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Topological Speedups

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Abstract

Given a dynamical system $T : X \to X$ one can define a speedup of (X, T) as another dynamical system conjugate to $S : X \to X$ where $S(x) = T^{p(x)}(x)$ for some function $p : X \to \mathbb{Z}^+$. In 1985 Arnoux, Ornstein, and Weiss showed that any aperiodic measure preserving system is isomorphic to a speedup of any ergodic measure preserving system. In this thesis we study speedups in the topological category. Specifically, we consider minimal homeomorphisms on Cantor spaces. Our main theorem gives conditions on when one such system is a speedup of another. Moreover, the main theorem serves as a topological analogue of the Arnoux, Ornstein, and Weiss speedup theorem, as well as a one-sided analogue of Giordano, Putnam, and Skau's characterization of orbit equivalence. Further, this thesis explores the special case of speedups when the p function is bounded. In this case, we provide bounds on the entropy of bounded speedups.

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Chapter 1

Introduction

The origins of dynamical systems extend at least as far back as the time of Newton and the study of celestial bodies. In fact, some of the terminology is derived from this very example. A dynamical system, in its most abstract form, is nothing more than the deterministic movement of points through a fixed, closed space. One may think of a particle bouncing around inside a closed box and an observer tracking the location of the particle at fixed time intervals. Thus, the study of dynamical systems is the study of how these deterministic systems evolve. A mathematical model for the deterministic behavior we seek to study is simply given by a function, T, acting on the points of a space X. For this thesis, space will be one of two types, either a measure space or a topological space. More precisely, a *measure theoretic dynamical system* is the quadruple $(X, \mathcal{X}, \mu, T)$ where $(X, \mathcal{X}, \mu)$ is a measure space, T is a bimeasurable, bijective map, and μ is T-*invariant*: that is, for every $A \in \mathcal{X}$ we have $\mu(A) = \mu(T^{-1}A)$. We are interested in cases akin to the particle bouncing around in a box, hence we will make a bounded condition on our space by requesting that $\mu(X) = 1$. Thus, from now on all measure spaces $(X, \mathcal{X}, \mu)$ will be considered probability spaces, i.e. $\mu(X) = 1$. In order to say something quite general about these dynamical systems we need to rule out trivial systems like the identity map, so we further posit that our systems satisfy the following condition: the only sets $E \in \mathcal{X}$ with $T^{-1}E = E$ satisfy $\mu(E) = 0$ or $\mu(E) = 1$. Any measure preserving system which satisfies the previous condition is called an *ergodic dynamical system* and forms a highly fertile field of study of modern dynamical systems.

The other spaces we wish to consider are topological spaces. A *topological dynamical system* is a metric space X paired with a homeomorphism $T : X \to X$, written as (X, T). As in the measure theoretic case we will impose a bounded condition on our metric space by requiring it to be compact. Here we will restrict our attention to the situation were X is a Cantor space. A *Cantor space* is a compact, metrizable, perfect, zero-dimension space, and is universal in the sense that any two Cantor spaces are homeomorphic. Further, it is well known that any compact metric space is the continuous image of a Cantor space. Hence, it is natural to simply look at homeomorphisms on Cantor spaces, known as *Cantor systems*. Moreover, we impose a similar condition on our now Cantor systems as we did with our probability preserving dynamical systems. A homeomorphism, T, of a Cantor space X is called *minimal* when the only closed sets E that satisfy $T^{-1}E = E$ are $E = \emptyset$ and $E = X$. Minimality is an apt description as minimal Cantor systems have no smaller dynamical systems sitting inside of the space.

There is an intimate relationship between ergodic dynamical systems and minimal Cantor systems. First, we will describe how to make any minimal

Cantor system into a measure preserving system. Second, we will show that minimal Cantor systems serve as a topological model for the most natural class of ergodic measure preserving systems through the Jewett-Krieger Theorem.

To make a Cantor space into a measure space is quite classical. Let (X, T) be a minimal Cantor system and let $\mathcal{B}(X)$ denote the Borel sigma algebra of X. By taking δ_x to be the Dirac measure of a point $x \in X$ we easily make our Cantor space into a probability space $(X, \mathcal{B}(X), \delta_x)$. Unfortunately, T cannot preserve δ_x; indeed, suppose $\delta_x = \delta_{T^{-1}x}$, this immediately implies that $x = T^{-1}x$ and we have a closed invariant set which contradicts the minimality of T. The following theorem guarantees the existence of an invariant measure for a topological dynamical system.

Theorem 1.0.1 (Bogolioubov-Krylov)**.** *Let (X, T) be a topological dynamical system and let $M(X, T)$ denote the collection of T-invariant Borel probability measures. The set $M(X, T) \neq \emptyset$.*

From the above theorem we can make the following definition.

Definition 1.0.2. *A minimal Cantor system (X, T) is called* **uniquely ergodic** *whenever $M(X, T) = \{\mu\}$.*

The following theorem shows not only the prevalence of uniquely ergodic minimal Cantor systems, but also shows that minimal Cantor systems serve as a topological model for ergodic systems.

Theorem 1.0.3 (Jewett-Krieger)**.** *Let $(Y, \mathscr{S}, \nu, S)$ be an ergodic automorphism of a non-atomic Lebesgue probability space. There exists a uniquely ergodic, minimal Cantor system (X, T), with a unique invariant Borel probability measure μ, such that $(Y, \mathscr{S}, \nu, S)$ is measurably conjugate to $(X, \mathscr{B}(X), \mu, T)$.*

A fundamental question in many mathematical fields is the classification question, that is, given two objects when are they the equivalent? In the topological category we have the following definition.

Definition 1.0.4. *Two minimal Cantor systems* (X_1, T_1) *and* (X_2, T_2) *are* **conjugate** *if there exists a homeomorphism* $\varphi : X_1 \to X_2$ *such that for every* $x \in X_1$

$$(\varphi \circ T_1)(x) = (T_2 \circ \varphi)(x).$$

In the measurable category we have a similar definition.

Definition 1.0.5. *Two measurable dynamical systems* $(X_1, \mathcal{X}_1, \mu, T_1)$ *and* $(X_2, \mathcal{X}_2, \nu, T_2)$ *are* **measurably conjugate** *if there exists* $M_1 \in \mathcal{X}_1$ *and* $M_2 \in \mathcal{X}_2$ *such that*

1. $T_1 M_1 \subseteq M_1$, $T_2 M_2 \subseteq M_2$ *and*

2. *there is an invertible measure-preserving transformation* $\phi : M_1 \to M_2$ *such that for every* $x \in M_1$

$$(\phi \circ T_1)(x) = (T_2 \circ \phi)(x).$$

In general, classifying when two dynamical systems are conjugate, in either the measurable or topological category, is formidable. Thus, it is natural to weaken the notion of equivalence. Many significant classification theorems have been proven with a slight weakening of conjugacy to *orbit equivalence.* Most relevant to this thesis will be orbit equivalence in the topological category.

