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UNIFORM l^1 BEHAVIOR OF A TIME DISCRETIZATION METHOD FOR A VOLTERRA INTEGRODIFFERENTIAL EQUATION WITH CONVEX KERNEL; STABILITY*

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Abstract. We study stability of a numerical method in which the backward Euler method is combined with order one convolution quadrature for approximating the integral term of the linear Volterra integrodifferential equation $\mathbf{u}'(t) + \int_0^t \beta(t-s)\mathbf{A}\mathbf{u}(s)\,ds = 0$, $t \geq 0$, $\mathbf{u}(0) = \mathbf{u}_0$, which arises in the theory of linear viscoelasticity. Here $\mathbf{A}$ is a positive self-adjoint densely defined linear operator in a real Hilbert space, and $\beta(t)$ is locally integrable, nonnegative, nonincreasing, convex, and $-\beta'(t)$ is convex. We establish stability of the method under these hypotheses on $\beta(t)$. Thus, the method is stable for a wider class of kernel functions $\beta(t)$ than was previously known. We also extend the class of operators $\mathbf{A}$ for which the method is stable.

Key words. Volterra integrodifferential equation, convolution quadrature, convex kernel, l^1-stability

AMS subject classifications. 45D05, 45K05, 65R20, 64D05

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1. Introduction.

Let $\mathbf{A}$ be a positive self-adjoint linear operator defined on a dense subspace $\mathcal{D}(\mathbf{A})$ of a real Hilbert space $\mathbf{H}$ with spectral decomposition

$$\mathbf{A}\mathbf{x} = \int_{-\infty}^{\infty} \lambda \, d\mathbf{E}_\lambda \, \mathbf{x} \tag{1}$$

for $\mathbf{x} \in \mathcal{D}(\mathbf{A})$. We assume that the spectrum of $\mathbf{A}$ is contained in $[\lambda_0, \infty)$, where $\lambda_0 > 0$. Xu established stability results in 2002 (see [21]) and convergence results in 2008 (see [22]) for a numerical method for approximating the initial value problem

$$\mathbf{u}'(t) + \int_0^t \beta(t-s)\mathbf{A}\mathbf{u}(s)\,ds = 0, \qquad t \geq 0, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{2}$$

Here $\mathbf{u} = \mathbf{u}(t)$ is a function in the Hilbert space $\mathbf{H}$ and $' = d/dt$. Xu assumes in both papers that the kernel $\beta(t) : (0, \infty) \to \mathbb{R}$ satisfies

$$\beta \in C(0,\infty) \cap L^1(0,1) \text{ and } 0 \leq \beta(\infty) < \beta(0+) \leq \infty, \tag{3}$$

and

$$(-1)^n \beta^{(n)}(t) \geq 0, \qquad t > 0, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

In Theorems 1 and 2 we substantially enlarge the class of functions $\beta(t)$ for which the stability results are valid by weakening the completely monotone hypotheses (4) on $\beta(t)$ to the assumption

$$\beta \text{ is nonnegative, nonincreasing, convex, and } -\beta' \text{ is convex.} \tag{5}$$

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We also note that our results hold for the wider class of operators $\mathbf{A}$ defined via a spectral family $\{\mathbf{E}_\lambda\}$, as in (1), whereas [21] employed the more restrictive condition that $\mathbf{A}$ possess a countable complete eigensystem.

Xu utilized a discrete analogue of the Payley–Wiener theorem in [23] to obtain results similar to those in the present paper for a class of quadratures and for certain kernels displaying log convexity. Although the hypotheses in [23] overlap ours, our results hold for kernels lacking log convexity, such as if $\beta(t) = 0$ for some $t > 0$. As an example,

$$f(x) = \begin{cases} (x_0 - t)^2 & \text{for } 0 \le x \le x_0, \\ 0 & \text{for } x_0 < x \end{cases}$$

for any fixed $x_0 > 0$.

Denote the Laplace transform of a function f by $\widehat{f}(t)$. Thus,

$$\widehat{\beta}(t) = \int_0^\infty e^{-ts}\beta(s)\,ds, \qquad t > 0. \tag{6}$$

By Bernstein's theorem [20, Chapter 8], a function $a = a(t)$ is completely monotonic iff there exists an associated nonnegative, increasing function $\alpha : [0,\infty) \to [0,\infty)$ with

$$a(t) = \int_0^\infty e^{-xt}\,d\alpha(x), \qquad t > 0. \tag{7}$$

From (7) we see that the Laplace transform of $a(t)$ may be analytically extended to the slit plane $\mathbb{C}' \equiv \mathbb{C} \setminus (-\infty, 0]$ via the formula

$$\widehat{a}(t) = \int_0^\infty \frac{d\alpha(s)}{s+t} \qquad t \in \mathbb{C}'. \tag{8}$$

Here a Stieltjes integral is used. Xu makes extensive use of this representation in his analysis.

A convex function will only be guaranteed to have a second derivative almost everywhere [18, Chapter 7]. In particular, the representation (8) does not hold. Without this representation we are still able to obtain the same conclusions as Xu by doing detailed estimates on the function $\widehat{\beta}(t)$ using the representation (6).

Let k denote the constant time step, $t_n = kn$ the nth time level, and $\mathbf{U}^n$ the approximation of $\mathbf{u}(t_n)$. The backward Euler method is used with $\bar{\partial}\mathbf{U}^n = \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}$ approximating the derivative $\mathbf{u}'$ in (2) at the nth time level. For the integral we apply the first-order convolution quadrature introduced by Lubich [7]:

$$q_n(\varphi) = \sum_{j=1}^{n} \beta_{n-j}(k)\varphi^j, \tag{9}$$

where $\varphi^j = \varphi(t_j)$ and the quadrature weights $\beta_{n-j}(k)$ are the coefficients of the power series

$$\widehat{\beta}\left(\frac{1-z}{k}\right) = \sum_{j=0}^{\infty} \beta_j(k) z^j. \tag{10}$$

This leads to the time discrete problem

$$\bar{\partial}\mathbf{U}^n + q_n(\mathbf{A}\mathbf{U}) = 0, \qquad \mathbf{U}^0 = \mathbf{u}_0. \tag{11}$$

Our first theorem generalizes Theorem 1 of [21] by replacing the completely monotonic assumption (4) with (5).

THEOREM 1. *If* (3) *and* (5) *hold, then*

$$k\sum_{n=1}^{\infty}||\mathbf{U}^n|| \le C||\mathbf{A}\mathbf{u}_0||. \tag{12}$$

In order to state our next theorem we must first define some auxiliary functions. For $\sigma + i\tau \notin (-\infty, 0]$, set $\beta(t) = c(t) + \beta(\infty)$, and then let

$$\phi(\sigma,\tau) = \int_0^\infty \cos(\tau t)e^{-\sigma t}\beta(t)\,dt \quad \text{and} \quad \theta(\sigma,\tau) = \frac{1}{\tau}\int_0^\infty \sin(\tau t)e^{-\sigma t}\beta(t)\,dt,$$
$$\phi_c(\sigma,\tau) = \int_0^\infty \cos(\tau t)e^{-\sigma t}c(t)\,dt \quad \text{and} \quad \theta_c(\sigma,\tau) = \frac{1}{\tau}\int_0^\infty \sin(\tau t)e^{-\sigma t}c(t)\,dt,$$

and for $0 < \tau < \infty$, set

$$\phi_c(\tau) = \lim_{\sigma\to 0+}\phi_c(\sigma,\tau) = \int_0^\infty \cos(\tau t)c(t)\,dt$$

and

$$\theta_c(\tau) = \lim_{\sigma\to 0+}\theta_c(\sigma,\tau) = \frac{1}{\tau}\int_0^\infty \sin(\tau t)c(t)\,dt.$$

So, for $\sigma + i\tau \notin (-\infty, 0]$, we have

$$\phi(\sigma,\tau) = \phi_c(\sigma,\tau) + \frac{\sigma\beta(\infty)}{\sigma^2+\tau^2} \quad \text{and} \quad \theta(\sigma,\tau) = \theta_c(\sigma,\tau) + \frac{\beta(\infty)}{\sigma^2+\tau^2},$$

and then, for $0 < \tau < \infty$, we may set

$$\phi(\tau) = \lim_{\sigma\to 0+}\phi(\sigma,\tau) = \phi_c(\tau) \quad \text{and} \quad \theta(\tau) = \lim_{\sigma\to 0+}\theta(\sigma,\tau) = \theta_c(\tau) + \frac{\beta(\infty)}{\tau^2}.$$

We see then that the Fourier transform of $\beta(t)$,

$$\widetilde{\beta}(\tau) = \int_0^\infty e^{-i\tau t}\beta(t)\,dt, \tag{13}$$

obeys the relation

$$\widetilde{\beta}(\tau) = \phi(\tau) - i\tau\theta(\tau), \tag{14}$$

and further, the Laplace transform obeys

$$\widehat{\beta}(\sigma + i\tau) = \phi(\sigma,\tau) - i\tau\theta(\sigma,\tau). \tag{15}$$

As a consequence of Theorem 2.2 and Corollary 2.1 of Carr and Hannsgen [1], (3) and (5) imply

$$\limsup_{\tau\to\infty}\frac{\theta_c(\tau)}{\phi_c(\tau)} < \infty. \tag{16}$$

By (4.3) of [1], we see that $\tau^{-2} = o(\theta_c(\tau))$ $(\tau \to \infty)$, so it follows that (3) and (5) imply that

$$\limsup_{\tau\to\infty}\frac{\theta(\tau)}{\phi(\tau)} < \infty. \tag{17}$$

If instead our kernel $\beta(t)$ is such that

$$\limsup_{\tau\to\infty} \frac{\tau^{\frac{1}{3}}\theta(\tau)}{\phi(\tau)} < \infty \tag{18}$$

holds, then we can obtain the following theorem which generalizes Theorem 2 of [21].

