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Dynamics of a cosine-root family

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DYNAMICS OF A COSINE-ROOT FAMILY

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ABSTRACT

We will investigate the chaotic dynamics of the complex cosine-root family, $C_{\alpha,\beta}(z) = \alpha \cos \sqrt{z+\beta}$, where $\alpha, \beta, z \in \mathbb{C}$. We show how to approximate the Julia set of each $C_{\alpha,\beta}$ and discuss some of its features. We also plot parameter space pictures, highlighting phenomena resembling that of quadratic polynomials. Finally, we make an analysis of super-attracting cycles for this family.

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CHAPTER 1

INTRODUCTION

1.1 Background

1.1.1 Basics of Real Dynamics

In studying the dynamics of a function, we wish to understand the asymptotic behavior of points under iteration of the function. Suppose we have a set S and a function $f : S \to S$.

Definition 1. *If n is a positive integer, then the n-fold composition of f, $f^{\circ n}(x) = f(f(...(f(x)))$ is called the nth iterate of f.*

For any $x \in S$, we want to understand the the sequence of points $\{x, f(x), f(f(x)), ...\}$. Does this sequence converge? If not, does it diverge? Does it contain any convergent subsequences? We call the set of numbers in this sequence the forward orbit of x under f.

Definition 2. *The forward orbit of x under f is the set of points $\{x, f(x), f(f(x)), ...\}$.*

We can see that if $f(x) = x$, then the forward orbit of x is just the set $\{x\}$. If $f^{\circ k}(x) = x$ for some integer k but $f^{\circ i}(x) \neq x$ for $i < k$, then the forward orbit of x is the set $\{x, f(x), ..., f^{\circ(k-1)}(x)\}$.

Definition 3. *The point x is a periodic point of prime period k if k is the smallest positive integer such that $f^{\circ k}(x) = x$. If $k = 1$ then x is a fixed point of f.*

So, if x is a periodic point of prime period k, then x is a fixed point of the function $f^{\circ k}$. If x is a periodic point of prime period k, then f has a k-cycle, containing the set of points $\{x, f(x), f^{\circ 2}(x), ..., f^{\circ(k-1)}(x)\}$. Choosing any element of this k-cycle and iterating f will produce the cycle.

If f is differentiable at a fixed point x, then the derivative of f at x provides some information about the orbits of points near x:

Definition 4. *If $\left|f'(x)\right| < 1$ then x is an attracting fixed point of f. That is, there is an open interval U around x such that $f(U) \subset U$. If $f'(x) = 0$ then x is a superattracting fixed point of f.*

Definition 5. *If $\left|f'(x)\right| > 1$ then x is a repelling fixed point of f.*

Definition 6. *If $\left|f'(x)\right| = 1$ then x is a non-hyperbolic, or indifferent, fixed point of f.*

We can think of repelling fixed points as “sources” and attracting fixed points as “sinks”; points near an attracting fixed point are drawn into the fixed point, while points near a repelling fixed point are pushed away from the fixed point. Note that cycles can also be sources and sinks: to determine if a k-cycle of f is attracting or repelling, we look at the value of $(f^{\circ k})'$ at any point in the cycle.

Indifferent fixed points and cycles may be sources, sinks, or neither.

1.1.2 Basics of Complex Dynamics

Working in the complex plane, it is difficult to graph functions, since a graph would require 4-dimensional space. However, we can observe the effects of attracting fixed points. One useful image is the Julia set.

Definition 7. *A set $\{f_n\}$ of analytic functions defined on an open set U is a normal family on U if every sequence of the f_n's contains a subsequence which converges uniformly on compact subsets of U. [1]*

As an example, we look at the quadratic function $Q(z) = z^2$. The set $\{Q^{\circ n}(z)\}$ is normal on the open unit disk $\{z_0 : |z_0| < 1\}$, and also on the set $\{z_0 : |z_0| > 1\}$, but not on any set containing some z_0 with $|z_0| = 1$.

Definition 8. *The Fatou set of f (also called the normal set or stable set) is the subset of $\mathbb{C}$ on which $\{f^{\circ n}\}$ is normal.*

The orbits of points in the Fatou set are similar to those of nearby points. If f is a polynomial, all points in the Fatou set will either be attracted to a fixed point (or cycle) or will diverge to infinity.

Definition 9. *The filled Julia set of a polynomial f, denoted K_f, is the set of all points with bounded orbits under f.*

Notice that K_f is not necessarily a subset of the Fatou set. In fact, the boundary of K_f is not in the Fatou set.

Definition 10. *The Julia set of f, denoted J_f, is the complement of the Fatou set.*

The Julia set J_f has the following properties:

1. If f is a polynomial, then J_f is the boundary of K_f.

2. The Julia set is completely invariant. That is, $f(J_f) \subseteq J_f$ and $f^{-1}(J_f) \subseteq J_f$.

3. If f is rational, then J_f is either nowhere dense or is the entire Riemann sphere $\hat{\mathbb{C}}$.

4. If $z_0 \in J_f$ and U is some open set containing z_0 then

$$\bigcup_{n=1}^{\infty} f^n(U)$$

is the entire complex plane, with the exception of at most one point.

We can see from the above properties that the Julia set is chaotic. That is, the orbit of $z \in J_f$ is very different from the orbits of nearby points.

Studying calculus, we learn the importance of critical points - the points at which the derivative of a function is equal to 0. In complex dynamics these points and their images will tell us a lot about the behavior of a function.

Definition 11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Then z is a critical value of f if there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = z$ and $f'(z_0) = 0$. That is, critical values are the images of critical points.*

Critical values play an important role in the study of dynamics. If f has an attracting fixed point, it must attract a critical value of f.

