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COMPETING RISKS IN PARALLEL AND SERIES SYSTEMS

by

Lloyd R. Jaisingh
Department of Mathematical Sciences
Morehead State University
Morehead, KY 40351

Key Words - Reliability, Hazard function, Probability density function, Probability distribution function, Competing risks, Series system, Parallel system.

Abstract - The reliability of a system is computed when failure is due to multiple modes. These failure modes are considered as competing risks. Both series and parallel systems are considered when these competing risks are acting independent of one another. A system with dependent competing risks is also considered.

1. INTRODUCTION

In any system, several classes of hazard may be present depending on the types of physical degradation the system experiences. The hazards may act independently of each other or in conjunction with each other to cause failure. For example, in a machining operation, stresses of magnitudes sufficient to cause tool failure because of crack hazards in the tool can occur independently of cratering type hazards. Also, crack hazards occuring because of crater hazards can cause failure; in which case both types of hazards contribute to failure. In the case where joint actions of the different hazards are necessary to induce tool failure, the resulting tool-life distribution will be different than when the hazards act independently of each other to produce failure.

If the life distribution of the entire system was available, it would in principle be possible to determine the marginal distributions for each class of failure hazard.

In this study, we consider series and parallel systems in which hazards are acting independently of each other to cause failure. A case with dependent hazards is also considered. In the case of a series system, the entire system will fail when the first component fails; i.e. the time to failure of the entire system is the minimum of the failure times of the respective components. Similarly, in a parallel system, the entire system will fail when the last component fails for single components in parallel or the entire system will fail when the last series string in parallel fails. In both the cases of parallel and series systems, the components may fail because of several hazards. This type of failure where the components are failing because of several hazards is usually referred to as *competing risks*. Several authors [1], [2], [3], [4] and [5] have discussed the problem of competing risks.

2. SERIES SYSTEM WITH SINGLE RISK IN EACH COMPONENT

In this section a general formula for the failure density function will be derived and interpreted.

Assume that the system may fail due to any hazard. That is, no two hazards contribute to the failure of the system at the same time. Let $h_1(t)$, $h_2(t), \ldots,$ $h_n(t)$ be the hazards that may cause failure. The joint failure density is given in [6] as

$$f_n(t) = \sum_{i=1}^{n} [\ f_i(t)\ \{ \prod_{i \neq j}^{n} R_j(t)\ \}] \tag{1}$$

where $f_i(t)$ is the failure density associated with the i th hazard $h_i(t)$ and $R_j(t)$ is the reliability associated with the j th hazard. If the source of failure is due to hazard $h_i(t)$, then the system must not have failed because of hazard $h_j(t)$, $i \neq j$.

If $R_j(t)$ is defined as the survival probability associated with the j th hazard, then

$$R_n(t) = \exp[- \int_0^t h_j(z)dz] \qquad (2)$$

Also, the relationship between $f_j(t)$ and $h_j(t)$ is given as

$$f_j(t) = h_j(t) \exp[- \int_0^t h_j(z)dz] \qquad (3)$$

Using equations (2) and (3), equation (1) can be written as

$$f_n(t) = \sum_{i=1}^{n} [\{ h_i(t) \exp(- \int_0^t h_i(z)dz)\}\{ \prod_{i\neq j}^{n} \exp(- \int_0^t h_j(z)dz)\}]$$

$$= \sum_{i=1}^{n} [\{ h_i(t) \exp(- \int_0^t h_i(z)dz)\}\{\exp(- \sum_{i\neq j} (\int_0^t h_j(z)dz))\}]$$

$$=[\sum_{i=1}^{n} h_i(t)][\exp(- \int_0^t \sum_{i=1}^{n} h_i(z)dz)] \qquad (4)$$

Equation (4) is very revealing in the sense that it implies that the hazards are additive when they act independently of one another. This is sometimes called the "addition law for failure rates" [1]. Therefore, if the n hazards act independent of one another, the overall hazard is

$$H_n(t) = \sum_{i=1}^{n} h_i(t) \qquad (5)$$

and the joint failure density is

$$f_n(t) = H_n(t)\exp\left(- \int_0^t H_n(z)dz\right) \qquad (6)$$

from which the reliability is

$$R_n(t) = 1 - \int_0^t f_n(z)dz \qquad (7)$$

Note that equation (6) is a general result since it holds for any $H_n(t)$ as given in equation (5).