THEOREM 2. *If* (3), (5), *and* (18) *hold, then*

$$k\sum_{n=1}^{\infty} ||\mathbf{U}^n|| \le C||\mathbf{u}_0||. \tag{19}$$

We note that (18) is a significantly weaker frequency condition upon $\beta(t)$ than is employed in Theorem 2 of [21], namely, that

$$\limsup_{\tau\to\infty} \frac{\tau\theta(\tau)}{\phi(\tau)} < \infty. \tag{20}$$

For example, if $\beta(t)$ satisfies (5) and behaves like $(-\log(t))^p$ $(p > 0)$ near the origin, then an easy calculation utilizing the relations (37) and (39) shows that (18) is satisfied, but not (20). We see that in Theorem 1 we are allowed a wider class of kernel functions $\beta(t)$, but we have the more restrictive requirement that $\mathbf{u}_0 \in \mathcal{D}(\mathbf{A})$, whereas Theorem 2 places greater restrictions upon $\beta(t)$, yet allows $\mathbf{u}_0$ to be any element of $\mathbf{H}$.

The resolvent kernel of (2) is defined by the formula

$$\mathbf{U}(t) = \int_{-\infty}^{\infty} u(t,\lambda)\, d\mathbf{E}_\lambda, \tag{21}$$

where $u(t,\lambda)$ is the solution of the scalar Volterra integrodifferential equation

$$u'(t) + \lambda \int_0^t \beta(t-s)u(s)\, ds = 0, \quad u(0) = 1; \tag{22}$$

the parameter λ satisfies $\lambda_0 \le \lambda$ and $0 \le t$.

It is clear from (21) that

$$\sup_{\lambda_0\le\lambda} |u(t,\lambda)| \to 0, \quad t\to\infty \tag{23}$$

and

$$\int_0^{\infty} \sup_{\lambda_0\le\lambda} |u(t,\lambda)|\, dt < \infty \tag{24}$$

imply, respectively,

$$\|\mathbf{U}(t)\| \to 0, \quad t\to\infty \tag{25}$$

and

$$\int_0^{\infty} \|\mathbf{U}(t)\|\, dt < \infty. \tag{26}$$

Then the resolvent formula

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{y}_0 + \int_0^t \mathbf{U}(t-s)\mathbf{f}(s)\, ds \tag{27}$$

can be used to obtain precise asymptotic information ($t \to \infty$) about the solution $\mathbf{y}(t)$ of the initial value problem

$$\mathbf{u}'(t) + \int_0^t \beta(t-s)\mathbf{A}\mathbf{u}(s)\,ds = \mathbf{f}(t), \qquad t \geq 0, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{28}$$

In [1] several sufficient conditions are given on $\beta(t)$ such that (23) and (24) hold. One easily stated consequence of [1] which is relevant here is that (23) and (24) both hold, and, as a consequence, (25) and (26) when $\beta(t)$ satisfies (5).

In [21] the stability of a numerical scheme for approximating the solution of (2) is a discrete analogue of (26). Let $\{U^n(\lambda)\}_{n=0}^{\infty}$ be a real sequence satisfying the difference equation

$$\frac{U^n(\lambda) - U^{n-1}(\lambda)}{k} + \lambda q_n(U(\lambda)) = 0, \qquad n \geq 1, \qquad U^0(\lambda) = 1. \tag{29}$$

It follows from the functional calculus for spectral decompositions (see [17]) that the solution to (11) may be representated as

$$\mathbf{U}^n = \int_{-\infty}^{\infty} U^n(\lambda)\,d\mathbf{E}_\lambda\,\mathbf{u}_0. \tag{30}$$

We note that Lemma 1 from [6] implies that $e^{-\sigma t}\beta(t)$ and $(e^{-\sigma t}\beta(t))'$ are convex for $\sigma > 0$. Also, from Theorem 2 and the comments following it in [13] we find that $\beta(t)$ is positive-definite, implying that $\mathrm{Re}(\widehat{\beta}(s)) > 0$ whenever $s = \sigma + i\tau$ with $\sigma > 0$. Then, by an argument similar to that in Lemma 3.1 of [8], we find that the quadrature (9) is positive-definite in the sense that for each function $\varphi : (0,\infty) \to \mathbf{H}$ and each positive integer N, we have

$$Q_N(\varphi) \equiv k\sum_{n=1}^{N}(q_n(\varphi), \varphi^n) \geq 0. \tag{31}$$

To see this, set

$$\widetilde{\varphi}(t) = \sum_{j=1}^{N}\varphi^j t^j, \quad \widetilde{\beta}(t) = \sum_{j=0}^{\infty}\beta_j(k)t^j \quad \text{and} \quad Q_{N,r}(\varphi) = k\sum_{n=1}^{N}\sum_{j=1}^{n}\beta_{n-j}(k)r^{n-j}(\varphi^j, \varphi^n)$$

for $0 < r < 1$. Then, it is straightforward to show that

$$Q_{N,r}(\varphi) = \frac{k}{2\pi}\int_0^{2\pi}\widetilde{\beta}(re^{i\theta})||\widetilde{\varphi}(e^{i\theta})||^2\,d\theta.$$

As $\mathbf{H}$ is a real Hilbert space, it follows from (10) that $Q_{N,r}(\varphi) \geq 0$. Then, by (9) we find that $Q_{N,r}(\varphi) \to Q_N(\varphi)$ $(r \uparrow 1)$, from which (31) follows.

By an argument very similar to that given in Lemma 3.1 of [10], it can be shown that (31) implies that

$$||\mathbf{U}^n|| \leq ||\mathbf{u}_0||. \tag{32}$$

Then (32) implies that

$$k\sum_{n=1}^{m}||\mathbf{U}^n|| \leq t_m||\mathbf{u}_0||. \tag{33}$$

Thus, by (30) and (33), we see that it is sufficient to show that

$$k \sum_{n=m+1}^{\infty} \sup_{\lambda \geq \lambda_0} |U^n(\lambda)\lambda^{-1}| \leq C \tag{34}$$

and

$$k \sum_{n=m+1}^{\infty} \sup_{\lambda \geq \lambda_0} |U^n(\lambda)| \leq C \tag{35}$$

to prove Theorems 1 and 2, respectively.

Equations (2) and (28) arise in the theory of linear viscoelasticity. A nice survey may be found in [16]. For a comprehensive treatment of Volterra equations see [5] or [15]. Another interesting work on the numerical approximation of the solution of (2) which assumes (3) and (5) is given by Xu in [24, Remark 2.3] in which a Galerkin method is studied. For a numerical solution utilizing quadrature applied to the inverse Laplace transform form of the solution, see [11]. For a second-order accurate finite difference solution, see [9]. A solution utilizing finite difference convolution quadrature is given in [3]. For a time-stepping discontinuous Galerkin solution, see [12].

In the next section we establish some preliminary results and in section 3 we present the proofs of our theorems. In all that follows we assume that $\varepsilon > 0$ is a sufficiently small fixed constant independent of k whose value will be specified later. We also note that C is a generic constant whose value may change at each appearance and which depends only upon ε and λ_0.

2. Preliminary estimates.

We begin with a lemma from [21, p. 139], which derives from a lemma in [1, p. 967].

