IMPROVING THE LOWER BOUND FOR THE RELIABILITY WHEN THE STRENGTH DISTRIBUTION IS GAMMA AND THE STRESS DISTRIBUTION IS CHI-SQUARE

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ABSTRACT

This paper investigates the lower bound for the reliability of a system when the strength distribution is gamma with parameters α and θ and the stress distribution is chi-square with parameter r. It is shown that the lower bound is a function of α when θ and r are fixed. The moment estimator and the maximum likelihood estimator for α are determined and the lower bound for the reliability using these estimators is computed and compared.

1. INTRODUCTION

In this paper, the lower bound for the reliability is estimated when the strength distribution is gamma and the stress distribution is chi-square. In this discussion, it is assumed that the random variables representing the strength and stress are independent. In the literature, there are several papers and books that deal with this subject. Kapur and Lamberson [1] have derived the relationship for the reliability of a system for different combinations of random variables. Disney and Sheth [2] derived some general formulas for computing the reliability and gave some examples using different random variables for strength and stress. Mischke [3] derived the lower bound for the reliability of a system using the Bienayme-Chebyshev and the Camp-Meidell theorems. Also, Kapur and Lamberson [1], using the Bienayme-Chebyshev theorem gave a different derivation for the lower bound for the reliability for a system. This derivation is similar to the one given by Thien and Massoud [4]. In the above mentioned literature, the gamma strength and the chi-square stress random variables were not considered. The lower bound derived by Kapur and Lamberson [1] will be analyzed in this discussion.

2. ASSUMPTIONS

1. The strength random variable S_1 and the stress random variable S_2 are independent.
2. The strength distribution is gamma with probability density function (pdf) given by

$$f_1(s_1) = \frac{1}{\Gamma(\alpha)\ \theta^{\alpha}}\ s_1^{\alpha-1}\ e^{-s_1/\theta}\ ,\ s_1 \geq 0,\ \theta > 0,\ \alpha > 0$$

   where α is the shape parameter and θ is the scale parameter.
3. The stress distribution is chi-square with pdf given by

$$f_2(s_2) = \frac{1}{\Gamma(r/2)\ 2^{r/2-1}}\ s_2^{r/2-1}\ e^{-s_2/2}\ ,\ s_2 \geq 0,\ r>0$$

   where r is the number of degrees of freedom for the random variable S_2.

4. The degrees of freedom r and the scale parameter θ are known.

3. THE REQUIRED RELIABILITY

The required reliability, R, is given by

$$R = P(S_1 > S_2)$$
$$= P(S_1/S_2 > 1)$$
$$= P(N > 1) \tag{1}$$

where $N = S_1/S_2$. The random variable N is usually defined as the safety factor. We first will determine the pdf for N.

The joint pdf of S_1 and S_2 is given by

$$f(s_1,s_2) = f_1(s_1).f_2(s_2) \tag{2}$$

since S_1 and S_2 are independent. That is

$$f(s_1,s_2) = \frac{1}{\Gamma(\alpha)\ \theta^{\alpha}\ \Gamma(r/2)\ 2^{r/2}} s_1^{\alpha-1}\ e^{-s_1/\theta}\ s_2^{r/2-1}\ e^{-s_2/2} \tag{3}$$

Let $N = S_1 / S_2$ and $W = S_2$; this implies that $S_1 = NW$ and $S_2 = W$.

Now the joint pdf of N and W is given by

$$g(n,w) = f(nw,w).|J|$$

where $|J|$ is the absolute value of the jacobian of transformation

$$J = \begin{vmatrix} \partial S_1/\partial N & \partial S_1/\partial W \\ \\ \partial S_2/\partial N & \partial S_2/\partial W \end{vmatrix}$$

$$= W$$

Therefore,

$$g(n,w) = K_1 (nw)^{\alpha-1} \; e^{-nw/\theta} \; w^{r/2-1} \; e^{-w/2} . w \tag{4}$$

where

$$K_1 = \frac{1}{\Gamma(\alpha)\ \theta^{\alpha}\ \Gamma(r/2)\ 2^{r/2}}$$

Now, the pdf of N, h(n) say, is

$$h(n) = \int_0^\infty g(n,w)dw$$

$$= K_1 n^{\alpha-1} \int_0^\infty w^{\alpha+r/2-1} \; e^{-(n/\theta+1/2)w} \; dw \tag{5}$$

Let $z = (n/\theta+1/2)w$, then we can write h(n) as

$$h(n) = K_1 n^{\alpha-1} \frac{1}{(n/\theta+1/2)^{\alpha+r/2}} \int_0^\infty z^{\alpha+r/2-1} \; e^{-z} \; dz \tag{6}$$

