain the following theorem.

THEOREM 2.2. *Let x_0 be an isolated solution of equation* (1.1) *and let x_n be the solution of equation* (2.5) *in a neighborhood of x_0. Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *If $x_0 \in W_\infty^l$ $(0 \le l \le r)$, then*

$$\|x_0 - x_n\|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W_p^l$ $(0 < l \le r,\ 1 \le p < \infty)$, then

$$\|x_0 - x_n\|_\infty = O(h^\nu),$$

where $\nu = \min\{l - 1, r\}$.

We remark that a similar result of Galerkin's method for Urysohn equations was obtained by Atkinson and Potra [6]. Hence, Theorem 2.2 may be derived by specializing their result to Hammerstein equations.

In the remaining portion of this section, we investigate the order of convergence of the Galerkin method for Hammerstein equations with weakly singular kernels. For this purpose, we define some necessary notation. For any $\epsilon \in R$, let $[0, 1]_\epsilon = \{t \in [0, 1] : t + \epsilon \in [0, 1]\}$. Let Δ_h denote the forward difference operator with step size h. For $\alpha > 0$ and $1 \le p \le \infty$, we define the Nikol'skii space $N_p^\alpha[0, 1]$ by

$$N_p^\alpha[0, 1] = \left\{x \in L_p[0, 1] : |x|_{\alpha,p} := \sup_{h\ne 0} \frac{1}{|h|^{\alpha_0}} \|\Delta_h^2 x^{[\alpha]}\|_{L_p[0,1]_{2h}} < \infty\right\}, \tag{2.13}$$

where $[\alpha]$ is an integer and $0 < \alpha_0 \le 1$ are chosen so that $\alpha = [\alpha] + \alpha_0$. Clearly, $N_p^\alpha[0, 1]$ is a Banach space with the norm $\|x\|_{\alpha,p} = \|x\|_p + |x|_{\alpha,p}$. We remark that the function $t^{\alpha-1}$ is in $N_1^\alpha[0, 1]$ but is not in $N_1^\beta[0, 1]$, for any $\beta > \alpha$, and $\log t \in N_1^1[0, 1]$. It is known from Graham [12] that

$$N_p^{m+\epsilon}[0, 1] \subseteq W_p^m[0, 1] \subseteq N_p^m[0, 1] \subseteq N_p^{m-\epsilon}[0, 1] \tag{2.14}$$

for $m \in N, 0 < \epsilon < 1$, and $1 \le p \le \infty$, and

$$N_p^\alpha[0, 1] \subseteq N_q^\beta[0, 1] \tag{2.15}$$

for $\alpha > 0$, $1 \le p \le q \le \infty$, and $\beta = \alpha - (1/p - 1/q) > 0$. We consider Hammerstein equations with kernels given by

$$k(t, s) = m(t, s)k(t - s), \quad t, s \in [0, 1], \tag{2.16}$$

with $k \in N_1^\alpha[0, 1]$ for some $0 < \alpha < 1$ and $m \in C^2([0, 1] \times [0, 1])$, and ψ as defined in the previous section.

Again, we let $X_n = S_r^\nu(\Pi_n)$. When no further conditions are made on the partition Π_n other than the one given by (2.11), the next theorem gives the best possible order of convergence of the Galerkin approximation to the solution of equation (1.1) with a weakly singular kernel defined by (2.16).

THEOREM 2.3. *Let x_0 be an isolated solution of equation* (1.1) *with a kernel given by* (2.16). *Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *If $f \in N_1^{\beta+1}[0, 1]$ for some $0 < \beta < 1$, then*

$$\|x_0 - x_n\|_\infty = O(h^\gamma),$$

with $\gamma = \min\{\alpha, \beta\}$.

Proof. By Theorem 2.1, we have

$$\|x_0 - x_n\|_\infty \le C \inf_{u \in S_{r,n}^\nu(\Pi_n)} \|x_0 - u\|_\infty. \tag{2.17}$$

A proof similar to the one given for Theorem 3 (ii) of Graham [12] shows that if $f \in N_1^{\beta+1}[0, 1]$ then $x_0 \in N_1^{\min\{\alpha+1,\beta+1\}}[0, 1] \subseteq N_\infty^{\min\{\alpha,\beta\}}[0, 1]$. In addition, (2.14) implies that $f \in W_1^1[0, 1]$. Hence f is equal to an absolutely continuous function almost everywhere. Without loss of generality, we have $f \in W_1^1[0, 1] \cap C[0, 1]$. It can be shown that $x_0 \in C[0, 1]$. Thus, $x_0 \in N_\infty^\gamma[0, 1] \cap C[0, 1]$. It was proved in Graham [12] that if $\phi \in N_\infty^\eta[0, 1] \cap C[0, 1]$ for some $0 < \eta < 1$, then there exists a spline $v \in S_r^\nu(\Pi_n)$ such that $\|\phi - v\|_\infty \le Ch^\eta$, where C is a constant independent of h. The result of this theorem follows immediately from (2.17) and the above argument. □

Now we consider a special form of (2.16). Namely, we assume

$$k(t, s) = m(t, s)g_\alpha(|t - s|), \tag{2.18}$$

where $m \in C^{\mu+1}([0, 1] \times [0, 1])$ and

$$g_\alpha(s) = \begin{cases} s^{\alpha-1}, & 0 < \alpha < 1, \\ \log s, & \alpha = 1. \end{cases} \tag{2.19}$$