1.1.3 Transcendental Dynamics

There is a great difference between the dynamics of polynomial and transcendental functions. Picard's Theorem tells us that for a transcendental function f, given any "neighborhood of infinity" U (in other words, $U = \{z : |z| > r\}$ for any $r \in \mathbb{R}$), $f(U)$ covers $\mathbb{C}$ with the exception of at most one point. This is certainly not true for polynomials: if p is a polynomial then we can find a neighborhood of infinity, U, so that $f(U) \subset U$.

For transcendental functions, such as e^z, points with unbounded orbits are not in the Fatou set; they must therefore be in the Julia set. So instead of defining the Julia set to be the boundary of the filled Julia set, we define it to be the set of points with unbounded orbits.

For exponential functions, a point is in the Julia set if its orbit contains a point whose real part exceeds 50. Some manipulation gives us that for complex sines and cosines, a point is in the Julia set if its orbit contains a point z with $|\mathrm{Im}(z)| \geq 50$.

1.2 Preliminaries

1.2.1 Julia Sets

We define the cosine-root function $C_{\alpha,\beta}(z) = \alpha \cos \sqrt{z+\beta}$. We use Marilyn Durkin's [2] criterion for determining if a point "escapes" under iteration of our function $C_{\alpha,\beta}$. Since $C_{\alpha,\beta}$ is basically a complex cosine function, we can manipulate the criteria for the Julia set of $\cos(z)$. A point z is in the Julia set of $C_{\alpha,\beta}$ if $|\mathrm{Im}(z)| \geq 50$, and one can easily check that $|\mathrm{Im}(z)| > 50$ if and only if $\mathrm{Re}(z) \geq \frac{(\mathrm{Im}(z))^2}{10{,}000} - 2500$. If $\mathrm{Re}(z) \geq \frac{(\mathrm{Im}(z))^2}{10{,}000} - 2500$, we say that z is inside our bounding parabola. If for all points z_k in the forward orbit of z_0 we have z_k inside our bounding parabola, then we say that the orbit of z_0 is bounded under iteration of $C_{\alpha,\beta}$, and so z_0 is not in the Julia set of z.

When $\alpha, \beta \in \mathbb{R}$, $C_{\alpha,\beta}$ can be written as a differentiable function $f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, such that

$$f_{\alpha,\beta}(x) = \begin{cases} \alpha \cos \sqrt{x+\beta}, & x \geq -\beta \\ \alpha \cosh \sqrt{|x+\beta|}, & x < -\beta \end{cases} . \tag{1.1}$$

We use the following algorithm to draw computer-generated Julia sets in order to observe the dynamics of a given function $C_{\alpha,\beta}$:

1. Choose a rectangle in the complex plane.

2. For each point z in the rectangle, examine the forward orbit of z; that is, examine the set $O(z) = \{z, f(z), f^{\circ 2}(z), ..., f^{\circ N}(z)\}$, where N is some positive integer (normally we choose N between 50 and 1000).

3. If every value in $O(z)$ is inside our bounding parabola, leave z black. Otherwise, color z some color according to the smallest k such that $f^{\circ k}$ is not inside our parabola.

Once the picture is complete, we have a set of colored points that are in the Julia set; the black points represent those points whose orbit is bounded under $C_{\alpha,\beta}$. We must keep in mind that the calculations made to generate the image are numerical and finite, and so can only give us an approximation of what the Julia set looks like.

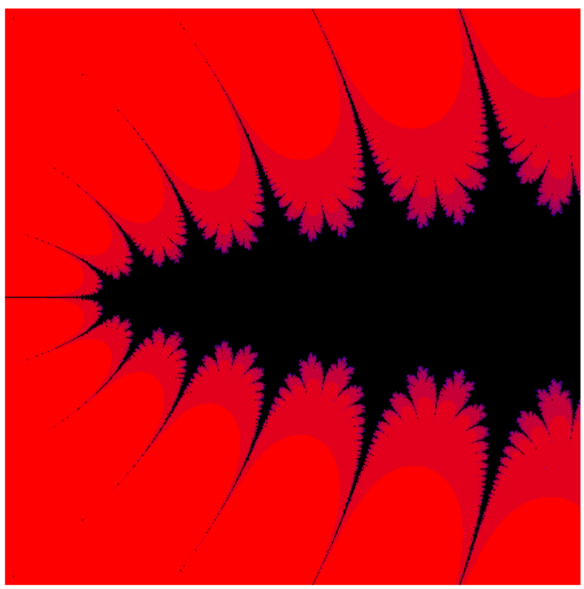

Figure 1.1: Julia Set for $C_{1,0}$, with $-400 \leq \text{Re}(z) \leq 2000$ and $|\text{Im}(z)| \leq 1200$

1.2.2 Tracking the Critical Values

When $\alpha \neq 0$, $C_{\alpha,\beta}$ has two distinct critical values, $\pm\alpha$. As with quadratic maps, which only have a single critical value, we can gain some insight into the behavior of $C_{\alpha,\beta}$ by knowing the behavior of its critical values under iteration. For each critical value, there are two possible outcomes of iteration under $C_{\alpha,\beta}$: it will either escape or remain bounded. We would like to generate some parameter-space bifurcation set images; the problem is that our parameter space, $\mathbb{C} \times \mathbb{C}$, is four-dimensional. So, we restrict ourselves by choosing two of the following real-valued parameters: $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Im}(\beta)$. Then we use the following algorithm to generate an image of the parameter space:

1. Choose a rectangle in the chosen parameter-space plane.
2. Each point in our rectangle represents some function $C_{\alpha,\beta}$. For each point, examine the forward orbit of $\pm\alpha$ under $C_{\alpha,\beta}$, using the same method as when we generate Julia set images.
3. Use the following table to determine the color of the point corresponding $C_{\alpha,\beta}$.

