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Minimization via the Subway Metric

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MINIMIZATION VIA THE SUBWAY METRIC

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ABSTRACT

The Subway Metric is a variant of the well-known Taxicab Metric in which a subway, in the form of a line in the plane, is used to alter walking distance within a city grid. In this work, we discuss where to place such a subway line in order to miminize the greatest walking distance within various city layouts.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Introduction

Imagine yourself traveling in a city with only perpendicular roads that run north to south and east to west in a evenly spaced grid-like fashion. The only way to travel from point $A = (a_1, a_2)$ to point $B = (b_1, b_2)$ is along these roads. The distance between two points is then the difference between the $x-$coordinates added to the difference between the y-coordinates of the two points. This method of measuring distance is called the Taxicab metric and the equation to find the distance between two points using this metric is as follows:

$$d_T(A, B) = |b_1 - a_1| + |b_2 - a_2|$$

Unlike in the Euclidean metric, the shortest distance between two points in the Taxicab metric is not always a straight line. For example, the distance traveling from the origin to the point $(3, 4)$ in the Euclidean metric is five, but in the Taxicab metric the distance is seven, as you must first travel horizontally and then vertically (see Figure 1.1).

Another method of measuring distance is called the Subway metric, denoted d_S. This metric is very similar to the Taxicab metric with a slight variation. The difference is the addition of a Euclidean line to the grid of roadways. Travel along this line does not add to the distance between two points. If you think of the distance between two

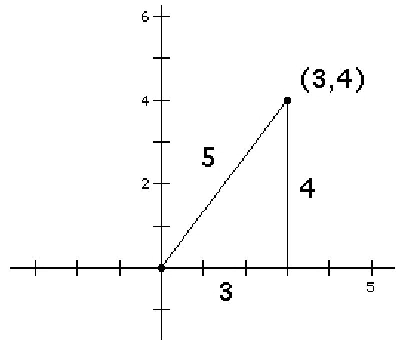

Figure 1.1: Taxicab versus Euclidean distance.

points as the amount of walking one must do to travel from one point to another, the subway line should not count as part of the distance as one does not walk while on the subway. With the addition of the subway line, the distance between two points is either simply the taxicab distance (if the subway line is not used) or the distance from one point to the line and then from the line to the other point, depending on which is shorter. The equation for this distance then becomes:

$$d_S(A, B) = \min[d_T(A, B), d_T(A, L) + d_T(B, L)]$$

where A and B are two points and L is a Euclidean line with the equation $y = mx+b$. If we look at the same example as above, travel from the origin to the point $(3, 4)$, with the addition of the subway line with the equation $y = 4x - 4$, we see that the distance now becomes two when using the subway metric (see Figure 1.2)

In our study of the Subway metric, we analyze different city shapes: squares, rectangles, and squares with an obstacle to subway placement, and determine the best placement for the subway line. The line that minimizes the longest walking distance determines the best placement. The addition of any subway line makes the distance between some points shorter, but our goal is to find the line that makes the distance between any two points in the city the shortest. In the case of the square city,

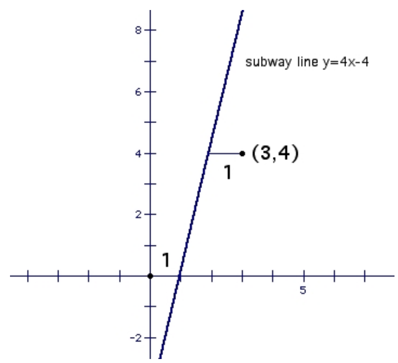

Figure 1.2: Taxicab versus Euclidean distance.

the best placement for the subway line is either of the perpendicular bisectors, as the city is symmetrical, it does not matter which line you choose. In the rectangular city, the best placement is the perpendicular bisector that runs parallel to the longer of the two sides. With the addition of a square obstacle in the middle of the square city, the perpendicular bisector is no longer an option. In this case, the best placement for the subway line is tangent to one of the sides of the square hole. This paper thoroughly discusses all of our results from the semester with careful proofs and informative pictures.

1.2 Background

As the Subway metric utilizes the Taxicab metric in the definition, we must verify that the proposed Taxicab method for measuring distances is in fact a metric.

Definition 1.1. *A* ***metric*** $d : \mathbb{R}^2 \to \mathbb{R}$ *is a function that satisfies three conditions:*

1. *For all* $A, B \in \mathbb{R}^2$, $d(A,B) \geq 0$ *and* $d(A,B) = 0$ *if and only if* $A = B$.