With these kernels, certain regularities of the solutions of (1.1) are known. Let S be a finite set in $[0, 1]$ and define the function $\omega_S(t) = \inf\{|t - s| : s \in S\}$. A function x is said to be of *Type*(α, k, S) for $-1 < \alpha < 0$ if

$$|x^{(k)}(t)| \le C[\omega_S(t)]^{\alpha-k}, \quad t \notin S,$$

and for $\alpha > 0$ if the above condition holds and $x \in Lip(\alpha)$. Kaneko, Noren, and Xu [14] proved that if f is of $Type(\beta, \mu, \{0, 1\})$, then a solution of equation (1.1) is of $Type(\gamma, \mu, \{0, 1\})$, where $\gamma = \min\{\alpha, \beta\}$. In order to recover the optimal rate of convergence of numerical solutions, we define a partition Π_n^γ of [0, 1] corresponding to the regularity of a solution. The knots of this partition Π_n^γ are given by

$$(2.20)\qquad \begin{aligned} t_i &= (1/2)(2i/n)^q, \quad & 0 \le i \le n/2, \\ t_i &= 1 - t_{n-i}, & n/2 < i \le n, \end{aligned}$$

where $q = \frac{r}{\gamma}$. Let $S_{r,n}^{\nu,\gamma} = S_r^\nu(\Pi_n^\gamma)$, with $r = 1$ and $\nu = 0$ or $r \ge 2$ and $\nu \in \{0, 1\}$. The following theorem gives the order of convergence of the Galerkin approximations to the solution of Hammerstein equations with kernels defined by (2.18) and (2.19). It should be noted that the technique of approximating a solution of the type described above by elements from the nonlinear spline space has been used on many occasions when dealing with the weakly singular Fredholm integral equations. For example, Vainikko and Uba [31] describe the collocation method, whereas in Vainikko, Pedas, and Uba [32] they describe the Galerkin method. In addition, Schneider [25] establishes the product-integration method based on the idea of the nonlinear spline approximation with nonuniform knots. A piecewise continuous collocation method is studied by Atkinson, Graham, and Sloan [5].

THEOREM 2.4. *Let x_0 be an isolated solution of* (1.1) *with kernels* (2.18) *and* (2.19) *and let x_n be the Galerkin approximation to x_0. Let $m \in C^{\mu+1}([0, 1] \times [0, 1])$ and f be of $Type(\beta, \mu, \{0, 1\})$. Assume that $\psi \in C^{(0,1)}([0, 1] \times (-\infty, \infty))$ for $\mu = 0, 1$ and $\psi \in C^{\mu-1}([0, 1] \times (-\infty, \infty))$ for $\mu \ge 2$. We also assume* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *Then*

$$\|x_0 - x_n\|_\infty = O\left(\frac{1}{n^r}\right).$$

Proof. This follows from Theorem 2.1, the regularity of the solution x_0, and from the results of Rice [24]. □

3. The iterated Galerkin method. In this section, we study the superconvergence of the iterated Galerkin method for the Hammerstein equation (1.1). Generalizing the linear case, we first define the iterated scheme. Assume that x_0 is an isolated solution of (1.1). As in §2, let P_n be the orthogonal projection from $L_2[0, 1]$ onto X_n with conditions (2.1) and (2.2) satisfied. Assume that x_n is the unique solution of (2.5) in the sphere $B(x_0, \delta)$ for some $\delta > 0$. Define

$$(3.1)\qquad x_n' = f + K\Psi x_n.$$

Applying P_n to both sides of (3.1), we obtain

$$(3.2)\qquad P_n x_n' = P_n f + P_n K\Psi x_n.$$

Comparing (3.2) with (2.5), we see that

$$(3.3)\qquad P_n x_n' = x_n.$$

Upon substituting (3.3) into (3.1), we find that the function x_n' satisfies the new Hammerstein equation

$$(3.4)\qquad x_n' = f + K\Psi P_n x_n'.$$

By letting $S_n \equiv f + K\Psi P_n$, we may rewrite (3.4) as $x_n' = S_n x_n'$. We first study the invertibility of the linear operators $I - S_n'(x_0)$ in the following lemma, which will be used to prove the main results of this section.

LEMMA 3.1. *Let $x_0 \in C[0, 1]$ be an isolated solution of* (1.1). *Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *Then for sufficiently large n, the operators $I - S'_n(x_0)$ are invertible and there exists a constant $L > 0$ such that*

$$\|(I - S'_n(x_0))^{-1}\|_\infty \le L \text{ for sufficiently large } n.$$

Proof. Recalling the definition of Fréchet derivatives $S'_n(x_0)$ and $\hat{T}'(x_0)$, we have, for each $x \in C[0, 1]$,

$$\|\hat{T}'(x_0)(x) - S'_n(x_0)(x)\|_\infty \le \sup_{0\le t\le 1} \int_0^1 |k(t,s)||\psi^{(0,1)}(s, x_0(s))|ds\|x - P_n x\|_\infty$$
$$+C \sup_{0\le t\le 1} M\|P_n\|_\infty\|x\|_\infty\|x_0 - P_n x_0\|_\infty.$$

By (2.3), the last two terms can be made arbitrarily small as $n \to \infty$. This implies that $S'_n(x_0) \to \hat{T}'(x_0)$ pointwise in $C[0, 1]$, as $n \to \infty$. By assumptions 1, 2, and 6, $\hat{T}'(x_0)$ is a compact operator in $C[0, 1]$. Notice that by assumptions 5 and 6 and condition (2.2), there exists a constant $C > 0$ such that

$$|\psi^{(0,1)}(s, P_n x_0(s))| \le C_2\|P_n x_0 - x_0\|_\infty + \|\psi^{(0,1)}(., x_0(.))\|_\infty \le C \text{ for all } n.$$