We next consider a proposition.

<u>Proposition 1</u>: Let $T_1, T_2, \ldots, T_n$ be n independent random variables. If $T = \min(T_1, T_2, \ldots, T_n)$ then the distribution function for T, $F_T(t)$, is

$$F_T(t) = 1 - \prod_{i=1}^{n} (1 - F_{T_i}(t)) \qquad (8)$$

where $F_{T_i}(t)$ is the distribution function for the random variable T_i.

<u>Proof</u>: From definition,

$$F_T(t) = P(T \le t)$$

$$= 1 - P(T > t)$$

$$= 1 - P[\min(T_1, T_2, \ldots, T_n)>t]$$

$$= 1 - P(T_1>t, T_2>t, \ldots, T_n>t)$$

since $T>t$ if and only if $T_i>t$ for every i.

Since the T_i's are independent, then

$$F_T(t) = 1 - \prod_{i=1}^{n} P(T_i>t)$$

$$= 1 - \prod_{i=1}^{n} [1 - F_{T_i}(t)] \qquad (10)$$

If the hazards are acting independently then the failure times will be independent also. It follows from Proposition 1 that if the the T_i's are the failure times then the distribution function for the minimum time to failure is

$$F_T(t) = 1 - \prod_{i=1}^{n} [1 - F_{T_i}(t)]$$

Let $R_i(t) = 1 - F_{T_i}(t)$ be the reliability associated with hazard $h_i(t)$. Then

$$F_T(t) = 1 - \prod_{i=1}^{n} R_i(t)$$

$$= 1 - \prod_{i=1}^{n} \exp(- \int_0^t h_i(z)dz)$$

$$= 1 - \exp\left(- \int_0^t H_n(z)dz \right) \quad (11)$$

Therefore, the probability density function for the minimum time to failure when the hazards are acting independent of each other is

$$\frac{d}{dt}F_T(t) = f_T(t) = - \frac{d}{dt}\exp\left(- \int_0^t H_n(z)dz \right)$$

$$= H_n(t)\exp\left(- \int_0^t H_n(z)dz \right) \quad (12)$$

Comparing equation (12) and equation (6) we see that the probability density for the pooled failure times when the hazards are independent is the same as the probability density for the minimum time to failure. It follows that the hazards are acting in series with one another or that we have a series system. This system will completely fail when any of the contributing components fail and hence the minimum time to failure will be the failure time of the component that fails first. Also, in this series system the respective components are failing because of a single hazard as shown in Figure 1.

This concept of minimum time to failure with independent hazards is often referred to as "independent competing risks" [1].

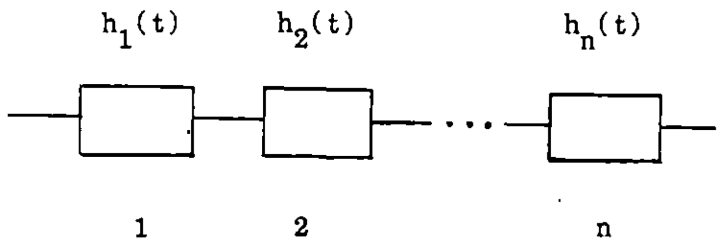

Figure 1. Series System with n components and single risk per component.