$\alpha \backslash -\alpha$	escapes	bounded
escapes	White	Red
bounded	Green	Black

Table 1.1: Coloring of the Bifurcation Sets

1.3 General Statements

Proposition 1. *The Julia set for* $C_{\alpha,\beta}$ *is a reflection, across the real axis, of the Julia set for* $C_{\overline{\alpha},\overline{\beta}}$.

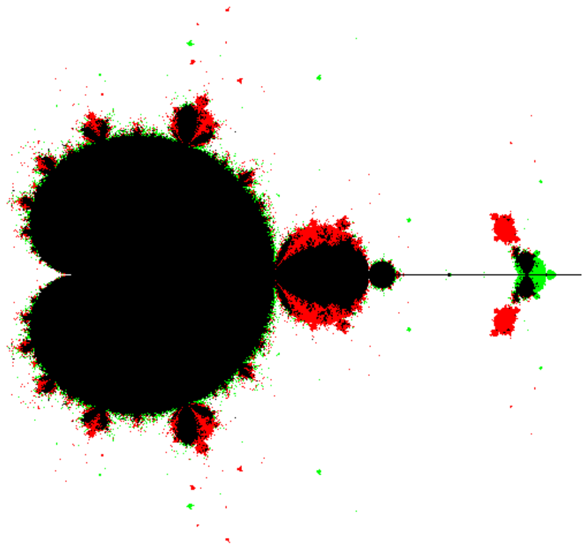

Figure 1.2: Parameter Space of $C_{\alpha,0}$, with $2.2 \le \mathrm{Re}(\alpha) \le 7.8$ and $|\mathrm{Im}(\alpha)| \le 5$

Proof. Let α, β be any complex numbers. Note that for every $z \in \mathbb{C}$,

$$\begin{aligned}\overline{C_{\alpha,\beta}(z)} &= \overline{\alpha \cos \sqrt{z+\beta}}\\ &= \overline{\alpha} \cos \sqrt{\overline{z}+\overline{\beta}}\\ &= C_{\overline{\alpha},\overline{\beta}}(\overline{z}).\end{aligned}$$

Let z_0 be any complex number. We see that for any $k \in \mathbb{N}$, we have $\overline{C^{\circ k}_{\alpha,\beta}(z_0)} = C^{\circ k}_{\overline{\alpha},\overline{\beta}}(\overline{z_0})$. Our test for boundedness of any z is a comparison between $\mathrm{Re}(z)$ and $\mathrm{Im}(z)^2$; so z is inside our bounding parabola if and only if $\overline{z}$ is inside our bounding parabola. Therefore, z_0 escapes under iteration of $C_{\alpha,\beta}$ if and only if $\overline{z_0}$ escapes under iteration of $C_{\overline{\alpha},\overline{\beta}}$. $\square$

CHAPTER 2

THE α SLICE

2.1 Real Dynamics when $\beta = 0$

Here we simplify $C_{\alpha,\beta}$ by using a special case of Equation (1.1). As $\beta = 0$, the parameter plane is a copy of $\mathbb{C}$ with axes $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$. Denote

$$f_\alpha(x) = f_{\alpha,0}(x) = \begin{cases} \alpha \cos \sqrt{x}, & x \geq 0 \\ \alpha \cosh \sqrt{|x|}, & x < 0 \end{cases}.$$

Proposition 2. *If $\alpha \geq 0$, then the orbit of every $x \in \mathbb{R}$ under $f_\alpha(x) = \alpha \cos \sqrt{x}$ is bounded.*

Proof. If $x \geq -\alpha$, then $-\alpha \leq f_\alpha(x) \leq \alpha \cosh \sqrt{\alpha}$. If $x < -\alpha$, then $f_\alpha(x) > 0 \geq -\alpha$; then, $-\alpha \leq f_\alpha^{\circ n}(x) \leq \alpha \cosh \sqrt{\alpha}$ for all $n \geq 1$. □

This confirms an observation we make while examining the parameter plane: when $\beta = 0$, the positive real α-axis is black.

Proposition 3. *If $\alpha < -2$, then the orbit under f_α of every $x < 0$ is unbounded. Moreover, if for any $k \in \mathbb{N}$, x falls in the closed interval $[(\frac{4k-3}{2}\pi)^2, (\frac{4k-1}{2}\pi)^2]$, then the orbit if x is unbounded.*

Proof. First, suppose $x < 0$. Then

$$\begin{aligned} f_\alpha(x) &= \alpha \cosh \sqrt{|x|} \\ &= \frac{\alpha}{2}(e^{\sqrt{|x|}} + e^{-\sqrt{|x|}}) \\ &< -(e^{\sqrt{|x|}} + e^{-\sqrt{|x|}}) \\ &< -e^{\sqrt{|x|}} \\ &< x - 1. \end{aligned}$$

We see, then, that the sequence $\{x, f_\alpha(x), f_\alpha^{\circ 2}(x), f_\alpha^{\circ 3}(x), ...\}$, the forward orbit of x under f_α, is decreasing and unbounded.

The second case, with $x \in [(\frac{4k-3}{2}\pi)^2, (\frac{4k-1}{2}\pi)^2]$ for some $k \in \mathbb{N}$, follows directly from the first: let $y = f_\alpha(x) = \alpha \cos \sqrt{x} < 0$. The orbit of y is unbounded, and therefore the orbit of x is unbounded. □

This also confirms an observation stemming from the bifurcation sets. When $\alpha < -2$, the orbit of α under f_α is unbounded. Therefore, we should never see black or green points on the negative real α-axis (to the left of -2).

The following lemma proves that as x increases, the increasing period of f_α causes the derivative to "flatten out".

Lemma 1. *Suppose p_1, p_2, and α are real numbers satisfying:*

1. $0 < p_1 < p_2$,
2. $f_\alpha(p_1)$ *and* $f_\alpha(p_2)$ *are positive, and*
3. $\left|f'_\alpha(p_1)\right| \leq \left|f'_\alpha(p_2)\right| \neq 0$.