Therefore, $\|S'_n(x_0)(x)\|_\infty \le MCP\|x\|_\infty$, and

$$|S'_n(x_0)(x)(t) - S'_n(x_0)(x)(t')| \le CP\|k_t - k_{t'}\|_1\|x\|_\infty.$$

This implies that $\{S'_n(x_0)\}$ is collectively compact. It follows from the theory of collectively compact operators in Anselone [1] and Atkinson [2] that $(I - S'_n(x_0))^{-1}$ exists for sufficiently large n and there exists a constant $L > 0$ such that $\|(I - S'_n(x_0))^{-1}\| \le L$ for sufficiently large n. □

For simplicity, from Lemma 3.1 we assume without loss of generality that $I - S'_n(x_0)$ is invertible for each $n \ge 1$ and

$$L = \sup\{\|(I - S'_n(x_0))^{-1}\|_\infty : n \ge 1\} < \infty.$$

Throughout the rest of this section, we assume without further mention that $\delta > 0$ satisfies $LC_2MP\delta < 1$ and δ_1 is chosen so that $C_1M\delta_1 \le \delta$. The following lemma establishes that x'_n defined in (3.1) is a unique solution of (3.4) in some neighborhood of x_0 and provides an error bound for x'_n approximating x_0.

LEMMA 3.2. *Let $x_0 \in C[0, 1]$ be an isolated solution of equation* (1.1) *and x_n be the unique solution of* (2.5) *in the ball $B(x_0, \delta_1)$. Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *Then for sufficiently large n, x'_n defined by the iterated scheme* (3.1) *is the unique solution of* (3.4) *in the ball $B(x_0, \delta)$. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\beta_n}{1+q} \le \|x'_n - x_0\|_\infty \le \frac{\beta_n}{1-q},$$

where $\beta_n = \|(I - S'_n(x_0))^{-1}[S_n(x_0) - \hat{T}(x_0)]\|_\infty$. *Finally,*

$$\|x'_n - x_0\|_\infty \le CE_n(x_0).$$

Proof. This follows easily using Lemma 2.1 and Theorem 2 of Vainikko [30]. □

One way to ensure a superconvergence of the iterated Galerkin method is to assume

$$\|(K\Psi)'(x_0)(I-P_n)|_{C[a,b]}\|_\infty \to 0 \qquad \text{as } n\to\infty. \tag{3.5}$$

In this case, using the identity (see Theorem 2.3 of Atkinson and Potra [6])

$$\begin{aligned}(I-(K\Psi)'(x_0))(x_n'-x_0)\\ &= [I-(K\Psi)'(x_0)(I-P_n)][K\Psi(x_n)-K\Psi(x_0)-(K\Psi)'(x_0)(x_n-x_0)]\\ &\quad -(K\Psi)'(x_0)(I-P_n)((K\Psi)'(x_0)-I)(x_n-x_0),\end{aligned}$$

we obtain

$$\begin{aligned}\|x_n'-x_0\|_\infty \le \|(I-(K\Psi)'(x_0))^{-1}\|_\infty \big\{&\|I-(K\Psi)'(x_0)(I-P_n)\|_\infty\\ &\times \sup_{0\le\theta\le1}\|(K\Psi)'(x_0+\theta(x_n-x_0))-(K\Psi)'(x_0)\|_\infty\|x_0-x_n\|_\infty\\ &+\|(K\Psi)'(x_0)(I-P_n)((K\Psi)'(x_0)-I)(x_n-x_0)\|_\infty\big\}.\end{aligned}$$

This and (3.5) give a superconvergence of x_n' to x_0. In the next theorem, we establish superconvergence of the iterated Galerkin method in a general setting. In establishing superconvergence of the iterates of the Fredholm equations, many authors assumed the condition $\|K(I-P_n)\|\to 0$ as $n\to\infty$ with K being a compact linear operator (e.g., Theorem 5 of Graham [12] and Theorem 3.1 of Sloan [28]). In our current problem, this is equivalent to assuming condition (3.5). However, the next theorem is proved without assumption (3.5). First, we apply the mean-value theorem to $\psi(s,y)$ to conclude

$$\psi(s,y)=\psi(s,y_0)+\psi^{(0,1)}(s,y_0+\theta(y-y_0))(y-y_0), \tag{3.6}$$

where $\theta := \theta(s,y_0,y)$ with $0<\theta<1$. The boundedness of θ is essential for the proof of the next theorem, although it may depend on s, y_0, and y. Let

$$g(t,s,y_0,y,\theta)=k(t,s)\psi^{(0,1)}(s,y_0+\theta(y-y_0)),$$

$$(G_nx)(t)=\int_0^1 g(t,s,P_nx_0(s),P_nx_n'(s),\theta)x(s)ds,$$

and $(Gx)(t)=\int_0^1 g_t(s)x(s)ds$, where $g_t(s)=k(t,s)\psi^{(0,1)}(s,x_0(s))$.

THEOREM 3.3. *Let $x_0\in C[0,1]$ be an isolated solution of equation* (1.1) *and x_n be the unique solution of* (2.5) *in the ball $B(x_0,\delta_1)$. Let x_n' be defined by the iterated scheme* (3.1). *Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$. Then for all $1\le p\le\infty$,*

$$\|x_0-x_n'\|_\infty \le C\left\{\|x_0-P_nx_0\|_\infty^2+\sup_{0\le t\le1}\inf_{u\in X_n}\|k(t,.)\psi^{(0,1)}(.,x_0(.))-u\|_q\|x_0-P_nx_0\|_p\right\},$$

where $1/p+1/q=1$ and C is a constant independent of n.

