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An Approach to Discrete Quantum Gravity

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Abstract

This article presents a simplified version of the author's previous work. We first construct a causal growth process (CGP). We then form path Hilbert spaces using paths of varying lengths in the CGP. A sequence of positive operators on these Hilbert spaces that satisfy certain normalization and consistency conditions is called a quantum sequential growth process (QSGP). The operators of a QSGP are employed to define natural decoherence functionals and quantum measures. These quantum measures are extended to a single quantum measure defined on a suitable collection of subsets of a space of all paths. Continuing our general formalism, we define curvature operators and a discrete analogue of Einstein's field equations on the Hilbert space of causal sets. We next present a method for constructing a QSGP using an amplitude process (AP). We then consider a specific AP that employs a discrete analogue of a quantum action. Finally, we consider the special case in which the QSGP is classical. It is pointed out that this formalism not only gives a discrete version of general relativity, there is also emerging a discrete analogue of quantum field theory. We therefore have discrete versions of these two theories within one unifying framework.

1 Introduction

In some previous articles, the author developed a model for discrete quantum gravity by first constructing a classical sequential growth process (CSGP) [1, 2, 5–11]. Roughly speaking, the CSGP corresponded to the classical configuration space for a system of multi-universes. As is frequently done in quantum theory, we then quantized the CSGP to form a Hilbert space. The role of a quantum state was played by a sequence of probability operators on an increasing sequence of Hilbert spaces. In the present article we give a simplified version in which we dispense with the CSGP and immediately begin with a quantum sequential grown process (QSGP). This approach appears to be cleaner and more direct.

We begin by constructing a causet growth process (CGP) for the collection of causal sets (causets) $\mathcal{P}$ [7, 9, 10]. We then define the space of paths Ω and the space Ω_n of n-paths of length n in $\mathcal{P}$. Letting $\mathcal{A}$ be the σ-algebra of subsets of Ω generated by the cylinder sets $\mathcal{C}(\Omega)$, we obtain the measurable space $(\Omega, \mathcal{A})$. For a set $A \in \mathcal{A}$, its n-th approximation $A^n \subseteq \Omega_n$ is the set of n-paths that can be extended to a path in Ω. Forming the Hilbert spaces $H_n = L_2(\Omega_n)$, a sequence of positive operators ρ_n on H_n satisfying a normalization and consistency condition is called a quantum sequential growth process (QSGP) [5]. The probability operators ρ_n are employed to define natural decoherence functionals $D_n(A, B)$ and quantum measures [8] $\mu_n(A) = D_n(A, A)$ on Ω_n. A set $A \in \mathcal{A}$ is called *suitable* if $\lim \mu_n(A^n)$ exists and in this case we define $\mu(A)$ to be this limit. Denoting the collection of suitable sets by $\mathcal{S}(\Omega)$ we have that $\mathcal{C}(\Omega) \subseteq \mathcal{S}(\Omega) \subseteq \mathcal{A}$ and, in general, the inclusions are proper. In a certain sense, μ becomes a quantum measure on $\mathcal{S}(\Omega)$ so $(\Omega, \mathcal{S}(\Omega), \mu)$ becomes a quantum measure space [8]. Continuing our general formalism, we present a discrete analogue to Einstein's field equations. This is accomplished by defining curvature operators $\mathcal{R}_{\omega, \omega'}$ on $L_2(\mathcal{P}) \otimes L_2(\mathcal{P})$ for every $\omega, \omega' \in \Omega$.

We next present a method for constructing a QSGP using an amplitude process (AP). An AP is essentially a Markov chain with complex-valued transition amplitudes. We show that an AP simplifies our previous formalism. We then consider a specific AP that employs a discrete analogue of a quantum action. We use this AP to illustrate some of our previous theory. Finally, we consider the special case in which the QSGP is classical. In this case the decoherence matrices $D_n(\omega, \omega')$ become diagonal and the quantum measure μ_n becomes ordinary measures. We end by studying the possibility that the

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curvature operator vanishes in this case.

2 The Causet Growth Process

In this article we call a finite partially ordered set a *causet*. Two isomorphic causets are considered to be identical. Let $\mathcal{P}_n$ be the collection of al causets of cardinality n, $n = 1, 2, \ldots$, and let $\mathcal{P} = \cup \mathcal{P}_n$ be the collection of all causets. If a, b are elements of a causet x, we interpret the order $a < b$ as meaning that b is in the causal future of a and a is in the causal past of b. An element $a \in x$, for $x \in \mathcal{P}$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$, $y \in \mathcal{P}_{n+1}$, then x *produces* y if y is obtained from x by adjoining a single maximal element a to x. In this case we write $y = x \uparrow a$ and use the notation $x \to y$. If $x \to y$, we also say that x is a *producer* of y and y is an *offspring* of x. Of course, x may produce many offspring and y may be the offspring of many producers. Moreover, x may produce y in various isomorphic ways [7,9,11]. We denote by $m(x \to y)$ the number of isomorphic ways that x produces y and call $m(x \to y)$ the *multiplicity* of $x \to y$.

If $a, b \in x$ with $x \in \mathcal{P}$, we say that a and b are *comparable* if $a \leq b$ or $b \leq a$. A *chain* in x is a set of mutually comparable elements of x and an *antichain* is a set of mutually incomparable elements of x. By convention, the empty set in both a chain and an antichain and clearly singleton sets also have this property. A chain is *maximal* if it is not a proper subset of a larger chain. The following result was proved in [5]. We denote the cardinality of a finite set A by $|A|$.

Theorem 2.1. *The number of offspring r of a causet x, including multiplicity, is the number of distinct antichains in x. We have that $|x| + 1 \leq r \leq 2^{|x|}$ with both bounds achieved.*

The transitive closure of $\to$ makes $\mathcal{P}$ into a poset itself and we call $(\mathcal{P}, \to)$ the causet growth process (CGP) [6,7,9,10]. Figure 1 illustrates the first three steps of the CGP. The 2 in Figure 1 denotes the fact that $m(x_2 \to x_6) = 2$. It follows from Theorem 2.1 that the number of offspring for x_4, x_5, x_6, x_7 and x_8 are $4, 5, 6, 5, 8$, respectively. In this case, Theorem 2.1 tells us that $4 \leq r \leq 8$.

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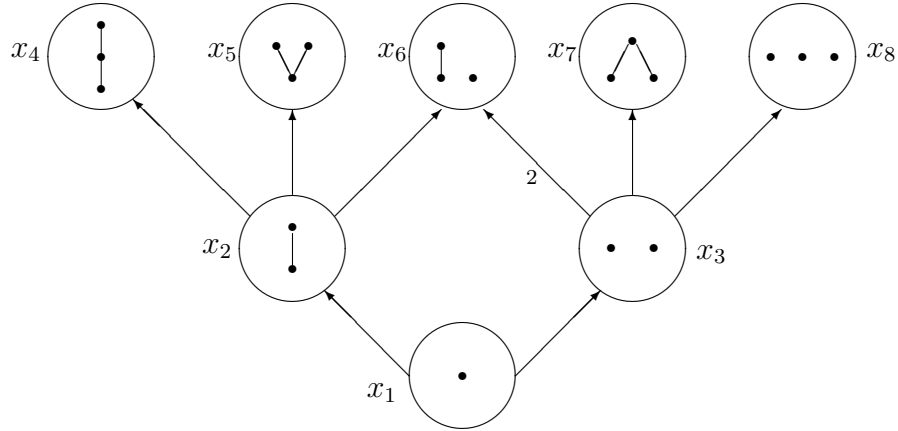
Figure 1

Theorem 2.1 has a kind of dual theorem that we now discuss. Two maximal chains C_1, C_2 in a causet x are *equivalent* if, when the maximal elements a_1 and a_2 of C_1, C_2, respectively, are deleted, then the resulting causets $C_1 \smallsetminus \{a_1\}$, $C_2 \smallsetminus \{a_2\}$ are isomorphic. For example, in Figure 1, the maximal chains in x_4, x_5, x_7, x_8 are mutually equivalent while x_6 has two inequivalent maximal chains. Notice that x_4, x_5, x_7, x_8 each have one producer, while x_6 has two producers. This observation motivates Theorem 2.3. But first we prove a lemma.