Then $f_\alpha(p_1) > f_\alpha(p_2)$.

Proof. We have

$$\left|\frac{-\alpha\sin\sqrt{p_1}}{2\sqrt{p_1}}\right| = \left|f'_\alpha(p_1)\right| \leq \left|f'_\alpha(p_2)\right| = \left|\frac{-\alpha\sin\sqrt{p_2}}{2\sqrt{p_2}}\right|,$$

and since $f'_\alpha(p_2) \neq 0$,

$$\left|\sin\sqrt{p_1}\right| \leq \frac{\left|-2\alpha\sqrt{p_1}\right|}{\left|-2\alpha\sqrt{p_2}\right|}\left|\sin\sqrt{p_2}\right| < \left|\sin\sqrt{p_2}\right|.$$

Now suppose $\left|\cos\sqrt{p_1}\right| \leq \left|\cos\sqrt{p_2}\right|$. This tells us that

$$1 = \sin^2\sqrt{p_1} + \cos^2\sqrt{p_1} < \sin^2\sqrt{p_2} + \cos^2\sqrt{p_2} = 1;$$

thus $\left|\cos\sqrt{p_1}\right| > \left|\cos\sqrt{p_2}\right|$. Then

$$\begin{aligned} f_\alpha(p_1) &= |f_\alpha(p_1)| \\ &= |\alpha\cos\sqrt{p_1}| \\ &> |\alpha\cos\sqrt{p_2}| \\ &= |f_\alpha(p_2)| \\ &= f_\alpha(p_2). \end{aligned}$$

□

The next result classifies all fixed points of f_α for $\alpha = (2n\pi)^2$.

Theorem 4. *If $\alpha = (2n\pi)^2$, with $n \in \mathbb{N}$, then f_α has exactly $2n+1$ fixed points $\{x_i \mid 0 < x_1 < x_2 < x_3 < \ldots < x_{2n+1} = \alpha\}$, where x_{2n+1} is superattracting and the rest are repelling. Similarly, if $\alpha = -((2n-1)\pi)^2$, with $n \in \mathbb{N}$, then f_α has exactly*

$2n$ fixed points $\{x_i \mid \frac{\pi^2}{4} < x_1 < x_2 < ... < x_{2n} = |\alpha|\}$, where x_{2n} is superattracting and the rest are repelling.

Proof. Suppose $\alpha = (2n\pi)^2$. It is easy to check that $f_\alpha(\alpha) = \alpha$ and $f'_\alpha(\alpha) = 0$, giving us a superattracting fixed point at $x = \alpha$. Since $f_\alpha(x) > 0$ when $x \leq 0$ and $f_\alpha(x) \leq x$ when $x \geq \alpha$, any other fixed points of f_α must be contained in the interval $I = (0, \alpha)$.

For $j = 1, 2, 3, ..., 2n-1$, let $p_j = (j\pi)^2$. So $f_\alpha(p_j) = \pm\alpha$ for all j; thus p_j cannot be a fixed point. So we can break I into $2n$ disjoint open intervals $\{(0, p_1), (p_1, p_2), ..., (p_{2n-1}, \alpha)\}$, knowing that every fixed point in I has to fall into one of these. Let $I_j = (p_{j-1}, p_j)$ be any of these intervals. We see that f_α is either strictly increasing or strictly decreasing, with a range of $(-\alpha, \alpha)$, on I_j. Let $g(x) = f_\alpha(x) - x$ be a function defined on I_j. Then the range of g must contain the interval $(-\alpha - p_{j-1}, \alpha - p_j)$; thus the Intermediate Value Theorem tells us that there is some $x_j \in I_j$ such that $g(x_j) = 0$, and so x_j is a fixed point of f_α.

Since $f'_\alpha(x_{2n+1}) = 0$, we know the graph of f_α is above the line $y = x$ for all x in the open interval (x_{2n}, x_{2n+1}); therefore $f'_\alpha(x_{2n}) \geq 1$. Now suppose some fixed point $0 < x_i < x_{2n}$ is non-repelling; that is, $\left|f'_\alpha(x_i)\right| \leq 1$. Then Lemma 1 tells us that $f_\alpha(x_i) > f_\alpha(x_{2n}) = x_{2n} > x_i$, and so x_i cannot be a fixed point.

We need to show that f_α has no more than $2n$ fixed points on the open interval $(0, |\alpha|)$. Suppose there are more than $2n$ fixed points; that is, some open I_j, as defined above, must contain two fixed points $t_1 < t_2$. We know that f_α is either increasing or decreasing on I_j. If it were monotonically decreasing, we would have no fixed points. So f_α is increasing on I_j. The Mean Value Theorem now provides us with a $t_3 \in (t_1, t_2)$ such that $f'_\alpha(t_3) = 1$. Since t_2 cannot be an attracting fixed point, we know that $\left|f'_\alpha(t_2)\right| > 1$. Then the points t_2 and t_3 fit the criteria for Lemma 1, which tells us that $f_\alpha(t_2) < f_\alpha(t_3)$; but $t_2 > t_3$ and f_α is increasing. Therefore f_α has no more than $2n+1$ fixed points.

Now suppose $\alpha = -((2n-1)\pi)^2$. It is easy to check that $f_\alpha(|\alpha|) = |\alpha|$ and $f'_\alpha(|\alpha|) = 0$, giving us a superattracting fixed point at $x = |\alpha|$. Since $f_\alpha(x) < x$ when $x \leq \frac{\pi^2}{4}$ (this follows from the proof of Proposition 3) and $f_\alpha(x) < x$ when $x > |\alpha|$, any other fixed points of f_α must be contained in the interval $I = (\frac{\pi^2}{4}, |\alpha|)$.