LEMMA 2.1. *If $\beta(t)$ satisfies (3) and (5), then*

$$\frac{1}{2\sqrt{2}} \int_0^{\frac{1}{\tau}} \beta(t)\,dt \leq |\widetilde{\beta}(\tau)| \leq 4 \int_0^{\frac{1}{\tau}} \beta(t)\,dt, \qquad \tau > 0, \tag{36}$$

$$\frac{1}{5} \int_0^{\frac{1}{\tau}} t\beta(t)\,dt \leq \theta(\tau) \leq 12 \int_0^{\frac{1}{\tau}} t\beta(t)\,dt, \qquad \tau > 0, \tag{37}$$

$$|\widetilde{\beta}'(\tau)| \leq 40 \int_0^{\frac{1}{\tau}} t\beta(t)\,dt, \qquad \tau > 0. \tag{38}$$

Here, recall that $\widetilde{\beta}(\tau)$ is the Fourier transform of $\beta(t)$. We note that these results hold without the convexity of $-\beta'(t)$ assumed. As we know that $e^{-\sigma t}\beta(t)$ and $(e^{-\sigma t}\beta(t))'$ are convex for $\sigma > 0$, then with only slight modifications to the proof we obtain results similar to those in Noren (see [14, eq. (4.14)]):

$$\frac{1}{C} \int_0^{\frac{1}{\tau}} -t\beta'(t)\,dt \leq \phi(\tau) \leq C \int_0^{\frac{1}{\tau}} -t\beta'(t)\,dt, \qquad \tau > 0, \tag{39}$$

and

$$\frac{1}{C} \int_0^{\frac{1}{\tau}} -t(e^{-\sigma t}\beta(t))'\,dt \leq \phi(\sigma,\tau) \leq C \int_0^{\frac{1}{\tau}} -t(e^{-\sigma t}\beta(t))'\,dt, \qquad \sigma,\tau > 0. \tag{40}$$

One consequence of (39) and (40) in the case where $0 < \sigma \leq \varepsilon\tau$ is that

$$\phi(\sigma,\tau) \geq C \int_0^{\frac{1}{\tau}} -t(e^{-\sigma t}\beta(t))'\,dt \geq Ce^{-\frac{\sigma}{\tau}} \int_0^{\frac{1}{\tau}} -t\beta'(t)\,dt \geq C\phi(\tau). \tag{41}$$

As $e^{-\sigma t}\beta(t)$ satisfies the hypotheses of Lemma 2.1, we obtain the following variants of (36) and (38):

$$\frac{1}{2\sqrt{2}}\int_0^{\frac{1}{\tau}} e^{-\sigma t}\beta(t)\,dt \le |(e^{-\sigma t}\beta(t))^{\sim}(\tau)| \le 4\int_0^{\frac{1}{\tau}} e^{-\sigma t}\beta(t)\,dt, \qquad \sigma,\tau>0, \tag{42}$$

and

$$\left|\frac{d}{d\tau}(e^{-\sigma t}\beta(t))^{\sim}(\tau)\right| \le 40\int_0^{\frac{1}{\tau}} te^{-\sigma t}\beta(t)\,dt, \qquad \sigma,\tau>0. \tag{43}$$

Defining the functions $A(x)=\int_0^x \beta(t)\,dt$ and $A_1(x)=\int_0^x t\beta(t)\,dt$, we also recall a result from Shea and Wainger [19, eq. (1.21)]):

$$\int_0^{\varepsilon}\frac{A_1(\tau^{-1})}{A^2(\tau^{-1})}\,d\tau<\infty. \tag{44}$$

Define the notations

$$\sigma=\sigma(k,\nu)=\frac{1-\cos(k\nu)}{k},\ \ \tau=\tau(k,\nu)=\frac{\sin(k\nu)}{k},\ \ s=s(k,\nu)=\sigma+i\tau=\frac{1-e^{-ik\nu}}{k},$$

and

$$D(s,\lambda)=D(\sigma+i\tau)=\frac{s}{\lambda}+\widehat{\beta}(s)=\frac{\sigma+i\tau}{\lambda}+\widehat{\beta}(\sigma+i\tau).$$

Following [1] and [21], we wish to define a strictly increasing function $\omega:[\lambda_0,\infty)\to[\varepsilon,\infty)$ with $\omega(\lambda)\to\infty$ $(\lambda\to\infty)$ and such that $\theta(\omega(\lambda))=\frac{1}{\lambda}$ for $\lambda\ge\lambda_1\ge\lambda_0$ and, if necessary, $\omega(\lambda)=\varepsilon$ for $\lambda_1>\lambda\ge\lambda_0$. We note that ω was continuous in [1], owing to the choice of $\rho=\frac{6}{t_1}$ in that paper, and in [21] by the analytic nature of a completely monotonic function. We do not require that ω be continuous. In this case, slight modification to the proof given in [1, Lemma 5.2] and [2, Lemma 8.1] gives us the following lemma.

Lemma 2.2. *If $\beta(t)$ satisfies* (3) *and* (5), *then*

$$|D(i\tau,\lambda)|\ge\begin{cases}C_1\frac{|\tau-\omega|}{\lambda} & (\tau\ge\frac{\omega}{2}),\\ C_1(\tau\int_0^{\frac{1}{\tau}}t\beta(t)\,dt+\int_0^{\frac{1}{\tau}}\beta(t)\,dt) & (\tau\in[\frac{\varepsilon}{2},\frac{\omega}{2}]).\end{cases} \tag{45}$$

This result also holds if $-\beta'(t)$ convex is dropped. We also note that [1] gives us

$$\omega(\lambda)\le C\lambda, \tag{46}$$

and it follows from (6.8) of [1] that, for $\tau\ge\frac{\omega}{2}$, we have

$$\theta(\tau)\le C\lambda^{-1}. \tag{47}$$

We now wish to establish a generalization of (2.9) from [21].

Lemma 2.3. *If $\beta(t)$ satisfies* (3) *and* (5) *and $0<\sigma\le\varepsilon\tau<\tau$, then*

$$|\theta(\sigma,\tau)-\theta(\tau)|\le 29\varepsilon\theta(\tau). \tag{48}$$

Proof. Beginning with the formulas

$$\theta(\sigma,\tau)=\theta_c(\sigma,\tau)+\frac{\beta(\infty)}{\sigma^2+\tau^2} \qquad\text{and}\qquad \theta(\tau)=\theta_c(\tau)+\frac{\beta(\infty)}{\tau^2},$$

we see that

$$\begin{aligned}|\theta(\sigma,\tau)-\theta(\tau)| &\le |\theta_c(\sigma,\tau)-\theta_c(\tau)| + \beta(\infty)\left|\frac{1}{\tau^2}-\frac{1}{\sigma^2+\tau^2}\right| \\ &\le |\theta_c(\sigma,\tau)-\theta_c(\tau)| + \beta(\infty)\frac{\varepsilon}{\tau^2},\end{aligned}$$

so it suffices for us to show that $|\theta_c(\sigma,\tau)-\theta_c(\tau)| \le 29\varepsilon\theta_c(\tau)$. Integrating by parts twice, we get

$$\begin{aligned}\theta_c(\sigma,\tau) &= \frac{1}{\sigma^2+\tau^2}\int_0^\infty \left\{(1-e^{-\sigma t}\cos(\tau t)) - \frac{\sigma}{\tau}e^{-\sigma t}\sin(\tau t)\right\}(-c'(t))\,dt \\ &= \frac{1}{(\sigma^2+\tau^2)^2}\int_0^\infty \left\{\left((\sigma^2+\tau^2)t + (\sigma^2-\tau^2)e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right)\right. \\ &\qquad \left. - 2\sigma(1-e^{-\sigma t}\cos(\tau t))\right\}c''(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\theta_c(\tau) &= \frac{1}{\tau^2}\int_0^\infty (1-\cos(\tau t))(-c'(t))\,dt \\ &= \frac{1}{\tau^2}\int_0^\infty \left(t-\frac{\sin(\tau t)}{\tau}\right)c''(t)\,dt.\end{aligned}$$

The boundary terms vanish due to the relations $tc(t) = t^2c'(t) = o(1)$ $(t \to 0+)$ and $tc'(t) = o(1)$ $(t \to \infty)$ from [1]. Then, setting

$$f(t) = \frac{1}{(\sigma^2+\tau^2)^2}\left\{\left((\sigma^2+\tau^2)t + (\sigma^2-\tau^2)e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right) - 2\sigma(1-e^{-\sigma t}\cos(\tau t))\right\}$$

and

$$g(t) = \frac{1}{\tau^2}\left(t-\frac{\sin(\tau t)}{\tau}\right),$$

we see that we need only show that

$$(1-29\varepsilon)g(t) \le f(t) \le (1+29\varepsilon)g(t) \tag{49}$$

to have our result. Since $f'(0) = f(0) = g'(0) = g(0) = 0$ and $(1-\varepsilon)g''(t) \le f''(t) \le g''(t)$ for $t \in [0, \frac{1}{\tau}]$, we may integrate twice over $[0,t]$ for $t \in [0, \frac{1}{\tau}]$ to obtain $(1-\varepsilon)g(t) \le f(t) \le g(t)$ for $t \in [0, \frac{1}{\tau}]$. First we show that

$$(1-29\varepsilon)g(t) \le f(t) \quad \left(t > \frac{1}{\tau}\right). \tag{50}$$

Note first that as $0 < \sigma \le \varepsilon\tau < \tau$ and $t > \frac{1}{\tau}$, we have

$$\frac{-2\sigma}{(\sigma^2+\tau^2)^2}(1-e^{-\sigma t}\cos(\tau t)) \ge \frac{-2\sigma}{\tau^4}(1-e^{-\sigma t}\cos(\tau t)) \ge \frac{-4\varepsilon}{\tau^3} \ge \frac{-26\varepsilon}{\tau^2}\left(t-\frac{\sin(\tau t)}{\tau}\right).$$

So, we must show that

$$\frac{1}{(\sigma^2+\tau^2)^2}\left((\sigma^2+\tau^2)t + (\sigma^2-\tau^2)e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right) \ge (1-3\varepsilon)\left(\frac{1}{\tau^2}\left(t-\frac{\sin(\tau t)}{\tau}\right)\right).$$

