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Superfluid Turbulence from Quantum Kelvin Wave to Classical Kolmogorov Cascades

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The main topological feature of a superfluid is a quantum vortex with an identifiable inner and outer radius. A novel unitary quantum lattice gas algorithm is used to simulate quantum turbulence of a Bose-Einstein condensate superfluid described by the Gross-Pitaevskii equation on grids up to 5760^3 . For the first time, an accurate power-law scaling for the quantum Kelvin wave cascade is determined: k^{-3} . The incompressible kinetic energy spectrum exhibits very distinct power-law spectra in 3 ranges of k space: a classical Kolmogorov $k^{-(5/3)}$ spectrum at scales greater than the outer radius of individual quantum vortex cores and a quantum Kelvin wave cascade spectrum k^{-3} on scales smaller than the inner radius of the quantum vortex core. The k^{-3} quantum Kelvin wave spectrum due to phonon radiation is robust, while the $k^{-(5/3)}$ classical Kolmogorov spectrum becomes robust on large grids.

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Introduction.—Superfluid dynamics is an intriguing subject: Quantum turbulence occurs in liquid helium II and should occur in Bose-Einstein condensates (BECs) of cold atomic vapors. Yet it is all-important for comparing to classical fluid turbulence to help us solve one of the grand challenge problems of the millennium [1,2]. There is a broadly acknowledged need for high resolution quantum turbulence simulations—in this Letter, we strive to meet this with a unitary simulation of the Gross-Pitaevskii (GP) equation on large spatial grids up to 5760^3 .

Fundamental to superfluid turbulence is the quantum vortex: a topological singularity with the superfluid density exactly zero at the vortex core in the simplest case [3]. Furthermore, in the simplest case, all of the quantum vortices are discrete and have the same charge (i.e., quantized circulation in multiples of $\pm 2\pi$), and the flow is inviscid. This stands in sharp contrast to classical incompressible fluid turbulence, where the concept of a vortex tube or eddy is imprecise and where viscosity plays an essential role. Moreover, in classical turbulence there are two strongly competing effects: sweeping of small scale eddies (advection) by large scale eddies and straining of eddies (deformation) by eddies of similar scales. Building on Richardson's local cascade of energy from large to smaller and smaller eddies until viscosity dissipates the smallest ones into heat [4], Kolmogorov [5] assumed there is an inertial energy spectrum that depends only on the energy input and wave number. Assuming the energy transfer and the interacting scales are purely local and sweeping is not important for energy transfer, he derived the inertial energy spectrum for classical incompressible turbulence: $E(k) = C_K \mathcal{E}^{2/3} k^{-(5/3)}$, for some constant C_K , where \mathcal{E} is the energy dissipation and k is the wave number magnitude.

Phenomenology of quantum turbulence.—Quantum turbulence is envisaged to arise from dense quantum vortex tangles [6], and this is borne out by numerical simulation, for example, as shown in Fig. 1. The coherence length ξ defines the inner radius of a quantum vortex core, while $\pi\xi$ approximates its outer radius. Since the flow outside a quantum vortex core is simple potential flow, it is thought that for large scales $\gg \pi\xi$ the discrete nature of the quantum vortices is lost and the superfluid density is approximately constant while supporting phonon radiation. Large eddies can form as aggregated quantum vortices, concomitant with sweeping and straining (the latter important to

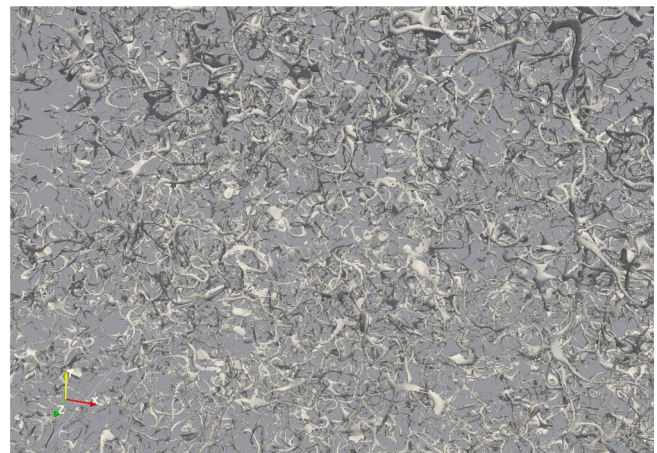


FIG. 1 (color online). Quantum turbulence (zoom-in online to view tangles). These are vortex core isosurfaces from our quantum lattice simulation on a 1024^3 grid at the $t = 50000$ time step, with parameters $a = 0.03$ and $g = 1.004$ as defined in (15). The initial conditions are the same as for the 5760^3 run for a repulsive nonlinear interaction.