Definition 1.0.6. *Let* (X_1, T_1) *and* (X_2, T_2) *be minimal Cantor systems. We say* (X_1, T_1) *and* (X_2, T_2) *are* **orbit equivalent** *if there exists a homeomorphism* $F : X_1 \to X_2$ *which preserves orbits, i.e. there exists functions* $m, n : X_1 \to \mathbb{Z}$ *such that*

$$(F \circ T_1^{n(x)})(x) = (T_2 \circ F)(x) \quad \textit{and} \quad (F \circ T_1)(x) = (T_2^{m(x)} \circ F)(x).$$

In this thesis we characterize, in the topological setting, when one minimal Cantor system is a speedup of another. Broadly speaking, given a dynamical system (X, T) a *speedup* is a new transformation $S : X \to X$ of the form $S(x) = T^{p(x)}(x)$ where $p : X \to \mathbb{Z}^+$. Hence, we speed up the evolution of the system in varying ways throughout the space. This theorem builds upon two different theorems in dynamics: one theorem from the measure theoretic category, the other from the topological category. Our main theorem is a topological analogue of the measure theoretic speedup theorem of Arnoux, Ornstein, and Weiss [AOW '85]. Their theorem shows that the realization of a measure preserving system as a speedup of another is very general, however there are restrictions that arise in the topological category. The form of our characterization is very similar to the remarkable theorem of Giordano, Putnam, and Skau [GPS '95, Theorem 2.2] in that both theorems the dynamical relations are characterized by associated ordered groups or associated simplices of invariant measures. Whereas in [GPS '95] they have bijective morphisms from one object onto the other, in our characterization theorem we obtain surjective and injective morphisms, respectively. Furthermore, through the similarity of these theorems we can relate topological speedups to topological orbit equivalence. For example, given a pair of minimal Cantor systems, both of which are uniquely ergodic, if one or both systems is a speedup of the other then the two systems are orbit equivalent.

These results follow in a long line of results coming from several different research areas of dynamics. The first, and perhaps most general, is that of finding topological analogues for results stemming from ergodic theory. One example, which we will mention a few times throughout this thesis, is the topological analogue to the classical ergodic theory result of Dye [Dye '59]. Recall Dye's theorem says that any two ergodic transformations on non-atomic Lebesgue probability spaces are orbit equivalent. Over 35 years later Giordano, Putnam, and Skau gave a complete characterization of when two minimal Cantor systems are orbit equivalent in the topological category. Unlike in the measure theoretic category, not all minimal Cantor systems are orbit equivalent in the topological category.

Another line of research we follow is that of speedups themselves, which have mostly been studied in the measurable category. By a *speedup* of a fixed aperiodic measure preserving transformation $(X, \mathscr{B}, \mu, T)$ we mean an automorphism of the form $S(x) = T^{p(x)}(x)$, $p : X \to \mathbb{Z}^+$. One of the earliest people to study speedups -though they were not called this until later- was Neveu in 1969. He had two papers [N1 '69],[N2 '69]; the latter, [N2 '69], would eventually give restrictions on what systems can be speedup to each other assuming integrability of p. The first major result, after Neveu, came in 1985 with Arnoux, Ornstein, and Weiss, when they showed: for any ergodic measure preserving transformation $(X, \mathscr{B}, \mu, T)$ and any aperiodic, not necessarily ergodic, $(Y, \mathscr{C}, \nu, S)$ there is a $\mathscr{B}-$measurable function $p : X \to \mathbb{Z}^+$ such that $\bar{S}(x) = T^{p(x)}(x)$ is invertible μ-a.e. and $(X, \mathscr{B}, \mu, \bar{S})$ is measurably conjugate to $(Y, \mathscr{C}, \nu, S)$. Finding a topological analogue to this theorem was the inspiration and impetus for this thesis. Interest in measure theoretic speedups has been rekindled as evidenced by the papers by [BBF '13], [JM '14].

The final line of research our thesis follows is that of topological orbit equivalence. Recall that (X_1, T_1) and (X_2, T_2) are orbit equivalent if there exists a space isomorphism $F : X_1 \to X_2$ such that for every $x \in X_1$, $F(\text{orbit}_{T_1}(x)) = \text{orbit}_{T_2}(F(x))$. Again Dye's theorem says that in the measurable category any two ergodic transformations on non-atomic Lebesgue probability spaces are orbit equivalent. This is not the case in the topological category. In 1995, Giordano, Putnam, and Skau completely characterized orbit equivalence in the topological category. In doing so, they introduced two new orbit equivalence invariants, namely: the dimension group, and having the simplices of invariants measures be affinely isomorphic via a space homeomorphism. We restate their characterization theorem here:

Theorem 1.0.7 ([GPS '95] Theorem 2.2)**.** *Let (X_i, T_i) be Cantor systems $(i = 1, 2)$. The following are equivalent:*

1. *(X_1, T_1) and (X_2, T_2) are orbit equivalent.*
2. *The dimension groups $K^0(X_i, T_i)/Inf(K^0(X_i, T_i))$, $i = 1, 2$, are order isomorphic by a map preserving the distinguished order units.*
3. *There exits a homeomorphism $F : X_1 \to X_2$ carrying the T_1-invariant probability measures onto the T_2-invariant probability measures.*

Above $K^0(X_i, T_i)/Inf(K^0(X_i, T_i))$ is the group of continuous functions from X_i to the integers modulo the subgroup of functions which integrate to 0 against every T_i-invariant Borel probability measure.

We can view speedups through the lens of orbit equivalence by observing that if (X_2, T_2) is a speedup of (X_1, T_1), then there exists a homeomorphism $F : X_1 \to X_2$ such that for every $x \in X_1$ we have

$$F(\text{orbit}^+_{T_1}(x)) \supseteq \text{orbit}^+_{T_2}(F(x)).$$

Our main theorem, stated below, has a very similar form to Theorem 1.0.7 above.

Main Theorem. *Let (X_1, T_1) and (X_2, T_2) be minimal Cantor systems. The following are equivalent:*

1. *(X_2, T_2) is a speedup of (X_1, T_1).*
2. *There exists*

   $$\varphi : K^0(X_2, T_2)/Inf(K^0(X_2, T_2)) \twoheadrightarrow K^0(X_1, T_1)/Inf(K^0(X_1, T_1))$$

   a surjective group homomorphism such that

   $$\varphi(K^0(X_2, T_2)/Inf(K^0(X_2, T_2))^+) = K^0(X_1, T_1)/Inf(K^0(X_1, T_1))^+$$

   and φ preserves the distinguished order units.
3. *There exists homeomorphism $F : X \to Y$, such that $F_* : M(X_1, T_1) \hookrightarrow M(X_2, T_2)$ is an injection.*

Here we can see the one-sided and reciprocal nature of our main theorem. Instead of having bijective morphims, as is the case in Giordano, Putnam, and

Skau's result, we alternatively have either surjective or injective morphisms from one object to the other: surjective morphism preserving the order unit and taking one positive cone onto the other in the dimension group setting, and an injection, arising from a space homeomorphism, from one simplex of invariant measures to the other. In Chapter 3 we will prove our main theorem and what's more having a surjective morphism on the dimension groups induces an injective morphism on the simplices of invariant measures (or states associated to the dimension group); hence illustrating the reciprocal nature of speedups. Furthermore, as a consequence of both the Main Theorem and Theorem 1.0.7, in the case of uniquely ergodic minimal Cantor system speedups characterize orbit equivalence. That is, given two uniquely ergodic minimal Cantor systems if one is a speedup of the other, then the systems are orbit equivalent. In Chapter 3 of the thesis we will define speedup equivalence and show speedup equivalence and orbit equivalence are the same in systems with finitely many ergodic measures. We conclude Chapter 3 by presenting an example which shows that speedups can leave the orbit equivalence class of a given minimal transformation.