Proof. Note that from equations (1.1) and (3.4) we have

$$x_0-x_n'=K(\Psi x_0-\Psi P_nx_n')=K(\Psi x_0-\Psi P_nx_0)+K(\Psi P_nx_0-\Psi P_nx_n'). \tag{3.7}$$

After replacing y by P_nx_n' and y_0 by P_nx_0 in equation (3.6), the last term of (3.7) can be written as

$$K(\Psi P_nx_0-\Psi P_nx_n')(t)=(G_nP_n(x_0-x_n'))(t).$$

Equation (3.7) now becomes

$$x_0 - x_n' = K(\Psi x_0 - \Psi P_n x_0) + G_n P_n (x_0 - x_n'). \tag{3.8}$$

By using condition (1.2) and the fact that $0 < \theta < 1$, we have, for all $x \in C[0, 1]$,

$$\|(G_n x) - (Gx)\|_\infty \leq \sup_{0\leq t\leq 1} \int_0^1 |k(t,s)|ds\|x\|_\infty(\|P_n x_0 - x_0\|_\infty + \|P_n\|_\infty \|x_n' - x_0\|_\infty).$$

Consequently, by assumption (2.1) and Lemma 3.2,

$$\|G_n - G\|_\infty \leq M(\|P_n x_0 - x_0\|_\infty + P\|x_n' - x_0\|_\infty) \to 0 \text{ as } n \to \infty.$$

That is, $G_n \to G$ in the norm of $C[0, 1]$ as $n \to \infty$. Moreover, for each $x \in C[0, 1]$,

$$\sup_{0\leq t\leq 1} |(GP_n x)(t) - (Gx)(t)| = \sup_{0\leq t\leq 1} \left| \int_0^1 g_t(s)[P_n x(s) - x(s)]ds \right| \leq MM_1 \|P_n x - x\|_\infty,$$

where

$$M_1 = \sup_{0\leq t\leq 1} |\psi^{(0,1)}(t, x_0(t))| < +\infty.$$

It follows that $GP_n \to G$ pointwise in $C[0, 1]$ as $n \to \infty$. Again since P_n is uniformly bounded, we have for each $x \in C[0, 1]$,

$$\|G_n P_n x - Gx\|_\infty \leq \|G_n - G\|_\infty \|P_n\|_\infty \|x\|_\infty + \|GP_n x - Gx\|_\infty.$$

Thus, $G_n P_n \to G$ pointwise in $C[0, 1]$ as $n \to \infty$. By assumptions 2, 5, and 6, we see that there exists a constant $C > 0$ such that for all n

$$|\psi^{(0,1)}(s, P_n x_0(s) + \theta(P_n x_n'(s) - P_n x_0(s)))| \leq C_2\|P_n x_0 - x_0\|_\infty + \theta C_2 P\|x_n' - x_0\|_\infty + M_1 \leq C.$$

By a proof similar to that for Lemma 3.1, we can show that $\{G_n P_n\}$ is collectively compact. Since $G = (K\Psi)'(x_0)$ is compact and $(I - G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I - G_n P_n)^{-1}$ exists and is uniformly bounded for sufficiently large n. By (3.8), we have the following estimate

$$\sup_{0\leq t\leq 1} |(x_0 - x_n')(t)| \leq C \sup_{0\leq t\leq 1} |K(\Psi x_0 - \Psi P_n x_0)(t)|.$$

Next, we estimate the function $d(t) \equiv |K(\Psi x_0 - \Psi P_n x_0)(t)|$. Using (3.6) with $y = P_n x_0$ and $y_0 = x_0$, we obtain, for $0 < \theta < 1$,

$$d(t) = \left| \int_0^1 g(t, s, x_0(s), P_n x_0(s), \theta)(x_0(s) - P_n x_0(s))ds \right|.$$

Note that $\int_0^1 u(s)[x_0(s) - P_n x_0(s)]ds = 0$ for all $u \in X_n$. Thus, for all $u \in X_n$,

$$\begin{aligned} d(t) &= \left| \int_0^1 [g(t, s, x_0(s), P_n x_0(s), \theta) - u(s)](x_0(s) - P_n x_0(s))ds \right| \\ &\leq \int_0^1 |g(t, s, x_0(s), P_n x_0(s), \theta) - g_t(s)|ds\|x_0 - P_n x_0\|_\infty \\ &\quad + \left| \int_0^1 [g_t(s) - u(s)](x_0(s) - P_n x_0(s))ds \right|. \end{aligned}$$

Now, by condition (1.2), we have

$$\int_0^1 |g(t,s,x_0,P_nx_0(s),\theta)-g_t(s)|ds \le C_1\theta \int_0^1 |k(t,s)|ds\,\|x_0-P_nx_0\|_\infty \le C_1M\|x_0-P_nx_0\|_\infty.$$

Moreover, for $1/p+1/q=1$,

$$\left|\int_0^1 [g_t(s)-u(s)][x_0(s)-P_nx_0(s)]ds\right| \le \|g_t-u\|_q\|x_0-P_nx_0\|_p.$$

Therefore,

$$d(t)\le C_1M\|x_0-P_nx_0\|_\infty^2+\|g_t-u\|_q\|x_0-P_nx_0\|_p \text{ for all } u\in X_n.$$

Hence the desired result follows. □

In the next two theorems, we consider the case $X_n = S_r^\nu(\Pi_n)$, where Π_n is an arbitrary partition of [0, 1] satisfying (2.11). First, we consider the case when both the kernels and the solutions of equation (1.1) are smooth.