The remainder of this proof is identical to the first case; the only difference is that $p_{2n-1} = |\alpha|$, and so there are only $2n-1$ intervals, resulting in exactly $2n$ fixed points. $\square$

Corollary 5. *Let $\beta = 0$ and let α_0 be of the form $(2n\pi)^2$ or $-((2n-1)\pi)^2$, where n is an integer. Then there exists a neighborhood D around $|\alpha_0|$ such that when $\alpha \in D$, the orbit of at least one critical value of the function $C_{\alpha,\beta}$ is bounded.*

Corollary 5 tells us that in the parameter space for f_{α_0}, there should be a region around $|\alpha_0|$ without any white.

2.1.1 Saddle-Node Bifurcations and 2-Cycles

For ease of notation, define two sets S^+ and S^-, representing the positive and negative superattracting fixed points of $f_{\alpha,\beta}$; so $S^+ = \{(2\pi)^2, (4\pi)^2, (6\pi)^2, ...\}$ and $S^- = \{-\pi^2, -(3\pi)^2, -(5\pi)^2, ...\}$.

Theorem 4 uses the periodicity of the cosine function. As we increase $|\alpha|$, the amplitude of our cosine wave, we see more phases reaching and crossing the line $y = x$. When α takes some value in S^+ or S^-, we see that $y = x$ crosses the peak of one of our cosine phases, representing a superattracting fixed point of our function f_α. Looking at the graphs of f_α near $x = |\alpha|$, we see that the function is nearly tangent to the line $y = x$; there is a repelling fixed point very close to α. This leads to the observation that for every α in S^+ or S^-, there is a nearby $\hat{\alpha}$, with $|\hat{\alpha}| < |\alpha|$, such that $f_{\hat{\alpha}}$ has a fixed point and a saddle-node bifurcation at $\hat{\alpha}$. However, the value of $\hat{\alpha}$ is not easy to find; we would need to solve the simultaneous equations

$$f_{\hat{\alpha}}(x_0) = \hat{\alpha}\cos\sqrt{x_0} = x_0$$

and

$$f'_{\hat{\alpha}}(x_0) = \frac{-\hat{\alpha}\sin\sqrt{x_0}}{2\sqrt{x_0}} = 1,$$

where x_0 is some value slightly less than $|\alpha|$. This is not easy to do by hand, so we use a computer to find a numerical approximation for $\hat{\alpha}$. First, we combine the equations to find that $x_0 = -2 \pm \sqrt{4+\hat{\alpha}^2}$; we then replace this for x_0 to see that $\hat{\alpha}$ must be a solution to both of the equations

$$-2 \pm \sqrt{4+\hat{\alpha}^2} = \hat{\alpha}\cos\sqrt{-2 \pm \sqrt{4+\hat{\alpha}^2}}$$

and

$$2\sqrt{-2 \pm \sqrt{4+\hat{\alpha}^2}} = -\hat{\alpha}\sin\sqrt{-2 \pm \sqrt{4+\hat{\alpha}^2}}.$$

In addition to saddle-node bifurcations; we conjecture that every α in S^+ or S^- has a nearby α_2, with $|\alpha_2| > |\alpha|$ such that $|\alpha_2|$ is in a superattracting period 2 cycle under f_{α_2}. To find these values of α_2, we use a computer to find numerical estimates of the solutions to the simultaneous equations

$$f_{\alpha_2}^{\circ 2}(|\alpha_2|) = \alpha_2\cos\sqrt{\alpha_2\cos\sqrt{|\alpha_2|}} = |\alpha_2|$$

and

$$(f_{\alpha_2}^{\circ 2})'(|\alpha_2|) = \frac{\alpha_2^2 \left(\sin\sqrt{|\alpha_2|}\right)\left(\sin\sqrt{\alpha_2 \cos\sqrt{|\alpha_2|}}\right)}{4\sqrt{\alpha_2\,|\alpha_2|\cos\sqrt{|\alpha_2|}}} = 0$$

$$\Downarrow$$

$$\left(\sin\sqrt{|\alpha_2|}\right)\left(\sin\sqrt{\alpha_2 \cos\sqrt{|\alpha_2|}}\right) = 0.$$

The following tables show the approximate values of $\hat{\alpha}$ and α_2 associated with the α values which give superattracting fixed points:

α	$\hat{\alpha}$	α_2
$(2\pi)^2 \approx$ 39.47841760	37.46100104	47.02763326
$(4\pi)^2 \approx$ 157.91367042	155.90941637	165.78516725
$(6\pi)^2 \approx$ 355.30575844	353.30387576	363.24708948
$(8\pi)^2 \approx$ 631.65468167	629.65362423	639.62131805
$(10\pi)^2 \approx$ 986.96044011	984.95976381	994.93902676

Table 2.1: Saddle-node bifurcations and 2-cycles for $\alpha > 0$

α	$\hat{\alpha}$	α_2
$-\pi^2 \approx$ -9.86960440	-7.79271815	-16.61749015
$-(3\pi)^2 \approx$ -88.82643961	-86.81883152	-96.60621288
$-(5\pi)^2 \approx$ -246.74011003	-244.73739492	-254.65643566
$-(7\pi)^2 \approx$ -483.61061565	-481.60923370	-491.56721848
$-(9\pi)^2 \approx$ -799.43795649	-797.43712132	-807.41154236

Table 2.2: Saddle-node bifurcations and 2-cycles for $\alpha < 0$

It appears that the attracting 2-cycle for f_{α_2} is the sequence $\{|\alpha|, |\alpha_2|, |\alpha|, ...\}$, where α is the element of $S^+ \cup S^-$ which is closest to α_2.