In Chapter 4 we consider the special case of bounded speedups: that is, given a minimal Cantor system (X, T) a *bounded speedup* is a minimal transformation $S : X \to X$ with $S(x) = T^{p(x)}(x)$ and $p : X \to \mathbb{Z}^+$ is bounded. The principal result in this chapter, which is a part of joint work with Lori Alvin and Nic Ormes, is bounding the entropy of bounded speedups. Along the way we will prove several results regarding the structure of p. Highlighting these structural results is the fact that p is a constant function plus a T co-boundary.

Chapter 2

Minimal Cantor Systems

As a general reference for dynamics we recommend: [W],[BS],[Pe]. Throughout this thesis X will always be taken to be a Cantor space, that is a compact, metrizable, perfect, zero-dimensional space. A *Cantor system* will consist of a pair (X, T) where X is a Cantor space and $T : X \to X$ is a homeomorphism. In addition, we require that our homeomorphism be *minimal*, by which we mean that every orbit is dense. Specifically, for every x in X we have that

$$\overline{\mathcal{O}_T(x)} = \overline{\{T^n(x) : n \in \mathbb{Z}\}} = X$$

where $\mathcal{O}_T(x)$ denotes the orbit of the point x. We call such systems (X, T) *minimal Cantor systems*. It is well-known (see [W]) that we can replace the density of all full orbits with the density of just the forward orbits. Thus, a homeomorphism T is minimal if for every $x \in X$ we have that

$$\overline{\mathcal{O}_T^+(x)} = \overline{\{T^n(x) : n \in \mathbb{N}\}} = X$$

where $\mathcal{O}_T^+(x)$ denotes the forward orbit of the point x.

A helpful example which will be referenced throughout the thesis is the dyadic odometer. Here we take $X = \{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ is endowed with the discrete topology, making X into a Cantor space. We define T to be "$+1$ and carry to the right", so for example

$$.000\ldots \overset{T}{\mapsto} .100\ldots \overset{T}{\mapsto} .010\ldots \overset{T}{\mapsto} .110\ldots \overset{T}{\mapsto} 001\ldots.$$

Formally, T can be defined as

$$T(x)(i) \begin{cases} 0 & \text{if } i < n \\ 1 & \text{if } i = n \\ x(i) & \text{if } i > n \end{cases}$$

where n is the least non-negative integer such that $x(n) = 0$, and T maps the constantly 1 sequence to the constantly 0 sequence. The triadic odometer, which is mentioned later in the thesis, is similarly defined on $\{0,1,2\}^{\mathbb{N}}$.

Minimal Cantor systems exhibit a wonderful structure, namely the existence of a refining sequence of Kakutani-Rokhlin tower partitions. These tower partitions, defined below, were instrumental in relating minimal Cantor systems to Bratteli diagrams, and hence dimension groups, AF-Algebras, and many other beautiful results.

Definition 2.0.8. *A* **Kakutani-Rokhlin tower partition** *of a minimal Cantor system* (X, T) *is a clopen partition* $\mathcal{P}$ *of* X *of the form*

$$\mathcal{P} = \{T^j C_k : k \in V,\, 0 \leq j < h_k\}$$

where V *is a finite set,* C_k *is a clopen set, and* h_k *is a positive integer.*

By fixing a k we may refer to a *column* of the partition $\{T^j C_k : 0 \leq j < h_k\}$, and h_k is referred to the *height* of the column. The set $T^j C_k$ is the j^{th} *level* of the k^{th} column. Furthermore, we refer to

$$C = \bigcup_{k \in V} C_k$$

as the *base* of the Kakutani-Rokhlin tower partition. A visualization of a Kakutani-Rokhlin tower partition is provided below.

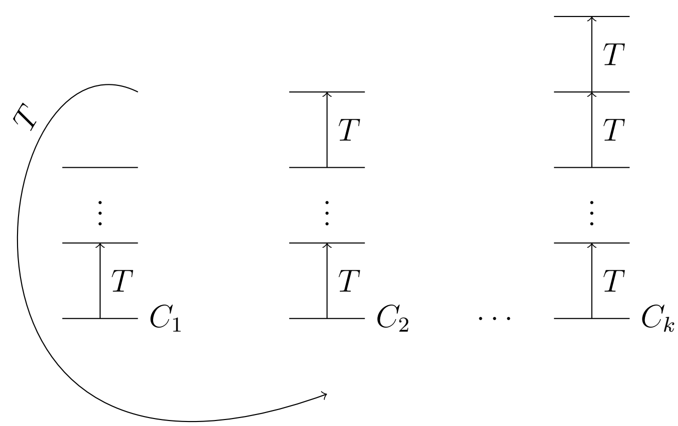

Figure 2.1: K-R Towers

Note T maps the top of each column into the base, and in only special cases does the top of any column map onto the first level of that column.

A nice property of these Kakutani-Rokhlin tower partitions is that they can have arbitrarily high columns heights and can refine any clopen partition the space. We summarize these properties in the following two propositions.

Proposition 2.0.9. *Let* (X, T) *be a minimal Cantor system, and* $n \in \mathbb{Z}^+$ *be given. There exists a Kakutani-Rokhlin tower partition of* X*,*

$$\{T^j(C_i) : 1 \leq i \leq t,\, 0 \leq j < h_i\}$$

such that for $i = 1, 2, \ldots, t$*,* $h_i > n$*.*

Proposition 2.0.10. *Let* (X, T) *be a minimal Cantor system,* $\mathcal{Q}$ *a clopen partition of* X*, and* $\mathscr{P}$ *a Kakutani-Rokhlin tower partition of* X*. Specifically,*

$$\mathscr{P} = \{T^j(C_i) : 1 \leq i \leq t,\, 0 \leq j < h_i\}.$$

Then we can refine $\mathscr{P}$ *into* $\mathscr{P}'$ *such that* $\mathscr{P}'$ *refines* $\mathcal{Q}$*, and* $\mathscr{P}'$ *maintains its tower structure: that is*

$$\mathscr{P}' = \{T^j(C_i') : 1 \leq i \leq t',\, 0 \leq j < h_i'\}$$

where t' *is the new number of columns and* h_i' *is the new height of the* i^{th} *column.*

Putting the following definition and propositions together, we get a fundamental theorem not only for this thesis, but for the study of minimal Cantor systems in general.