Lemma 2.2. *The definition of equivalent maximal chains gives an equivalence relation.*
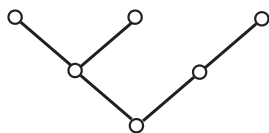
Proof. If C_1, C_2 are equivalent maximal chains in a causet x, we write $C_1 \sim C_2$. It is clear that $C_1 \sim C_1$ and $C_1 \sim C_2$ implies $C_2 \sim C_1$. To prove transitivity, suppose that $C_1 \sim C_2$ and $C_2 \sim C_3$. Let a_1, a_2, a_3 be the maximal elements of C_1, C_2, C_3, respectively. Then $C_1 \smallsetminus \{a_1\}$ and $C_2 \smallsetminus \{a_2\}$ are isomorphic and so are $C_2 \smallsetminus \{a_2\}$ and $C_3 \smallsetminus \{a_3\}$. Since being isomorphic is an equivalence relation, we conclude that $C_1 \smallsetminus \{a_1\}$ and $C_3 \smallsetminus \{a_3\}$ are isomorphic. Hence, $C_1 \sim C_3$. $\square$

Theorem 2.3. *The number of producers of a causet y is the number of inequivalent maximal chains in y.*

Proof. If x produces y, then $y = x \uparrow a$ and a is the maximal element of at least one maximal chain in y. Conversely, if C is a maximal chain in y with maximal element a, then $y \smallsetminus \{a\}$ produces y with $y = (y \smallsetminus \{a\}) \uparrow a$. If C_1 is another maximal chain that is inequivalent to C and b is the maximal

element of C_1, then $y \smallsetminus \{a\}$ and $y \smallsetminus \{b\}$ are nonisomorphic producers of y. Hence, the correspondence between producers of y and inequivalent maximal chains in y is a bijection. $\qquad\square$

**Example 1**. The following causet has two inequivalent maximal chains.

and the two producers
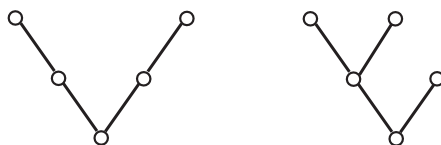
Notice that equivalent maximal chains may appear to be quite different. In particular, if two (or more) maximal chains have the same maximal element , they are equivalent.

Example 2. The three maximal chains in the following causet are equivalent, so this causet has just one producer.
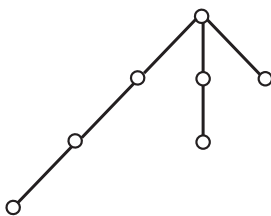
A *path* in $\mathcal{P}$ is a sequence (string) $\omega_1\omega_2\ldots$, where $\omega_i \in \mathcal{P}_i$ and $\omega_i \to \omega_{i+1}$, $i = 1, 2, \ldots$. An *n-path* in $\mathcal{P}$ is a finite string $\omega_1\omega_2\ldots\omega_n$, where again $\omega_i \in \mathcal{P}_i$ and $\omega_i \to \omega_{i+1}$. We denote the set of paths by Ω and the set of n-paths by Ω_n. If $\omega = \omega_1\omega_2\ldots\omega_n \in \Omega_n$ we define $(\omega \to) \subseteq \Omega_{n+1}$ by

$$(\omega \to) = \{\omega_1\omega_2\ldots\omega_n\omega_{n+1} \colon \omega_n \to \omega_{n+1}\}$$

Thus, $(\omega \to)$ is the set of one-step continuations of ω. If $A \in \Omega_n$ we define $(A \to) \subseteq \Omega_{n+1}$ by

$$(A \to) = \bigcup\{(\omega \to) \colon \omega \in A\}$$

The set of all paths beginning with $\omega \in \Omega_n$ is called an *elementary cylinder set* and is denoted by $\mathrm{cyl}(\omega)$. If $A \in \Omega_n$, then the *cylinder set* $\mathrm{cyl}(A)$ is defined by

$$\mathrm{cyl}(A) = \bigcup\{\mathrm{cyl}(\omega) \colon \omega \in A\}$$

Using the notation

$$\mathcal{C}(\Omega_n) = \{\mathrm{cyl}(A) \colon A \subseteq \Omega_n\}$$

we see that

$$\mathcal{C}(\Omega_1) \subseteq \mathcal{C}(\Omega_2) \subseteq \ldots$$

is an increasing sequence of subalgebras of the *cylinder algebra* $\mathcal{C}(\Omega) = \cup\,\mathcal{C}(\Omega_n)$. Letting $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}(\Omega)$, we have that $(\Omega, \mathcal{A})$ is a measurable space. For $A \subseteq \Omega$ we define the sets $A^n \subseteq \Omega_n$ by

$$A^n = \{\omega_1\omega_2\ldots\omega_n \colon \omega_1\omega_2\ldots\omega_n\omega_{n+1}\ldots \in A\}$$

That is, A^n is the set of n-paths that can be continued to a path in A. We think of A^n as the n-step approximation to A. We have that

$$\mathrm{cyl}(A^1) \supseteq \mathrm{cyl}(A^2) \supseteq \mathrm{cyl}(A^3) \supseteq \ldots \supseteq A$$

so that $A \subseteq \cap\,\mathrm{cyl}(A^n)$ but $A \neq \cap\,\mathrm{cyl}(A^n)$ in general even if $A \in \mathcal{A}$.

3   Quantum Sequential Growth Processes

Let $H_n = L_2(\Omega_n)$ be the *n-path Hilbert space* $\mathbb{C}^{\Omega_n}$ with the usual inner product

$$\langle f, g \rangle = \sum \left\{ \overline{f(\omega)} g(\omega) \colon \omega \in \Omega_n \right\}$$

For $A \subseteq \Omega_n$, the characteristic function $\chi_A \in H_n$ has norm $\|\chi_A\| = \sqrt{|A|}$. In particular, $1_n = \chi_{\Omega_n}$ satisfies $\|1_n\| = \sqrt{|\Omega_n|}$. A positive operator ρ on H_n that satisfies $\langle \rho 1_n, 1_n \rangle = 1$ is called a *probability operator*. Corresponding to a probability operator ρ we define the *decoherence functional* $D_\rho \colon 2^{\Omega_n} \times 2^{\Omega_n} \to \mathbb{C}$ by

$$D_\rho(A, B) = \langle \rho \chi_B, \chi_A \rangle$$

Notice that D_ρ has the usual properties of a decoherence functional. That is, $D_\rho(\Omega_n, \Omega_n) = 1$, $D_\rho(A, B) = \overline{D_\rho(B, A)}$, $A \mapsto D_\rho(A, B)$ is a complex measure on 2^{Ω_n} for every $B \subseteq \Omega_n$ and if $A_i \subseteq \Omega_n$, $i = 1, 2, \ldots, r$, then the $r \times r$ matrix with components $D_\rho(A_i, A_j)$ is positive semidefinite [6, 8–10].

We interpret $D_\rho(A, B)$ as a measure of the interference between the events A, B when the system is described by ρ. We also define the *q-measure* $\mu_\rho \colon 2^{\Omega_n} \to \mathbb{R}^+$ by $\mu_\rho(A) = D_\rho(A, A)$ and interpret $\mu_\rho(A)$ as the quantum propensity of the event $A \subseteq \Omega_n$. In general, μ_ρ is not additive on 2^{Ω_n} so μ_ρ is not a measure. However, μ_ρ is *grade-2 additive* [8, 9] in the sense that if $A, B, C \in 2^{\Omega_n}$ are mutually disjoint, then

$$\mu_\rho(A \cup B \cup C) = \mu_\rho(A \cup B) + \mu_\rho(A \cup C) + \mu_\rho(B \cup C) - \mu_\rho(A) - \mu_\rho(B) - \mu_\rho(C)$$

Let ρ_n be a probability operator on H_n, $n = 1, 2, \ldots$. We say that the sequence $\{\rho_n\}$ is *consistent* [5] if

$$D_{\rho_{n+1}}(A \to, B \to) = D_{\rho_n}(A, B)$$

for every $A, B \subseteq \Omega_n$. We call a consistent sequence $\{\rho_n\}$ a *quantum sequential growth process* (QSGP). Let $\{\rho_n\}$ be a QSGP and denote the corresponding *q*-measures by μ_n. A set $A \in \mathcal{A}$ is called *suitable* if $\lim \mu_n(A^n)$ exists (and is finite) in which case we define $\mu(A) = \lim \mu_n(A^n)$. We denote the collection of suitable sets by $\mathcal{S}(\Omega)$. Of course, $\emptyset, \Omega \in \mathcal{S}(\Omega)$ with $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$. If $A \in \mathcal{C}(\Omega)$, then $A = \text{cyl}(B)$ where $B \subseteq \Omega_n$ for some $n \in \mathbb{N}$. Since $A^n = B$, $A^{n+1} = B \to$, $A^{n+2} = (B \to) \to, \ldots$, it follows from consistency that $\lim \mu_n(A^n) = \mu_n(B)$. Hence, $A \in \mathcal{S}(\Omega)$ and $\mu(A) = \mu_n(B)$. It follows

that $\mathcal{C}(\Omega) \subseteq \mathcal{S}(\Omega) \subseteq \mathcal{A}$ and it can be shown that the inclusions are proper, in general. In a certain sense, μ is a quantum measure on $\mathcal{S}(\Omega)$ that extends the q-measures μ_n.