Observe that the integral is a gamma function, so

$$h(n) = \frac{K_1 \; n^{\alpha-1}}{(n/\theta+1/2)^{\alpha+r/2}} \; \Gamma(\alpha+r/2) \tag{7}$$

That is, the pdf for the safety factor when the strength distribution is gamma (with parameters α and θ) and the stress distribution is chi-square (with parameter r) is

$$h(n) = \frac{\Gamma(\alpha+r/2)\ n^{\alpha-1}}{\Gamma(\alpha)\ \Gamma(r/2)\ \theta^{\alpha}\ 2^{r/2}\ (n/\theta+1/2)^{\alpha+r/2}}\ ,n>0 \qquad (8)$$

Now, the reliability, R, of the system is given by

$$R = P(N>1)\ ,\ \text{so}$$

$$R = K_2 \int_1^{\infty} \frac{n^{\alpha-1}}{(n/\theta+1/2)^{\alpha+r/2}}\, dn \qquad (9)$$

where $K_2 = \{\Gamma(\alpha+r/2)\} \;/\; \{\Gamma(\alpha)\ \Gamma(r/2)\ \theta^{\alpha}\ 2^{r/2}\}$.

Let $x = n/\theta+1/2$, then we can transform (9) to

$$R = \frac{K_2\ \theta^{\alpha}}{2^{\alpha-1}} \int_{(1/\theta+1/2)}^{\infty} \frac{(2x-1)^{\alpha-1}}{x^{\alpha+r/2}}\, dx \qquad (10)$$

Next, if we let $v = 2x$ and then $u = 1/v$, then (10) transforms to

$$R = K_2\ \theta^{\alpha}\ 2^{r/2} \int_0^{\theta/(2+\theta)} u^{r/2-1}(1-u)^{\alpha-1}\ du\ ,\ 0<u<1 \tag{11}$$

The above integral is recognized to be the well known incomplete beta function. That is,

$$R = \frac{\Gamma(\alpha+r/2)}{\Gamma(\alpha)\ \Gamma(r/2)}\ \mathbb{B}_{\theta/(\theta+2)}(\alpha, r/2) \tag{12}$$

where $\mathbb{B}_{\theta/(\theta+2)}(\alpha, r/2)$ is the integral in (11).

Special Cases for the Parameters

1. If $\alpha = 1$ and $r = 2$, then

$$f_1(s_1) = 1/\theta\ e^{-s_1/\theta}\ ,\ s_1>0$$

and

$$f_2(s_2) = 1/2\ e^{-s_2/2}\ ,\ s_2>0$$

which are both exponential pdf and

$$R = \frac{\Gamma(2)}{\Gamma(1)\ \Gamma(1)} \int_0^{\theta/(\theta+2)} (1-u)^0\, u^0\, du$$

$$= \theta/(\theta+2) \qquad (13)$$

2. If $\alpha = 1$ and $r>2$,

$$f_1(s_1) = 1/\theta\ e^{-s_1/\theta}\ , \ s_1>0$$

and

$$f_2(s_2) = \frac{1}{\Gamma(r/2)\ 2^{r/2}}\, s_2^{r/2-1}\ e^{-s_2/2}\ , \ s_2>0,\ r>2$$

which are exponential and chi-square pdfs respectively. The reliability, R, is given by

$$R = \frac{1}{\Gamma(1)\ \Gamma(r/2)} \int_0^{\theta/(\theta+2)} (1-u)^0\, u^{r/2-1}\, du$$

$$= \{\theta/(\theta+2)\}^{r/2} \tag{14}$$

3. If $\alpha>1$ and $r = 2$, then

$$f_1(s_1) = \frac{1}{\Gamma(\alpha)\ \theta^{\alpha}}\, s_1^{\alpha-1}\, e^{-s_1/\theta}\ ,\ s_1>0,\ \alpha>1$$

and

$$f_2(s_2) = 1/2\ e^{-s_2/2}\ ,\ s_2>0,\ r=2$$

which are gamma and exponential pdfs respectively. So,

$$R = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\ \Gamma(1)} \int_0^{\theta/(\theta+2)} (1-u)^{\alpha-1}\, u^0\, du$$

$$= 1 - \{2/(\theta+2)\}^{\alpha} \tag{15}$$

4. If $\theta=2$ and $\alpha=r/2$, where r is a positive integer, then both $f_1(s_1)$ and $f_2(s_2)$ are chi-square distributions. This case can be investigated by itself so it will not be discussed here.