2. *For all* $A, B \in \mathbb{R}^2$, $d(A,B) = d(B,A)$

3. *For all* $A, B, C \in \mathbb{R}^2$, $d(A,B) + d(B,C) \geq d(A,C)$. *(This is the triangle inequality.)*

Proposition 1.2. *$d_T[(a_1,a_2),(b_1,b_2)] = |b_1 - a_1| + |b_2 - a_2|$ is a metric.*

Proof. 1. Show $d_T(A,B) \geq 0$ and $d_T(A,B) = 0$ if and only if $A = B$.

As the absolute value is a metric on $\mathbb{R}$, $|b_1 - a_1| \geq 0$ and $|b_2 - a_2| \geq 0$ and hence,

$$d_T(A,B) = |b_1 - a_1| + |b_2 - a_2| \geq 0.$$

Now, suppose $|b_1 - a_1| + |b_2 - a_2| = 0$. Then, $|b_1 - a_1| = 0$ and $|b_2 - a_2| = 0$. Thus, $b_1 = a_1$ and $b_2 = a_2$, so that $A = B$.

Suppose $A = B$. Then, $b_1 = a_1,\ b_2 = a_2$, and so, $b_1 - a_1 = 0$ and $b_2 - a_2 = 0$. Thus, $|b_1 - a_1| + |b_2 - a_2| = 0$.

2. Show $d_T(A,B) = d_T(B,A)$.

$$d_T(A,B) = |b_1 - a_1| + |b_2 - a_2| = |a_1 - b_1| + |a_2 - b_2| = d_T(B,A),$$

by the properties of absolute values.

3. Show $d_T(A,B) + d_T(B,C) \geq d_T(A,C)$.

$$\begin{aligned} d_T(A,B) + d_T(B,C) &= |b_1 - a_1| + |b_2 - a_2| + |c_1 - b_1| + |c_2 - b_2| \\ &= (|c_1 - b_1| + |b_1 - a_1|) + (|c_2 - b_2| + |b_2 - a_2|) \\ &\geq |c_1 - a1| + |c_2 - a_2| \\ &= d_T(A,C). \end{aligned}$$

□

The Subway metric relies on the knowledge of the Taxicab distance from a point to a line. This distance may be found by using the following theorem.

Theorem 1.3. *If A is a point in the real plane and $L = mx+b$ is a line in the plane, then*

1. *$d_T(A, L)$ is the horizontal distance from A to the line if $m \geq 1$, and*

2. *$d_T(A, L)$ is the vertical distance from A to the line if $m \leq 1$.*

Proof. As the taxicab metric only allows for horizontal or vertical travel, the shortest distance from a point to a line is either going to be the horizontal distance or the vertical distance between them. When the slope of the line is greater than one, then the horizontal distance is shorter. When the slope of the line is less than one, then the vertical distance is shorter. When the slope equals one, then the vertical and horizontal distances are equal. □

It is now necessary to verify that the proposed Subway metric is actually not a metric, but a pseudometric.

Definition 1.4. *A **pseudometric** $d : \mathbb{R}^2 \to \mathbb{R}$ satisfies every condition of being a metric, except the strict requirement $d(A, B) = 0$ implies $A = B$.*

In the case of the Subway metric, this stipulation is slightly modified to say that if the distance from A to B is zero, then $A = B$ or both A and B are on the same subway line.

Proposition 1.5. *The subway distance is a pseudometric.*

Proof. 1. Show $d_S(A, B) \geq 0$ and $d_S(A, B) = 0$ if and only if $A = B$ or $A, B \in L$. By Proposition 1.2, $d_T(A, B) \geq 0$, $d_T(A, L) \geq 0$ and $d_T(B, L) \geq 0$, and so $d_T(A, L) + d_T(B, L) \geq 0$. Then,

$$d_S(A, B) = \min[d_T(A, B), d_T(A, L) + d_T(B, L)] \geq 0.$$

If $d_S(A,B) = 0$, then either $d_T(A,B) = 0$ or $d_T(A,L) + d_T(B,L) = 0$. If $d_T(A,B) = 0$, then $A = B$ by Proposition 1.2. If $d_T(A,L) + d_T(B,L) = 0$, then $d_T(A,L) = 0$ and $d_T(B,L) = 0$ so $A \in L$ and $B \in L$.