We see that as we move α along the real axis, f_α, near its attracting fixed points, behaves much like the quadratic map $Q_c(z) = z^2 + c$, with $c \in \mathbb{R}$: we see a saddle-node bifurcation (at $\hat{\alpha}$) and attracting 2-cycles (near α_2). So it is not too surprising that we should see Mandelbrot-like sets in the parameter space of our complex function $C_{\alpha,\beta}$ if we vary α off of the real axis (see Section "The α Slice")

2.1.2 Superattracting 3-Cycles

We know from Sarkovskii's theorem that cycles of prime period 3 are the "strongest;" the existence of a period-3 cycle implies the existence of cycles of any other period. We examine a special case of these: the set of superattracting period-3 cycles, specifically those that contain $-\alpha$. That is, we would like to find those values of α for which $f_\alpha^{\circ 3}(-\alpha) = -\alpha$, but $f_\alpha(-\alpha) \neq -\alpha$.

Definition 12. *Let T denote the set of all positive α such that f_α has a superattracting 3-cycle containing $-\alpha$.*

Notice that when $\alpha > 0$, $f_\alpha(-\alpha) = \alpha \cosh \sqrt{\alpha} > 0 > -\alpha$; therefore $\alpha \in T$ if and only if $\alpha > 0$ and $f_\alpha^{\circ 3}(-\alpha) = -\alpha$.

Proposition 6. *For positive values of α, $\alpha \in T$ if and only if $\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}} = (2n+1)^2\pi^2$, where n is an integer.*

Proof. ($\Rightarrow$) Suppose $\alpha > 0$ and $f_\alpha^{\circ 3}(-\alpha) = f_\alpha(f_\alpha(f_\alpha(-\alpha))) = -\alpha$. So

$$\begin{aligned} -\alpha &= f_\alpha\left(f_\alpha\left(\alpha \cosh \sqrt{\alpha}\right)\right) \\ &= f_\alpha\left(\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}}\right) \\ &= \alpha \cos \sqrt{\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}}}, \end{aligned}$$

with the last line coming from the fact that $f_\alpha(x)$ is always positive when $x < 0$. Dividing by α, we see that $\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}} = (2n+1)^2\pi^2$, where $n \in \mathbb{N}$.

($\Leftarrow$) Suppose $\alpha > 0$ and $\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}} = (2n+1)^2\pi^2$ for some integer n. Then

$$\begin{aligned} f_\alpha^{\circ 3}(-\alpha) &= f_\alpha\left(\alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}}\right) \\ &= f_\alpha\left((2n+1)^2\pi^2\right) \\ &= \alpha \cos \sqrt{(2n+1)^2\pi^2} \\ &= \alpha \cos\left((2n+1)\pi\right) \\ &= -\alpha. \end{aligned}$$

□

We now define the continuous real-valued function $g_n(\alpha) = \alpha \cos \sqrt{\alpha \cosh \sqrt{\alpha}} - (2n+1)^2\pi^2$. Each g_n is defined for all non-negative real numbers. Also, we can look at the infinite set of these functions, $G = \{g_n \mid n = 0, 1, 2, 3, ...\}$. We see that $\alpha \in T$ if and only if α is a root of some $g_n \in G$.

We will identify several properties of g_n.

Proposition 7. *The family of functions $G = \{g_n\}$ has the following properties:*

1. *Each g_n is bounded by the lines $y_1 = \alpha - (2n+1)^2\pi^2$ and $y_2 = -\alpha - (2n+1)^2\pi^2$.*
2. *Each g_n exhibits cosine-like "cycling," touching y_1 and y_2 on each cycle.*
3. *The g_n functions "nest" inside each other, never intersecting.*

Proof. Properties 1 and 3 follow directly from the definition of g_n. Property 2 follows because $g_n(\alpha) = \alpha \cos(h(\alpha)) - (2n+1)^2\pi^2$, where $h(\alpha) = \sqrt{\alpha \cosh \sqrt{\alpha}}$ is a continuous, strictly increasing function. □

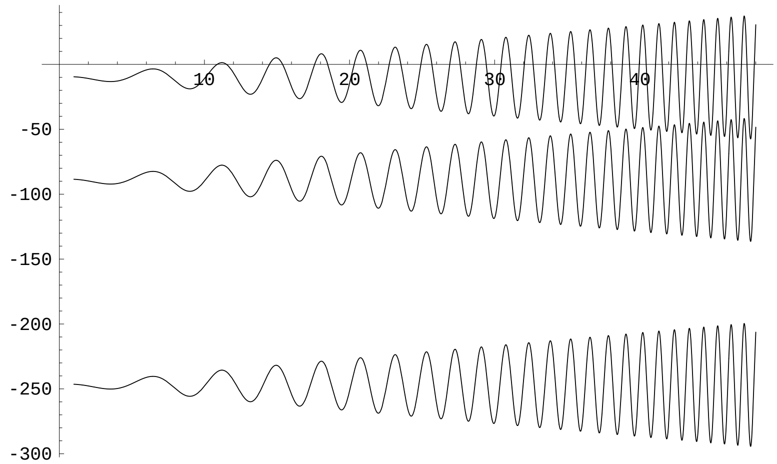

Figure 2.1: The first few g_n functions: g_1 (top), g_2 (middle), g_3 (bottom)

(Note: Since $h(0) = 0$ and h is strictly increasing when $\alpha > 0$, we can speak of "the" solution to $h(\alpha) = \sqrt{\alpha \cosh \sqrt{\alpha}} = k$, for any $k \geq 0$.) We wish to examine the roots of our g_n functions, so it is helpful to know some facts about them:

Proposition 8. *g_n has no roots on the interval $\left(0, (2n+1)^2\, \pi^2\right)$.*