Theorem 2.0.11. *Let (X, T) be a minimal Cantor system and fix $x \in X$. There exists a sequence of Kakutani-Rokhlin tower partitions $(\mathscr{P}(n))_{n\in\mathbb{N}}$ with*

$$\mathscr{P}(n) := \{T^j C_i(n) : 1 \leq i \leq t(n), 0 \leq j < h_i(n)\}$$

satisfying

1. $\bigcap_{n\in\mathbb{N}} \bigcup_{1\leq i\leq t(n)} C_i(n) = \{x\}$
2. *For every n we have that $\mathscr{P}(n+1)$ is finer than $\mathscr{P}(n)$ i.e. $\mathscr{P}(n) \leq \mathscr{P}(n+1)$ for every n.*
3. $\bigcup_{n\in\mathbb{N}} \mathscr{P}(n)$ *generates the topology of X.*

We will make extensive use of this theorem throughout the proof of the main result.

2.1 Invariant measures associated to minimal Cantor systems

In this section, we will review some standard facts about invariant measures associated to topological dynamical systems and fix notation. Then we will introduce the definition of a dynamical simplex, or D-Simplex, which is due to

Heidi Dahl, and was inspired by and extended the notion of a good measure introduced by Ethan Akin in [A '05].

Recall that the Bogolioubov-Krylov Theorem says that any continuous transformation of a compact metric space has an invariant Borel probability measure. Fix a minimal Cantor system (X, T) and let $M(X)$ denote the collection of all Borel probability measures on X. We are interested in the measures in $M(X)$ which are T-invariant, and we denote the collection of all T-invariant Borel probability measures by $M(X, T)$, i.e.

$$M(X,T) = \{\mu \in M(X) : \mu(T^{-1}(A)) = \mu(A) \text{ for every Borel subset } A\}$$

Again by Bogolioubov-Krylov, $M(X, T) \neq \emptyset$.

The set $M(X, T)$ has a very nice structure as it is a Choquet simplex with respect to the weak* topology; that is, $M(X, T)$ is a compact, convex subset of $M(X)$ in which every measure μ can be uniquely represented as an integral against a measure τ which is fully supported on the extreme points, denoted by $\partial_e(M(X, T))$. Furthermore recall that a measure μ is *full or has full support* if μ gives positive measure to every nonempty open set. Also, we say that a measure μ is *non-atomic* if μ gives measure 0 to singletons. We are now ready to define a D-simplex.

Definition 2.1.1 (Dahl)**.** *Let $K \subseteq M(X)$ be a Choquet simplex consisting of non-atomic probability measures with full support. We say that K is a* **dynamical simplex (abbreviated D-simplex)** *if it satisfies the following two conditions:*

1. *For clopen subsets A and B of X with $\mu(A) < \mu(B)$ for all $\mu \in K$, there exists a clopen subset $B_1 \subseteq B$ such that $\mu(A) = \mu(B_1)$ for all $\mu \in K$ (this is known as the subset condition).*

2. *If $\mu, \nu \in \partial_e K$, $\mu \neq \nu$, then μ and ν are mutually singular, i.e. there exists a measurable set $A \subseteq X$ such that $\mu(A) = 1$ and $\nu(A) = 0$.*

It is well known that for any minimal Cantor system (X, T), $M(X, T)$ is a Choquet simplex whose extreme points are mutually singular, see [W, Chapter 6]. The fact that all measures are non-atomic and full both follow from X being uncountable coupled with T being a minimal transformation. Showing $M(X, T)$ is actually a D-simplex follows immediately from a proof of Lemma 2.5 from Glasner and Weiss [GW '95]. From this we have the following theorem.

Theorem 2.1.2. *Let (X, T) be a minimal Cantor system. The set $M(X, T)$ is a D-simplex.*

The fact that $M(X, T)$ is a D-simplex will play a role in the proof of the main theorem.

2.2 Ordered groups and dimension groups

One of the more recent tools in the study of minimal Cantor systems and ,in particular, the study of topological orbit equivalence, is the dimension group. Dimensions groups were first defined by Elliot in [Ell '76] using inductive limits of groups. However, the definitions which follow are an equivalent and more abstract way of defining dimension groups which is due to Effros, Handelman, and Shen [EHS '80].

Before we can define a dimension group, we must first introduce partially ordered groups. A general reference for parially ordered Abelian groups is [G], for references specifically related to dynamics we refer the reader to [HPS '92],[GPS '95], and for a summary see [D].

In this thesis we will deal exclusively with countable Abelian groups.

Definition 2.2.1. *A* **partially ordered group** *is a countable, Abelain group G together with a special subset denoted G^+, referred to as the* **positive cone**, *satisfying the following:*

1. $G^+ + G^+ \subseteq G^+$
2. $G^+ - G^+ = G$
3. $G^+ \cap (-G^+) = \{0\}$

Since we are calling these groups partially ordered given $a, b \in G$ we will write

$$a \leq b \text{ if } b - a \in G^+$$

and we can define a strict inequality, $a < b$ by requesting that $b - a \in G^+ \backslash \{0\}$. We will further require that our partially ordered Abelian groups be *unperforated* by which we mean: if $a \in G$ and $na \in G^+$ for some $n \in \mathbb{Z}^+$ then $a \in G^+$. We press on towards defining what a dimension group is with the final condition: the Riesz interpolation property.

Definition 2.2.2. *A partially ordered group is said to satisfy the* **Riesz interpolation property** *if given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ for $i, j = 1, 2$, then there exists $c \in G$ such that*

$$a_i \leq c \leq b_j \text{ for } i, j = 1, 2.$$

Finally, we have enough background to define a dimension group.

Definition 2.2.3. *A* **dimension group** *is an unperforated, partially ordered group (G, G^+) which satisfies the Riesz interpolation property.*

An example of a dimension group, which will appear multiple times in this thesis, is $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+)$ where

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{\frac{a}{2^b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\} \text{ and } \mathbb{Z}\left[\frac{1}{2}\right]^+ = \left\{x \in \mathbb{Z}\left[\frac{1}{2}\right] : x \geq 0\right\}.$$

In fact, this dimension group is the exact dimension group associated to the dyadic odometer. Furthermore, a theorem by Giordano, Putnam, and Skau, which we will give later in the thesis, showed that nearly all dimension groups arise from minimal Cantor systems.

There are two other properties of dimension groups we must discuss before moving forward. The first being the notion of an order unit.

Definition 2.2.4. *Let (G, G^+) be a partially ordered group, we call $u \in G^+$ an* **order unit** *if for every $a \in G$ there exists an $n \in \mathbb{N}$ such that $a \leq nu$. Furthermore, any dimension group with an order unit will be called a* **unital dimension group.**

Note 1 plays the role of an ordered unit in our example above, which makes $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, \mathbf{1})$ a unital dimension group.

Second, when dealing with minimal Cantor systems we only encounter *simple dimension groups*, defined below. Since our groups are Abelian, simple does not refer to the group being simple, but rather that the order ideal structure is simple.