incompressible classical turbulence), so large scale quantum turbulence could resemble a Kolmogorov energy cascade $E(k) \approx k^{-(5/3)}$, for $k \ll (\pi\xi)^{-1}$. The dissipation wave number $k_{\text{diss}} = \mathcal{E}^{1/4} \nu^{-(3/4)}$ cuts off the Kolmogorov energy cascade in classical turbulence—in quantum turbulence, one expects that $k_{\text{diss}} \sim (\pi\xi)^{-1}$.

For length scales on the order of the coherence length, one needs to consider the effects of vortex reconnection—a reconnection that occurs in superfluids without the need for viscous dissipation, unlike classical vortex tube reconnection. During the quantum vortex-vortex reconnection or collision and vortex self-interaction, the vortex lines are sharply distorted, supporting large amplitude Kelvin waves (large relative to the wavelength). The Kelvin wave modes couple to generate Kelvin waves of smaller and still smaller wavelength, emitting phonon radiation in the process [1,2]. This Kelvin wave energy cascade continues until one reaches the shortest operable scale, e.g., a k -space cutoff on the order of half the inverse the mean-free path length in helium II or ultracold quantum gases or the grid scale in simulations. For the k dependence of the Kelvin wave cascade in quantum turbulence, one anticipates a power law in the incompressible kinetic energy spectrum: $E(k) \approx k^{-\alpha}$, for $k \gg \xi^{-1}$, where the exponent is here determined to be $\alpha = 3.00$. There has been considerable effort to devise theories and methods [1,2,7] to predict this exponent's value as well as to predict the incompressible kinetic energy spectrum in the transition region $(\pi\xi)^{-1} \leq k \leq \xi^{-1}$ between the Kolmogorov and the Kelvin wave cascade spectra.

Gross-Pitaevskii equation.—At sufficiently low temperatures, the ground state wave function φ of a BEC can be described by the (normalized) GP equation

$$i\partial_t \varphi = -\nabla^2 \varphi + a(g|\varphi|^2 - 1)\varphi. \quad (1)$$

We have introduced two parameters in (1) that are useful in our numerical simulations: a is simply a spatial rescaling parameter to enhance the grid resolution of the vortex core, and g is a measure of the strength of the nonlinear coupling term in GP. Unlike the Navier-Stokes equation for classical turbulence, the GP equation is a Hamiltonian system: the total energy $E_{\text{TOT}} = \text{const}$. It is well known [8] that the Madelung transformation $\varphi = \sqrt{\rho}e^{i\theta}$ on the GP equation results in compressible inviscid fluid equations for the density $\rho = |\varphi|^2$ and velocity $\mathbf{v} = 2\nabla\theta$, with the appearance of quantum pressure terms in the momentum and energy equations. The E_{TOT} can be split into incompressible and compressible kinetic energies, an internal energy, and a quantum energy [9]:

$$E_{\text{TOT}} = E_{\text{kin}}^{\text{comp}}(t) + E_{\text{kin}}^{\text{incomp}}(t) + E_{\text{int}}(t) + E_{\text{qu}}(t) = \text{const}. \quad (2)$$

Typically, the GP equation has been solved numerically by the split Fourier time method [10]. In the seminal work of Nore, Abid, and Brachet [11], the GP equation was solved on a 512^3 grid and their incompressible kinetic

energy spectra while not incompatible with the Kolmogorov $k^{-(5/3)}$ scaling did not unequivocally prove the $k^{-(5/3)}$ scaling. Barenghi [2] and Kobayashi and Tsubota [1] attributed this to the presence of the Kelvin wave cascade. The Tsubota group [1] introduced a wave-number-dependent dissipative term into their simulations to damp out wave numbers on the order of the vortex core. While this suppresses the Kelvin wave cascade on the quantum turbulence, it also leads to a time decay in both the total number and total energy E_{TOT} . To circumvent the decay in the total number, they add a time-varying chemical potential in the GP equation although the total energy still decays. Most of their simulations were restricted to a 512^3 grid and did not yield a convincing incompressible kinetic energy spectrum of $k^{-(5/3)}$ for this augmented GP equation. These earlier simulations could not (nor did not want) to resolve the Kelvin wave cascade regime and its spectral power.