THEOREM 3.4. *Let* $x_0 \in W_p^l$ $(0 < l \le r)$ *be an isolated solution of* (1.1), x_n *be the unique solution of* (2.5) *in* $B(x_0,\delta_1)$, *and* x_n' *be defined by the iterated scheme* (3.1). *Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$. *Assume that for all* $t\in[0,1]$, $k_t(.)\psi^{(0,1)}(.,x_0(.))\in W_q^m (0\le m\le r)$. *Then*

$$\|x_0-x_n'\|_\infty = O(h^{\mu+\min\{\mu,\nu\}}),$$

where $\mu=\min\{l,r\}$ *and* $\nu=\min\{m,r\}$.

Proof. Since the partition Π_n of [0, 1] satisfies condition (2.11), we conclude that

$$P := \sup_n \|P_n\|_\infty < \infty.$$

Hence,

$$\|x_0-P_nx_0\|_p \le \|x_0-P_nx_0\|_\infty \le (1+P)\inf_{u\in S_r^\nu(\Pi_n)}\|x_0-u\|_\infty \le Ch^\mu.$$

In addition,

$$\sup_{0\le t\le 1}\inf_{u\in S_r^\nu(\Pi_n)}\|k_t(.)\psi^{(0,1)}(.,x_0(.))-u\|_q \le Ch^\nu.$$

The result of this theorem follows from Theorem 3.3 with $X_n = S_r^\nu(\Pi_n)$. □

We remark that Theorem 3.4 may be obtained from Theorem 5.2 of Atkinson and Potra [6], Theorem 3.4 being a special case of Atkinson and Potra's theorem extended to Hammerstein equations.

In the following theorem, we assume that $k(t,s)$ is a kernel given by (2.19), i.e., $k(t,s)=m(t,s)k(t-s)$, with $k\in N_1^\alpha[0,1]$ for some $0<\alpha<1$ and $m\in C^2([0,1]\times[0,1])$. Also, we assume that $S_r^\nu(\Pi_n)$ is such that $\nu\ge 1$.

THEOREM 3.5. *Let* x_0 *be an isolated solution of equation* (1.1) *with kernels given by* (2.16), x_n *be the unique solution of equation* (2.5) *in* $B(x_0,\delta_1)$, *and* x_n' *be defined by iterated scheme* (3.1). *Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$, $f\in N_1^{\beta+1}[0,1]$ *for some* $0<\beta<1$, $\psi^{(0,1)}(.,x(.))\in W_1^1$ *for* $x\in W_1^1$. *Then*

$$\|x_0-x_n'\|_\infty = O(h^{2\gamma}),$$

with $\gamma=\min\{\alpha,\beta\}$.

Proof. Following the proof of Theorem 3.4, we have

$$\|x_0 - P_n x_0\|_\infty \leq (1 + P) \inf_{u \in S_r^\nu(\Pi_n)} \|x_0 - u\|_\infty. \tag{3.9}$$

As stated in the proof of Theorem 2.4, we know that

$$x_0 \in N_\infty^\gamma[0, 1] \cap C[0, 1] \cap W_1^1. \tag{3.10}$$

Using (3.9) and an argument similar to the one used in the proof of Theorem 2.4, we obtain $\|x_0 - P_n x_0\|_\infty \leq Ch^\gamma$. Now, by Theorem 4(i) of Graham [12], we find that there exists $v_t \in S_r^\nu(\Pi_n)$ such that $\|k_t - v_t\|_1 = O(h^\alpha)$. Since $\nu \geq 1$, it follows that $S_r^\nu(\Pi_n) \subset W_1^1$. Thus, $v_t \in W_1^1$. From (3.10), $x_0 \in W_1^1$. This yields that $\psi^{(0,1)}(., x_0(.)) \in W_1^1$. Consequently, $v_t(.)\psi^{(0,1)}(., x_0(.)) \in W_1^1$. The remark made before Theorem 2.2 implies that there exists $u_t \in S_r^\nu(\Pi_n)$ for which

$$\|v_t(.)\psi^{(0,1)}(., x_0(.)) - u_t(.)\|_1 = O(h).$$

Therefore,

$$\begin{aligned}
\|g_t - u_t\|_1 &= \int_0^1 |m(t,s)k(t-s)\psi^{(0,1)}(s, x_0(s)) - u_t(s)|ds \\
&\leq \int_0^1 |m(t,s)k(t-s)\psi^{(0,1)}(s, x_0(s)) - v_t(s)\psi^{(0,1)}(s, x_0(s))|ds \\
&\quad + \int_0^1 |v_t(s)\psi^{(0,1)}(s, x_0(s)) - u_t(s)|ds \\
&\leq \|k_t - v_t\|_1 \|\psi^{(0,1)}(., x_0(.))\|_\infty + \|v_t(.)\psi^{(0,1)}(., x_0(.)) - u_t\|_1 \\
&= O(h^\alpha) + O(h) = O(h^\alpha).
\end{aligned}$$

Now, applying Theorem 3.3 with $q = 1$, $p = \infty$, and $X_n = S_r^\nu(\Pi_n)$, we conclude that

$$\begin{aligned}
\|x_0 - x_n'\|_\infty &\leq C\left\{\|x_0 - P_n x_0\|_\infty^2 + \inf_{u \in S_r^\nu(\Pi_n)} \|g_t - u_t\|_1 \|x_0 - P_n x_0\|_\infty\right\} \\
&= O(h^{\alpha+\gamma}) + O(h^{2\gamma}) = O(h^{2\gamma}).
\end{aligned}$$

The proof is complete. □

Next, we apply Theorem 3.3 to equation (1.1) with kernels given by (2.18) and (2.19) and use $X_n = S_r^\nu(\Pi_n^\gamma)$ as approximate spaces, where $S_r^\nu(\Pi_n^\gamma)$ of splines with nonuniform knots are defined as in §2 such that $r \geq 2$ and $\nu = 1$.