Definition 2.2.5. *An* **order ideal** *is a subgroup J so that*

1. $J = J^+ - J^+$ *where* $J^+ = J \cap G^+$
2. *if* $0 \leq a \leq b \in J$*, then* $a \in J$*.*

A dimension group is **simple** *if it has no non-trivial order ideals.*

From now on we will only concern ourself with simple dimension groups. There are many connections between dimension groups and minimal Cantor systems and we will highlight some of these connections later in the thesis. We need another definition.

Definition 2.2.6. *Let G be a simple dimension group with a fixed order unit $u \in G^+ \backslash \{0\}$. We say that a homomorphism $p : G \to \mathbb{R}$ is a* **state** *if p is positive (i.e. $p(G^+) \subseteq [0, \infty)$) and $p(u) = 1$.*

States play an important role in the order structure of these dimension groups. To see this, let (G, G^+, u) be a unital simple dimension group (i.e. (G, G^+) is a simple dimension group and u is an order unit) and let $S_u(G)$ denote the collection of all states on G. It is known that states always exists and so $S_u(G) \neq \emptyset$. Paraphrasing a result of Effros[E, Cor. 4.2] we have that

$$G^+ = \{a \in G : p(a) > 0 \text{ for all } p \in S_u(G)\} \cup \{0\}.$$

This tells us that by knowing the states we know the order structure of G. Furthermore, we can make at least one connection with minimal Cantor systems, which we will make explicit once we have more notation, in that states on the dimension group correspond exactly to invariant measures for the minimal Cantor system associated to this dimension group. Hence there always exists at least one state, just as there always exists at least one invariant measure.

We now would like to single out special elements of any simple dimension group (G, G^+). First, fix (G, G^+) a simple unital dimension group with $u \in G^+ \backslash \{0\}$ an ordered unit. We say that $a \in G$ is an *infinitesimal* if $p(a) = 0$ for every $p \in S_u(G)$. We will let $Inf(G)$ denote the collection of all infinitesimals of G, and we note that it is a subgroup of G. Furthermore, if we start with a dimension group G and form the quotient group $G/Inf(G)$, then the quotient has a natural order structure coming from G in that $[a] > 0$ if $a > 0$. From this it can be seen that $G/Inf(G)$ becomes a dimension group in its own right and has no infintesimals other than $[0]$.

2.3 Dimension groups and dynamical system

In the section we will give a brief introduction to some basic definitions, notation, and theorems about dimension groups associated to minimal Cantor systems. For a more detailed and motivational exploration of these links we suggest [GPS '95],[HPS '92].

Given a minimal Cantor system (X, T), let $C(X, \mathbb{Z})$ denote the collection of all continuous $\mathbb{Z}$ valued functions on X. This is a countable Abelian group under addition. Furthermore, define

$$K^0(X,T) = C(X,\mathbb{Z})/\{f - f \circ T : f \in C(X,\mathbb{Z})\}.$$

We denote by $B_T = \{f - f \circ T : f \in C(X,\mathbb{Z})\}$ and call it collection of co-boundaries. Define the positive cone, the positive elements, to be

$$K^0(X,T)^+ = \{[f] : f \geq 0,\ f \in C(X,\mathbb{Z})\}$$

also let $\mathbf{1}$ denote the constantly 1 function on X. We now have the following theorem relating dimension groups arising from minimal Cantor systems.

Theorem 2.3.1 ([GPS '95] Theorem 1.12)**.** *Let (X,T) be a minimal Cantor system. Then $K^0(X,T)$ with positive cone $K^0(X,T)^+$ is a simple, acyclic (i.e. $G \not\cong \mathbb{Z}$) dimension group with (canonical) distinguished order unit $\mathbf{1}$. Furthermore, if (G,G^+) is a simple, acyclic dimension group with distinguished order unit u, there exists a minimal Cantor system (X,T) so that*

$$(G, G^+, u) \cong (K^0(X,T), K^0(X,T)^+, \mathbf{1})$$

meaning that there exists an order isomorphism $\alpha : G \to K^0(X,T)$ so that $\alpha(u) = \mathbf{1}$.

Giordano, Putnam, and Skau used dimension groups to completely classify both strong orbit equivalence and orbit equivalence, see [GPS '95]. The dimension group we concern ourselves with in this thesis are dimension groups modulo their infinitesimals. As mentioned previously, there is a lovely connection between states of a dimension group and invariant measures which we will make explicit now. We then can give a simple characterization of the dimension groups that will appear in this thesis. First we present the following theorem.

Theorem 2.3.2 ([GPS '95] Theorem 1.13)**.** *Let (X, T) be a minimal Cantor system.*

1. *Every T-invariant probability measure μ on X induces a state $T(\mu)$ on $(K^0(X,T), K^0(X,T)^+, \mathbf{1})$ by $f \to \int f d\mu$, $f \in C(X, \mathbb{Z})$.*

2. *The map T is a bijective correspondence between the set of T-invariant probability measures on X and the set of states on*

   $(K^0(X,T), K^0(X,T)^+, \mathbf{1})$.

States and T-invariant measures have a similar relationship with the dimension group $(K^0(X,T)/Inf(K^0(X,T)), K^0(X,T)/Inf(K^0(X,T)), \mathbf{1})$.

We now have seen states arise as integration against an invariant measure, hence we can simplify the representation of $K^0(X,T)/Inf(K^0(X,T))$. Let $Z_T = \{f \in C(X, \mathbb{Z}) : \int f d\mu = 0,\ \mu \in M(X,T)\}$, we then have

$$Inf(K^0(X,T)) = Z_T/B_T = \{f \in C(X, \mathbb{Z}) : \int f d\mu = 0,\ \mu \in M(X,T)\}/B_T.$$

Thus,

$$K^0(X,T)/Inf(K^0(X,T)) \cong C(X, \mathbb{Z})/Z_T$$

and the order unit $\mathbf{1}$ is preserved when $C(X, \mathbb{Z})/Z_T$ is endowed with the induced order of $[f] \geq 0$ if $f \geq 0$ in $C(X, \mathbb{Z})$.

Chapter 3

Speedups

In this section we will define what we mean by a speedup of a minimal Cantor system (X, T). Furthermore, we explore some of its basic properties which will lead up to the main theorem of the thesis. First, we define a speedup.

Definition 3.0.3. *Let (X_1, T_1) and (X_2, T_2) be minimal Cantor systems. We say (X_2, T_2) is a* **speedup** *of (X_1, T_1) if (X_2, T_2) is conjugate to (X_1, S) where S is a minimal homeomorphism of X defined by*

$$S(x) = T_1^{p(x)}(x)$$

where $p : X \to \mathbb{Z}^+$.

For example, if (X, T) is the dyadic odometer, then (X, T^3) would constitute a speedup of (X, T) as it is again a minimal Cantor system. We would like to point out that our definition of speedup is a bit more general in that any minimal Cantor system conjugate to (X, T^3) is also considered to be a speedup of (X, T). We remark that (X, T^2), or anything conjugate to (X, T^2), cannot be a speedup of (X, T) as T^2 is not minimal, as evidenced by $T^2([0]) = [0]$, where $[0] = \{x \in X : x(0) = 0\}$.