In this Letter, we, for the first time (to our knowledge), perform very high grid resolution runs (up to 5760^3) of the Hamiltonian GP equation to resolve the quantum Kelvin wave cascade. We use a novel unitary quantum algorithm whose solution naturally exhibits three power-law regions for $E_{\text{kin}}^{\text{incomp}}(k)$: for small k the Kolmogorov $k^{-(5/3)}$ spectrum while for high k a Kelvin wave spectrum of k^{-3} . A transitional power law on the order of k^{-6} to k^{-7} joins these two spectral regions. For very large k , the Kelvin wave power-law spectrum, along with consequent phonon emissions, is cutoff by the lattice.

Unitary quantum lattice gas algorithm.—To recover the GP equation in $3 + 1$ dimensions, we need only specify 2 complex amplitudes per point \mathbf{x} on a cubic lattice

$$\psi(\mathbf{x}, t) = \begin{pmatrix} \alpha(\mathbf{x}, t) \\ \beta(\mathbf{x}, t) \end{pmatrix}. \quad (3)$$

At each point, these excited state probability amplitudes are entangled by a collision operator of the form

$$C = e^{i(\pi/4)\sigma_x(1-\sigma_x)}, \quad (4)$$

where the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

With $n = \frac{1}{2}(1 - \sigma_z)$ and $\bar{n} = \frac{1}{2}(1 + \sigma_z)$, the local qubit entanglement is then spread throughout the lattice by streaming operators

$$S_{\pm\Delta x, 0} = n + e^{\pm\Delta x\partial_x}\bar{n}, \quad S_{\pm\Delta x, 1} = \bar{n} + e^{\pm\Delta x\partial_x}n, \quad (6)$$

unitarily shifting the components of ψ along $\pm\Delta\mathbf{x}$, respectively. In particular, let us consider the evolution operator for the γ th component of ψ . Our quantum algorithm interleaves the noncommuting collide and stream operators, i.e., $[S_{\Delta x, \gamma}, C] \neq 0$,

$$I_{x\gamma} = S_{-\Delta x, \gamma} C^\dagger S_{\Delta x, \gamma} C, \quad (7)$$

where C^\dagger is the adjoint of C and γ is either 0 or 1 corresponding to streaming either the α or β component of ψ in (3). Since $|\Delta x|$ is small, (7) is close to unity.

Consider the following evolution operator for the γ component of ψ :

$$U_\gamma[\Omega(x)] = I_{x\gamma}^2 I_{y\gamma}^2 I_{z\gamma}^2 e^{-i\varepsilon^2 \Omega(x)}, \quad (8)$$

where ε is a small perturbation parameter and Ω will be specified later. With this evolution operator, the time advancement of the state ψ is given by

$$\psi(x, t + \Delta t) = U_\gamma[\Omega] \psi(x, t). \quad (9)$$

After considerable algebra, it can be shown that, for the particular sequence of unitary collide, stream, and phase-rotation operators in (8), one obtains the following quantum lattice gas equation on expanding in ε :

$$\begin{aligned} \psi(x, t + \Delta t) = & \psi(x, t) - i\varepsilon^2 \left[-\frac{1}{2} \sigma_x \nabla^2 + \Omega \right] \psi(x, t) \\ & + \frac{(-1)^\gamma \varepsilon^3}{4} (\sigma_y + \sigma_z) \nabla^3 \psi(x, t) + \mathcal{O}(\varepsilon^4), \end{aligned} \quad (10)$$

where $\gamma = 0$ or 1. Since the order ε^3 term in (10) changes sign with γ , one can eliminate this term by using a symmetrized evolution operator

$$U[\Omega(x)] = U_1 \left[\frac{\Omega}{2} \right] U_0 \left[\frac{\Omega}{2} \right]. \quad (11)$$

Under diffusion ordering $\Delta t \sim \varepsilon^2$, in the scaling limit $[\psi(x, t + \Delta t) - \psi(x, t)] \rightarrow \varepsilon^2 \partial_t \psi(x, t)$, the quantum map $\psi(x, t + \Delta t) = U[\Omega(x)] \psi(x, t)$ leads to a representation of the two-component parabolic equation

$$i \partial_t \psi = \left(-\frac{1}{2} \sigma_x \nabla^2 + \Omega \right) \psi + \mathcal{O}(\varepsilon^2), \quad (12)$$

where we still have not specified the local generator Ω . To see an emergent GP equation (1), one simply rescales the spatial grid $\nabla \rightarrow a^{-1} \nabla$, contracts the two-component field ψ to the (scalar) BEC wave function φ

$$\varphi = (1, 1) \cdot \psi = \alpha + \beta, \quad (13)$$

and chooses $\Omega = g|\varphi|^2 - 1$:

$$i \partial_t \varphi = -\nabla^2 \varphi + a(g|\varphi|^2 - 1)\varphi + \mathcal{O}(\varepsilon^2). \quad (14)$$