There are physically relevant sets in $\mathcal{A}$ that are not in $\mathcal{C}(\Omega)$. In this case it is important to know whether such a set A is in $\mathcal{S}(\Omega)$ and to find $\mu(A)$. For example, if $\omega \in \Omega$ then

$$\{\omega\} = \bigcap_{n=1}^{\infty} \{\omega\}^n \in \mathcal{A}$$

but $\{\omega\} \notin \mathcal{C}(\Omega)$. It is of interest whether $\{\omega\} \in \mathcal{S}(\Omega)$ and if so, to find $\mu(\{\omega\})$. As another example, the complement $\{\omega\}' \notin \mathcal{C}(\Omega)$. Even if $\{\omega\} \in \mathcal{S}(\Omega)$, since $\mu_n(A') \neq 1 - \mu_n(A)$ for $A \subseteq \Omega_n$, it does not immediately follow that $\{\omega\}' \in \mathcal{S}(\Omega)$. For this reason we would have to treat $\{\omega\}'$ as a separate case.

4   Discrete Einstein Equations

Let $\{\rho_n\}$ be a QSGP with corresponding decoherence matrices

$$D_n(\omega, \omega') = D_n(\{\omega\}, \{\omega'\})$$

$\omega, \omega' \in \Omega_n$. If $\omega = \omega_1 \omega_2 \ldots \omega_n \in \Omega_n$ and $\omega_i = x$ for some i, then ω *contains* x. Notice that ω contains x if and only if $\omega_{|x|} = x$. For $x, y \in \mathcal{P}$ with $|x|, |y| \leq n$ we define

$$D(x, y) = \sum \{D_n(\omega, \omega') \colon \omega \text{ contains } x, \omega' \text{ contains } y\}$$

If $A_x \subseteq \Omega_n$ is the set $A_x = \{\omega \in \Omega_n \colon \omega_{|x|} = x\}$ and similarly for $A_y \subseteq \Omega_n$, then $D(x, y) = D_n(A_x, A_y)$. Due to the consistency of $\{\rho_n\}$, $D(x, y)$ is independent of n if $|x|, |y| \leq n$. Also, $D(x, y)$ for $|x|, |y| \leq n$, are the components of a positive semidefinite matrix. We think of $\mathcal{P}$ as a discrete analogue of a differentiable manifold and $D(x, y)$ as a discrete analogue of a metric tensor.

Let $K = L_2(\mathcal{P})$ be the Hilbert space of square summable complex-valued functions on $\mathcal{P}$ with the standard inner product

$$\langle f, g \rangle = \sum_{x \in \mathcal{P}} \overline{f(x)} g(x)$$

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Let $L = K \otimes K$ which we can identify with the space of square summable complex-valued functions on $\mathcal{P} \times \mathcal{P}$. For $\omega, \omega' \in \Omega$ we define the *covariant bidifference operator* $\nabla_{\omega,\omega'}$ on L [4] by

$$\nabla_{\omega,\omega'} f(x, y) = \left[D(\omega_{|x|-1}, \omega'_{|y|-1}) f(x, y) - D(x, y) f(\omega_{|x|-1}, \omega'_{|y|-1}) \right] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}}$$

In general, $\nabla_{\omega,\omega'}$ may be unbounded but it is densely defined. The covariant designation stems from the fact that $\nabla_{\omega,\omega'} D(x, y) = 0$ for every $x, y \in \mathcal{P}$, $\omega, \omega' \in \Omega$.

In analogy to the curvature tensor on a manifold, we define the *discrete curvature operator* $\mathcal{R}_{\omega,\omega'}$ on L by

$$\mathcal{R}_{\omega,\omega'} = \nabla_{\omega,\omega'} - \nabla_{\omega',\omega}$$

We also define the *discrete metric operator* $\mathcal{D}_{\omega,\omega'}$ on L by

$$\begin{aligned}
\mathcal{D}_{\omega,\omega'} & f(x, y) \\
&= D(x, y) \left[f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right]
\end{aligned}$$

and the *discrete mass-energy* operator $\mathcal{T}_{\omega,\omega'}$ on L by

$$\begin{aligned}
\mathcal{T}_{\omega,\omega'} & f(x, y) \\
&= \left[D(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} - D(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y)
\end{aligned}$$

It is not hard to show that

$$\mathcal{R}_{\omega,\omega'} = \mathcal{D}_{\omega,\omega'} + \mathcal{T}_{\omega,\omega'} \tag{4.1}$$

We call (4.1) the *discrete Einstein equations* [4].

If we can find $D(x, y)$ such that the classical Einstein equations are an approximation to (4.1) then it would give information about $D(x, y)$. Moreover, an important problem in discrete quantum gravity theory is to test whether general relativity is a close approximation to the theory. Whether Einstein's classical equations are an approximation to (4.1) would provide such a test.

We can obtain a simplification by considering the *contracted discrete operators* $\widehat{\mathcal{R}}_{\omega,\omega'}$, $\widehat{\mathcal{D}}_{\omega,\omega'}$, $\widehat{\mathcal{T}}_{\omega,\omega'}$, from their domains in L into K, respectively, given by $\widehat{\mathcal{R}}_{\omega,\omega'} f(x) = \mathcal{R}_{\omega,\omega'} f(x, x)$, $\widehat{\mathcal{D}}_{\omega,\omega'} f(x) = \mathcal{D}_{\omega,\omega'} f(x, x)$, $\widehat{\mathcal{T}}_{\omega,\omega'} f(x) = \mathcal{T}_{\omega,\omega'} f(x, x)$. We then have the *contracted discrete Einstein equations*

$$\widehat{\mathcal{R}}_{\omega,\omega'} = \widehat{\mathcal{D}}_{\omega,\omega'} + \widehat{\mathcal{T}}_{\omega',\omega}$$

Defining $\mu(x) = \mathcal{D}(x,x)$ for every $x \in \mathcal{P}$ we have

$$\widehat{\mathcal{D}}_{\omega,\omega'} f(x) = \mu(x) \left[f(\omega'_{|x|-1}, \omega_{|x|-1}) - f(\omega_{|x|-1}, \omega'_{|x|-1}) \right] \delta_{x,\omega'_{|x|}} \delta_{x,\omega_{|x|}}$$

$$\widehat{\mathcal{T}}_{\omega,\omega'} f(x) = \left[2i \, Im \, D(\omega_{|x|-1}, \omega'_{|x|-1}) \right] \delta_{x,\omega_{|x|}} \delta_{x,\omega'_{|x|}} f(x,x)$$

We can obtain a better understanding of these operators by seeing how they are realized on basis vectors. For $x \in \mathcal{P}$ define the unit vector $e_x = \chi_{\{x\}}$ in K. Then $\{e_x \colon x \in \mathcal{P}\}$ forms an orthonormal basis for K and $S = \{e_x \otimes e_y \colon x, y \in \mathcal{P}\}$ forms an orthonormal basis for L. The subspace spanned by S is a dense subspace of L on which the operators are defined. Let $\omega, \omega' \in \Omega$ and $x, y \in \mathcal{P}$. We call (x, y) an (ω, ω') *pair* if ω contains x and ω' contains y. If (x, y) is both an (ω, ω') pair and an (ω', ω) pair, then (x, y) is an $(\omega, \omega') - (\omega', \omega)$ *pair*. Then all our operators vanish except possibly at points (x, y) that are (ω, ω') or (ω', ω) pairs. We conclude that these operators are local in the sense that they vanish except along the two paths ω, ω'.