In the paper by Arnoux, Ornstein, and Weiss [AOW '85], p is a measurable map. When (X, T) is in the topological category, we make the observation that if $T^{p(\cdot)}$ is to be continuous, then p must be lower semicontinuous.

Proposition 3.0.4. *Let $p : X \to \mathbb{Z}^+$ and suppose that $T^{p(x)}(x) = S(x)$ is a minimal Cantor system, then p is lower semicontinuous, hence a Borel map.*

Proof. First, we show that for every $n \in \mathbb{Z}^+$ we have

$$p^{-1}(\{n\}) \text{ is closed.}$$

Let $n \in \mathbb{Z}^+$, $\{x_m\}_{m \geq 1} \subseteq p^{-1}(\{n\})$, and $x \in X$ such that $x_m \to x$; since both S and T^n are continuous, we have that

$$S(x_m) \to S(x) \text{ and } T^n(x_m) \to T^n(x).$$

Since for every m, $S(x_m) = T^n(x_m)$ and by uniqueness of limits we have that

$$S(x) = T^n(x).$$

We may conclude $p(x) = n$ as a result of T being aperiodic by virtue of being a minimal transformation on a Cantor space.

Recall that a real valued function is *lower semicontinuous* on a topological space if

$$\{x \in X : f(x) > \alpha\}$$

is open for every real α. Now let $\alpha \in \mathbb{R}$ be given. Observe that for any α there are only finitely many $n \in \mathbb{Z}^+$ such that $n \leq \alpha$; thus,

$$\{x : p(x) \leq \alpha\} = \bigcup_{n \leq \alpha} p^{-1}(\{n\})$$

is a finite union of closed sets whence is closed. Consequently $\{x : p(x) > \alpha\}$ is open, therefore p is lower semicontinuous as desired. □

Remark 3.0.5. *If p is continuous, then p must be bounded as X is compact. However, the converse is true as well; that is, if p is bounded and defines a speedup S, then p is continuous. This follows almost immediately from the proof of the previous proposition. In this case, where p is bounded, finitely valued, or continuous and $S(x) = T^{p(x)}(x)$ is a speedup of T, we call these* **bounded speedups.** *Bounded speedups will be discussed in Chapter* 4.

One important aspect of speedups is how they interact with the invariant measures of the original system. The following proposition gives the relationship between the invariant measures of the original system and speedups of it. Furthermore, we have an example which shows the relationship below can be strict; thus showing that speedups can leave the conjugacy class of the original system. We will discuss this more later in the chapter. Before we prove this relationship, it will be useful to be able to refer to the following proposition.

Proposition 3.0.6. *Suppose (X, T) is a minimal Cantor system, then*

$$M(X, T) = M(X, T^{-1}).$$

We now show how speedups interact with the invariant measures of the original system.

Proposition 3.0.7. *Let (X, T) be a minimal Cantor system. If (X, S) is a speedup of (X, T), then $M(X, T) \subseteq M(X, S)$.*

Proof. Let $p : X \to \mathbb{Z}^+$ be such that $S(x) = T^{p(x)}(x)$ is a minimal homeomorphism of X and let $\mu \in M(X, T)$. Observe by Proposition 3.0.6 it suffices to simply show that $\mu \in M(X, S^{-1})$. Let $A \in \mathscr{B}(X)$, we then have

$$\begin{aligned}
\mu(S(A)) &= \mu\left(S\left(\bigsqcup_{n\in\mathbb{Z}^+} A \cap p^{-1}(\{n\})\right)\right) \\
&= \mu\left(\bigsqcup_{n\in\mathbb{Z}^+} S(A \cap p^{-1}(\{n\}))\right) \\
&= \mu\left(\bigsqcup_{n\in\mathbb{Z}^+} T^n(A \cap p^{-1}(\{n\}))\right) \\
&= \sum_{n\in\mathbb{Z}^+} \mu(T^n(A \cap p^{-1}(\{n\}))) \\
&= \sum_{n\in\mathbb{Z}^+} \mu(A \cap p^{-1}(\{n\})) \text{ as } \mu \in M(X, T). \\
&= \mu(A).
\end{aligned}$$

□

Notice that this proposition gives us an immediate restriction on when one system can be a speedup of another. For example, the previous proposition rules out the possibility of the triadic odometer being a speedup of the dyadic odometer, and vice versa, as both systems are uniquely ergodic and do not share the same clopen value set. The natural question to ask is: is this the only such restriction? We answer this and more with the statement of the main theorem of the thesis.

Theorem 3.0.8. *Let (X_1, T_1) and (X_2, T_2) be minimal Cantor systems and let*

$$G_1 = C(X_1, \mathbb{Z})/Z_{T_1} \text{ and } G_2 = C(X_2, \mathbb{Z})/Z_{T_2}.$$

Where $Z_{T_i} = \{g \in C(X, \mathbb{Z}) : \int g d\mu = 0 \, \forall \mu \in M(X_i, T_i)\}$.

The following are equivalent:

1. *(X_2, T_2) is a speedup of (X_1, T_1).*

2. *There exists*

   $$\varphi : (G_2, G_2^+, \mathbf{1}) \twoheadrightarrow (G_1, G_1^+, \mathbf{1})$$

   a surjective group homomorphism such that $\varphi(G_2^+) = G_1^+$ and $\varphi(\mathbf{1}) = \mathbf{1}$.

3. *There exists homeomorphism $F : X_1 \to X_2$, such that $F_* : M(X_1, T_1) \hookrightarrow M(X_2, T_2)$ is an injection.*

We will break up the proof of the main theorem into three sections, as each part of the proof requires a different set of lemmas. The main difficulty is proving (3) implies (1).

3.1 Proof of (1) implies (2)

Here we present the proof of (1) $\Rightarrow$ (2).

Proof. Since (X_2, T_2) is a speedup of (X_1, T_1), (X_2, T_2) is conjugate, through a conjugacy κ, to (X_1, S) where $S : X_1 \to X_1$

$$S(x) = T_1^{p(x)}(x)$$

and $p : X_1 \to \mathbb{Z}^+$. Let $H_1 = C(X_1, \mathbb{Z})/Z_S$ and $(H_1, H_1^+, \mathbf{1})$ be the unital dimension group associated to (X_1, S). Hence, right composition of k induces a unital dimension group isomorphism $\varphi_1 : (H_1, H_1^+, \mathbf{1}) \to (G_2, G_2^+, \mathbf{1})$. Define $\varphi_2 : (G_2, G_2^+, \mathbf{1}) \to (G_1, G_1^+, \mathbf{1})$ by

$$\varphi_2([g]_S) = [g]_{T_1}.$$

Observe, Proposition 3.0.7 gives us

$$Z_S \subseteq Z_{T_1}$$

whence φ_2 is well defined. It is standard to check that φ_2 is a surjective group homomorphism (see the Third Isomorphism Theorem for groups). One can verify

$$\varphi_2(G_2^+) = G_1^+ \text{ and } \varphi_2(\mathbf{1}) = \mathbf{1}.$$

Therefore $\varphi = \varphi_2 \circ \varphi_1$ is our desired group homomorphism. □

3.2 Proof of (2) implies (3)

In order to proceed from (2) to (3), we would like to make use of Giordano, Putnam, and Skau's characterization of topological orbit equivalence [GPS '95, Thm 2.2]. To do so we will need to extend the First Isomorphism Theorem from groups to partially ordered Abelian groups with interpolation. We recall for the reader one of the main theorems from [GPS '95].