3. PARALLEL SYSTEM WITH SINGLE RISK IN EACH OF THE SINGLE COMPONENTS

If a system fails because the hazards are acting independent of one another then the joint probability distribution function of failure, $F_n(t)$, must obey the product law of probabilities. That is, given hazards $h_i(t)$, with associated failure probability distribution functions $F_i(t)$, i=1, 2,..., n, the joint probability distribution function of failure is given by

$$F_n(t) = \prod_{i=1}^{n} F_i(t) \tag{13}$$

Assuming that the failure densities are continuous, we differentiate equation (13) with respect to t to obtain the joint failure probability density, $f_p(t)$:

$$f_p(t) = \frac{d}{dt}\prod_{i=1}^{n} F_i(t)$$

$$= f_1(t) \prod_{i=2}^{n} [\ F_i(t)\ ] + f_2(t) \prod_{\substack{i=1 \\ i \neq 2}}^{n} [\ F_i(t)\ ] + \ldots$$

$$+ f_n(t) \prod_{i=1}^{n-1} [\ F_i(t)\ ]$$

$$= f_1(t) \prod_{i=2}^{n} [\ 1 - R_i(t)\ ] + f_2(t) \prod_{\substack{i=1 \\ i \neq 2}}^{n} [\ 1 - R_i(t)\ ] + \ldots$$

$$+ f_n(t) \prod_{i=1}^{n-1} [\ 1 - R_i(t)\ ]$$

$$= h_1(t)\ \exp[\ -\int_0^t h_1(z)dz\ ] \bullet \prod_{i=2}^{n} [\ 1 - \exp(\ -\int_0^t h_i(z)dz)] +$$

$$h_2(t)\ \exp[\ -\int_0^t h_2(z)dz\ ] \bullet \prod_{\substack{i=1 \\ i \neq 2}}^{n} [\ 1 - \exp(\ -\int_0^t h_i(z)dz)] +$$

$$\ldots +$$

$$h_n(t)\ \exp[\ -\int_0^t h_n(z)dz\ ] \bullet \prod_{i=1}^{n-1} [\ 1 - \exp(\ -\int_0^t h_i(z)dz)] \tag{13}$$

From equation (13), the joint failure density when the hazards are acting independently can be written as

$$f_p(t) = \sum_{i=1}^{n} \{h_i(t) \exp[- \int_0^t h_i(z)dz] \bullet \prod_{\substack{j=1 \\ i\neq j}}^{n} [1 - \exp(- \int_0^t h_j(z)dz)]\} \tag{14}$$

A less complex form will be

$$f_p(t) = \frac{d}{dt}\{ \prod_{i=1}^{n} [1 - \exp(- \int_0^t h_i(z)dz)]\} \tag{15}$$

Note that equation (14) involves only hazard functions; thus if the respective hazards are simple in nature then $f_p(t)$ will not be as complex as it looks. Knowing $f_p(t)$ in equation (14), then $R_n(t)$ as given in equation (7) can be computed. Note again, equation (14) and (15) are general results for the failure density when the hazards are acting independent of each other.

We will consider another proposition.

Proposition 2: Let T_1, T_2, ..., T_n be n independent random variables. If $T = \max(T_1, T_2, \ldots, T_n)$ then the distribution function for T, $F_T(t)$, is given by

$$F_T(t) = \prod_{i=1}^{n} F_{T_i}(t) \tag{16}$$

where $F_{T_i}(t)$ is the distribution function for T_i.

Proof: From the definition of the distribution function, we see

$$F_T(t) = P(T \le t)$$

$$= P[\ \max(T_1,\ T_2,\ \ldots,\ T_n) \le t\ ]$$

$$= P(T_1 \le t,\ T_2 \le t,\ \ldots,\ T_n \le t) \qquad (17)$$

since the largest of the T_i's is less than or equal to t. Since the T_i's are independent, then

$$F_T(t) = \prod_{i=1}^{n} P(T_i \le t)$$

$$= \prod_{i=1}^{n} F_{T_i}(t) \qquad (18)$$

Again, when the hazards are acting independent of each other the failure times will be independent. So, from Proposition 2, if the T_i's are the failure times, the probability density function for the maximum failure time is

$$\frac{d}{dt}F_T(t) = f_T(t) = \frac{d}{dt}[\ \prod_{i=1}^{n} F_{T_i}(t)\ ] \qquad (19)$$