Several comments are in order here. (i) Our unitary algorithm precisely reproduces the Hamiltonian nature of (1). No artificial numerical dissipation is introduced by our reversible algorithm. (ii) While the quantum lattice algorithm is ostensibly second-order accurate, it turns out that its actual accuracy approaches pseudospectral accuracies. Unitarity causes detailed-balanced collisions in the mesoscopic representation and yields Onsager reciprocity. (iii) This quantum lattice gas code with an interleaved unitary collide-stream sequence and operator $e^{-i\varepsilon^2 \Omega}$ was benchmarked against exact collisions of scalar and of vector soliton solutions of the 1D and 2D nonlinear Schrödinger equation [12,13] as well as Korteweg–

de Vries solitons [14]. (iv) Since our quantum algorithm consists of local collisions and information streaming to the nearby grid points, it is ideally parallel. In fact, we have seen no saturation in performance up to the maximum number of cores available to us: 12 288 cores on the CRAY XT-5 and 163 840 cores on the IBM Blue Gene/P. (v) While not stressed here, our quantum algorithm is an exact quantum simulation suited to quantum computing when it becomes available; two qubits represent the ψ field at a point instead of two complex amplitudes in (3) and the

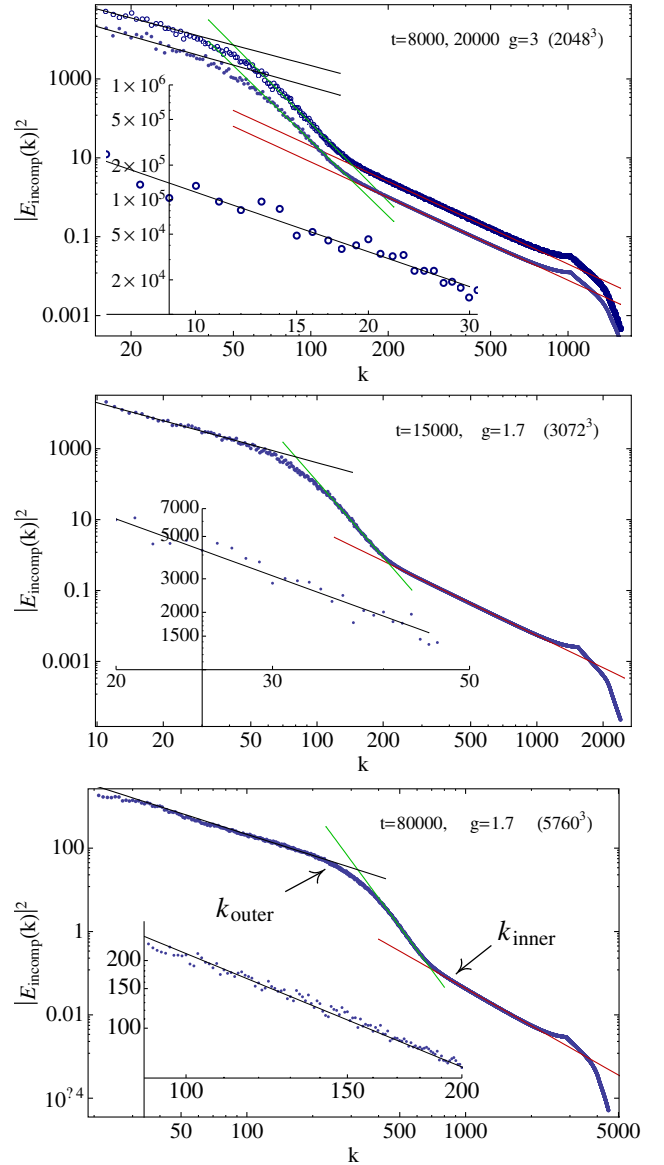


FIG. 2 (color online). The incompressible kinetic energy spectra for a periodic 12-vortex set with $a = 0.02$, and an initial core inner radius is approximately $\xi = 10$ lattice units. The linear regression fits for power-law $k^{-\alpha}$ yield α 's given in Table I. There are 3 distinct spectral regions: (a) $k^{-5/3}$ Kolmogorov energy cascade for small k , (b) steep semiclassical transition region for intermediate k , and (c) k^{-3} Kelvin wave cascade for large k . The Kolmogorov cascade becomes robust for large grids, as seen by the insets.