2. Show $d_S(A,B) = d_S(B,A)$.

$$\begin{aligned} d_S(A,B) &= \min[d_T(A,B),\ d_T(A,L) + d_T(B,L)] \\ &= \min[d_T(B,A),\ d_T(B,L) + d_T(A,L)] \quad \text{by Proposition 1.2,} \\ &= d_S(B,A). \end{aligned}$$

3. To prove that $d_S(A,B) + d_S(B,C) \geq d_S(A,C)$, we need to show

$$d_S(A,B) + d_S(B,C) \geq d_T(A,C)$$

or

$$d_S(A,B) + d_S(B,C) \geq d_T(A,L) + d_T(C,L).$$

Case 1: $d_S(A,B) = d_T(A,B)$ and $d_S(B,C) = d_T(B,C)$.

Then

$$d_S(A,B) + d_S(B,C) = d_T(A,B) + d_T(B,C) \geq d_T(A,C)$$

by the triangle inequality for the taxicab metric.

Case 2: $d_S(A,B) = d_T(A,B)$ and $d_S(B,C) = d_T(B,L) + d_T(C,L)$.

Then

$$\begin{aligned} d_S(A,B) + d_S(B,C) &= d_T(A,B) + d_T(B,L) + d_T(C,L) \\ &\geq d_T(A,L) + d_T(C,L), \end{aligned}$$

as $d_T(A,B) + d_T(B,L)$ is the length of the path from A to L passing through B while $d_T(A,L)$ the length of the shortest path from A to L.

Case 3: $d_S(A,B) = d_T(A,L) + d_T(B,L)$ and $d_S(B,C) = d_T(B,C)$.

Then, similar to the previous case,

$$\begin{aligned} d_S(A,B) + d_S(B,C) &= d_T(A,L) + d_T(B,L) + d_T(B,C) \\ &\geq d_T(A,L) + d_T(C,L). \end{aligned}$$

Case 4: $d_S(A,B) = d_T(A,L) + d_T(B,L)$ and $d_S(B,C) = d_T(B,L) + d_T(C,L)$.

Then,

$$\begin{aligned} d_S(A,B) + d_S(B,C) &= d_T(A,L) + 2d_T(B,L) + d_T(C,L) \\ &\geq d_T(A,L) + d_T(C,L), \qquad \text{as } 2d_T(B,L) \geq 0. \end{aligned}$$

□

When minimizing the longest walking distance, it becomes important to determine the longest walking distance prior to the introduction of the subway. In the Euclidean metric, one method of determining the distance between two points is by measuring the radius of the smallest Euclidean circle that is centered at one of the points and contains the other point. The longest distance would then be the radius of this circle. This method can also be used in the Taxicab metric, but a taxicab circle must be used. The definition of a circle is the set of all points P that are equidistance from a fixed point. In the Taxicab metric a circle then looks like a square rotated 45 degrees (see Figure 1.4).

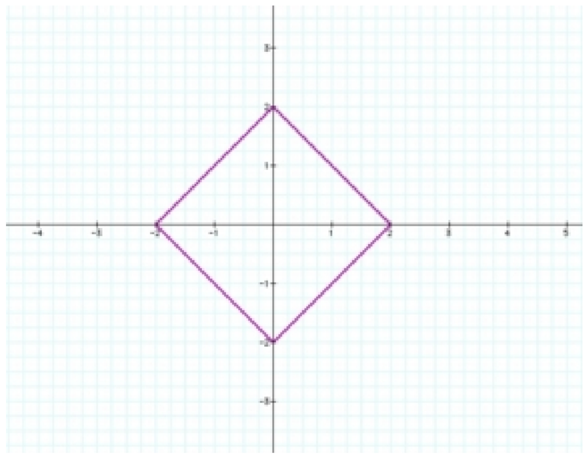

Figure 1.3: Taxicab Circle.

When dealing with the square city, we can see that the two points furthest away without the use of a subway line would be either of the two opposite vertices. If we were to create a taxicab circle centered at one corner with the edge of the circle touching the opposite corner, we would create a circle that encompasses the entire square. Thus, any other two points could have a taxicab circle created in the prescribed way smaller than the one with the opposite vertices, thus implying that their distance from each other would be less. We make use of this method throughout the paper when determining the two points of farthest distance.

In order to minimize the number of proofs required for each city shape, we used the following theorem by Doris Schattschneider [1].