Theorem 4.1. (a) *If (x, y) is an (ω, ω') pair but not an (ω', ω) pair, then*

$$\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = -D(\omega_{|x|+1}, \omega'_{|y|+1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|y|+1}}$$

$$\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = D(\omega_{|x|-1}, \omega'_{|y|-1}) e_x \otimes e_y$$

(b) *If (x, y) is an (ω', ω) pair but not an (ω, ω') pair, then*

$$\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = D(\omega'_{|x|+1}, \omega_{|y|+1}) e_{\omega'_{|x|+1}} \otimes e_{\omega_{|y|+1}}$$

$$\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = -D(\omega'_{|x|-1}, \omega_{|y|-1}) e_x \otimes e_y$$

(c) *If (x, y) is an $(\omega, \omega') - (\omega', \omega)$ pair, then*

$$\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = D(\omega'_{|x|+1}, \omega_{|y|+1}) e_{\omega'_{|x|+1}} \otimes e_{\omega_{|y|+1}}$$
$$- D(\omega_{|x|+1}, \omega'_{|y|+1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|y|+1}}$$

$$\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = \left[D(\omega_{|x|-1}, \omega'_{|y|-1}) - D(\omega'_{|x|-1}, \omega_{|y|-1}) \right] e_x \otimes e_y$$

Proof. We shall prove Part (a) and the other parts are similar. If (x, y) is an (ω, ω') pair but not an (ω', ω) pair, then

$$(\mathcal{D}_{\omega,\omega'} e_x \otimes e_y)(u, v)$$
$$= D(u, v) \left[e_x \otimes e_y(\omega'_{|u|-1}, \omega_{|v|-1}) \delta_{u,\omega'_{|u|}} \delta_{v,\omega_{|v|}} \right.$$
$$\left. - e_x \otimes e_y(\omega_{|u|-1}, \omega'_{|v|-1}) \delta_{u,\omega_{|u|}} \delta_{v,\omega'_{|v|}} \right] \tag{4.2}$$

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The right side of (4.2) vanishes unless $\omega'_{|u|-1} = x$ and $\omega_{|v|-1} = y$ or $\omega_{|u|-1} = x$ and $\omega'_{|v|-1} = y$. Since (x,y) is not an (ω', ω) pair, the second alternative applies. This term does not vanish only if $\omega_{|u|-1} = \omega_{|x|}$ and $\omega'_{|v|-1} = \omega_{|y|}$ so we have

$$u = \omega_{|u|} = \omega_{|x|+1}$$

and

$$v = \omega'_{|v|} = \omega'_{|y|+1}$$

In this case we have

$$\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = -D(\omega_{|x|+1}, \omega'_{|y|+1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|y|+1}}$$

In a similar way, when (x,y) is an (ω, ω') pair but not an (ω', ω) pair we have

$$(\mathcal{T}_{\omega,\omega'} e_x \otimes e_y)(u,v) = \left[D(\omega_{|u|-1}, \omega'_{|v|-1}) \delta_{u,\omega_{|u|}} \delta_{v,\omega'_{|v|}} \right] e_x \otimes e_y(u,v) \qquad (4.3)$$

The right side of (4.3) vanishes unless $u = x$ and $v = y$. In this case we have

$$\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = -D(\omega_{|x|-1}, \omega'_{|y|-1}) e_x \otimes e_y \qquad \square$$

An interesting special case is when (x,x) is an (ω, ω') pair. We then have

$$\begin{aligned}
\mathcal{D}_{\omega,\omega'} e_x \otimes e_x &= D(\omega'_{|x|+1}, \omega_{|x|+1}) e_{\omega'_{|x|+1}} \otimes e_{\omega_{|x|+1}} \\
&\quad - D(\omega_{|x|+1}, \omega'_{|x|+1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|x|+1}}
\end{aligned}$$

Notice that this is an entanglement of $e_{\omega_{|x|+1}}$ and $e_{\omega'_{|x|+1}}$. We also have

$$\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = 2i \, Im \, D(\omega_{|x|-1}, \omega'_{|x|-1}) e_x \otimes e_x$$

For the contracted operators we have that $\widehat{\mathcal{D}}_{\omega,\omega'} e_x \otimes e_y = 0$ except for the cases

$$\widehat{\mathcal{D}}_{\omega,\omega'} e_{\omega'_{|x|-1}} \otimes e_{\omega_{|x|-1}} = \mu(x) e_x$$
$$\mathcal{D}_{\omega,\omega'} e_{\omega_{|x|-1}} \otimes e_{\omega'_{|x|-1}} = -\mu(x) e_x$$

when $\omega_{|x|} = \omega'_{|x|} = x$ and $\omega_{|x|-1} \neq \omega'_{|x|-1}$. Moreover, $\widehat{\mathcal{T}}_{\omega,\omega'} e_x \otimes e_y = 0$ except for the case

$$\widehat{\mathcal{T}}_{\omega,\omega'} e_x \otimes e_x = 2i \, Im \, D(\omega_{|x|-1}, \omega'_{|x|-1}) e_x$$

when $\omega_{|x|} = \omega'_{|x|} = x$.

It is clear that $\mathcal{T}_{\omega,\omega'}$ is a diagonal operator and hence $\mathcal{T}_{\omega,\omega'}$ is a normal operator. We shall show shortly that $\mathcal{D}_{\omega,\omega'}$ is not normal. However, from Theorem 4.1 we see that $\mathcal{D}_{\omega,\omega'}$ is a type of shift operator. More physically, we see that $\mathcal{D}_{\omega,\omega'}$ can be thought of as a creation operator because it takes $e_x \otimes e_y$ corresponding to $(|x|, |y|)$ "particles" to a scalar multiple of $e_{\omega_{|x|+1}} \otimes e_{\omega_{|y|+1}}$ corresponding to $(|x|+1, |y|+1)$ "particles" (Theorem 4.1(a),(b)). The next result shows that the adjoint $\mathcal{D}^*_{\omega,\omega'}$ can be thought of as an annihilation operator. We conclude that this formalism not only gives a discrete version of general relativity, there is also emerging a discrete analogue of quantum field theory.

Theorem 4.2. (a) *If (x,y) is an (ω, ω') pair but not an (ω', ω) pair, then*

$$\mathcal{D}^*_{\omega,\omega'} e_x \otimes e_y = -\overline{D}(x,y) e_{\omega_{|x|-1}} \otimes e_{\omega'_{|y|-1}}$$

(b) *If (x,y) is an (ω', ω) pair but not an (ω, ω') pair, then*

$$\mathcal{D}^*_{\omega,\omega'} e_x \otimes e_y = \overline{D}(x,y) e_{\omega'_{|x|-1}} \otimes e_{\omega_{|y|-1}}$$

(c) *If (x,y) is an $(\omega, \omega') - (\omega', \omega)$ pair, then*

$$\mathcal{D}^*_{\omega,\omega'} e_x \otimes e_y = \overline{D}(x,y) \left[e_{\omega'_{|x|-1}} \otimes e_{\omega_{|y|-1}} - e_{\omega_{|x|-1}} \otimes e_{\omega'_{|y|-1}} \right]$$

Proof. We shall prove Part (a) and the other parts are similar. Let T be the operator on L satisfying

$$T e_x \otimes e_y = -\overline{D}(x,y) e_{\omega_{|x|-1}} \otimes e_{\omega'_{|y|-1}}$$

We then have

$$\begin{aligned}
\langle T e_x \otimes e_y, e_u \otimes e_v \rangle &= -D(x,y) \Big\langle e_{\omega_{|x|-1}} \otimes e_{\omega'_{|y|-1}}, e_u \otimes e_v \Big\rangle \\
&= -D(x,y) \delta_{\omega_{|x|-1},u} \delta_{\omega'_{|y|-1},v}
\end{aligned}$$