THEOREM 3.6. *Let x_0 be an isolated solution of* (1.1) *with weakly singular kernels given by* (2.18) *and* (2.19). *Let x_n be the unique solution of* (2.5) *in $B(x_0, \delta_1)$, and x_n' be defined by the iterated scheme* (3.1). *Assume that* 1 *is not an eigenvalue of $(K\Psi)'(x_0)$ and that the hypotheses of Theorem* 2.4 *are satisfied with $\mu \geq 1$. Also assume that $\psi^{(0,1)}(\cdot, x_0(\cdot))$ is of Type$(\alpha, r, \{0, 1\})$ for $\alpha > 0$ whenever x_0 is of the same type. Then*

$$\|x_0 - x_n'\|_\infty = O\left(\frac{1}{n^{\alpha+r}}\right).$$

Proof. The proof of this theorem is similar to that of Theorem 3.5. We apply Theorem 3.3 with $q = 1$, $p = \infty$, and $X_n = S_r^\nu(\Pi_n^\gamma)$. By Rice [24], we have $\|x_0 - P_n x_0\|_\infty = O(\frac{1}{n^r})$.

It can be proved that there exists $u \in S_r^\nu(\Pi_n^\gamma)$ such that $\|g_t - u\|_1 = O(\frac{1}{n^\alpha})$. From this, the result of this theorem follows. □

As the last application of Theorem 3.3, we consider equation (1.1) with kernels having singularity at the four corners of the square $[0, 1] \times [0, 1]$, a problem that arises from boundary integration for the harmonic Dirichlet problem in plane domains with corners (see Kress [19]). In the following theorem, we assume $k_s(t) = k(t, s)$ is of *Type*$(\alpha, \mu, \{0, 1\})$ for $\alpha > 0$, and $k_t(s) = k(t, s)$ is of *Type*$(\alpha, \mu, \{0, 1\})$ for $\alpha > -1$, e.g., $k(t, s) = m(t, s)\sqrt{t}$, and $k(t, s) = m(t, s)\frac{1}{\sqrt{1-s}}$, etc., with $m(t, s)$ smooth, and assume f is of *Type*$(\beta, \mu, \{0, 1\})$ for $\alpha, \beta > 0$ and a positive integer μ. It is not difficult to prove that an isolated solution x_0, of the corresponding equation (1.1), is of *Type*$(\gamma, \mu, \{0, 1\})$, where $\gamma = \min\{\alpha, \beta\}$ if $\alpha > 0$ and $\gamma = \min\{\alpha + 1, \beta\}$ if $-1 < \alpha < 0$ by modifying the proofs of theorems in Kaneko, Noren, and Xu [15]. We again let $q = \frac{r}{\gamma}$ and define the Galerkin subspaces $S_r^\nu(\Pi_n^\gamma)$ as in §2 with $r = 1$ or $\nu = 0$, or $r \geq 2$ and $\nu \in \{0, 1\}$, where partition Π_n^γ is defined as in (2.20). The following theorem describes the order of convergence of the Galerkin approximation x_n and that of superconvergence of the iterated Galerkin approximation x_n'. To the best of our knowledge, this result is not known in the literature even for Fredholm integral equations of the second kind.

THEOREM 3.7. *Let x_0 be an isolated solution of* (1.1) *with kernels of the type defined in the paragraph preceding this theorem. Let x_n be the unique solution of* (2.5) *in $B(x_0, \delta_1)$ and x_n' be defined by the iterated scheme* (3.1). *Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x_0)$ *and that f is of Type$(\beta, r, \{0, 1\})$. Also assume that $\psi^{(0,1)}(\cdot, x_0(\cdot))$ is of Type$(\gamma, r, \{0, 1\})$ whenever x_0 is of the same type. Then*

$$\|x_0 - x_n\|_\infty = O\left(\frac{1}{n^r}\right)$$

and

$$\|x_0 - x_n'\|_\infty = O\left(\frac{1}{n^{2r}}\right).$$

Proof. We present the proof for the case when $\alpha > 0$, since the proof for the other case is similar. The proof of the first estimate is similar to that for Theorem 2.6. Thus, we omit the details. Since P_n in this theorem is defined to be the Galerkin projection from $C[0, 1]$ onto $S_r^\nu(\Pi_n^\gamma)$, where $\gamma = \min\{\alpha, \beta\}$, and since x_0 is of *Type*$(\gamma, r, \{0, 1\})$, we have $\|x_0 - P_n x_0\|_\infty = O(\frac{1}{n^r})$. Meanwhile, since $k_t(s) = k(t, s)$ is of *Type*$(\alpha, r, \{0, 1\})$ and $\gamma \leq \alpha$, we find that $k_t(s) = k(t, s)$ is also of *Type*$(\gamma, r, \{0, 1\})$. By the assumption on $\psi^{(0,1)}$, we conclude that $\psi^{(0,1)}(., x_0(.))$ is of *Type*$(\gamma, r, \{0, 1\})$. Hence, $k(t, .)\psi^{(0,1)}(., x_0(.))$ is of *Type*$(\gamma, r, \{0, 1\})$. It follows that

$$\inf_{u \in S_r^\nu(\Pi_n^\gamma)} \|k(t, .)\psi^{(0,1)}(., x_0(.)) - u\|_1 = O\left(\frac{1}{n^r}\right).$$

Therefore, the result of this theorem follows from Theorem 3.3. □

4. The iterated Galerkin–Kantorovich method. In this section, we extend the classical Kantorovich regularization (see Kantorovich [18]) and the iterated Galerkin–Kantorovich method for Fredholm integral equations of the second kind to Hammerstein equations. These extensions will be made on equations with both singular kernels and singular forcing terms. The superconvergence of the corresponding iterated solution is also investigated.