Theorem 1.6. *The group of isometries of the plane, $\mathbb{R}^2$, with respect to the taxicab metric consists of all translations of $\mathbb{R}^2$ and the symmetries of the taxicab circle (D_4, the dihedral group of order eight.)*

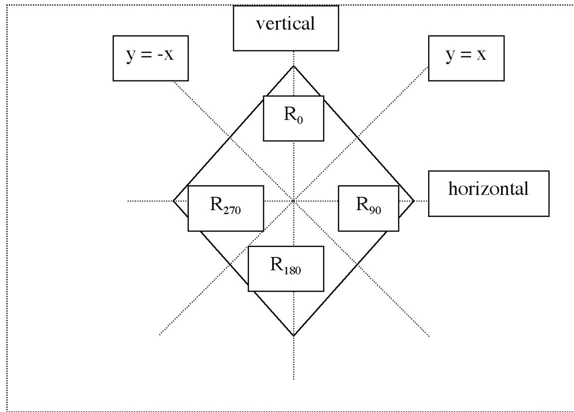

Figure 1.4: Isometries of the Taxicab Metric.

The symmetries of the taxicab circle are shown in Figure 1.3 and are as follows:

1. R_0 (a rotation of 0 degrees)

2. R_{90} (a rotation of 90 degrees)

3. R_{180} (a rotation of 180 degrees)

4. R_{270} (a rotation of 270 degrees)

5. H (a reflection on the horizontal axis)

6. V (a reflection on the vertical axis)

7. D (a reflection in $y = x$)

8. D_1 (a reflection on $y = -x$)

The Subway metric consists of sums of Taxicab metric distances, thus this theorem for the Taxicab metric can be applied to the Subway metric. By using this theorem, it is possible to reduce the total number of calculations needed, as it becomes necessary to only look at one portion of symmetrical cities.

CHAPTER 2

SUBWAY PLACEMENT FOR MINIMIZATION

2.1 Square City

The first city type that we will analyze is the simple square. We will let the side of the square have length S. Without a subway line, the longest walking distance is $2S$, which results from a person walking from one corner to the opposite vertex. We find that with the subway line in the best place, the longest walking distance becomes S, and thus we were able to cut the longest walking distance in half.

Theorem 2.1. *The best placement for a subway line in a square city is the perpendicular bisector, which is the line that runs perpendicular to a side of the square and cuts the square in half.*

Proof. We begin by demonstrating that for all vertical subway lines, the perpendicular bisector of the square is the most efficient. Let S be the side of the square that has two sides along the coordinate axes and lies in the first quadrant. Let A be the top left corner of the square. Without loss of generality, assume that x, the placement of the subway line, is restricted as follows: $S/2 \leq x \leq S$. Let F be the point with the farthest walking distance from A. We can see that this point must lie on the x-axis; see Figure 2.1. If we let F be a point not on the $x-$axis, then the point on the $x-$axis with the same $x-$coordinate as this chosen point will have a distance greater than or equal to the point F. We will also let y be the distance from the origin to

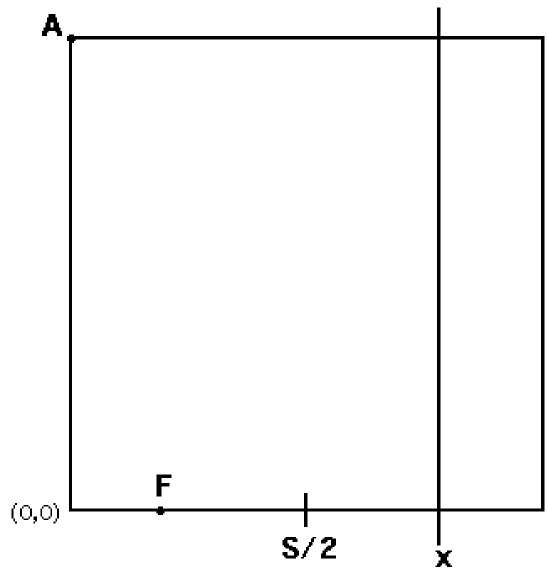

Figure 2.1: Subway placement for a square city.

the point F. Thus, the walking distance from A to F would be $S + y$, without the use of the subway line, or $x + (x - y)$ if one were to take the subway. By equating these two equations, we have $y = x - S/2$. Given x, we then have that distance from A to F would be $S + (x - S/2) = x + S/2$. We want to minimize this distance. As $S/2 \leq x \leq S$, the minimum occurs when $x = S/2$.