4. THE LOWER BOUND FOR THE RELIABILITY

If $\bar{S}_1$ and $\bar{S}_2$ denote the mean strength and stress of the system respectively, and N denote the expected value of the safety factor, then by the Bienayme-Chebyshev inequality [1], [4],

$$P(|N-b|\le\epsilon) \ge 1 - \frac{E\{(N-b)\}^2}{\epsilon^2} \tag{16}$$

where b is any positive constant and $\epsilon>0$. Using this inequality, Kapur and Lamberson [1] and Thien and Massoud [4] showed that

$$P(1\le N\le 2k\bar{N}-1) \ge 1 - \frac{\bar{N}^2\,[V_N^2 + (1-k)^2]}{(k\bar{N} - 1)^2} \tag{17}$$

where $k=b/\bar{N}$ and $V_N^2 = \sigma_N^2 / \bar{N}^2 = [Var(N)]/[E(N)]^2$. By definition,

$$R = P(N>1),$$

so

$$R \geq 1 - \frac{\bar{N}^2 [V_N^2 + (1-k)^2]}{(k\bar{N} - 1)^2} \tag{18}$$

The largest lower bound can be computed if

$$\frac{\bar{N}^2 [V_N^2 + (1-k)^2]}{(k\bar{N}-1)^2} \tag{19}$$

is minimized with respect to k. The critical value for k which minimizes (19) is

$$\frac{\bar{N} (V_N^2 + 1) - 1}{(\bar{N} - 1)} \tag{20}$$

from which

$$R \geq 1 - \frac{\bar{N}^2 V_N^2}{[\bar{N}^2 V_N^2 + (\bar{N} - 1)^2]} \tag{21}$$

Equation (21) is a function of σ_N^2 and $\bar{N}$, since $V_N = \sigma_N^2 / \bar{N}^2$. Thus we can write the lower bound for R as

$$R \geq 1 - \frac{\sigma_N^2}{\sigma_N^2 + (\bar{N} - 1)^2} \qquad (22)$$

Now,

$$\bar{N} = E(N) = K_2 \int_0^\infty \frac{n^\alpha}{(n/\theta + 1/2)^{\alpha + r/2}} \, dn \qquad (23)$$

where K_2 is defined in section 2. With several transformations, we can write E(N) as

$$E(N) = K_2 \, \theta^\alpha \, 2^{r/2-1} \int_0^1 y^{r/2-1} \, (1-y)^\alpha \, dy \ , \ 0<y<1 \qquad (24)$$

Observe that this integral is a beta integral, so

$$E(N) = K_2 \, \theta^\alpha \, 2^{r/2-1} \, \frac{\Gamma(r/2-1) \, \Gamma(\alpha+1)}{\Gamma(r/2+\alpha)}$$

$$= \frac{\alpha\,\Gamma(r/2-1)}{2\,\Gamma(r/2)} \quad , \; r>2 \qquad (25)$$

Also,

$$\sigma_N^2 = E(N^2) - [E(N)]^2 \qquad (26)$$

where

$$E(N^2) = K_2 \int_0^\infty \frac{n^{\alpha+1}}{(n/\theta+1/2)^{\alpha+r/2}}\, dn \qquad (27)$$

Again, with several transformations, we can write $E(N^2)$ as

$$E(N^2) = K_2\, \theta^{\alpha+2}\, 2^{r/2-2} \int_0^1 y^{r/2-3}\, (1-y)^{\alpha+1}\, dy, \quad 0<y<1 \qquad (28)$$

Now the above integral is recognized to be a beta integral, so

$$E(N^2) = K_2\, \theta^{\alpha+2}\, 2^{r/2-2}\, \frac{\Gamma(r/2-2)\,\Gamma(\alpha+2)}{\Gamma(r/2+2)}$$

$$= \frac{\theta^2 \alpha(\alpha+1)\, \Gamma(r/2-2)}{2^2\, \Gamma(r/2)} \tag{29}$$

Therefore,

$$\sigma_N^2 = E(N^2) - [E(N)]^2$$

$$= \frac{\theta^2 \alpha(\alpha+1)\, \Gamma(r/2-2)}{2^2\, \Gamma(r/2)} - \left[\frac{\alpha\, \Gamma(r/2-1)}{2\, \Gamma(r/2)}\right]^2 \tag{30}$$

Now, if we can reduce the factor

$$\frac{\sigma_N^2}{\sigma_N^2 + (\bar{N} - 1)^2}$$

in the lower bound for the reliability, then we can improve the lower bound for the reliability.