In equation (2.4) we put

$$z = K\Psi x \tag{4.1}$$

so that

$$x = f + z. \tag{4.2}$$

Upon applying $K\Psi$ on both sides of (4.2), we obtain

$$z = K\Psi(f + z). \tag{4.3}$$

Now we define the operators by $\Psi_0(x)(t) \equiv \psi(t, x(t))$ and

$$\Psi_1(x)(t) \equiv \Psi_0(f + x)(t) - \Psi_0(f)(t). \tag{4.4}$$

In addition, define f_1 by

$$f_1(t) \equiv K\Psi_0(f)(t) = \int_0^1 k(t, s)\psi(s, f(s))ds. \tag{4.5}$$

From (4.4), we have $K\Psi_0(f + z)(t) = K\Psi_1(z)(t) + K\Psi_0(f)(t)$ so that (4.3) becomes

$$z - K\Psi_1(z) = K\Psi_0(f) \equiv f_1. \tag{4.6}$$

Equation (4.6) will be called the "regularized" equation for the original Hammerstein equation (1.1). It is interesting to note that

$$|\Psi_1(x_1)(t) - \Psi_1(x_2)(t)| = |\Psi_0(f + x_1)(t) - \Psi_0(f + x_2)(t)| \le C_1|x_1(t) - x_2(t)|.$$

Thus, Ψ_1 is also Lipschitz continuous with the same Lipschitz constant C_1 as Ψ_0. Hence the solvability of equation (4.6) is guaranteed by the solvability of the original equation (1.1).

The Galerkin method described in §2 is now applied to equation (4.6). Namely, we find $z_n \in X_n$ that satisfies

$$z_n - P_n K\Psi_1 z_n = P_n f_1. \tag{4.7}$$

The Galerkin-Kantorovich regularization solution for (1.1) is now given by

$$x_n^K = f + z_n. \tag{4.8}$$

Note that x_n^K inherits the singularity of f. From equations (4.2) and (4.8), we have $x - x_n^K = z - z_n$. Since $z, z_n \in C[0, 1]$, we see that $x - x_n^K \in C[0, 1]$, although neither x nor x_n^k may be in $C[0, 1]$. Denote $T_n z_n \equiv P_n f_1 + P_n K\Psi_1 z_n$ and $Tz \equiv f_1 + K\Psi_1 z$.

THEOREM 4.1. *Let x_0 be an isolated solution of equation* (1.1) *such that $z_0 = K\Psi_0 x_0 \in C[0, 1]$. Assume that* 1 *is not an eigenvalue of the linear operator $(K\Psi_1)'(z_0)$. Then equation* (4.7) *has a unique solution $z_n \in B(z_0, \delta)$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \|x_n^K - x_0\|_\infty \le \frac{\alpha_n}{1-q}, \tag{4.9}$$

where $x_n^K = f + z_n$ and

$$\alpha_n = \|(I - T_n'(z_0))^{-1}(T_n(z_0) - T(z_0))\|_\infty. \tag{4.10}$$

Finally,

$$E_n(z_0) \le \|x_0 - x_n^K\|_\infty \le C E_n(z_0), \tag{4.11}$$

where $E_n(z_0) = \inf_{y \in X_n} \|y - z_0\|_\infty$ *and* C *is a constant independent of* n.

Proof. The inequalities (4.9) follow again from Theorem 2 of Vainikko [30]. It is also noted that

$$z_0 - z_n = x_0 - x_n^K. \tag{4.12}$$

Since $z_n \in X_n$, (4.10) holds and $E_n(z_0) \le \|z_0 - z_n\|_\infty = \|x_0 - x_n^K\|_\infty$. This gives the first inequality in (4.11). Since $T_n(z_0) - T(z_0) = P_n(f_1 - K\Psi_1 z_0) - z_0 = P_n z_0 - z_0$, we find

$$\|z_n - z_0\|_\infty \le \frac{\|(I - T_n'(z_0))^{-1}\|_\infty \|T_n(z_0) - T(z_0)\|_\infty}{1-q} = \frac{\|(I - T_n'(z_0))^{-1}\|_\infty}{1-q} \|P_n z_0 - z_0\|_\infty.$$

Also for $u \in X_n$,

$$\|z_0 - P_n z_0\| = \|z_0 - u - P_n(z_0 - u)\| \le (1 + \|P_n\|)\|z_0 - u\|.$$

Therefore, we have $\|x_0 - x_n^K\| \le C E_n(z_0)$, where C is a constant independent of n. □

We next consider the iterated Galerkin-Kantorovich method and investigate its superconvergence property. Assume that z_0 is an isolated solution of (4.6) and z_n is the unique solution of (4.7) in $B(z_0, \delta)$ for some $\delta > 0$. Define

$$z_n' = K\Psi_1(z_n) + f_1 \tag{4.13}$$

and $x_n^{K'} = f + z_n'$. The element $x_n^{K'}$ is called the iterated Galerkin-Kantorovich approximate solution of equation (1.1). Applying P_n to both sides of (4.13) gives

$$P_n z_n' = P_n K\Psi_1(z_n) + P_n f_1. \tag{4.14}$$

Again, by using (4.7), we have $P_n z_n' = z_n$. Upon substituting this equation into (4.13), we find that z_n' satisfies the following new Hammerstein equation $z_n' = K\Psi_1 P_n z_n' + f_1$. In view of the fact that Ψ_1 is Lipschitz continuous with the same Lipschitz constant as Ψ_0, the same proofs given for Theorems 3.1, 3.2, and 3.3 can be applied to $S_n \equiv K\Psi_1 P_n + f_1$ to obtain the following theorem. Here δ_1 is chosen as in §3. As in Theorem 3.3, the assumption that $\|(K\Psi)'(x_0)(I - P_n)\|_\infty \to 0$ as $n \to \infty$ is no longer needed.