Thus the best placement for the subway line when dealing solely with vertical lines, is the line $x = S/2$, or the perpendicular bisector. By rotation of $\pi/2$, a horizontal line bisecting the square would also work to reduce the longest walking distance to S. Next we will show that a vertical line, as opposed to an oblique line, is again the most efficient. Let $y = mx + b$ be the equation of the subway line. The most promising option for an oblique subway line would be to intersect two opposite sides of the square, as opposed to two adjacent. If the line were to intersect two adjacent sides, we would be left with a portion of the square that contains three vertices, unless the line intersected the square at two opposite vertices. Ignoring this case, as it can be included in the subway lines which intersect two opposite sides, we can conclude that within the larger portion of the square, two points exists along the perimeter that have a walking distance of S between them. The reason for this is that the subway line is greater than S units away from the vertex that is formed by the two sides of the square that the subway line does not intersect.

From this we can assume the subway line intersects the $x-$axis and the line $y = S$, the bottom and top edge of the square, respectively. If we were to draw a vertical line dropping down from the intersection of the top of the square and the subway, we would create a distance, W, from the $x-$intercept of the subway line to the point where this dropped down line crosses the $x-$axis. See Figure 2.2. We would then have a walking distance of $S + W$ from A to C. We know from above that with the vertical line the longest path is S. Thus, a longer path would result with an oblique line, unless $W = 0$. In other words, when the line has infinite slope (a vertical line). We conclude, then, that the vertical subway line is more efficient than an oblique line, and thus the perpendicular bisector gives rise to the shortest longest-walking-distance in a square city. □

2.2 Rectangular City

We will now move from the square city to a rectangular city. Assume the rectangle rests in the plane with sides of length S and R. Assume that $R > S$ and that the vertices of the rectangle are $A = (0, S)$, $B = (R, S)$, $C = (R, 0)$ and the origin.

Theorem 2.2. *The best placement for a subway line in a rectangular city is the perpendicular bisector that runs parallel to the longer side.*

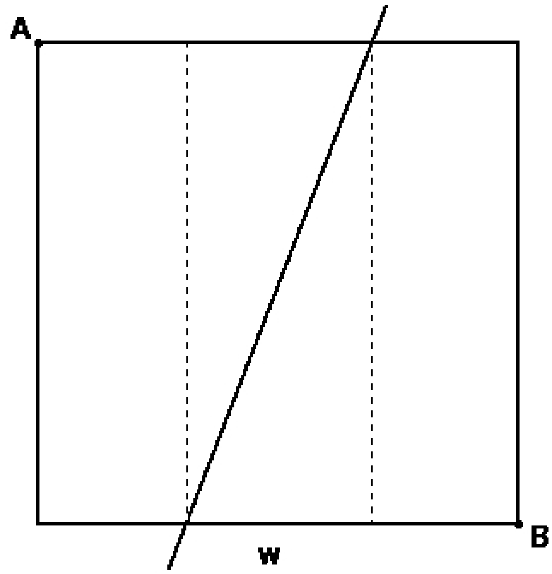

Figure 2.2: Oblique subway line does not minimize.

Proof. First we will show that dealing only with vertical lines, the perpendicular bisector is the best placement. Assume that x, the placement of the subway line, is restricted as follows: $R/2 \leq x \leq R$. Let y be the distance from the origin to F, the farthest point from A, which lies on the $x-$axis. See Figure 2.3.Thus $d_S(A, F) = S+y$, as well as $x + x - y$ if one were to take the subway. If we equate these two equations, we find that $y = x - S/2$. We want to minimize y so as to minimize $d_S(A, F)$. As $R/2 \leq x \leq R$, the least value for y would be $R/2 - S/2$. So the best placement for the vertical line would be $x = R/2$, or the vertical perpendicular bisector. The longest walking distance with this placement is found by traveling from one corner to the opposite corner. This distance is R.

We now must investigate what happens with horizontal lines. With the horizontal lines, we know that $d_S(A, C) = S$, no matter where the line is placed. It becomes necessary to see if there exists a possibly longer walking distance. Let us examine the distance from A to B. We will assume that the subway line is restricted as follows:

Figure 2.3: Subway placement in a rectangular city.