TABLE I. Spectral exponents found by linear regression for the 3 distinct $k^{-\alpha}$ regions. The 1st row for a 2048^3 grid is at time $t = 8000$ (upper curve in the top panel of Fig. 2), while the 2nd row for a 2048^3 grid is at $t = 20000$ (lower curve in the same panel of Fig. 2). The 3rd row for a 3072^3 grid and 4th for a 5760^3 grid correspond to the middle and bottom panels of Fig. 2, respectively.

| Grid | Kolmogorov | Semiclassical | Kelvin wave |
|----------|-------------------------|--------------------------|----------------------------|
| 2048^3 | 1.73 ($6 < k < 30$) | 6.59 ($60 < k < 140$) | 2.96 ($250 < k < 600$) |
| 2048^3 | 1.84 ($6 < k < 30$) | 6.34 ($60 < k < 140$) | 2.97 ($250 < k < 600$) |
| 3072^3 | 1.69 ($7 < k < 45$) | 7.11 ($120 < k < 200$) | 3.01 ($220 < k < 1000$) |
| 5760^3 | 1.68 ($90 < k < 230$) | 7.12 ($430 < k < 600$) | 3.00 ($1000 < k < 1650$) |

operator (4) would be implemented by a 2-qubit $\sqrt{\text{SWAP}}$ quantum gate creating pairwise entanglement between the on-site qubits. This entanglement spreads through the qubits by (6).

Quantum turbulence simulations.—Most of our simulations had as initial conditions a set of 12 straight line vortices consisting of three groups of 4 vortices [9], with the group axes in the x , y , and z directions. Because the space is periodic, these lines are loops on a torus. The groupings by 4 ensures periodicity. Asymptotically, one can determine the form of a straight line vortex with a unit winding number using a Padé approximate to the steady state solution of (14), following Berloff [15]. In polar coordinates (r, ϕ, z) , one such straight line z vortex (centered at the origin) is

$$\varphi(r) = e^{i\phi} \sqrt{\frac{11ar^2(12 + ar^2)}{g[384 + ar^2(128 + 11ar^2)]}} \quad (15)$$

Asymptotically, $|\varphi| \rightarrow 1/\sqrt{g}$ as $r \rightarrow \infty$ and $|\varphi| \sim r\sqrt{a/g}$ as $r \rightarrow 0$. For a typical isolated core, its *inner radius* scales as the quantum coherence length $\xi \approx (2/a)^{1/2}$, and its *outer radius* scales as $\approx \pi\xi$. Two or more perpendicularly oriented line vortices are unstable, and vortex entanglement ensues from such initial conditions.

The incompressible kinetic energy spectrum can be extracted from the conserved total energy, following Nore, Abid, and Brachet [8]. On the top of Fig. 2, we present such spectra from our simulation of the GP equation on a 2048^3 grid for $a = 0.02$ and $g = 3$ at evolution time $t = 8000$ and $t = 20000$ (in lattice units). The power laws are determined by linear regression. Thus, within a single simulation run, we find that the incompressible kinetic energy spectrum has three distinct power-law $k^{-\alpha}$ regions that range from the classical turbulent regime of Kolmogorov for “large” scales (greater than the outer radius of quantized vortex cores) to the quantum Kelvin wave cascades at the “small” scales (smaller than the inner radius of the individual quantized cores). There is a semiclassical region adjoining the Kolmogorov and Kelvin spectra. On an L^3 grid, the wave number corresponding to the core’s inner radius is $k_{\text{inner}} \approx (\sqrt{D}/2)\frac{1}{\xi}$, while the outer radius wave number is $k_{\text{outer}} \approx k_{\text{inner}}/\pi$, where $D = 3$ is the number of spatial dimensions [9]. These three power-law regions are quite robust as shown in the middle and

bottom of Figs. 2 from simulations on larger grids: 3072^3 and 5760^3 and different initial conditions. The exponent α for the spectral $k^{-\alpha}$ are given in Table I, together with the range of k for these regions. An inset linear regression fit for the Kolmogorov range is shown in Fig. 2. The Kelvin wave cascade is cut off by the lattice.

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