Let $F_{T_i}(t) = 1 - R_i(t)$ where $R_i(t)$ is the reliability associated with T_i. Then

$$F_{T_i}(t) = 1 - \exp(\ -\int_0^t h_i(z)dz)$$

so

$$f_T(t) = \frac{d}{dt}\{\prod_{i=1}^{n}[1 - \exp(-\int_0^t h_i(z)dz)]\} \qquad (20)$$

Comparing equations (15) and (20) we see that the probability density for the pooled failure times when the hazards are independent is the same as the probability density for the maximum time to failure. This implies that the hazards are acting in parallel; i.e we have a parallel system. This system will completely fail when the last component fails and hence the maximum time to failure will be the failure time of the component that fails last. Again, in this parallel system, the respective components are failing because of a single hazard and each branch of the parallel system has only one component as shown in Figure 2.

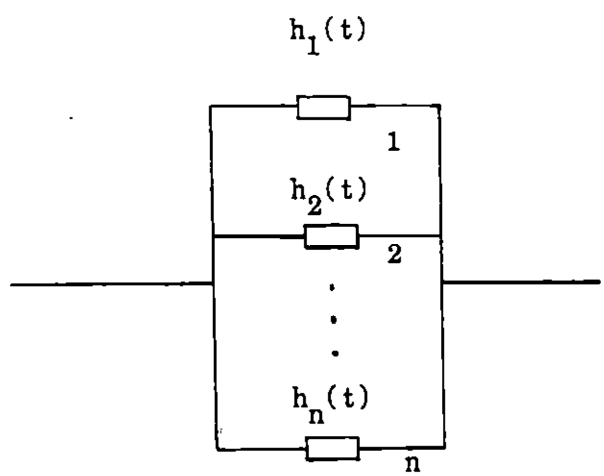

Figure 2. Parallel System with n components and single risk per component.

4. SERIES SYSTEM WITH COMPETING RISKS IN EACH COMPONENT

Suppose we have a series system with n components in which each component can fail because of j causes or hazards, where $j=1, 2, \ldots, m_i$, and $i=1, 2, \ldots, n$. This would imply that each component has a minimum time to failure because of these competing risks and hence the time to failure of the entire system will be the minimum of these failure times. Also, suppose the hazards causing failure in each component are acting independent of each other. Figure 3 shows the system we are considering.

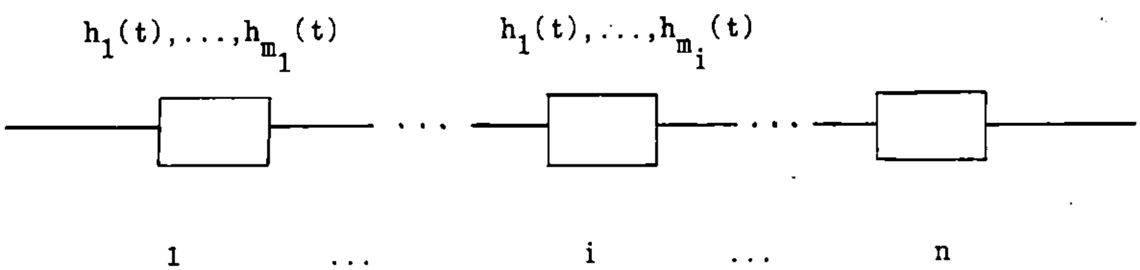

Figure 3. Series System with n components and multiple risks per component.

Let

$h_{ij}(t)$ = j th hazard for the i th component, $i=1, 2, \ldots, n$ and $j=1, 2, \ldots, m_i$,

$R_i(t)$ = reliability for the i th component,

T_{ij} = time to failure of the i th component due to hazard j, and T = time to failure of the entire system.