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Now $\omega_{|x|-1} = u$ if and only if $\omega_{|u|+1} = x$ and $\omega'_{|y|-1} = v$ if and only if $\omega_{|v|+1} = y$. We conclude that

$$
\begin{aligned}
\langle e_x \otimes e_y, \mathcal{D}_{\omega,\omega'} e_u \otimes e_v \rangle &= -D(\omega_{|u|+1}, \omega'_{|v|+1}) \Big\langle e_x \otimes e_y, e_{\omega_{|u|+1}} \otimes e_{\omega'_{|v|+1}} \Big\rangle \\
&= -D(\omega_{|u|+1}, \omega'_{|v|+1}) \delta_{x,\omega_{|u|+1}} \delta_{y,\omega'_{|v|+1}} \\
&= -D(x,y) \delta_{\omega_{|x|-1},u} \delta_{\omega'_{|y|-1},v} \\
&= \langle T e_x \otimes e_y, e_u \otimes e_v \rangle
\end{aligned}
$$

Hence,

$$
T e_x \otimes e_y = \mathcal{D}^*_{\omega,\omega'} e_x \otimes e_y
$$

and the result holds. $\qquad\square$

In the case when Theorems 4.1(a) and 4.2(a) are both applicable we have that

$$
\mathcal{D}_{\omega,\omega'} \mathcal{D}^*_{\omega,\omega'} e_x \otimes e_y = |D(x,y)|^2 \, e_x \otimes e_y
$$

and

$$
\mathcal{D}^*_{\omega,\omega'}, \mathcal{D}_{\omega,\omega'} e_x \otimes e_y = \left| D(\omega_{|x|+1}, \omega'_{|y|+1}) \right|^2 e_x \otimes e_y
$$

so $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{D}^*_{\omega,\omega'}$ do not commute in general. Again, $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{T}_{\omega,\omega'}$ do not commute because

$$
\mathcal{D}_{\omega,\omega'}, \mathcal{T}_{\omega,\omega'} e_x \otimes e_y = -D(\omega_{|x|+1}, \omega'_{|y|+1}) D(\omega_{|x|-1}, \omega'_{|y|-1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|y|+1}}
$$

while

$$
\mathcal{T}_{\omega,\omega'} \mathcal{D}_{\omega,\omega'} e_x \otimes e_y = -D(\omega_{|x|+1}, \omega'_{|y|+1}) D(x,y) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|y|+1}}
$$

5 Amplitude Processes

Various ways of constructing QSGP have been considered [2, 5]. Here we introduce a method called an amplitude process (AP). Although a QSGP need not be generated by an AP, we shall characterize those that are so generated. In the next section we shall present a concrete realization of an AP in terms of a natural quantum action

A *transition amplitude* is a map $\widetilde{a}\colon \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ such that $\widetilde{a}(x,y) = 0$ if $x \not\to y$ and $\sum_y \widetilde{a}(x,y) = 1$ for all $x \in \mathcal{P}$. This is similar to a Markov chain except $\widetilde{a}(x,y)$ may be complex. The *amplitude process* (AP) corresponding to $\widetilde{a}$ is given by the maps $a_n\colon \Omega_n \to \mathbb{C}$ where

$$a_n(\omega_1\omega_2\cdots\omega_n) = \widetilde{a}(\omega_1,\omega_2)\widetilde{a}(\omega_2,\omega_3)\cdots\widetilde{a}(\omega_{n-1},\omega_n)$$

We can consider a_n to be a vector in $H_n = L_2(\Omega_n)$. Notice that

$$\langle 1_n, a_n \rangle = \sum_{\omega \in \Omega_n} a_n(\omega) = 1$$

and

$$\|a_n\| = \sqrt{\sum_{\omega \in \Omega_n} |a_n(\omega)|^2}$$

Define the rank-1 positive operator $\rho_n = |a_n\rangle\langle a_n|$ on H_n. The norm of ρ_n is

$$\|\rho_n\| = \|a_n\|^2 = \sum_{\omega \in \Omega_n} |a_n(\omega)|^2$$

Since

$$\langle e_n 1_n, 1_n \rangle = |\langle 1_n, a_n \rangle|^2 = 1$$

we conclude that ρ_n is a probability operator.

The corresponding decoherence functional becomes

$$D_n(A,B) = \langle \rho_n \chi_B, \chi_A \rangle = \langle \chi_B, a_n \rangle \langle a_n, \chi_A \rangle$$
$$= \sum_{\omega \in A} \overline{a_n}(\omega) \sum_{\omega \in B} a_n(\omega)$$

In particular, for $\omega, \omega' \in \Omega_n$, $D_n(\omega, \omega') = \overline{a_n}(\omega) a_n(\omega')$ are the matrix elements of ρ_n in the standard basis. Defining the complex-valued measure ν_n on 2^{Ω_n} by $\nu_n(A) = \sum_{\omega \in A} a_n(\omega)$ we see that

$$D_n(A,B) = \overline{\nu_n(A)}\nu_n(B)$$

The q-measure $\mu_n\colon 2^{\Omega_n} \to \mathbb{R}^2$ becomes

$$\mu_n(A) = D_n(A,A) = |\nu(A)|^2 = \left|\sum_{\omega \in A} a_n(\omega)\right|^2$$

In particular, $\mu_n(\omega) = |a_n(\omega)|^2$ for every $\omega \in \Omega_n$ and $\mu_n(\Omega_n) = 1$.

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Theorem 5.1. *The sequence of operators $\rho_n = |a_n\rangle\langle a_n|$ forms a QSGP.*

Proof. We only need to show that $\{\rho_n\}$ is a consistent sequence. Using the notation $\omega\omega_{n+1} = \omega_1\omega_2\ldots\omega_n\omega_{n+1}$, for $A, B \in 2^{\Omega_n}$ we have

$$
\begin{aligned}
D_{n+1}(A, B) &= \sum_{\omega \in A\to} \overline{a_n(\omega)} \sum_{\omega \in B\to} a_n(\omega) \\
&= \sum \{\overline{a_n}(\omega\omega_{n+1}) \colon \omega \in A, \omega_n \to \omega_{n+1}\} \\
&\quad \times \sum \{a_n(\omega\omega_{n+1}) \colon \omega \in B, \omega_n \to \omega_{n+1}\} \\
&= \sum \{\overline{a_n}(\omega)\overline{\widetilde{\widetilde{a}}}(\omega_n, \omega_{n+1}) \colon \omega \in A, \omega_n \to \omega_{n+1}\} \\
&\quad \times \sum \{a_n(\omega)\widetilde{a}(\omega_n, \omega_{n+1}) \colon \omega \in B, \omega_n \to \omega_{n+1}\} \\
&= \sum_{\omega \in A} \overline{a_n}(\omega) \sum_{\omega \in B} a_n(\omega) = D_n(A, B)
\end{aligned}
$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 5.1 shows that an AP $\{a_n\}$ *generates* a QSGP $\{\rho_n\}$. Not all QSGP are generated by an AP and the next theorem characterizes those that are so generated.