Theorem 3.2.1 ([GPS '95] Theorem 2.2). *Let* (X_i, T_i) *be Cantor systems* $(i = 1, 2)$. *The following are equivalent:*

(i) (X_1, T_1) *and* (X_2, T_2) *are orbit equivalent.*

(ii) The dimension groups $K^0(X_i, T_i)/Inf(K^0(X_i, T_i))$, $i = 1, 2$, *are order isomorphic by a map preserving the distinguished order units.*

(iii) There exits a homeomorphism $F : X_1 \to X_2$ *carrying the* T_1-*invariant probability measures onto the* T_2-*invariant probability measures.*

Furthermore, recall what an isomorphism is in the category of unital partially ordered Abelian groups with interpolation.

Definition 3.2.2. *An* **isomorphism** *between two unital partially ordered Abelian groups say* (G, G^+, u) *and* (H, H^+, v) *is a map* $\varphi : G \to H$ *a group and order isomorphism and* $\varphi(u) = v$. *In such a case we say that* (G, G^+, u) *is isomorphic to* (H, H^+, v), *written* $(G, G^+, u) \cong (H, H^+, v)$.

We now proceed with a short proof of the First Isomorphism Theorem in the category of partially ordered Abelian groups with interpolation.

Theorem 3.2.3. *Let* (G, G^+, u) *and* (H, H^+, v) *be unital dimension groups. If* $\varphi : H \to G$ *is a surjective, order and order unit preserving homomorphism with* $\varphi(H^+) = G^+$, *then*

$$(H/\ker(\varphi), H^+/\ker(\varphi), [v]) \cong (G, G^+, u)$$

as unital dimension groups.

Proof. Define $\hat{\varphi} : H/\ker(\varphi) \to G$ by

$$\hat{\varphi}([h]) = \varphi(h)$$

for $h \in H$. By the First Isomorphism Theorem for groups $\hat{\varphi}$ is a group isomorphism; thus it suffices to show that $\hat{\varphi}\left(H^+/\ker(\varphi)\right) = G^+$, and $\hat{\varphi}([v]) = u$. These follow immediately as $\varphi(H^+) = G^+$ and $\varphi(v) = u$. □

We will need one more proposition before tackling (2) ⇒ (3) and it begins to illustrate the reciprocal nature of the main theorem.

Proposition 3.2.4. *Let* $\varphi : G_2 \twoheadrightarrow G_1$ *be as in* (2) *of Theorem* 3.0.8*. Then there exists an injection* $\varphi_* : M(X_1, T_1) \hookrightarrow M(X_2, T_2)$

Proof. We will show that φ induces an injective map on $M(X_1, T_1)$ into the state space of G_2, from there we appeal to Theorem 2.3.2, which says that the states and invariant measures are in bijective correspondence. Composing these two functions gives us our injection from $M(X_1, T_1)$ into $M(X_2, T_2)$.

Let $\mu \in M(X_1, T_1)$, $h \in C(X_2, \mathbb{Z})$ and define

$$\begin{aligned} \varphi_*\mu[h] &= \int_X \varphi([h])\, d\mu \\ &= \int_X g\, d\mu \qquad \text{where } g \in C(X_1, \mathbb{Z}) \text{ and } g \in \varphi([h]) \end{aligned}$$

Let us first show that φ_* is well-defined. Let $h \in C(X_2, \mathbb{Z})$ and $g_1, g_2 \in C(X_1, \mathbb{Z})$ be such that $g_1, g_2 \in \varphi([h])$; thus there exists $i \in Inf(G_1)$ such that

$g_1 + i = g_2$. Now we calculate

$$\begin{aligned}\int_{X_1} g_2 \, d\mu &= \int_{X_1} (g_1 + i) \, d\mu \\ &= \int_{X_1} g_1 \, d\mu\end{aligned}$$

so φ_* is well-defined. Since φ is order unit preserving we see that

$$\varphi_*\mu[1] = \int_{X_1} 1 \, d\mu = 1.$$

To see that $\varphi_*\mu$ is positive, let $h \in C(X_2, \mathbb{Z})$ be such that for every x, $h(x) \geq 0$, thus $[h] \in G_2^+$, and whence $\varphi([h]) \geq 0$ as φ is positive. So there exists $g \in C(X_1, \mathbb{Z})$ such that for every x, $g(x) \geq 0$ and $g \in \varphi([h])$. Thus,

$$\varphi_*\mu[h] = \int_{X_1} g \, d\mu \geq 0.$$

Finally, to see that $\varphi_*\mu$ is a homomorphism, let $h_1, h_2 \in C(X_2, \mathbb{Z})$. Observe,

$$\begin{aligned}\varphi_*\mu[h_1 + h_2] &= \int_{X_1} \varphi([h_1 + h_2]) \, d\mu \\ &= \int_{X_1} \varphi([h_1] + [h_2]) \, d\mu \\ &= \int_{X_1} (\varphi([h_1]) + \varphi([h_2])) \, d\mu \\ &= \int_{X_1} \varphi([h_1]) \, d\mu + \int_{X_1} \varphi([h_2]) \, d\mu \\ &= \varphi_*\mu[h_1] + \varphi_*\mu[h_2].\end{aligned}$$

Therefore, $\varphi_*\mu$ is a state on G_2 as desired.

Now we will show that φ_* is injective. Let $\mu, \nu \in M(X_1, T_1)$ such that $\mu \neq \nu$. So there exists a nonempty clopen set C such that,

$$\int_{X_1} \mathbb{1}_C \, d\mu = \mu(C) \neq \nu(C) = \int_{X_1} \mathbb{1}_C \, d\nu$$

Since $\varphi(G_2^+) = G_1^+$ there exists $h \in C(X_2, \mathbb{Z})$, for every x, $h(x) \geq 0$ such that $\varphi([h]) = [\mathbb{1}_C]$, rather $\mathbb{1}_C \in \varphi([h])$. Now we compute,

$$\varphi_*\mu([h]) = \int_{X_1} \mathbb{1}_C \, d\mu = \mu(C) \neq \nu(C) = \int_{X_1} \mathbb{1}_C \, d\nu = \varphi_*\nu([h]).$$

So φ_* is injective. Recall [E, Cor. 4.2], which says that the set of states is in bijective correspondence with the set of invariant measures; so, we get our desired injection by composing φ_* with this bijection. □

With Theorem 3.2.3, Proposition 3.2.4 and Theorem 2.2 of [GPS '95] at our disposal, we wish to dispense of (2) ⇒ (3).