Then the time to failure of component i is

$$T_i = \min(T_{i1}, T_{i2}, \ldots, T_{im_i}), \ i=1, 2, \ldots, n \qquad (21)$$

and the time to failure of the entire system is

$$T = \min(T_1, T_2, \ldots, T_n) \qquad (22)$$

If we assume that the components are failing independent of each other, i.e. the T_i's are independent, then the probability distribution for the failure time for the entire series system, $F_T(t)$, is given by

$$\begin{aligned} F_T(t) &= P(T \le t) \\ &= 1 - P(T \ge t) \\ &= 1 - P[\ \min(T_1, T_2, \ldots, T_n) > t\ ] \\ &= 1 - P(T_1 > t, T_2 > t, \ldots, T_n > t) \\ &= 1 - \prod_{i=1}^{n} P(T_i > t) \\ &= 1 - \prod_{i=1}^{n} P[\ \min(T_{i1}, T_{i2}, \ldots, T_{im_i}) > t\ ] \\ &= 1 - \prod_{i=1}^{n} P(T_{i1} > t, T_{i2} > t, \ldots, T_{im_i} > t) \end{aligned}$$

$$= 1 - \prod_{i=1}^{n} \{ \prod_{j=1}^{m_i} P(T_{ij} > t) \}$$

$$= 1 - \prod_{i=1}^{n} \prod_{j=1}^{m_i} [1 - F_{T_{ij}}(t)] \qquad (23)$$

and $f_T(t) = \frac{d}{dt} F_T(t)$ is the probability density function for the failure time for the entire system.

The reliability of the series system is then

$$R_T(t) = 1 - F_T(t)$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m_i} [1 - F_{T_{ij}}(t)]$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m_i} R_i(t)$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m_i} \{ \exp(- \int_0^t h_{ij}(z)dz \}$$

$$= \exp[- \int_0^t H_{nm_i}(z)dz] \qquad (24)$$

where

$$H_{nm_i}(z) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} h_{ij}(z). \tag{25}$$

Equations (24) and (25) indicate that the hazards are additive even though we have competing risks in each component.

5. COMPETING RISKS IN A COMBINED SERIES AND PARALLEL SYSTEM

Consider the system shown in Figure 4 in which there are p series systems acting in parallel.

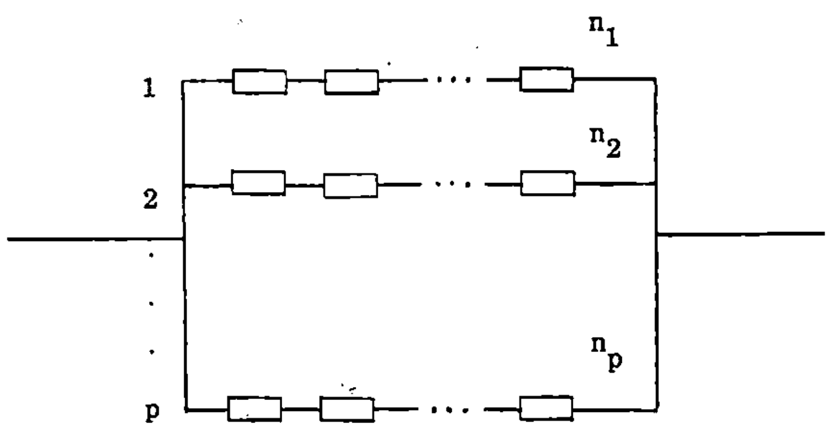

Figure 4. Combined Series and Parallel Systems with competing risks in each component.

Let there be n_k components in the k th series system where k=1, 2, ... , p. Here we will consider two cases: (1) when each component has only one hazard causing failure, and (2) when each component has more than one hazard causing failure and the hazards are acting independently of each other.

Case (1): Single hazards for each component

Let

$h_{ijk}(t)$ = j th hazard for the i th component in the k th series system, where i=1, 2, ..., n_k, and j=1, 2, ... , p.(Note:j=1, since we are considering single hazards for each component.

T_{ik} = failure time for the i th component in the k th series system.

T_k = failure time for the k th series system, k=1, 2, ... ,p.

T = time to complete failure of the system.