$0 \leq y \leq S/2$. Then, $d_S(A, B) = \min[2(S - y), R]$; these distances represent either travel straight across the top of the rectangle or walking down to the subway, riding the entire subway line, and then walking up to B. With the restrictions we have in place on y, the minimum value of $2(S - y)$ is S. We know $S < R$, thus the shortest walking distance from A to B would be S, which happens when $y = S/2$, or when the subway is the horizontal perpendicular bisector. As the shortest, longest walking distance for the vertical subway lines was R, and we know $R > S$, it becomes apparent that the between all vertical or horizontal lines, the best placement for the subway would be the horizontal perpendicular bisector. From here, we can easily generalize this result to all possible subway lines by using a similar argument to the one used in the square cities to demonstrate that oblique lines would not reduce the longest walking distance to less than S. Thus, in a rectangular city, the best subway line is the perpendicular bisector that runs parallel to the longer side. $\square$

2.3 Square City with an Obstacle

We will now investigate what will happen if there is a lake or a pond of some sort in the middle of the city which denies the passage of a subway line. We begin again dealing only with square cities, and ponds that are also square. Let the city be a square of side length S, with a square obstruction (the pond) in the exact middle with side length R. We know $0 < R < S$. We will place the square in the first quadrant of the plane with the bottom left corner resting on the origin. The four corners then become $O = (0, 0)$, $A = (0, S)$, $B = (S, S)$ and $C = (S, 0)$.

Theorem 2.3. *The best placement for the subway line in square city with a square obstruction denying subway lines is a vertical or horizontal line tangent to one of the edges of the center obstruction.*

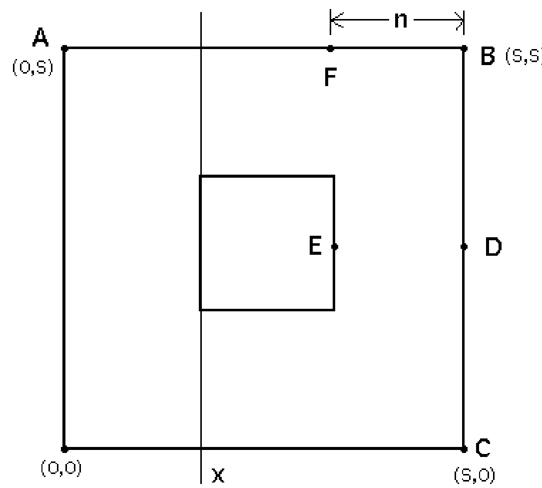

Figure 2.4: Subway placement with an obstacle.

Proof. Without a subway line, the longest walking distance in this city would be $2S$, from one corner to the opposite vertex. First we will show that using only vertical lines, the best possible placement would be the vertical line tangent to the hole. Without loss of generality, we will let this line be $x = (S-R)/2$. With this placement, $d_S(A, B) = S$, $d_S(A, C) = S$, $d_S(A, D) = S + R/2$, and $d_S(A, E) = S/2 + R$, where D is the point on the edge of the city, $(S, S/2)$, and E is the point on the edge of the obstruction $(S - (S - R)/2, S/2)$. We also create the point F along the top of the city. This point represents the farthest distance from C, whether you use the subway or not. See Figure 2.4.F has a $y-$coordinate value of S, and an $x-$coordinate value of $S - n$, where n is an arbitrary length. The route not taking the subway would simply be walking straight up to the top of the city and then moving west a length of n. Thus the total length would be $S + n$. The route taking the subway would be $S - (S - R)/2 + S - n - (S - R)/2$. To find the distance to this point we equated the two routes. So we have,

$$\begin{aligned} S + n = S - \frac{S-R}{2} + S - n - \frac{S-R}{2} &\implies 2n = -S + R + S \\ &\implies 2n = R \\ &\implies n = R/2. \end{aligned}$$

From this we know that $d_S(C, F) = S + R/2$. This is the same distance as A to D, and is thus the longest distance between any two points.

If the subway remains a vertical line, but has an equation $x < (S - R)/2$, then there would be a point G along the top of the city with a distance greater than $S + R/2$ from C. This can be demonstrated using a similar proof as above. Let q be the distance from x to $(S - R)/2$. We will still let n be the distance moving west along the top of the city to the point G. So not taking the subway, the distance from C to G will still be $S + n$. The formula for the distance to G if taking the subway would be $S - (S - R)/2 + q + S - n - (S - R)/2 + q$. By equating the two routes we have:

$$S + n = S - \frac{S-R}{2} + q + S - n - \frac{S-R}{2} + q \implies 2n = -S + R + S + 2q$$
$$\implies 2n = R + 2q$$
$$\implies n = \frac{R}{2} + q.$$

From this we have that the distance from B to G would be $S + R/2 + q$, with $q \geq 0$. So if the vertical subway line is not tangent to the obstruction, than we can find two points that have a further subway distance than $S + R/2$.