THEOREM 4.2. *Let* x_0 *be an isolated solution of equation* (1.1) *such that* $z_0 = K\Psi_0 x_0 \in C[0, 1]$. *Let* z_n *be the unique solution of equation* (4.7) *in* $B(z_0, \delta_1)$. *Let* $x_n^{K'}$ *be the corresponding iterated Galerkin-Kantorovich approximate solution. Assume that* 1 *is not an eigenvalue of* $(K\Psi_1)'(z_0)$. *Then, for all* $1 \le p \le \infty$,

$$\|x_0 - x_n^{K'}\| \le C\left\{\|z_0 - P_n z_0\|_\infty^2 + \sup_{0\le t\le 1} \inf_{u \in X_n} \|k(t, \cdot)\psi_1^{(0,1)}(\cdot, z_0(\cdot)) - u\|_q \|z_0 - P_n z_0\|_p\right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. □

Results parallel to Theorems 3.4–3.7, for smooth and weakly singular kernels can be obtained also by using Theorem 4.2 for the iterated Kantorovich method. The iterated Kantorovich regularization method for the Fredholm equations of the second kind was investigated by Sloan [26].

5. Numerical examples. In this section, some numerical examples are given to illustrate the theory established in the previous sections.

Example 1. Consider

$$x(t) - \int_0^1 \frac{x^2(s)}{\sqrt{|t-s|}} ds = f(t), \qquad 0 \le t \le 1, \tag{5.1}$$

where f is selected so that $x(t) = \sqrt{t}$ is the solution. The splines of orders 1 ($q = 2$) and 2 ($q = 4$) with knots defined by equation (2.23) in terms of q, are used in computations. To establish the Galerkin matrix, we must compute the integral of the form

$$\int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \frac{\varphi_i(s)\varphi_j(t)}{\sqrt{|t-s|}} dt ds, \tag{5.2}$$

where φ_i's are respective B-splines of the above mentioned spline space. It can be proved that $\varphi_i(s) \int_{t_{j-1}}^{t_j} \frac{\varphi_j(t)}{\sqrt{|t-s|}} dt$ belongs to $Type(\frac{1}{2}, k, \{t_{j-1}, t_j\})$. Consequently, we employ the recently developed Gauss-type quadrature formula of Kaneko and Xu [17] to approximate integrals (5.2). This brings to our attention the problem of the discrete Galerkin method for Hammerstein equations with weakly singular kernels. This will be dealt with in a future paper. In the ensuing data, $e_n \equiv \|x - x_n\|_\infty$ and $e_n' \equiv \|x - x_n'\|_\infty$ were approximated, respectively, by

$$\max\left\{\left|x\left(\frac{i}{100}\right) - x_n\left(\frac{i}{100}\right)\right| : i = 0, 1, \ldots, 100\right\}$$

and

$$\max\left\{\left|x\left(\frac{i}{100}\right) - x_n'\left(\frac{i}{100}\right)\right| : i = 0, 1, \ldots, 100\right\}.$$

Data 1. $q = 2$.

n	e_n	decay exp.	e_n'	decay exp.
16	$1.60D-2$		$3.01D-3$	
32	$7.26D-3$	1.14	$9.10D-4$	1.73
64	$3.34D-3$	1.12	$2.88D-4$	1.66
128	$1.64D-3$	1.03	$9.50D-5$	1.60

Data 2. $q = 4$.

n	e_n	decay exp.	e_n'	decay exp.
16	$4.01D-3$		$8.04D-4$	
32	$9.93D-4$	2.01	$1.30D-4$	2.61
64	$2.46D-4$	2.01	$2.28D-5$	2.51
128	$6.06D-5$	2.02	$3.90D-6$	2.55

It can be seen clearly that the iterated Galerkin approximation has superconvergence by an order $\frac{1}{2}$.

Example 2. To illustrate the use of Theorem 3.7, we consider

$$x(t) - \int_0^1 \frac{x^2(s)}{\sqrt[3]{s}} ds = f(t), \qquad 0 \le t \le 1, \tag{5.3}$$

where f is selected so that $x(t) = \sqrt{t}$ is the solution of equation (5.3). As in the first example, the splines of orders 1 and 2 are used. Since the solution is of $Type(\frac{1}{2}, k, \{0, 1\})$ for any positive integer k, the partition is formed according to $\alpha = \frac{1}{2}$.

Data 1. $q = 2$.

n	e_n	decay exp.	e'_n	decay exp.
16	$1.12D-2$		$2.10D-3$	
32	$5.15D-3$	1.12	$5.21D-4$	2.01
64	$2.22D-3$	1.21	$1.30D-4$	2.00
128	$1.08D-3$	1.04	$3.25D-5$	2.00

Data 2. $q = 4$.

n	e_n	decay exp.	e'_n	decay exp.
16	$3.12D-3$		$5.12D-4$	
32	$7.53D-4$	2.05	$3.05D-5$	4.07
64	$1.74D-4$	2.11	$1.85D-6$	4.04
128	$4.26D-5$	2.03	$1.14D-7$	4.02

The iteration process doubles the rate of convergence.

Acknowledgments. The authors would like to thank Professor K. Atkinson and the referee, who made many useful suggestions that improved this paper.

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