As $(1+\varepsilon)(1-3\varepsilon) \le (1-2\varepsilon)$ and

$$\frac{1}{(\sigma^2+\tau^2)^2}\left((\sigma^2+\tau^2)t + (\sigma^2-\tau^2)e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right) \ge \frac{1}{(1+\varepsilon)\tau^2}\left(t - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right),$$

it suffices to show that (after some rearrangement)

$$\left(1 - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\right)\frac{\sin(\tau t)}{\tau} \ge 2\varepsilon\left(\frac{\sin(\tau t)}{\tau} - t\right).$$

This clearly holds if $\sin(\tau t) \ge 0$, so assume otherwise. Then, as

$$\begin{aligned}\left(1 - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\right)\frac{\sin(\tau t)}{\tau} &\ge \left(1 - \frac{1-\varepsilon}{1+\varepsilon}(1-\varepsilon\tau t)\right)\frac{\sin(\tau t)}{\tau} \\ &= \left(\frac{2\varepsilon}{1+\varepsilon} + \left(\frac{1-\varepsilon}{1+\varepsilon}\right)\varepsilon\tau t\right)\frac{\sin(\tau t)}{\tau},\end{aligned}$$

(50) follows since $2\varepsilon \le 2\varepsilon(1+\varepsilon)$ and $(\frac{1-\varepsilon}{1+\varepsilon})\sin(\tau t) \ge -1$. We now show that

$$f(t) \le (1+29\varepsilon)g(t) \qquad \left(t > \frac{1}{\tau}\right). \tag{51}$$

Note that as $0 < \sigma \le \varepsilon\tau < \tau$ and $t > \frac{1}{\tau}$, we have

$$f(t) \le \frac{1}{\sigma^2+\tau^2}\left(t - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right) \le \frac{1}{\tau^2}\left(t - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\frac{\sin(\tau t)}{\tau}\right).$$

So, we need only show that (after some rearrangement)

$$\left(1 - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\right)\frac{\sin(\tau t)}{\tau} \le 29\varepsilon\left(t - \frac{\sin(\tau t)}{\tau}\right).$$

This clearly holds if $\sin(\tau t) \le 0$, so assume otherwise. Then, we see that

$$\begin{aligned}\left(1 - \frac{\tau^2-\sigma^2}{\tau^2+\sigma^2}e^{-\sigma t}\right)\frac{\sin(\tau t)}{\tau} &\le \left(1 - \frac{1-\varepsilon}{1+\varepsilon}(1-\varepsilon\tau t)\right)\frac{\sin(\tau t)}{\tau} \\ &\le \varepsilon(2+\tau t)\frac{\sin(\tau t)}{\tau} \\ &\le 3\varepsilon t,\end{aligned}$$

from which (51) follows, as $\sin(\tau t) \le \frac{26}{29}\tau t$ for $t > \frac{1}{\tau}$. Then, combining (50) and (51), we obtain (49), which proves the lemma. ∎

We next wish to prove the following estimate.

LEMMA 2.4. *If* $\beta(t)$ *satisfies* (3) *and* (5), $k \le 1$, $\frac{\pi-\varepsilon}{k} \le \nu \le \frac{\pi}{k}$ *and* $\varepsilon \le \frac{1}{5}$, *then*

$$\phi(\sigma,\tau) \ge \frac{1}{2}\widehat{\beta}(\sigma). \tag{52}$$

Proof. Beginning with the formulas

$$\phi(\sigma,\tau) = \int_0^\infty \cos(\tau t)e^{-\sigma t}c(t)\,dt + \frac{\sigma\beta(\infty)}{\sigma^2+\tau^2} \qquad \text{and} \qquad \widehat{\beta}(\sigma) = \int_0^\infty e^{-\sigma t}c(t)\,dt + \frac{\beta(\infty)}{\sigma},$$

we note that as $\frac{\pi-\varepsilon}{k} \le \nu \le \frac{\pi}{k}$ gives us $\frac{2-\varepsilon}{k} \le \sigma \le \frac{2}{k}$ and $0 \le \tau \le \frac{\varepsilon}{k}$, we find that $\frac{\sigma}{\sigma^2+\tau^2} \ge \frac{(2-\varepsilon)k}{4+\varepsilon^2} \ge \frac{45k}{101} > \frac{k}{4-2\varepsilon} \ge \frac{1}{2\sigma}$, so we need only show that $\phi_c(\sigma,\tau) \ge \frac{1}{2}\widehat{c}(\sigma)$. Integrating by parts, we obtain

$$\phi_c(\sigma,\tau) = \frac{1}{\sigma^2+\tau^2}\int_0^\infty [\sigma(1-e^{-\sigma t}\cos(\tau t)) + \tau e^{-\sigma t}\sin(\tau t)](-c'(t))\,dt$$

and

$$\widehat{c}(\sigma) = \frac{1}{\sigma}\int_0^\infty (1-e^{-\sigma t})(-c'(t))\,dt.$$

The boundary terms vanish as in the proof of Lemma 2.3. Then, setting

$$f(t) = \frac{1}{\sigma^2+\tau^2}[\sigma(1-e^{-\sigma t}\cos(\tau t)) + \tau e^{-\sigma t}\sin(\tau t)] \quad \text{and} \quad g(t) = \frac{1}{\sigma}(1-e^{-\sigma t}),$$

we see that for $t \le \frac{1}{\tau}$, we have $f'(t) = e^{-\sigma t}\cos(\tau t) \ge \frac{1}{2}e^{-\sigma t} = \frac{1}{2}g'(t)$, so as $f(0) = g(0) = 0$, we may integrate over $[0,t]$, for $t \le \frac{1}{\tau}$, to obtain $f(t) \ge \frac{1}{2}g(t)$ for $t \le \frac{1}{\tau}$. Thus, it is only a matter of showing that $f(t) \ge \frac{1}{2}g(t)$ for $t > \frac{1}{\tau}$ to establish (52). As $\frac{2-\varepsilon}{k} \le \sigma \le \frac{2}{k}$ and $0 \le \tau \le \frac{\varepsilon}{k}$, it follows that for $\varepsilon \le \frac{1}{5}$ we have $\frac{\sigma}{\tau} \ge \frac{2-\varepsilon}{\varepsilon} \ge 9$; so clearly we have $e^{\frac{-\sigma}{\tau}} \le (1-e^{-\sigma t})$ for $t > \frac{1}{\tau}$. Thus, for $t > \frac{1}{\tau}$, we see that

$$\begin{aligned} f(t) &\ge \frac{k^2}{4+\varepsilon^2}\left(\frac{2-\varepsilon}{k}(1-e^{-\sigma t}) - \frac{\varepsilon}{k}e^{\frac{-\sigma}{\tau}}\right) \\ &\ge \frac{k}{4+\varepsilon}\left((2-\varepsilon)(1-e^{-\sigma t}) - \varepsilon(1-e^{-\sigma t})\right) \\ &= k\left(\frac{2-2\varepsilon}{4+\varepsilon}\right)(1-e^{-\sigma t}) \\ &\ge \frac{8k}{21}(1-e^{-\sigma t}) \\ &\ge \frac{1}{2}g(t), \end{aligned}$$

which proves the lemma. □

We next wish to extend Lemma 3.1 in [21].

LEMMA 2.5. *If $\beta(t)$ satisfies* (3) *and* (5), $\lambda \ge \lambda_0$, *and* $k < 1$, *then*

$$k\sum_{n=1}^\infty |U^n(\lambda)| \le C\lambda. \tag{53}$$

Proof. Following [21], we define the generating function of $\{U^n(\lambda)\}_{n=0}^\infty$ to be

$$\widetilde{U}(z,\lambda) = \sum_{n=1}^\infty U^n(\lambda)z^n,$$

which may be shown to obey the relations

$$\widetilde{U}(z,\lambda) = \frac{z}{k}\frac{1}{\left(\frac{1-z}{k}\right) + \lambda\widehat{\beta}\left(\frac{1-z}{k}\right)} = \frac{z}{k}\widehat{u}\left(\frac{1-z}{k},\lambda\right).$$

Then, an application of Hardy's inequality [4, p. 48] gives us

$$\sum_{n=1}^{\infty}|U^n(\lambda)| \le 2\sum_{n=1}^{\infty}\frac{n|U^n(\lambda)|}{n+1} \le \int_{-\pi}^{\pi}|\widetilde{U}'(e^{i\nu},\lambda)|\,d\nu$$

$$\le 2k\int_0^{\frac{\pi}{k}}|\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu = 2k\left\{\int_0^{\varepsilon}+\int_{\varepsilon}^{\frac{\varepsilon}{k}}+\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}\right\}|\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu, \tag{54}$$

where

$$\widetilde{U}'(z,\lambda) = \frac{1}{k}\frac{1}{\left[\left(\frac{1-z}{k}\right)+\lambda\widehat{\beta}\left(\frac{1-z}{k}\right)\right]^2}\left[\frac{1}{k}+\lambda\widehat{\beta}\left(\frac{1-z}{k}\right)+\frac{\lambda z}{k}\widehat{\beta}'\left(\frac{1-z}{k}\right)\right].$$

Thus, our extension reduces to establishing the three estimates

$$k^2\int_0^{\varepsilon}|\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu \le C\lambda^{-1}, \tag{55}$$

$$k^2\int_{\varepsilon}^{\frac{\varepsilon}{k}}|\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu \le C\lambda, \tag{56}$$

$$k^2\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}|\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu \le Ck\lambda. \tag{57}$$

