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Constructing a Large Number of Nonisomorphic Graphs of Order n

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Abstract: We give a simple construction of a large number of nonisomorphic graphs of order n.

While it is easy to see that there are $T_n = 2^{\binom{n}{2}}$ labelled graphs of order n, the enumeration of nonisomorphic graphs is a different question altogether. A counting polynomial for such graphs can be obtained as an application of Pólya's theorem (*cf.* Harary [2, pp. 180 - 185]). We give a simple construction that yields approximately $\sqrt{T_n}$ nonisomorphic graphs of order n. We use the notation of [1].

Basic construction

We begin with a basic construction of a disconnected graph G of even order n, which will be a subgraph of each of the nonisomorphic graphs. Let $V(G) = A \cup B$ (disjoint union), where $A = \{a_1, ..., a_{\frac{n}{2}}\}$ and $B = \{b_1, ..., b_{\frac{n}{2}}\}$. Add edges such that $\langle\{a_1, a_2, a_3\}\rangle = K_3$, $a_3, ..., a_{\frac{n}{2}}$ is the vertex sequence of the path $P_{\frac{n}{2}-2}$, and $\langle B\rangle \cong \overline{\langle A\rangle}$ under the isomorphism $a_i \mapsto b_i$. Note that $\Delta\left(\langle A\rangle\right) = 3$ (with a_3 the only vertex of degree 3) and $\delta\left(\langle B\rangle\right) = \frac{n}{2} - 4$.

Construction of the nonisomorphic graphs

Let $A_1 = \{a_8, ..., a_{\frac{n}{2}}\}$ and $B_1 = \{b_8, ..., b_{\frac{n}{2}}\}$. There are $p = \left(\frac{n}{2} - 7\right)^2$ possible edges between A_1 and B_1. For each of the 2^p subsets $S_1, ..., S_{2^p}$ of these edges, let G_i be the graph obtained from G by adding the edges in S_i. We prove that no two of these graphs are isomorphic. To simplify the notation we use the same labels (that of G) for the vertices of all the graphs constructed; confusion is unlikely.

Theorem 1 *For all pairs $i, j \in \{1, ..., 2^p\}$ with $i \neq j$, $G_i \ncong G_j$.*

Proof. Suppose to the contrary that $f : G_i \to G_j$ is an isomorphism. By construction, $\Delta_{G_k}\left(\langle A\rangle\right) \leq \frac{n}{2} - 5$ and $\delta_{G_k}\left(\langle B\rangle\right) \geq \frac{n}{2} - 4$ for any $k \in \{1, ..., 2^p\}$. Thus for any $a \in A$ and $b \in B$, $f(a) \in A$ and $f(b) \in B$. Also, a_3 is the only vertex of degree three which lies on a triangle contained in $\langle A \rangle$; hence $f(a_3) = a_3$. Now suppose $f(a_k) = a_l$, where $k, l \geq 4$. The length of a shortest $a_3 - a_k$ path in G_i that consists entirely of vertices of degree at most $\frac{n}{2} - 5$ is $k - 3$. Hence the length of a similar $a_3 - a_l$ path in G_j is also $k - 3$. This is only possible if $k = l$. It thus follows that $f(a_k) = a_k$ for all $k \geq 3$.

A similar argument shows that $f(b_k) = b_k$ for all $k \geq 3$. But then $a_k b_l \in S_i$ if and only if $a_k b_l \in S_j$, a contradiction since $i \neq j$. ■

We have thus constructed $2^p \approx 2^{n^2/4} \approx \sqrt{T_n}$ nonisomorphic graphs of order n.

References

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall, London, 1996.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1969.