Theorem 5.2. *A QSGP $\{\rho_n\}$ is generated by an AP if and only if $\rho_n = |a_n\rangle\langle a_n|$, $a_n \in H_n$, are rank-1 operators and if $\omega = \omega_1, \ldots \omega_n$, $\omega' = \omega'_1 \ldots \omega'_n \in \Omega_n$ with $\omega_n = \omega'_n$ and $\omega_n \to x$, then*

$$a_n(\omega')a_{n+1}(\omega x) = a_n(\omega)a_{n+1}(\omega' x) \tag{5.1}$$

Proof. If $\{a_n\}$ is an AP, then it is straightforward to show that (5.1) holds. Conversely, suppose $\rho_n = |a_n\rangle\langle a_n|$ is a QSGP and $\rho_n = |a_n\rangle\langle a_n|$ where (5.1) holds. We now show that $\{a_n\}$ is an AP. If $x, y, \in \mathcal{P}$ with $x \to y$, suppose $|x| = n$ and let $\omega \in \Omega_{n+1}$ with $\omega = \omega_1 \cdots \omega_n\omega_{n+1}$ where $\omega_n = x$, $\omega_{n+1} = y$. If $a_n(\omega') = 0$ for every $\omega' \in \Omega_n$ with $\omega'_n = x$ define $\widetilde{a}(x, y) = 0$. Otherwise, we can assume that $a_n(\omega_1 \cdots \omega_n) \neq 0$ and define

$$\widetilde{a}(x, y) = \frac{a_{n+1}(\omega_1 \cdots \omega_n y)}{a_n(\omega_1 \cdots \omega_n)} \tag{5.2}$$

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By (5.1) this definition is independent of $\omega \in \Omega_n$ with $\omega_n = x$. If $x \not\to y$ we define $\widetilde{a}(x, y) = 0$. Since $\{\rho_n\}$ is consistent we have

$$\sum_y \widetilde{a}(x, y) = [a_n(\omega_1 \cdots \omega_n)]^{-1} \sum_y a_n(\omega_1 \cdots \omega_n y)$$

$$= [a_n(\omega_1 \cdots \omega_n)]^{-1} \left\langle \rho_{n+1} \chi_{(\omega \to)}, \chi_{(\Omega_n \to)} \right\rangle$$

$$= [a_n(\omega_1 \cdots \omega_n)]^{-1} \left\langle \rho_n \chi_{\{\omega\}}, \chi_{\Omega_n} \right\rangle = 1$$

so $\widetilde{a}$ is a transition amplitude. Applying (5.2) we have

$$a_n(\omega_1 \cdots \omega_n) = a_{n-1}(\omega_1 \cdots \omega_{n-1}) \widetilde{a}(\omega_{n-1}, \omega_n)$$

$$= a_{n_2}(\omega_1 \cdots \omega_{n-2}) \widetilde{a}(\omega_{n-2}, \omega_{n-1}) \widetilde{a}(\omega_{n-1}, \omega_n)$$

$$\vdots$$

$$= \widetilde{a}(\omega_1, \omega_2) \widetilde{a}(\omega_2, \omega_3) \cdots \widetilde{a}(\omega_{n-1}, \omega_n)$$

It follows that $\{a_n\}$ is derived from the transition amplitude $\widetilde{a}$ so $\{a_n\}$ is an AP. $\qquad \square$

6   Quantum Action

We now present a specific example of an AP that arises from a natural quantum action. For $x \in \mathcal{P}$, the *height* $h(x)$ of x is the cardinality of a longest chain in x. The *width* $w(x)$ of x is the cardinality of a largest antichain in x. Finally, the *area* $A(x)$ of x is given by $A(x) = h(x)w(x)$. Roughly speaking, $h(x)$ corresponds to an internal time in x, $w(x)$ corresponds to the mass or energy of x [3] and $A(x)$ corresponds to an action for x. If $x \to y$, then $h(y) = h(x)$ or $h(x) + 1$ and $w(y) = w(x)$ or $w(x) + 1$. In the case $h(y) = h(x) + 1$ we call y a *height offspring* of x, in the case $w(y) = w(x) + 1$ we call y a *width offspring* of x and if both $h(y) = h(x)$, $w(y) = w(x)$ hold, we call y a *mild offspring* of x. Let $H(x)$, $W(x)$ and $M(x)$ be the sets of height, width and mild offspring of x, respectively. It is shown in [5] that $H(x)$, $W(x)$, $M(x)$ partition the set $(x \to)$. It is easy to see that $H(x) \neq \emptyset$, $W(x) \neq \emptyset$ but examples show that $M(x)$ can be empty.

If $x \to y$ we have the following possibilities: $y \in M(x)$ in which case $A(y) - A(x) = 0$, $y \in H(x)$ in which case

$$A(y) - A(x) = [h(x) + 1] \, w(x) - h(x)w(x) = w(x)$$

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and $y \in W(x)$ in which case

$$A(y) - A(x) = h(x) [w(x) + 1] - h(x)w(x) = h(x)$$

We now define the transition amplitude $\widetilde{a}(x, y)$ in terms of the "action" change for x to y. We first define the *partition function*

$$z(x) = \sum_y \left\{ m(x \to y)e^{2\pi i[A(y)-A(x)]/|x|} : x \to y \right\}$$

If $x \not\to y$ define $\widetilde{a}(x, y) = 0$. If $x \to y$ and $z(x) = 0$ define $\widetilde{a}(x, y) = [(x \to)]^{-1}$ where $[(x \to)]$ means the cardinality of $(x \to)$ including multiplicity. If $x \to y$ and $z(x) \neq 0$ define

$$\widetilde{a}(x, y) = \frac{m(x \to y)}{z(x)} e^{2\pi i[A(y)-A(x)]/|x|}$$

As before, we have three possibilities when $z(x) \neq 0$. If $y \in M(x)$, then $\widetilde{a}(x, y) = m(x \to y)[z(x)]^{-1}$, if $y \in H(x)$ then

$$\widetilde{a}(x, y) = \frac{m(x \to y)}{z(x)} e^{2\pi i w(x)/|x|}$$

and if $y \in W(x)$, then

$$\widetilde{a}(x, y) = \frac{m(x \to y)}{z(x)} e^{2\pi i h(x)/|x|}$$

Notice that for $x \in \mathcal{P}_n$ we have

$$z(x) = [M(x)] + [H(x)] e^{2\pi i w(x)/n} + [W(x)] e^{2\pi i h(x)/n}$$

We now illustrate this theory by checking the first three steps in Figure 1. Since $M(x_1) = \emptyset$, $H(x) = \{x_2\}$, $W(x_1) = \{x_3\}$ and $w(x_1) = h(x_1) = 1$ we have $z(x_1) = 2e^{2\pi i} = 2$. Since $M(x_2) = \emptyset$, $H(x_2) = \{x_4\}$, $W(x_2) = \{x_5, x_6\}$ and $w(x_2) = 1$, $h(x_2) = 2$ we have

$$z(x_2) = e^{2\pi i/2} + 2e^{2\pi i} = -1 + 2 = 1$$

Since $M(x_3) = \emptyset$, $H(x_3) = \{x_6, x_7\}$, $W(x_3) = \{x_8\}$, $w(x_3) = 2$, $h(x_3) = 1$ and $[H(x_3)] = 3$ we have

$$z(x_3) = 3e^{2\pi i} + e^{2\pi i/2} = 3 - 1 = 2$$

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There are six 3-paths in Ω_3: $\gamma_1 = x_1 x_2 x_4$, $\gamma_2 = x_1 x_2 x_5$, $\gamma_3 = x_1 x_2 x_6$, $\gamma_4 = x_1 x_3 x_6$, $\gamma_5 = x_1 x_3 x_7$ and $\gamma_8 = x_1 x_3 x_8$. The amplitudes of the paths become:

$$a_3(\gamma_1) = \widetilde{a}(x_1, x_2)\widetilde{a}(x_2, x_4) = \tfrac{1}{2}e^{2\pi i}e^{\pi i} = -\tfrac{1}{2}$$
$$a_3(\gamma_2) = \widetilde{a}(x_1, x_2)\widetilde{a}(x_2, x_5) = \tfrac{1}{2}e^{2\pi i}e^{2\pi i} = \tfrac{1}{2}$$
$$a_3(\gamma_3) = \widetilde{a}(x_1, x_2)\widetilde{a}(x_2, x_6) = \tfrac{1}{2}e^{2\pi i}e^{2\pi i} = \tfrac{1}{2}$$
$$a_3(\gamma_4) = \widetilde{a}(x_1, x_3)\widetilde{a}(x_3, x_6) = \tfrac{1}{2}e^{2\pi i}e^{2\pi i} = \tfrac{1}{2}$$
$$a_3(\gamma_5) = \widetilde{a}(x_1, x_3)\widetilde{a}(x_3, x_7) = \tfrac{1}{2}e^{2\pi i}\tfrac{1}{2}e^{2\pi i} = \tfrac{1}{4}$$
$$a_3(\gamma_6) = \widetilde{a}(x_1, x_3)\widetilde{a}(x_3, x_8) = \tfrac{1}{2}e^{2\pi i}\tfrac{1}{2}e^{\pi i} = -\tfrac{1}{4}$$