We prove (55) first. For $\varepsilon, k < 1$, we see that when $0 \le \nu \le \varepsilon$, we get

$$\frac{\nu}{2} \le \tau(k,\nu) \le \nu, \quad \sigma(k,\nu) \le \varepsilon\tau \le \tau, \quad \cos(k\nu) \ge \frac{1}{2}, \quad \frac{\sin(k\nu)}{k} \le \varepsilon. \tag{58}$$

By (40) and (58), we see that

$$\mathrm{Re}\widehat{\beta}(\sigma+i\tau) \ge C\int_0^{\frac{1}{\tau}} -te^{-\sigma t}\beta'(t)\,dt \ge C\int_0^{\frac{1}{\varepsilon}} -t\beta'(t)\,dt \ge \sqrt{2}C\frac{\tau}{\lambda}. \tag{59}$$

We may obviously assume $C < 1$ in (59), giving

$$\begin{aligned}
|D(\sigma+i\tau,\lambda)|^2 &= \left|\phi(\sigma,\tau)+\frac{\sigma}{\lambda}\right|^2 + \left|\tau\theta(\sigma,\tau)-\frac{\tau}{\lambda}\right|^2 \\
&\ge \phi^2(\sigma,\tau) + C^2\left|\tau\theta(\sigma,\tau)-\frac{\tau}{\lambda}\right|^2 \\
&\ge \frac{\phi^2(\sigma,\tau)}{2} + C^2\frac{\tau^2}{\lambda^2} + C^2\left|\tau\theta(\sigma,\tau)-\frac{\tau}{\lambda}\right|^2 \\
&= \frac{\phi^2(\sigma,\tau)}{2} + C^2\frac{\tau^2}{2}\left(\theta(\sigma,\tau)-\frac{2}{\lambda}\right)^2 + C^2\frac{\tau^2}{2}\theta^2(\sigma,\tau) \\
&\ge C^2\left(\frac{\phi^2(\sigma,\tau)+\tau^2\theta^2(\sigma,\tau)}{2} + \frac{\tau^2}{2}\left(\theta(\sigma,\tau)-\frac{2}{\lambda}\right)^2\right) \\
&\ge C^2\left(\frac{\phi^2(\sigma,\tau)+\tau^2\theta^2(\sigma,\tau)}{2}\right),
\end{aligned}$$

so we obtain

$$|D(\sigma+i\tau,\lambda)| \ge C|\widehat{\beta}(\sigma+i\tau)|. \tag{60}$$

Then, (36), (42), and (58) give us

$$|\widehat{\beta}(\sigma+i\tau)| \geq C\int_0^{\frac{1}{\tau}} e^{-\sigma t}\beta(t)\,dt \geq C|\widetilde{\beta}(\tau)| \geq C\int_0^{\frac{1}{\varepsilon}} \beta(t)\,dt \geq C|\widetilde{\beta}(\varepsilon)|,$$

so (60) implies that

$$|D(\sigma+i\tau,\lambda)| \geq C|\widetilde{\beta}(\tau)| \geq C|\widetilde{\beta}(\varepsilon)|. \tag{61}$$

Note that (36) and (42) give us

$$|\widehat{\beta}(\sigma+i\tau)| = |(e^{-\sigma t}\beta(t))\widetilde{\ }(\tau)| \leq C|\widetilde{\beta}(\tau)|, \qquad \tau > 0. \tag{62}$$

We also see that (37) and (43) imply

$$|\widehat{\beta}'(\sigma+i\tau)| = \left|\frac{d}{d\tau}(e^{-\sigma t}\beta(t))\widetilde{\ }(\tau)\right| \leq C\theta(\tau), \qquad \sigma,\tau > 0. \tag{63}$$

Then it follows from (36), (37), (44), (61), (62), and (63) that

$$\begin{aligned} k^2\int_0^{\varepsilon} |\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu &\leq \int_0^{\varepsilon} \frac{1}{\lambda^2|D^2(\sigma+i\tau,\lambda)|}\left[1+\lambda k|\widehat{\beta}(\sigma+i\tau)|+\lambda|\widehat{\beta}'(\sigma+i\tau)|\right]d\nu \\ &\leq \frac{C}{\lambda}\int_0^{\varepsilon} \frac{1}{|\widetilde{\beta}(\varepsilon)|^2} + \frac{k|\widetilde{\beta}(\tau)|}{|\widetilde{\beta}(\tau)||\widetilde{\beta}(\varepsilon)|} + \frac{\theta(\tau)}{|\widetilde{\beta}(\tau)|^2}\,d\tau \\ &\leq C\lambda^{-1}, \end{aligned}$$

so estimate (55) holds. We now show (56). Here $\varepsilon \leq \nu \leq \frac{\varepsilon}{k}$ and for $\varepsilon, k \leq 1$ we have

$$\frac{\nu}{2} \leq \tau(k,\nu) \leq \nu, \quad \sigma(k,\nu) \leq \varepsilon\tau(k,\nu) \leq \tau(k,\nu), \quad \cos(k\nu) \geq \frac{1}{2}. \tag{64}$$

We shall establish the following estimates on $|D(\sigma+i\tau,\lambda)|$ when $\varepsilon \leq \min\left\{\frac{C_1}{1392}, \frac{5}{348}\right\}$:

$$|D(\sigma+i\tau,\lambda)| \geq C\left(\tau\theta(\tau) + \int_0^{\frac{1}{\tau}} \beta(t)\,dt\right), \qquad \tau \in \left[\frac{\varepsilon}{2},\frac{\omega}{2}\right]\bigcap\left[\frac{\varepsilon}{2},\frac{\varepsilon}{k}\right], \tag{65}$$

and

$$|D(\sigma+i\tau,\lambda)| \geq C\frac{\tau-\omega}{\lambda}, \qquad \tau \geq 2\omega. \tag{66}$$

We show (65) first. To establish

$$|D(\sigma+i\tau,\lambda)| \geq C\tau\theta(\tau), \qquad \tau \in \left[\frac{\varepsilon}{2},\frac{\omega}{2}\right]\bigcap\left[\frac{\varepsilon}{2},\frac{\varepsilon}{k}\right], \tag{67}$$

note that in the case where

$$\tau\left|\theta(\tau)-\frac{1}{\lambda}\right| < \frac{C_1\tau}{2}\int_0^{\frac{1}{\tau}} t\beta(t)\,dt$$

we can use (37), (41), (45), and (64) to show that

$$\begin{aligned} |D(\sigma+i\tau,\lambda)| &\geq C\phi(\tau) \geq C\left(C_1\tau\int_0^{\frac{1}{\tau}} t\beta(t)\,dt - \tau\left|\theta(\tau)-\frac{1}{\lambda}\right|\right) \\ &\geq C\left(\frac{C_1\tau}{2}\int_0^{\frac{1}{\tau}} t\beta(t)\,dt\right) \geq C\tau\theta(\tau). \end{aligned}$$

Similarly, when

$$\tau\left|\theta(\tau)-\frac{1}{\lambda}\right| \geq \frac{C_1\tau}{2}\int_0^{\frac{1}{\tau}} t\beta(t)\,dt,$$

we find that (37) and (48) give us

$$\tau\left|\theta(\tau)-\frac{1}{\lambda}\right| \geq \frac{C_1\tau}{2}\int_0^{\frac{1}{\tau}} t\beta(t)\,dt \geq \frac{C_1\tau}{24}\theta(\tau) \geq 2\tau|\theta(\sigma,\tau)-\theta(\tau)|.$$

Thus, it follows by (37) that

$$\begin{aligned}|D(\sigma+i\tau,\lambda)| &\geq \tau\left|\theta(\sigma,\tau)-\frac{1}{\lambda}\right| = \tau\left|\left(\theta(\tau)-\frac{1}{\lambda}\right)+(\theta(\sigma,\tau)-\theta(\tau))\right| \geq \frac{\tau}{2}\left|\theta(\tau)-\frac{1}{\lambda}\right| \\ &\geq \frac{C_1\tau}{2}\int_0^{\frac{1}{\tau}} t\beta(t)\,dt \geq C\tau\theta(\tau).\end{aligned}$$

This establishes (67). The estimate

$$|D(\sigma+i\tau,\lambda)| \geq C\int_0^{\frac{1}{\tau}}\beta(t)\,dt, \qquad \tau\in\left[\frac{\varepsilon}{2},\frac{\omega}{2}\right]\bigcap\left[\frac{\varepsilon}{2},\frac{\varepsilon}{k}\right], \tag{68}$$

follows from the same argument with $\tau\int_0^{\frac{1}{\tau}} t\beta(t)\,dt$ replaced by $\int_0^{\frac{1}{\tau}}\beta(t)\,dt$. Then, combining (67) and (68), we have (65). To prove (66), we note that (41) gives us

$$|D(\sigma+i\tau,\lambda)| \geq C\phi(\tau),$$

which establishes the estimate when $\phi(\tau) \geq \frac{C_1}{2}\frac{\tau-\omega}{\lambda}$. Thus, assume $\phi(\tau) < \frac{C_1}{2}\frac{\tau-\omega}{\lambda}$. Then, (45) gives us

$$C_1\frac{\tau-\omega}{\lambda} \leq |D(i\tau,\lambda)| \leq \phi(\tau)+\tau\left|\theta(\tau)-\frac{1}{\lambda}\right| \leq \frac{C_1}{2}\frac{\tau-\omega}{\lambda}+\tau\left|\theta(\tau)-\frac{1}{\lambda}\right|,$$

so

$$\left|\frac{1}{\lambda}-\theta(\tau)\right| \geq \frac{C_1(\tau-\omega)}{2\tau\lambda}, \qquad \tau\geq 2\omega. \tag{69}$$