Let

$$A_N = \frac{\sigma_N^2}{\sigma_N^2 + (N - 1)\;\bar{\ }^{2}} \; ; \tag{31}$$

therefore, for given θ and r, A_N is just a function of α, the shape parameter. That is, we can reduce (31) to

$$A_N(\alpha) = \frac{1}{1 + \dfrac{[\alpha\,\Gamma(r/2 - 1) - 2\,\Gamma(r/2)]^2}{\theta^2\;\alpha(\alpha + 1)\,\Gamma(r/2)\,\Gamma(r/2 - 2) - [\alpha\,\Gamma(r/2) - 1]^2}} \tag{32}$$

where $A_N(\alpha)$ denotes that the factor A_N is a function of α only for fixed θ and r. We will analyze $1 - A_N(\alpha)$ for both the moment estimator for α and the maximum likelihood estimator for α.

5. ESTIMATORS FOR α

The two estimators for α that are considered are the moment estimator and the maximum likelihood estimator.

<u>The Moment Estimator for α, $\tilde{\alpha}$</u>

Recall that

$$E(N) = \bar{N} = \frac{\alpha\,\Gamma(r/2 - 1)}{2\,\Gamma(r/2)}\,, \quad r>2, \tag{33}$$

thus the moment estimator for α, say $\tilde{\alpha}$, is

$$\tilde{\alpha} = \frac{2\,\bar{N}\,\Gamma(r/2)}{\Gamma(r/2 - 1)}\,, \quad r>2$$

$$= 2\,\bar{N}\,(r/2 - 1) \tag{34}$$

<u>The Maximum Likelihood Estimator for α, $\hat{\alpha}$</u>

Since

$$h(n) = \frac{\Gamma(\alpha + r/2)\; n^{\alpha-1}}{\Gamma(\alpha)\;\Gamma(r/2)\;\theta^{\alpha}\;2^{r/2}\;(n/\theta + 1/2)^{\alpha+r/2}}\,, \quad n>0$$

the likelihood function is

$$L(\alpha) = \prod_{i=1}^{p} \frac{\Gamma(\alpha+r/2)\; n_i^{\alpha-1}}{\Gamma(\alpha)\;\Gamma(r/2)\;\theta^{\alpha}\;2^{r/2}\;(n_i/\theta + 1/2)^{\alpha+r/2}}$$

$$= \left[\frac{\Gamma(\alpha+r/2)}{\Gamma(\alpha)\ \Gamma(r/2)\ \theta^{\alpha}\ 2^{r/2}}\right]^{p} \prod_{i=1}^{p} \frac{n_i^{\alpha-1}}{(n_i/\theta + 1/2)^{\alpha+r/2}} \tag{35}$$

for p=1, 2, 3, ... Let the loglikelihood function be $\mathfrak{L}(\alpha)$, thus

$$\mathfrak{L}(\alpha) = p \ln\left[\frac{\Gamma(\alpha+r/2)}{\Gamma(\alpha)\ \Gamma(r/2)\ \theta^{\alpha}\ 2^{r/2}}\right]$$

$$+ \sum_{i=1}^{p}\left[\ln(n_i^{\alpha-1}) - \ln(n_i/\theta + 1/2)^{\alpha+r/2}\right]$$

$$= p \ln[\ \Gamma(\alpha+r/2)] - p \ln[\ \Gamma(\alpha)] - p\ \alpha \ln(\theta)$$

$$+ \sum_{i=1}^{p} [(\alpha-1) \ln(n_i) - \sum_{i=}^{p} [(\alpha+r/2) \ln(n_i/\theta + 1/2)]$$

$$+ K_3 \tag{36}$$

where $K_3 = -p \ln[\ \Gamma(r/2)\ 2^{r/2}]$ which is not a function of α. Now,

$$\frac{\partial\, \mathcal{L}(\alpha)}{\partial \alpha} = p\, \frac{\Gamma^{\prime}(\alpha+r/2)}{\Gamma(\alpha+r/2)} - p\, \frac{\Gamma^{\prime}(\alpha)}{\Gamma(\alpha)} - p \ln(\theta)$$

$$+ \sum_{i=1}^{p} \ln(n_i) - \sum_{i=1}^{p} \ln(n_i/\theta + 1/2) \tag{37}$$

To solve for the maximum likelihood estimator, $\hat{\alpha}$, we need to equate (37) to zero and solve for α. The expression for $\frac{\partial\, \mathcal{L}(\alpha)}{\partial \alpha}$ however involves two terms which are digamma functions. These terms make it difficult (if not impossible) to obtain an expression for $\hat{\alpha}$. Thus $\frac{\partial\, \mathcal{L}(\alpha)}{\partial \alpha} = 0$ has to be solved numerically for $\hat{\alpha}$.