Notice that $\sum a(\gamma_i) = 1$ as it must. The amplitude decoherence matrix has components $D_3(\gamma_i, \gamma_j) = \overline{a(\gamma_i)}a(\gamma_j)$, $i, j = 1, \ldots, 6$. The q-measures $\mu_3(\gamma_i) = |a(\gamma_i)|^2$, $i = 1, \ldots 6$, are given by

$$\mu_3(\gamma_1) = \mu_3(\gamma_2) = \mu_3(\gamma_3) = \mu_3(\gamma_4) = 1/4$$
$$\mu_3(\gamma_5) = \mu_3(\gamma_6) = 1/16$$

Interference effects are evident if we consider the q-measures of various sets of sites (causets). For example,

$$\mu_3(\{x_6\}) = \left|\tfrac{1}{2} + \tfrac{1}{2}\right|^2 = 1$$
$$\mu_3(\{x_4, x_5\}) = \left|-\tfrac{1}{2} + \tfrac{1}{2}\right|^2 = 0$$
$$\mu_3(\{x_5, x_6\}) = \left|\tfrac{3}{2}\right|^2 = 9/4$$

We can easily compute the amplitudes for the sites $x_1, \ldots, x_8$ to get $a(x_1) = 1$ by convention and $a(x_2) = a(x_3) = a(x_5) = \tfrac{1}{2}, a(x_4) = -\tfrac{1}{2}$, $a(x_6) = 1$, $a(x_7) = \tfrac{1}{4}$, $a(x_8) = -\tfrac{1}{4}$. The site decoherence matrix is the positive semidefinite matrix with components $D(x_i, x_j) = \overline{a(x_i)}a(x_j)$, $i, j = 1, \ldots, 8$.

For this particular AP we conjecture that $\{\omega\} \in \mathcal{S}(\Omega)$ for every $\omega \in \Omega$ and $\mu(\{\omega\}) = 0$. Moreover, we conjecture that $\{\omega\}' \in \mathcal{S}(\Omega)$ for every $\omega \in \Omega$ and $\mu(\{\omega\}') = 1$. Since

$$\mu_n\left(\{\omega\}^n\right) = \left|a_n\left(\{\omega\}^n\right)\right|^2 = \left(\prod_{j=1}^{n-1} |z(\omega_j)|^2\right)^{-1}$$

$\mu_n(\{\omega\}) = 0$ would follow from

$$\lim_{|x| \to 0} \{|z(x)| : x \in \mathcal{P}\} = \infty \tag{6.1}$$

We also conjecture that (6.1) holds. Notice that $\{\omega\}' \in \mathcal{S}(\Omega)$ would follow from $\{\omega\} \in \mathcal{S}(\Omega)$ and $\mu(\{\omega\}) = 0$ because

$$\mu_n\left(\{\omega\}'^n\right) = \left|a_n\left(\{\omega\}'^n\right)\right|^2 = \left|1 - a_n\left(\{\omega\}^n\right)\right|^2$$
$$= 1 + \left|a_n\left(\{\omega\}^n\right)\right|^2 - 2\,Re\,a_n\left(\{\omega\}^n\right)$$

If $\mu(\{\omega\}) = 0$, then $\lim\left|a_n\left(\{\omega\}^n\right)^2\right| = 0$ so $\lim a_n\left(\{\omega\}^n\right) = 0$ and hence, $\lim \mu_n\left(\{\omega\}'^n\right) = 1$.

We can prove our conjectures in two extreme cases. Let $\omega \in \Omega$ be the path $\omega = \omega_1\omega_2\cdots$ for which ω_n is a chain, $n = 1, 2, \ldots$. Then $[H(\omega_n)] = 1$, $[W(\omega_n)] = n$ and $[M(\omega_n)] = 0$, $n = 1 =, 2, \ldots$. Then $\omega^n = \{\omega\}^n = \omega_1\omega_2\ldots\omega_n$ and for $j = 1, \ldots, n$ we have

$$z(\omega_j) = e^{2\pi i/j} + j e^{2\pi ij/j} = e^{2\pi i/j} + j$$

Hence,
$$|z(\omega_j)| = \left|e^{2\pi i/j} + j\right| \geq |j| - \left|e^{2\pi i/j}\right| = j - 1$$

We then have
$$|z(\omega^n)| = |z(\omega_1)\cdots z(\omega_{n-1})| \geq \prod_{j=1}^{n-2} j$$

Hence, $\lim_{n\to\infty} |z(\omega^n)| = \infty$ so $\{\omega\} \in \mathcal{S}(\Omega)$ and $\mu(\{\omega\}) = 0$.

The other extreme case is when $\omega = \omega_1\omega_2\cdots$ and each ω_n is an antichain. Then $[H(\omega_n)] = 2^{n-1}$, $[W(\omega_n)] = 1$ and $[M(\omega_n)] = 0$, $n = 1, 2, \ldots$. Again, $\omega^n = \omega_1\omega_2\cdots\omega_n$ and for $j = 1, \ldots, n$ we obtain

$$z(\omega_j) = (2^j - 1)e^{2\pi i} + e^{2\pi i/j} = 2^j - 1 + e^{2\pi i/j}$$

Hence, $|z(\omega_j)| \geq 2^j - 2$ and we obtain

$$|z(\omega^n)| = |z(\omega_1)\ldots z(\omega_{n-1})| \geq \prod_{j=1}^{n-2}(2^n - 2)$$

Again, we conclude that $\lim_{n\to\infty} |z(\omega^n)| = \infty$ so $\{\omega\} \in \mathcal{S}(\Omega)$ with $\mu(\{\omega\}) = 0$. As noted before, in both these extreme cases we have $\{\omega\}' \in \mathcal{S}(\Omega)$ and $\mu(\{\omega\}') = 1$.

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7   Classical Processes

A QSPG $\{\rho_n\}$ is *classical* if the decoherence matrix
$$D_n(\omega, \omega') = \langle \rho_n \chi_{\{\omega'\}}, \chi_\omega \rangle$$
is diagonal for all n. As an example, let $\widetilde{a} \colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ be a real-valued transition amplitude and define the AP $a_n \colon \Omega_n \to \mathbb{R}$ as in Section 5. Defining the diagonal operators
$$\rho_n(\omega, \omega') = a_n(\omega) \delta_{\omega, \omega'}$$
we conclude that $\{\rho_n\}$ is a classical QSGP.

Theorem 7.1. *The following statements are equivalent.* (a) $\{\rho_n\}$ *is classical.* (b) $D_n(A, B) = \mu_n(A \cap B)$ *for all* $A, B \subseteq \Omega_n$. (c) $D_n(A, B) = 0$ *if* $A \cap B = \emptyset$.

Proof. (a)$\Rightarrow$(b) If ρ_n is diagonal, then $D_n(\omega, \omega') = 0$ for $\omega \neq \omega'$. Hence,

$$
\begin{aligned}
D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle &= \left\langle \rho_n \sum_{\omega' \in B} \chi_{\{\omega'\}}, \sum_{\omega \in A} \chi_{\{\omega\}} \right\rangle \\
&= \sum_{\omega \in A} \sum_{\omega' \in B} \langle \rho_n \chi_{\{\omega'\}}, \chi_{\{\omega\}} \rangle = \sum_{\omega \in A} \sum_{\omega' \in B} D_n(\omega, \omega') \\
&= \sum_{\omega \in A \cap B} D_n(\omega, \omega) = D_n(A \cap B, A \cap B) = \mu_n(A \cap B)
\end{aligned}
$$

(b)$\Rightarrow$(c) If $D_n(A, B) = \mu_n(A \cap B)$ then $A \cap B = \emptyset$
$$D_n(A, B) = \mu_n(\emptyset) = 0$$

(c)$\Rightarrow$(a) If (c) holds and $\omega \neq \omega'$ then $\{\omega\} \cap \{\omega'\} = \emptyset$ so $D_n(\omega, \omega') = 0$. $\square$

Corollary 7.2. *If* $\{\rho_n\}$ *is classical, then* μ_n *is a measure,* $n = 1, 2, \ldots$.