As $\theta(\tau)=\theta_c(\tau)+\frac{\beta(\infty)}{\tau^2}$, we see by (4.4) of [1] that $\theta(\tau)$ is decreasing. It follows by our construction of $\omega(\lambda)$ that in the case where $\lambda\geq\lambda_1$, or by (69), in the case where $\lambda_0\geq\lambda>\lambda_1$ with $\frac{1}{\lambda}-\theta(\tau)\geq\frac{C_1(\tau-\omega)}{2\tau\lambda}$, that as $\varepsilon\leq\min\left\{\frac{C_1}{1392},\frac{5}{348}\right\}$, we have

$$\begin{aligned}\theta(\sigma,\tau) &\leq (1+29\varepsilon)\theta(\tau) = (1+29\varepsilon)\frac{1}{\lambda}-(1+29\varepsilon)\left(\frac{1}{\lambda}-\theta(\tau)\right) \\ &\leq \frac{1}{\lambda}-\frac{C_1(\tau-\omega)}{4\tau\lambda}\left(2+58\varepsilon-\frac{116\varepsilon\tau}{C_1(\tau-\omega)}\right) \\ &\leq \frac{1}{\lambda}-\frac{C_1(\tau-\omega)}{4\tau\lambda}\left(2-\frac{232\varepsilon}{C_1}\right) \\ &\leq \frac{1}{\lambda}-\frac{C_1(\tau-\omega)}{4\tau\lambda},\end{aligned}$$

so we find that

$$|D(\sigma+i\tau,\lambda)| \geq \tau\left(\frac{1}{\lambda}-\theta(\sigma,\tau)\right) \geq C\frac{\tau-\omega}{\lambda}. \tag{70}$$

Similarly, by (69), in the case where $\lambda_0 \geq \lambda > \lambda_1$ with $\frac{1}{\lambda}-\theta(\tau) < -\frac{C_1(\tau-\omega)}{2\tau\lambda}$, we see that as $\varepsilon \leq \min\left\{\frac{C_1}{1392}, \frac{5}{348}\right\}$ and as $\frac{\tau}{\tau-\omega} \leq 2$ for $\tau \geq 2\omega$, we have

$$\begin{aligned}
\theta(\sigma,\tau) &\geq (1-29\varepsilon)\theta(\tau) = (1-29\varepsilon)\frac{1}{\lambda} + (1-29\varepsilon)\left(\theta(\tau)-\frac{1}{\lambda}\right)\\
&\geq \frac{1}{\lambda} + \frac{C_1(\tau-\omega)}{4\tau\lambda}\left(2-58\varepsilon-\frac{116\varepsilon\tau}{C_1(\tau-\omega)}\right)\\
&\geq \frac{1}{\lambda} + \frac{C_1(\tau-\omega)}{4\tau\lambda}\left(2-58\varepsilon-\frac{232\varepsilon}{C_1}\right)\\
&\geq \frac{1}{\lambda} + \frac{C_1(\tau-\omega)}{4\tau\lambda},
\end{aligned}$$

so we obtain

$$|D(\sigma+i\tau,\lambda)| \geq \tau\left(\theta(\sigma,\tau)-\frac{1}{\lambda}\right) \geq C\frac{\tau-\omega}{\lambda}. \tag{71}$$

Combining (70) and (71) completes the proof of (66).

Next, let $E_1 = [\frac{\varepsilon}{2}, \frac{\omega}{2}] \cap [\frac{\varepsilon}{2}, \frac{\varepsilon}{k}], E_2 = [\frac{\omega}{2}, 2\omega] \cap [\frac{\varepsilon}{2}, \frac{\varepsilon}{k}]$, and $E_3 = [2\omega, \infty) \cap [\frac{\varepsilon}{2}, \frac{\varepsilon}{k}]$. Note that our construction of ω and the decreasing nature of $\theta(\tau)$ give us $\lambda\theta(\tau) \geq C$ for $\tau \in E_1$. Also, by (36), we see that $|\widetilde{\beta}(\tau)| \leq C$ for $\tau \in E_3$. Then by (17), (36), (41), (48), (62), (63), (65), and (66), we see that

$$\begin{aligned}
&k^2\int_{\varepsilon}^{\frac{\varepsilon}{k}} |\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu\\
&\quad\leq C\left\{\int_{E_1}+\int_{E_2}+\int_{E_3}\right\}\frac{1}{\lambda^2|D^2(\sigma+i\tau,\lambda)|}\left[1+\lambda k|\widehat{\beta}(\sigma+i\tau)|+\lambda|\widehat{\beta}'(\sigma+i\tau)|\right]d\tau\\
&\quad\leq C\left(\int_{\frac{\varepsilon}{2}}^{\frac{\omega}{2}}\frac{d\tau}{\tau^2}+k\int_{E_1}\frac{A(\tau^{-1})}{\lambda\tau\theta(\tau)A(\tau^{-1})}d\tau\right)\\
&\qquad+\frac{C}{\lambda}\left(\int_{E_2}\frac{d\tau}{\lambda\phi^2(\tau)}+k\frac{\phi(\tau)+\tau\theta(\tau)}{\phi^2(\tau)}+\frac{\theta(\tau)}{\phi^2(\tau)}\,d\tau\right)\\
&\qquad+C\left(\int_{E_3}\frac{1+\lambda(1+\theta(\tau))}{(\tau-\omega)^2}\,d\tau\right),
\end{aligned}$$

so by (46) and (47) we find that

$$k^2\int_{\varepsilon}^{\frac{\varepsilon}{k}} |\widetilde{U}'(e^{-ik\nu},\lambda)|\,d\nu \leq C(1+\lambda+k\lambda) \leq C\lambda,$$

which establishes (56). We now establish (57). Here $\frac{\varepsilon}{k} \leq \nu \leq \frac{\pi}{k}$, and we have

$$\sigma(k,\nu) \geq \frac{1-\cos(\varepsilon)}{k} \equiv \frac{C(\varepsilon)}{k}, \qquad |D(\sigma+i\tau)| \geq \frac{\sigma}{\lambda} \geq \frac{C(\varepsilon)}{k\lambda}. \tag{72}$$

Then, as $k \leq 1$, we have

$$|\widehat{\beta}(\sigma + i\tau)| = \left| \int_0^\infty e^{-(\sigma+i\tau)t} \beta(t)\, dt \right| \leq \int_0^\infty e^{-C(\varepsilon)t} \beta(t)\, dt = \widehat{\beta}(C(\varepsilon)) \tag{73}$$

and

$$|\widehat{\beta}'(\sigma + i\tau)| = \left| \int_0^\infty -te^{-(\sigma+i\tau)t} \beta(t)\, dt \right| \leq \int_0^\infty te^{-C(\varepsilon)t} \beta(t)\, dt = |\widehat{\beta}'(C(\varepsilon))|. \tag{74}$$

So, (72), (73), and (74) give us

$$\begin{aligned} k^2 \int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}} |\widetilde{U}'(e^{-ik\nu}, \lambda)|\, d\nu &\leq \int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}} \frac{1}{\lambda^2 |D^2(\sigma + i\tau, \lambda)|} \left[1 + \lambda k |\widehat{\beta}(\sigma + i\tau)| + \lambda |\widehat{\beta}'(\sigma + i\tau)| \right] d\nu \\ &\leq \int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}} \frac{k^2}{(C(\varepsilon))^2} \left(1 + \lambda k \widehat{\beta}(C(\varepsilon)) + \lambda |\widehat{\beta}'(C(\varepsilon))| \right) \\ &\leq Ck\lambda, \end{aligned}$$

so estimate (57) holds. This proves the lemma. ∎

3. Proof of Theorems 1 and 2. Here we adopt the overall strategy of Xu in proving our theorems, and we refer the reader to [21] for the preliminaries of the proof. We remark that Xu establishes the formula

$$U^n(\lambda) = \mathrm{Re} \left\{ \frac{1}{\pi t_{n-1} \lambda} \int_0^{\frac{\pi}{k}} e^{i\nu t_{n-2}} \frac{D_s(s(k,\nu), \lambda)}{D^2(s(k,\nu), \lambda)}\, d\nu \right\}. \tag{75}$$