Therefore,

$$T_k = \min(T_{1k}, T_{2k}, \ldots, T_{n_k k}), \ k=1, 2, \ldots, p \qquad (26)$$

and

$$T = \max(T_1, T_2, \ldots, T_p) \qquad (27)$$

If we assume that each component is failing independent of the other then the distribution function for the failure time for the entire system, $F_T(t)$, is given by

$$F_T(t) = P(T \leq t)$$

$$= P[\ \max(T_1, T_2, \ldots, T_p) \leq t\ ]$$

(from Proposition 2)

$$= P(T_1 \le t,\ T_2 \le t,\ \ldots\ ,\ T_p \le t)$$

$$= \prod_{k=1}^{p} P(T_k \le t)$$

$$= \prod_{k=1}^{p} [\ 1 - P(T_k > t)\ ]$$

$$= \prod_{k=1}^{p} \{\ 1 - P[\ \min(T_{1k},\ T_{2k},\ \ldots\ , T_{n_k k}) > t\ ]\}$$

(from Proposition 1)

$$= \prod_{k=1}^{p} \{\ 1 - P(T_{1k} > t,\ T_{2k} > t,\ \ldots\ ,\ T_{n_k k} > t)\}$$

$$= \prod_{k=1}^{p} \{\ 1 - \prod_{i=1}^{n_k} P(T_{ik} > t)\}$$

$$= \prod_{k=1}^{p} \{\ 1 - \prod_{i=1}^{n_k} [\ 1 - F_{T_{ik}}(t)\ ]\} \qquad (28)$$

The failure probability density is then

$$f_T(t) = \frac{d}{dt} F_T(t)$$

and the reliability of the system, $R_T(t)$, is given by

$$R_T(t) = 1 - F_T(t)$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [1 - F_{T_{ik}}(t)]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{k=1}^{n_k} [R_{ik}(t)]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\exp(- \int_0^t h_{ijk}(z)\, dz)]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \exp(- \int_0^t H_{jk}(z)\, dz)\} \qquad (29)$$

where

$$H_{jk}(z) = \sum_{i=1}^{n_k} h_{ijk}(z) \text{ for } j=1. \qquad (30)$$

Case 2: Multiple Independent Hazards for each component

Let

$h_{ijk}(t)$ = j th hazard for the i th component in the k th series system where i=1, 2, ... , n_k; j =1, 2, ... , n_i; and k=1, 2, ... , p,

T_{ijk} = failure time for the i th component due to hazard j in the k th series string,

T_{ik} = failure time for the i th component in the k th series string,

and T_k = time to failure of the entire system.

Therefore,

$$T_{ik} = \min(T_{i1k},\ T_{i2k},\ \ldots\ ,\ T_{in_ik}) \tag{31}$$

$$T_k = \min(T_{1k},\ T_{2k},\ \ldots\ ,\ T_{n_kk}) \tag{32}$$

and

$$T = \max(T_1,\ T_2,\ \ldots\ ,\ T_p) \tag{33}$$

Again if we assume that each component is failing independently of each other then the distribution function for the failure time for the entire system, $F_T(t)$, is given by

$$
\begin{aligned}
F_T(t) &= P(T \le t) \\
&= P[\ \max(T_1,\ T_2,\ \ldots\ ,\ T_p) \le t\ ] \\
&= P(T_1 \le t,\ T_2 \le t,\ \ldots\ ,\ T_p \le t) \\
&= \prod_{k=1}^{p} P(T_k \le t) \\
&= \prod_{k=1}^{p} \{\ 1 - P[\ \min(T_{1k},\ T_{2k},\ \ldots\ ,\ T_{n_kk}) > t\ ]\} \\
&= \prod_{k=1}^{p} \{\ 1 - P(T_{1k} > t,\ T_{2k} > t,\ \ldots\ ,\ T_{n_kk} > t)\} \\
&= \prod_{k=1}^{p} \{\ 1 - \prod_{i=1}^{n_k} P(T_{ik} > t)\}
\end{aligned}
$$

$$= \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} P[\min(T_{i1k}, T_{i2k}, \ldots , T_{in_k k})>t]\}$$

$$= \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [P(T_{i1k}>t, T_{i2k}>t, \ldots , T_{in_k k}>t)]\}$$

$$= \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\prod_{j=1}^{n_i} P(T_{ijk}>t)]\}$$

$$= \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\prod_{j=1}^{n_i} (1 - F_{T_{ijk}}(t))]\} \tag{34}$$

from which the probability density function for the failure time for the entire system can be derived.