It is now necessary to check that oblique subway lines would not prevail over the vertical or horizontal lines. If we were to draw an oblique subway line, we would create two lengths, lets say u and v, such that u and v are the distances from the intersection between the edge of the city and the vertical subway line tangent to the obstruction and the intersection between the oblique line and the edge of the city. See Figure 2.5.If we let the oblique line touch the top left edge of the obstruction, and let u be the length at the top of the city and v be the length on the bottom, we can create the following equivalent ratios:

$$\frac{S-R}{2u} = \frac{\frac{S-R}{2} + R}{v} \implies \frac{S-R}{2u} = \frac{S+R}{2v}$$
$$\implies u = \frac{S-R}{S+R}v.$$

So we know that $u < v$, as $0 < R < S$. We can now use the same method as used above and equate the equations for the two routes (either using the subway or not) between the points C and H, where H is again along the top of the city, with a distance n west of the point (S, S). By doing so, we have:

$$d_S(B, H) = S + n = S - \frac{S-R}{2} + v + S - n - \frac{S-R}{2} - u \implies 2n = R + v - u$$
$$\implies n = \frac{R}{2} + \frac{v-u}{2}.$$

We then have that $d_S(C, H) = S + \frac{R}{2} + \frac{v-u}{2}$. We would like to minimize this distance, and since $u < v$, the best we could do is let $\frac{v-u}{2} = 0$. So $u = v$. In other words, the line would have infinite slope. $\square$

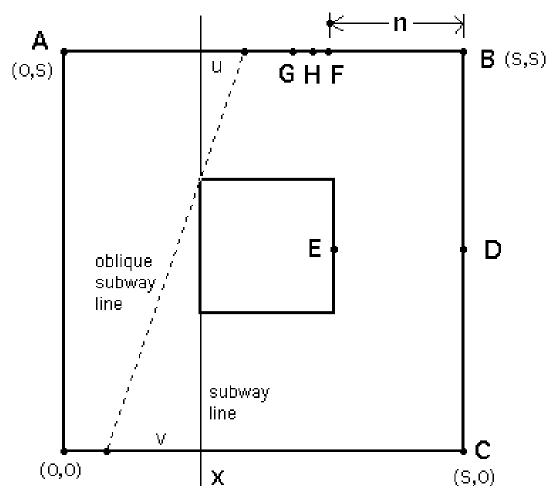

Figure 2.5: Oblique subway line does not minimize.

2.4 City for which No Subway Line Helps

In our quest to find a city that would result in an oblique line as the best placement for a subway line, we analyzed the city that is shaped and oriented like a taxicab circle. What we found was that with such a city shape, there is no subway line that makes the longest walking distance without a subway line any shorter.

Theorem 2.4. *If we have a city shaped like a Taxicab square with side S, the longest walking distance is $\sqrt{2}S$. With the placement of any subway line, there will still exist two points that have a walking distance between them of $\sqrt{2}S$.*

Proof. By the same reasoning we used earlier, we can conclude that the best possible placement for the line is going to intersect two opposite sides of the tilted square. If we place the subway line as the perpendicular bisector of the tilted square, the distance from either of the opposite vertices is still equal to $\sqrt{2}S$. If we place the subway line connecting either of the opposite sides of the square, with a slope less than one, then travel North to South is not reduced by the subway, and thus we still have a walking distance of $\sqrt{2}S$. If the line has a slope greater than one, then travel East to West is not reduced by the subway line, and again we still have a walking distance of $\sqrt{2}S$. If the slope is equal to one, then travel between either set of opposite vertices is not reduced by the introduction of the subway. We were able to restrict the slope of the line to be greater than zero as the city is symmetrical. Thus no subway line helps the worse off people. We are not concluding that the subway line would not reduce the walking distance for some people, simply that the longest walking distance is not reduced no matter what subway line is used. $\square$

REFERENCES

[1] Schattschneider, D. *The Taxicab Group*. American Mathematical Monthly, 1984.