Then, following [1], this integral is decomposed into the five parts:

$$U^n(\lambda) = \mathrm{Re}\{\lambda^{-1} U_1^n + \lambda^{-2} U_2^n + \lambda^{-3} U_3^n + U_4^n(\lambda) + U_5^n(\lambda)\}, \tag{76}$$

where

$$U_1^n = \frac{1}{\pi t_{n-1}} \int_0^\varepsilon e^{i\nu t_{n-2}} \frac{\widehat{\beta}'(s)}{[\widehat{\beta}(s)]^2}\, d\nu, \tag{77}$$

$$U_2^n = \frac{1}{\pi t_{n-1}} \int_0^\varepsilon e^{i\nu t_{n-2}} \frac{1}{[\widehat{\beta}(s)]^2} \left(1 - \frac{2s\widehat{\beta}'(s)}{\widehat{\beta}(s)} \right) d\nu, \tag{78}$$

$$U_3^n = \frac{-1}{\pi t_{n-1}} \int_0^\varepsilon e^{i\nu t_{n-2}} \frac{2s}{[\widehat{\beta}(s)]^3}\, d\nu, \tag{79}$$

$$U_4^n(\lambda) = \frac{1}{\pi t_{n-1\lambda^3}} \int_0^\varepsilon e^{i\nu t_{n-2}} \frac{s^2 D_s(s,\lambda)}{[\widehat{\beta}(s)]^2 D(s,\lambda)} \left(\frac{2}{\widehat{\beta}(s)} + \frac{1}{D(s,\lambda)} \right) d\nu, \tag{80}$$

$$U_5^n(\lambda) = \frac{1}{\pi t_{n-1\lambda}} \int_\varepsilon^{\frac{\pi}{k}} e^{i\nu t_{n-2}} \frac{D_s(s,\lambda)}{[D(s,\lambda)]^2}\, d\nu. \tag{81}$$

For $m > 2$, we will establish the estimates

$$|U_4^n(\lambda)| \leq Ct_{n-2}^{-2}, \qquad n \geq m, \qquad \lambda \geq \lambda_0 \tag{82}$$

and either

$$|U_5^n(\lambda)| \leq Ct_{n-2}^{-2}\, \lambda, \qquad n \geq m, \qquad \lambda \geq \lambda_0 \tag{83}$$

or

$$|U_5^n(\lambda)| \le C t_{n-2}^{-2}, \qquad n \ge m, \qquad \lambda \ge \lambda_0 \tag{84}$$

to prove Theorem 1 or 2, respectively. This follows, as then we could insert three different values of λ into (76) and, by utilizing (53), solve for the U_j^n ($j = 1, 2, 3$) to show

$$k \sum_{n=m+1}^{\infty} (|\mathrm{Re}\{U_1^n\}| + |\mathrm{Re}\{U_2^n\}| + |\mathrm{Re}\{U_3^n\}|) < \infty.$$

Then, (34) or (35) follow, giving us Theorem 1 and 2, respectively.

We show (82) first. Integrating (80) and (81) by parts, we obtain

$$\begin{aligned} U_4^n(\lambda) = {} & \frac{1}{i\pi t_{n-2} t_{n-1} \lambda^3} e^{i\nu t_{n-2}} \frac{s^2 D_s(s,\lambda)}{[\widehat{\beta}(s)]^2 D(s,\lambda)} \left(\frac{2}{\widehat{\beta}(s)} + \frac{1}{D(s,\lambda)} \right) \Bigg|_{\nu=0}^{\varepsilon} \\ & - \frac{1}{\pi t_{n-2} t_{n-1} \lambda^3} \int_0^{\varepsilon} e^{i\nu t_{n-3}} \left[\frac{2s D_s(s,\lambda) + s^2 \widehat{\beta}''(s)}{[\widehat{\beta}(s)]^2 D(s,\lambda)} \left(\frac{2}{\widehat{\beta}(s)} + \frac{1}{D(s,\lambda)} \right) \right. \\ & \qquad - \frac{s^2 D_s(s,\lambda)}{[\widehat{\beta}(s)]^2 D(s,\lambda)} \\ & \qquad \left. \left(\frac{6\widehat{\beta}'(s)}{[\widehat{\beta}(s)]^2} + \frac{4\widehat{\beta}'(s) + 2\lambda^{-1}}{\widehat{\beta}(s) D(s,\lambda)} + \frac{2D_s(s,\lambda)}{D^2(s,\lambda)} \right) \right] d\nu \end{aligned} \tag{85}$$

and

$$\begin{aligned} U_5^n(\lambda) = {} & \frac{1}{i\pi t_{n-2} t_{n-1} \lambda} e^{i\nu t_{n-2}} \frac{D_s(s,\lambda)}{D^2(s,\lambda)} \Bigg|_{\nu=\varepsilon}^{\frac{\pi}{k}} \\ & - \frac{1}{i\pi t_{n-2} t_{n-1} \lambda} \int_{\varepsilon}^{\frac{\pi}{k}} e^{i\nu t_{n-3}} \left[\frac{\widehat{\beta}''(s)}{D^2(s,\lambda)} - \frac{2D_s^2(s,\lambda)}{D^3(s,\lambda)} \right] d\nu. \end{aligned} \tag{86}$$

We see by (41) and (48) that for $0 < \nu \le \varepsilon$ with ε appropriately small,

$$|\widehat{\beta}(s)| = \sqrt{\phi^2(\sigma,\tau) + \tau^2\theta^2(\sigma,\tau)} \ge C\sqrt{\phi^2(\tau) + \tau^2\theta^2(\tau)} = C|\widehat{\beta}(i\tau)|. \tag{87}$$

Then, (36), (37), (58), (63), and (87) imply that the boundary term in (85) vanishes at $\nu = 0$. Note that [1, eq. (5.3)] and (37) give us, for $\sigma, \tau > 0$,

$$\begin{aligned} |\widehat{\beta}''(s)| &= \left| \frac{d^2}{d\tau^2} (e^{-\sigma t}\beta(t))\check{}(\tau) \right| \le C \int_0^{\frac{1}{\tau}} t^2 e^{-\sigma t} \beta(t)\, dt \\ &\le \frac{C}{\tau} \int_0^{\frac{1}{\tau}} t e^{-\sigma t} \beta(t)\, dt \le \frac{C}{\tau}\, \theta(\tau). \end{aligned} \tag{88}$$

Then, (36), (37), (58), (63), (87), and (88) allow us to establish the estimate (82) in a manner similar to (5.6) of [21].

To establish (83) and (84), we first note that as $\int_0^{\frac{2}{\sigma}} te^{-\sigma t}\, dt \ge \int_{\frac{2}{\sigma}}^{\infty} te^{-\sigma t}\, dt$ for all $\sigma > 0$, it follows from the decreasing nature of $\beta(t)$ and (36) that

$$\begin{aligned} |\widehat{\beta}'(\sigma)| &= \left(\int_0^{\frac{2}{\sigma}} + \int_{\frac{2}{\sigma}}^{\infty} \right) te^{-\sigma t} \beta(t)\, dt \le 2 \int_0^{\frac{2}{\sigma}} te^{-\sigma t}\beta(t)\, dt \\ &\le \frac{4}{\sigma} \int_0^{\frac{2}{\sigma}} e^{-\sigma t} \beta(t)\, dt \le \frac{C}{\sigma} \widehat{\beta}(\sigma). \end{aligned} \tag{89}$$

Also, as $\int_0^{\frac{1}{\tau}} e^{-\tau t}\,dt \geq \int_{\frac{1}{\tau}}^{\infty} e^{-\tau t}\,dt$ for all $\tau > 0$, we see by the decreasing nature of $\beta(t)$, (36), (42), (58), and (60) that, for $0 < \sigma \leq \varepsilon\tau < \tau$,

$$\begin{aligned} |D(s,\lambda)| &\geq C|\widehat{\beta}(\sigma+i\tau)| \geq C\int_0^{\frac{1}{\tau}} e^{-\sigma t}\beta(t)\,dt \geq C\int_0^{\frac{1}{\tau}} e^{-\tau t}\beta(t)\,dt \\ &\geq C_0\left(\int_0^{\frac{1}{\tau}} + \int_{\frac{1}{\tau}}^{\infty}\right) e^{-\tau t}\beta(t)\,dt = C\widehat{\beta}(\tau). \end{aligned} \tag{90}$$

Then, by (89) and (90), we are able to estimate the boundary terms in (86) as on pp. 148–149 of [21]. From this we obtain

$$|U_5^n(\lambda)| \leq Ct_{n-2}^{-2}\left[k^2 + \lambda^{-1} + \lambda^{-1}\left(\int_\varepsilon^{\frac{\pi}{k}} \frac{|\widehat{\beta}''(s)|}{|D^2(s,\lambda)|} + \frac{|D_s^2(s,\lambda)|}{|D^3(s,\lambda)|}\,d\nu\right)\right]. \tag{91}$$