Proof. This is obvious, since $g_n \leq \alpha - (2n+1)^2\pi^2$. $\square$

We wish to describe the distribution of the roots of G, since each one corresponds to a superattracting 3-cycle. In order to do so, we partition the real line into intervals, each one containing a certain finite number of roots. These intervals will be defined by values of x_b:

Definition 13. *Let b be a positive integer. Define x_b to be the (positive) solution to $\alpha \cosh \sqrt{\alpha} = b^2\pi^2$; that is, $x_b \cosh \sqrt{x_b} = b^2\pi^2$.*

Proposition 9. *If a and b are positive integers with $a < b$, then $x_a < x_b$.*

Proof. We have $x_a \cosh \sqrt{x_a} = a^2\pi^2 < b^2\pi^2 = x_b \cosh \sqrt{x_b}$. Since the cosh and square root functions are increasing, we have $x_a < x_b$. $\square$

We see that when b is even, we have $g_n(x_b) = x_b - (2n+1)^2\pi^2$; therefore g_n touches its upper-bounding line. Similarly, if b is odd then $g_n(x_b) = -x_b - (2n+1)^2\pi^2$; so g_n touches its lower-bounding line.

Theorem 10. *Let n and b be positive integers, with $x_b > (2n+1)^2\pi^2$. Then g_n has exactly one root on the interval (x_b, x_{b+1}).*

Proof. Suppose b is even. Then $g_n(x_b) = x_b - (2n+1)^2\pi^2 > 0$. and $g_n(x_{b+1}) = -x_{b+1} - (2n+1)^2\pi^2 < 0$. So there must be at least one root on the open interval (x_b, x_{b+1}). Note that we get the same result if b is odd. Uniqueness follows from the fact that $h(\alpha) = \sqrt{\alpha \cosh \sqrt{\alpha}}$ is a strictly increasing concave up function coupled with the periodicity of cosine.

$\square$

Corollary 11. *Let n, a, b be positive integers such that $a < b$ and $x_a > (2n+1)^2\pi^2$. Then g_n has exactly $b-a$ roots on the closed interval (x_a, x_b).*

Corollary 12. *Let a, b be positive integers with $a < b$. The number of roots of all $g_n \in G$ on the interval (x_a, x_b) is between $\left\lceil \frac{\sqrt{x_a}-\pi}{2\pi} \right\rceil (b-a)$ and $\left\lfloor \frac{\sqrt{x_b}-\pi}{2\pi} \right\rfloor (b-a)$, inclusive.*

Proof. Let $n_1 = \left\lceil \frac{\sqrt{x_a}-\pi}{2\pi} \right\rceil$ and $n_2 = \left\lfloor \frac{\sqrt{x_b}-\pi}{2\pi} \right\rfloor$. That is, for every $n < n_1$, g_n crosses the α axis to the left of x_a and so, by Corollary 11, has exactly $b-a$ roots on the interval. Similarly, for every $n > n_2$, $g_n < 0$ on (x_a, x_b) and so has no roots on the interval. $\square$

If we look at the α slice of the bifurcation sets with $\beta = 0$, we can see evidence of these superattracting 3-cycles. They appear as red or black Mandelbrot sets, symmetric across the real axis, pointing to the right. Since the 3-cycles become more

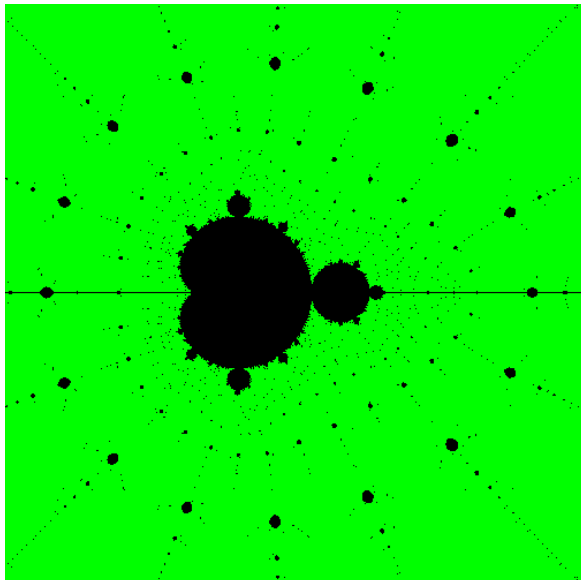

Figure 2.2: Parameter Space of $C_{\alpha,0}$, with $40.7675 \leq \mathrm{Re}(\alpha) \leq 40.7775$ and $|\mathrm{Im}(\alpha)| \leq .005$

"dense" as we increase α along the real axis, we can see that some of the centers of these Mandelbrot sets, which represent superattracting 3-cycles containing the critical value $-\alpha$, will eventually land in the main cardioids of the green Mandelbrot sets. These values of α are examples of "independent" behavior of critical values, which Durkin stated is possible but did not provide examples. At these points, one critical value is attracted to a fixed point while the other is attracted to a 3-cycle.

2.2 Real Dynamics when $\beta \neq 0$

Here we will use Equation (1.1):

$$f_{\alpha,\beta}(x) = \begin{cases} \alpha \cos \sqrt{x+\beta}, & x \geq -\beta \\ \alpha \cosh \sqrt{|x+\beta|}, & x < -\beta, \end{cases}$$

where $x, \alpha, \beta \in \mathbb{R}$.