6. COMPARISON OF THE LOWER BOUNDS AND RELIABILITY VALUES

In the simulation studies, values were generated for the safety factor random variable, N, from the distribution function for N. The parameter θ was taken to be one for simplicity. Values of r used were from the interval (6, 20). Table 1 shows some of the generated results for r, α_1 (the moment estimator for α), α_2 (the maximum likelihood estimator for α), LB1 (the lower bound for the reliability when using α_1), LB2 (the lower bound for the reliability when using α_2), REL1 (the reliability of the system when using α_1) and REL2 (the reliability of the system when using α_2). From this table, one can generally observe that the moment estimator for α produced a higher reliability value and lower bound value for the system than when the maximum likelihood estimator for α is used. Table 2 shows for a set of r values, the percentage of the times LB1 is greater than LB2 and REL1 is greater than REL2. Again, this table suggests that for this study, the moment estimator for α is producing better reliability and lower bound values.

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Table 1

Generated Values for r, the moment estimator(α_1), the MLE(α_2), the lower bound for the reliability using α_1 and α_2(LB1 and LB2), and the reliability of the system using α_1 and α_2(REL1 and REL2).

r	α_1	α_2	LB1	LB2	REL1	REL2
6.1	2.0999	2.5900	0.3253	0.0133	0.5708	0.2224
6.2	2.1818	3.6900	0.3241	0.0132	0.5589	0.2150
6.3	2.1772	3.6600	0.3549	0.0217	0.5737	0.2275
6.4	2.3112	3.8300	0.3343	0.0166	0.5463	0.2095
6.5	2.5334	4.1300	0.2851	0.0065	0.4949	0.1759
6.6	2.4632	4.0199	0.3359	0.0169	0.5278	0.1973
6.7	2.5371	4.0999	0.3375	0.0180	0.5195	0.1936
6.8	2.6193	4.1800	0.3361	0.0192	0.5090	0.1899
6.9	2.6379	4.1999	0.3559	0.0248	0.5165	0.1942
7.0	2.9133	4.5500	0.2929	0.0093	0.4543	0.1578
7.1	2.7699	4.3399	0.3635	0.0291	0.5052	0.1898
7.2	2.7882	4.3300	0.3825	0.0388	0.5130	0.1979
7.3	3.3731	5.1299	0.2317	0.0012	0.3774	0.1172
7.4	2.8272	4.3599	0.4186	0.0563	0.5278	0.2081
7.5	3.0986	4.7099	0.3577	0.0301	0.4671	0.1701
7.6	3.2046	4.8200	0.3493	0.0289	0.4521	0.1636
7.7	3.3878	5.0599	0.3188	0.0185	0.4188	0.1439
7.8	3.3143	4.9399	0.3631	0.0350	0.4484	0.1622
7.9	3.1892	4.7800	0.4226	0.0672	0.4928	0.1924
8.0	3.1746	4.7000	0.4489	0.0854	0.5093	0.2045
8.1	3.7947	5.5199	0.2955	0.0143	0.3705	0.1216
8.2	3.3155	4.8699	0.4509	0.0873	0.4969	0.1959
8.3	3.5593	5.1600	0.4034	0.0612	0.4472	0.1675
8.4	3.6571	5.2700	0.3976	0.0592	0.4352	0.1613

Table 2

Total entries generated and percentage of the times LB1>LB2 and REL1>REL2 for fixed r values.

r	Total Entries	% LB1>LB2	% REL1>REL2
6.1	20	85.00	100.00
6.2	20	80.00	100.00
6.3	21	66.67	100.00
6.4	35	68.57	94.29
6.5	20	65.00	95.00
6.6	26	80.77	100.00
6.7	18	61.11	100.00
6.8	23	60.87	91.30
6.9	20	80.00	100.00
7.0	20	55.00	90.00
7.1	17	52.94	100.00
7.2	16	50.00	100.00
7.3	11	72.73	100.00
7.4	30	56.67	93.33
7.5	7	85.71	100.00
7.6	17	70.59	100.00
7.7	14	64.29	100.00
7.8	21	80.95	100.00
7.9	11	54.55	100.00
8.0	15	53.33	100.00
8.1	6	66.67	100.00
8.2	15	86.67	100.00
8.3	13	46.15	84.62
8.4	10	60.00	100.00
8.5	13	69.23	100.00
8.9	13	84.62	100.00
9.1	6	83.33	100.00