We decompose the interval of integration into the three intervals $E_1 = \left[\varepsilon, \frac{\varepsilon}{k}\right), E_2 = \left[\frac{\varepsilon}{k}, \frac{\pi-\varepsilon}{k}\right)$, and $E_3 = \left[\frac{\pi-\varepsilon}{k}, \frac{\pi}{k}\right)$. To estimate the integral on E_1, we note that when $\nu \in E_1$, we have $0 < \sigma \leq \varepsilon\tau < \tau$. So by (17), (37), (41), (47), (63), (65), (66), and (88), it follows that

$$\begin{aligned} &\lambda^{-1}\int_\varepsilon^{\frac{\varepsilon}{k}} \left(\frac{|\widehat{\beta}''(s)|}{|D(s,\lambda)|^2} + \frac{|D_s(s,\lambda)|^2}{|D(s,\lambda)|^3}\right) d\nu \\ &\quad\leq C\lambda^{-1}\left(\int_{\frac{\varepsilon}{2}}^{\frac{\omega}{2}} \frac{\lambda^{-2}+\theta^2(\tau)}{\tau^3\theta^3(\tau)}\,d\tau + \int_{\frac{\omega}{2}}^{2\omega} \frac{\lambda^{-1}}{\tau\phi^2(\tau)}\,d\tau + \int_{\frac{\omega}{2}}^{2\omega} \frac{\lambda^{-2}}{\phi^3(\tau)}\,d\tau \right.\\ &\qquad\qquad \left. + \int_{2\omega}^{\infty} \frac{\lambda\,d\tau}{\tau|\tau-\omega|^2} + \int_{2\omega}^{\infty} \frac{\lambda\,d\tau}{|\tau-\omega|^3}\right). \end{aligned} \tag{92}$$

Then, by (37), (46), (47), and either (17) or (18), we obtain

$$\lambda^{-1}\int_\varepsilon^{\frac{\varepsilon}{k}} \left(\frac{|\widehat{\beta}''(s)|}{|D(s,\lambda)|^2} + \frac{|D_s(s,\lambda)|^2}{|D(s,\lambda)|^3}\right) d\nu \leq C(\lambda+1) \tag{93}$$

or

$$\lambda^{-1}\int_\varepsilon^{\frac{\varepsilon}{k}} \left(\frac{|\widehat{\beta}''(s)|}{|D(s,\lambda)|^2} + \frac{|D_s(s,\lambda)|^2}{|D(s,\lambda)|^3}\right) d\nu \leq C, \tag{94}$$

respectively.

To estimate the integrals on E_2 and E_3, we first note that as $\int_0^{\frac{3}{\sigma}} t^2e^{-\sigma t}\,dt \geq \int_{\frac{3}{\sigma}}^{\infty} t^2e^{-\sigma t}\,dt$ for all $\sigma > 0$, the decreasing nature of $\beta(t)$ gives us, for $\sigma > 0$,

$$\begin{aligned} |\widehat{\beta}''(s)| &= \left|\int_0^\infty t^2e^{-i\tau t}(e^{-\sigma t}\beta(t))\,dt\right| \leq \int_0^\infty t^2e^{-\sigma t}\beta(t)\,dt = \widehat{\beta}''(\sigma) \\ &= \left(\int_0^{\frac{3}{\sigma}} + \int_{\frac{3}{\sigma}}^{\infty}\right) t^2e^{-\sigma t}\beta(t)\,dt \leq 2\int_0^{\frac{3}{\sigma}} t^2e^{-\sigma t}\beta(t)\,dt \\ &\leq \frac{18}{\sigma^2}\int_0^{\frac{3}{\sigma}} e^{-\sigma t}\beta(t)\,dt \leq \frac{C}{\sigma^2}\widehat{\beta}(\sigma). \end{aligned} \tag{95}$$

Then, in the case where $\nu \in E_2$, we see that $\sigma \le \frac{2}{k}$ and $\tau \ge \frac{\sin(\varepsilon)}{k}$. Then (39) and (40) give, for $\nu \in E_2$,

$$\phi(\sigma,\tau) \ge -C\int_0^{\frac{1}{\tau}} te^{-\sigma t}\beta'(t)\,dt \ge -Ce^{\frac{-\sigma}{\tau}}\int_0^{\frac{1}{\tau}} t\beta'(t)\,dt$$

$$\ge -Ce^{\frac{-2}{\sin(\varepsilon)}}\int_0^{\frac{1}{\tau}} t\beta'(t)\,dt \ge C\phi(\tau). \tag{96}$$

So, we see that (17), (63), (88), and (96) give us

$$\lambda^{-1}\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon}{k}} \frac{|\widehat{\beta}''(s)|}{|D^2(s,\lambda)|} + \frac{|D_s^2(s,\lambda)|}{|D^3(s,\lambda)|}\,d\nu$$

$$\le C\lambda^{-1}\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon}{k}} \frac{k\lambda\theta(\tau)}{\tau\phi(\tau)} + k^3\lambda + \frac{k^2\lambda\theta(\tau)}{\phi(\tau)} + \frac{k\lambda\theta^2(\tau)}{\phi^2(\tau)}\,d\nu$$

$$\le C. \tag{97}$$

For the integral on E_3, note that since we have $\frac{2-\varepsilon}{k} \le \sigma \le \frac{2}{k}$ and $0 \le \tau \le \frac{\varepsilon}{k}$ for $\nu \in E_3$, we see by (52), (89), and (95) that

$$\lambda^{-1}\int_{\frac{\pi-\varepsilon}{k}}^{\frac{\pi}{k}} \frac{|\widehat{\beta}''(s)|}{|D^2(s,\lambda)|} + \frac{|D_s^2(s,\lambda)|}{|D^3(s,\lambda)|}\,d\nu$$

$$\le C\lambda^{-1}\int_{\frac{\pi-\varepsilon}{k}}^{\frac{\pi}{k}} \frac{\lambda\widehat{\beta}(\sigma)}{\sigma^3\widehat{\beta}(\sigma)} + \frac{\lambda}{\sigma^3} + \frac{\lambda\widehat{\beta}(\sigma)}{\sigma^3\widehat{\beta}(\sigma)} + \frac{\lambda[\widehat{\beta}(\sigma)]^2}{\sigma^3[\widehat{\beta}(\sigma)]^2}\,d\nu$$

$$\le C. \tag{98}$$

Then, by (97), (98), and either (93) or (94), we have established (83) or (84), respectively, and thus we have proven Theorems 1 or 2, respectively.

REFERENCES

[1] R. W. Carr and K. B. Hannsgen, *A nonhomogeneous integrodifferential equation in Hilbert space*, SIAM J. Math. Anal., 10 (1979), pp. 961–984.
[2] R. W. Carr and K. B. Hannsgen, *Resolvent formulas for a Volterra equation in Hilbert space*, SIAM J. Math. Anal., 13 (1982), pp. 459–483.
[3] E. Cuesta, C. Lubich, and C. Palencia, *Convolution quadrature time discretization of fractional diffusive-wave equations*, Math. Comp., 75 (2006), pp. 673–696.
[4] P. L. Duren, *The Theory of H^p Spaces*, Academic, New York, 1970.
[5] G. Gripenberg, S-O. Londen, and O. Staffens, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
[6] K. B. Hannsgen, *Indirect Abelian theorems and a linear Volterra equation*, Trans. Amer. Math. Soc., 142 (1969), pp. 539–555.
[7] C. Lubich, *Convolution quadrature and discretized operational calculus* I, Numer. Math., 52 (1988), pp. 129–145.
[8] C. Lubich, I. H. Sloan, and V. Thomée, *Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term*, Math. Comp., 65 (1996), pp. 1–17.
[9] W. Mclean and K. Mustapha, *A second-order accurate numerical method for a fractional wave equation*, Numer. Math., 105 (2007), pp. 481–510.
[10] W. McLean and V. Thomée, *Numerical solution of an evolution equation with a positive type memory term*, J. Aust. Math. Soc., 35 (1993), pp. 23–70.
[11] W. McLean and V. Thomee, *Numerical solution via Laplace transforms of a fractional order evolution equation*, J. Integral Equations Appl., 22 (2010), pp. 57–94.

[12] K. Mustapha and W. McLean, *Discontinuous Galerkin method for an evolution equation with a memory term of positive type*, Math. Comp., 78 (2009), pp. 1975–1995.
[13] J. A. Nohel and D. F. Shea, *Frequency domain methods for Volterra equations*, Adv. Math., 22 (1976), pp. 278–304.
[14] R. D. Noren, *Uniform L^1 behavior for the solution of a Volterra equation with a parameter*, SIAM J. Math. Anal., 19 (1988), pp. 270–286.
[15] J. Prüss, *Evolutionary Integral Equations and Applications*, Monogr. Math. 87, Birkhäuser, Basel, 1993.
[16] M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Longman, London, 1987.
[17] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover, New York, 1990.
[18] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, Boston, 1987.
[19] D. F. Shea and S. Wainger, *Variants of the Wiener–Levy theorem with applications to stability problems for some Volterra integral equations*, Amer. J. Math., 97 (1975), pp. 312–343.
[20] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, NJ, 1946.
[21] D. Xu, *Uniform l^1 behavior for time discretization of a Volterra equation with completely monotonic kernel:* I. *Stability*, IMA J. Numer. Anal., 22 (2002), pp. 133–151.
[22] D. Xu, *Uniform l^1 behavior for time discretization of a Volterra equation with completely monotonic kernel:* II. *Convergence*, SIAM J. Numer. Anal., 46 (2008), pp. 231–259.
[23] D. Xu, *Stability of the difference type methods for linear Volterra equations in Hilbert space*, Numer. Math., 109 (2008), pp. 571–595.
[24] D. Xu, *Uniform l^1 error bounds for the semidiscrete solution of a Volterra equation with completely monotonic convolution kernel*, Comput. Math. Appl., 43 (2002), pp. 1303–1318.