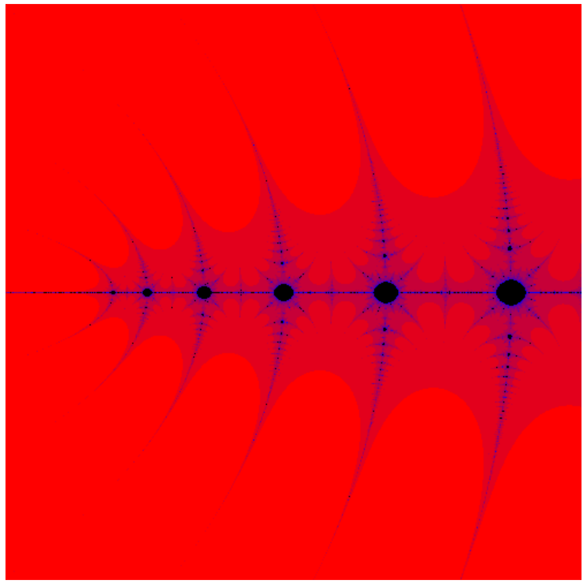

Figure 2.3: Julia Set of $C_{40.77235+.00007i,0}$, with $-400 \leq \mathrm{Re}(z) \leq 2000$ and $|\mathrm{Im}(z)| \leq 1200$

Proposition 13. *If $\beta \geq |\alpha|$, then the orbits of both critical values of $f_{\alpha,\beta}$ are bounded; the point corresponding to $f_{\alpha,\beta}$ in a bifurcation set will therefore be black.*

Proof. Let $x_0 = \pm\alpha$ be a critical value of $f_{\alpha,\beta}$. Then $x_0 \geq -\beta$, and so

$$\left|f_{\alpha,\beta}(x_0)\right| = \left|\alpha \cos \sqrt{x_0 + \beta}\right| \leq |\alpha| \leq \beta.$$

Inductively, we see that the inequality $\left|f^{\circ n}_{\alpha,\beta}(x_0)\right| \leq \beta$ holds true for all $n \in \mathbb{N}$. □

Proposition 14. *If $\beta < \alpha < -2$, then the orbit of α under $f_{\alpha,\beta}$ is unbounded; the point corresponding to $f_{\alpha,\beta}$ in a bifurcation set will be either red or white.*

Proof. Let $x_0 = \alpha < -\beta$. Then $f_{\alpha,\beta}(x_0) = \alpha \cosh \sqrt{|x_0 + \beta|}$ is negative, and

$$\left|f_{\alpha,\beta}(x_0)\right| \geq |\alpha| \left|\cosh \sqrt{|x_0|}\right| > |x_0 + 1|,$$

with the last inequality coming from the proof for Proposition 3. Inductively, we see that the inequality $f_{\alpha,\beta}^{\circ n}(x_0) < f_{\alpha,\beta}^{\circ(n-1)}(x_0) - 1$ will hold true for all $n \in \mathbb{N}$, thus showing that the orbit of α is unbounded. $\square$

These two propositions are very similar to Propositions 2 and 3, but they help explain the bifurcation set we see when we choose our x- and y-axes to be $\mathrm{Re}(\alpha)$ and $\mathrm{Re}(\beta)$, respectively. The image reflects these two facts: the solid black 'V' in the upper half-plane represents $\beta \geq |\alpha|$, while the third quadrant and the lower half of the second quadrant, which are entirely red and white when $\alpha < 2$, represent $\beta < \alpha < -2$.

2.3 Complex Dynamics when $\beta = 0$

Here we look at functions of the form $C_\alpha(z) = C_{\alpha,0}(z) = \alpha \cos \sqrt{z}$, with $\alpha, z \in \mathbb{C}$.

2.3.1 Bifurcation Sets

We examine the bifurcation set images, choosing $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ for the x- and y-axes, respectively. We know from Proposition 1 that these images are going to be symmetric across the real axis. Most of the bifurcation set is white, meaning that both critical points escape under iteration of C_α. However, along the α-real axis we see what appears to be an infinite line of Mandelbrot sets, though they are slightly distorted. Centered near the origin is a black Mandelbrot set pointing to the right. On the positive side of the axis there are numerous green Mandelbrot sets pointing to the right; while on the negative side, they are red and pointing to the left.

This is not at all surprising; we showed previously that in certain regions along the α-real axis, C_α behaves much like the quadratic map $Q_c(z) = z^2 + c$. Also, the positive (green) copies of $\mathcal{M}$ appear to be centered at $\alpha = (2n\pi)^2$, while the negative (red) copies are centered near $-((2n-1)\pi)^2)$. [This is consistent with Theorem 4; the centers of the green Mandelbrot sets coincide with the superattracting fixed points at $x = \alpha$ provided by the first case of the theorem, while the centers of the red Mandelbrot sets coincide with the superattracting fixed points at $x = |\alpha| = -\alpha$ provided by the second case of the theorem.] And, for each α, the cusp of each cardioid is near the corresponding $\hat{\alpha}$ as described in the section "The Real Slice." Similarly, the center of the period-2 lobe of each copy of $\mathcal{M}$ appears to be at the corresponding α_2.

The red Mandelbrots seem to be solid, with no black inside. This is not surprising, since Proposition 3 guarantees that, at least along the real axis, the orbit of $-\alpha$ is unbounded. However, we notice something surprising inside the green Mandelbrot sets. Inside the lobes that intersect the real line (which we know is black), there appear to be quadratic Julia sets, shaded in black. Also, given one of these "Julia sets", its shape is similar to that of the quadratic Julia set whose parameter corresponds to the same location within $\mathcal{M}$.

More formally, let G be our green copy of $\mathcal{M}$. Then G is similar to $\mathcal{M}$. Let B be one of our black "Julia sets" inside G, centered at α_0. Let F be the filled Julia set of $Q_c(z) = z^2 + c$, where c be the point in $\mathcal{M}$ corresponding to α_0. Then B is similar to F.

Another thing we see in the main cardioid of each of our green Mandelbrot sets is a tiny black copy of $\mathcal{M}$. It appears that inside the main black of this small set, the orbit of $-\alpha$ under $C_{\alpha,\beta}$ is attracted to a 2-cycle. Likewise, inside each period-k lobe of the tiny set we see that $-\alpha$ is attracted to a $2k$-cycle.

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