Proof. To show that μ_n is a measure, suppose that $A \cap B = \emptyset$. By Theorem 7.1 we have

$$
\begin{aligned}
\mu_n(A \cup B) = D_n(A \cup B, A \cup B) &= D_n(A, A \cup B) = D_n(B, A \cup B) \\
&= D_n(A, A) + D_n(B, B) + 2 \operatorname{Re} D_n(A, B) = \mu_n(A) + \mu_n(B)   \quad \square
\end{aligned}
$$

We now give a more general condition. A QSGP $\{\rho_n\}$ is *semiclassical* if $Re\, D_n(\omega, \omega')$ is diagonal. The fact that μ_n is a measure, $n = 1, 2, \ldots$, does not imply that $\{\rho_n\}$ is classical. However, we do have the following result whose proof is similar to that of Theorem 7.1 and Corollary 7.2.

Theorem 7.3. *The following statements are equivalent.* (a) $\{\rho_n\}$ *is semiclassical.* (b) $Re\, D_n(A, B) = \mu_n(A \cap B)$. (c) $Re\, D_n(A, B) = 0$ *if* $A \cap B = \emptyset$. (d) μ_n *is a measure.*

If $\{\rho_n\}$ is semiclassical, then by Theorem 7.3, μ_n are measures so for every $A \in \mathcal{A}$ we have

$$\mu_{n+1}(A^{n+1}) \leq \mu_{n+1}(A^n \rightarrow) = \mu_n(A^n)$$

Hence, $\mu_n(A^n)$ is a decreasing sequence so it converges. We conclude that $\mathcal{S}(\Omega) = \mathcal{A}$. However, the q-measure μ defined earlier need not be a measure on $\mathcal{A}$. One reason is that if $A \cap B = \emptyset$, $A, B \in \mathcal{A}$ then $A^n \cap B^n \neq \emptyset$, in general. Another way to see this is the following. Define $\nu(A) = \mu_n(A_1)$ for $A = \mathrm{cyl}(A_1)$, $A_1 \subseteq \Omega_n$. Then by the Hahn extension theorem ν extends to unique measure on $\mathcal{A}$. But $\nu = \mu$ on $\mathcal{C}(\Omega)$ so if μ were a measure, by uniqueness $\nu = \mu$ on $\mathcal{A}$. But this is impossible because $\cap A^n \neq A$ in general, so

$$\mu(A) = \lim \mu_n(A^n) \neq \nu(A)$$

Suppose $\{\rho_n\}$ is classical for the rest of this section unless specified otherwise. It is of interest to study the operators considered in Section 4 for this case. Since $D_n(\omega, \omega') = 0$ for $\omega \neq \omega'$ we have for all $x, y \in \mathcal{P}$ and $n \geq |x|, |y|$ that

$$\begin{aligned}
D(x, y) &= \sum \left\{ D_n(\omega, \omega') \colon \omega_{|x|} = x, \omega'_{|y|} = y \right\} \\
&= \sum \left\{ D_n(\omega, \omega) \colon \omega_{|x|} = x, \omega_{|y|} = y \right\} \\
&= \sum \left\{ \mu_n(\omega) \colon \omega_{|x|} = x, \omega_{|y|} = y \right\} \qquad (7.1)
\end{aligned}$$

Notice that the set

$$A = \left\{ \omega \in \Omega \colon \omega_{|x|} = x, \omega_{|y|} = y \right\} \in \mathcal{C}(\Omega)$$

and by (7.1) $\mu(A) = D(x, y)$. We conclude from (7.1) that $D(x, y) = 0$ except if x and y are comparable. It follows that $\widehat{\mathcal{T}}_{\omega, \omega'} = 0$ for every $\omega, \omega' \in \Omega$. Since

the mass-energy operators vanish when the process is classical, this indicates that mass-energy is generated by quantum interference. (Is this related to the Higgs boson?) The operators $\widehat{\mathcal{D}}_{\omega,\omega'}$ have the same form classically as quantum mechanically.

Now suppose we have a "flat" space $\mathcal{P}$ so that $\widehat{\mathcal{R}}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$. Since we already have that $\widehat{\mathcal{T}}_{\omega,\omega'} = 0$, it follows that $\widehat{\mathcal{D}}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$. Now $\widehat{\mathcal{D}}_{\omega,\omega'} e_u \otimes e_v = 0$ for every $\omega, \omega' \in \Omega$ unless possibly when $u \neq v$ and u and v have a common offspring x (from which it follows that $|u| = |v|$). In this case we have that $\widehat{\mathcal{D}}_{\omega,\omega'} e_u \otimes e_v = \pm \mu(x) e_x$ so that $\mu(x) = 0$. We conclude that $\widehat{\mathcal{R}}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$ if and only if $\mu(x) = 0$ whenever x has more than one producer.

The next theorem considers the general operators $\mathcal{D}_{\omega,\omega'}$ and $\mathcal{T}_{\omega,\omega'}$

Theorem 7.4. *The operators $\mathcal{D}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$ if and only if $\mu(x) = 0$ whenever x has more than one producer and the operators $\mathcal{T}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$ if and only if $\mu(x) = 0$ whenever x has more than one offspring.*

Proof. Suppose (x,y) is an (ω, ω') pair but not an (ω', ω) pair. Then by Theorem 4.1(a) we have that $\mathcal{D}_{\omega,\omega'} e_x \otimes e_y \neq 0$ if and only if $\omega_{|x|+1} = \omega'_{|y|+1}$ and $x \neq y$ with $\mu(\omega_{|x|+1}) \neq 0$. In this case $\omega_{|x|+1}$ has the two producers x, y and if $\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = 0$ then $\mu(\omega_{|x|+1}) = 0$. We get the same result if (x,y) is an (ω', ω) pair but not an (ω, ω') pair. Now suppose (x,y) is an $(\omega, \omega') - (\omega', \omega)$ pair. Then by Theorem 4.1(c) we have that $\mathcal{D}_{\omega,\omega'} e_x \otimes e_y \neq 0$ if and only if $\omega'_{|x|+1} = \omega_{|y|+1}$ or $\omega_{|x|+1} = \omega'_{|y|+1}$. If either of these hold, then $|x| = |y|$ so $x = y$. We then have by Theorem 4.1(c) that

$$
\begin{aligned}
\mathcal{D}_{\omega,\omega'} e_x \otimes e_y &= D(\omega'_{|x|+1}, \omega_{|x|+1}) e_{\omega'_{|x|+1}} \otimes e_{\omega_{|x|+1}} \\
&\quad - D(\omega_{|\omega|+1}, \omega'_{|x|+1}) e_{\omega_{|x|+1}} \otimes e_{\omega'_{|x|+1}} \\
&= 0
\end{aligned}
$$

We conclude that $\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = 0$ which is a contradiction. Hence, in this case $\mathcal{D}_{\omega,\omega'} e_x \otimes e_y = 0$ automatically.

Again, suppose (x,y) is an (ω, ω') pair but not an (ω', ω) pair. By Theorem 4.1(a) we have that $\mathcal{T}_{\omega,\omega'} e_x \otimes e_y \neq 0$ if and only if $\omega_{|x|-1} = \omega'_{|y|-1}$ and $x \neq y$ with $\mu(\omega_{|x|-1}) \neq 0$. In this case $\omega_{|x|-1}$ has the two offspring x, y and if $\mathcal{T}_{\omega,\omega'} e_x \otimes e_y = 0$, then $\mu(\omega_{|x|-1} = 0$. The rest of the proof is similar to that in the previous paragraph. $\square$

In very special conditions, it may be possible to arrange things so that $\mu(x) = 0$ whenever x has more than one producer in which case $\mathcal{D}_{\omega,\omega'} = 0$ for every $\omega, \omega' \in \Omega$. However, $\mu(x) = 0$ whenever x has more than one offspring is impossible. This is because all $x \in \mathcal{P}$ have more than one offspring so $\mu(x) = 0$ for all $x \in \mathcal{P}$. But this is a contradiction because μ is a probability measure on $\mathcal{P}_n$. We conclude that $\mathcal{T}_{\omega,\omega'} \neq 0$ for every ω, ω'. It follows that $\mathcal{R}_{\omega,\omega'} \neq 0$ for every ω, ω'. Thus in the classical case $\mathcal{P}$ is never "flat" in the sense that $\mathcal{R}_{\omega,\omega'} = 0$ for all $\omega, \omega' \in \Omega$ but it may be "flat" in the sense that $\widehat{\mathcal{R}}_{\omega,\omega'} = 0$ for all $\omega, \omega' \in \Omega$.

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