The reliability of the system is

$$R_T(t) = 1 - F_T(t)$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\prod_{j=1}^{n_i} (1 - F_{T_{ijk}}(t))]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\prod_{j=1}^{n_i} R_{ijk}(t))]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \prod_{i=1}^{n_k} [\prod_{j=1}^{n_i} (\exp(- \int_0^t h_{ijk}(z)\, dz))]\}$$

$$= 1 - \prod_{k=1}^{p} \{ 1 - \exp(- \int_0^t H_{ijk}(z)\, dz)\} \qquad (35)$$

where

$$H_{ijk}(z) = \sum_{i=1}^{n_k} \sum_{j=1}^{n_i} h_{ijk}(z) \qquad (36)$$

6. DEPENDENT COMPETING RISKS FOR A PARALLEL SYSTEM

In this section , we will consider a parallel system with each series string containing three components. We will not only assume that each component can fail due to its single hazard but whenever the second component in each series string fails, this will cause the single hazard in the other two components to increase such that we have instantaneous failure of all three components in the given string. That is, we have dependent hazards or dependent competing risks.

Let

T_{kj} = time to failure of the j th component in the k th series string ; k=1, 2, ... , p and j=1,2,3,

T_k = time to failure of the k th series string,

and T = time to failure of the entire system.

i.e.

$$T_1 = \min[\min(T_{11}, T_{12}), T_{12}, \min(T_{12}, T_{13})]$$

$$T_2 = \min[\min(T_{21}, T_{22}), T_{22}, \min(T_{22}, T_{23})]$$

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$$T_p = \min[\ \min(T_{p1},T_{p2}),\ T_{p2},\ \min(T_{p2},\ T_{p3})\ ]$$

and $$T = \max[\ T_1,\ T_2,\ \ldots\ ,T_p\ ].$$

Sinc the T_k's are independent, it follows that the distribution function for the time to failure, $F_T(t)$, is

$$F_T(t) = P(T \le t)$$

$$= \prod_{k=1}^{p} P(T_k \le t)$$

$$= \prod_{k=1}^{p} [\ 1 - P(T_k > t)\ ]$$

$$= \prod_{k=1}^{p} \{\ 1 - P[\ \min[\ \min(T_{k1},T_{k2}),\ T_{k2},\ \min(T_{k2},T_{k3})]>t]\}$$

$$= \prod_{k=1}^{p} \{\ 1 - P[\ \min(T_{k1},T_{k2})>t,\ T_{k2}>t,\ \min(T_{k2},T_{k3})>t]\} \qquad (37)$$

i.e. $$F_T(t) = \prod_{k=1}^{p} \{\ 1 - [\ \prod_{j=1}^{3} P(T_{kj}>t)\ + \prod_{j=1}^{2} P(T_{kj}>t)\ + \prod_{j=2}^{3} P(T_{kj}>t) + P(T_{k2}>t)]\} \qquad (38)$$

from which $f_T(t)$, the probability density function for the failure time of the entire system, can be found. Also, $R_T(t)$, the reliability of the system, can be computed. Note that equation (38) can be evaluated only if the densities, or the distributions for the T_{kj}'s are known.

If we let

$$X = \min(T_{k1}, T_{k2}),$$

$$Y = T_{k2},$$

and

$$Z = \min(T_{k2}, T_{k3}) \text{ , for } k=1, 2, \ldots, p,$$

we observe that these random variables are not independent so that $F_T(t)$ as given in equation (38) cannot be written as the product of the marginal distribution functions for X, Y and Z. This follows from the dependence of the competing risks.

7. CONCLUSIONS

The results derived in this discussion are very general in nature. Hence they can be used for any given hazard when competing risks are considered for the systems discussed. The general results are not very complex; and hence, for simple hazards, the end results will not be complex. The results derived for dependent competing risks are a little more involved than when the hazards are acting independent of each other. For more complex systems however, the results for both dependent and independent competing risks will be difficult to determine.

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