Proof. By assuming (2) and in conjunction with Theorem 3.2.3 we know $\hat{\varphi}$ is an unital dimension group isomorphism

$$\hat{\varphi} : (G_2/\ker(\varphi), G_2^+/\ker(\varphi), [\mathbf{1}]_\varphi) \to (G_1, G_1^+, \mathbf{1});$$

so, in particular $(G_2/\ker(\varphi), G_2^+/\ker(\varphi), [\mathbf{1}]_\varphi)$ is itself a unital dimension group. As a result of the isomorphism, $(G_2/\ker(\varphi), G_2^+/\ker(\varphi), [\mathbf{1}]_\varphi)$ must have one infinitesimal, namely $[0]_\varphi$. Observe, $G_2^+/\ker(\varphi)$ is determined by $\varphi_*(M(X_1, T_1))$ by Proposition 3.2.4. By [GPS '95, Thm. 2.2] there exists a homeomorphism $F : X_1 \to X_2$ such that the invariant measures associated to (X_1, T_1) are taken bijectively onto the g-invariant measures, where g is a minimal realization of

$$(G_2/\ker(\varphi), G_2^+/\ker(\varphi), [\mathbf{1}]_\varphi)$$

by Theorem 2.3.1. Finally, Proposition 3.2.4 also shows that the invariant measures associated to $G_2/\ker(\varphi)$ are a subset of $M(X_2, T_2)$, and we have our injection from $M(X_1, T_1)$ into $M(X_2, T_2)$ via a space homeomorphism from X_1 to X_2, as desired. Note that (X_2, g) and (X_1, T_1) are orbit equivalent as a result [GPS '95, Thm. 2.2], since their dimension groups modulo infinitesimals are isomorphic as dimension groups. □

3.3 Proof of (3) implies (1)

This is by far the most technical portion of the thesis. The idea of the proof is quite similar to the construction presented in the Arnoux, Ornstein, and Weiss' paper [AOW '85]. In fact our key lemma, Lemma 3.3.6, is a topological version of the key lemma from [AOW '85] and a modification of Proposition 2.6 from [GW '95]. Note that a key difference in our lemma is the range of our p map: $\mathbb{Z}^+$ instead of $\mathbb{Z}$. This lemma allows us to actually construct the speedup on the non-final levels on a Kakutani-Rokhlin tower partition.

Before moving forward with the construction to prove (3) implies (1), we will prove a short sequence of lemmas culminating with our key lemma, Lemma 3.3.6. Again, many of the following propositions and lemmas are similar to propositions and lemmas found in [GW '95].

Proposition 3.3.1. *Let (X,T) be a minimal Cantor system. For every $\varepsilon > 0$ there exists a nonempty clopen set C such that for all $\mu \in M(X,T)$, $\mu(C) < \varepsilon$.*

Proof. Suppose, towards a contradiction, there exists an $\varepsilon > 0$ such that for every nonempty clopen set C there exists $\nu \in M(X,T)$ such that $\nu(C) \geq \varepsilon$. Fix $x \in X$, let $C_n(x)$ be a clopen set of diameter less than $1/n$ containing x, and let μ_n be a measure in $M(X,T)$ such that $\mu_n(C_n) \geq \varepsilon$. By compactness of $M(X,T)$ there exists $\nu \in M(X,T)$ and $n_k \nearrow \infty$ such that

$$\mu_{n_k} \xrightarrow{\text{weak}^*} \nu.$$

Clearly,

$$\bigcap_{n=1}^{\infty} C_n = \bigcap_{k=1}^{\infty} C_{n_k} = \{x\}$$

and so

$$\nu(\{x\}) = \lim_{k\to\infty} \nu(C_{n_k})$$

and we claim that for all $k \in \mathbb{Z}^+$, $\nu(C_{n_k}) \geq \varepsilon$. Fix $k \in \mathbb{Z}^+$, since C_{n_k} is clopen we have by definition

$$\nu(C_{n_k}) = \lim_{j\to\infty} \mu_{n_j}(C_{n_k})$$

and for $k < j$ we have that

$$C_{n_k} \supseteq C_{n_j} \Rightarrow \mu_{n_j}(C_{n_k}) \geq \mu_{n_j}(C_{n_j}) \geq \varepsilon$$

thus $\nu(C_{n_k}) \geq \varepsilon$. So we see that

$$\nu(\{x\}) = \lim_{k\to\infty} \nu(C_{n_k}) \geq \varepsilon > 0$$

contradicting the fact that ν must be non-atomic. □

We immediately use this proposition to prove the following lemma.

Lemma 3.3.2. *Let* (X,T) *be a minimal Cantor system. For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* $A \in \mathscr{B}(X)$ *with* $\operatorname{diam}(A) < \delta$ *and every* $\mu \in M(X,T)$*, we have* $\mu(A) < \varepsilon$*.*

Proof. Let $\varepsilon > 0$ be given, by Proposition 3.3.1 there exists a nonempty clopen set C such that for all $\mu \in M(X,T)$, $0 < \mu(C) < \varepsilon$. Since C is nonempty, clopen, and as T is minimal there exists $N \in \mathbb{Z}^+$ such that

$$X = \bigcup_{i=-N}^{N} T^i(C).$$

Let $\delta > 0$ be the Lebesgue number for the open cover $\{T^iC\}_{i=-N}^{N}$ (recall that a Lebesgue number for an open covering $\mathcal{A}$ of a compact metric space X is a constant $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it). Now let $A \in \mathscr{B}(X)$ with $\operatorname{diam}(A) < \delta$, then

$$\begin{aligned} \operatorname{diam}(A) < \delta &\Rightarrow A \subseteq T^i(C) \qquad \text{for some } i \in \{-N, \dots, N\} \\ &\Rightarrow \mu(A) \leq \mu(T^i(C)) \qquad \text{for every } \mu \in M(X,T) \\ &\Rightarrow \mu(A) \leq \mu(C) \qquad \text{as } \mu \in M(X,T) \\ &\Rightarrow \mu(A) < \varepsilon. \end{aligned}$$

So for every $\mu \in M(X,T)$ and $A \in \mathscr{B}(X)$ with $\operatorname{diam}(A) < \delta$ we have $\mu(A) < \varepsilon$ as desired. □

Before we can state and prove one of our key lemmas, we need one more proposition.

Proposition 3.3.3. *Let (X, T) be a minimal Cantor system, and $f : X \to \mathbb{R}$ a continuous function. If*

$$\inf \left\{ \int_X f \, d\mu : \mu \in M(X,T) \right\} > c > 0,$$

then there exists a $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ and for all $x \in X$ we have

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) \geq c.$$

Proof. Fix $f \in C(X, \mathbb{R})$ and suppose, towards a contradiction, that our proposition is false; that is, there is no such $N_0 \in \mathbb{N}$. So there exists $\{N_k\}_{k \geq 0}$ and $\{x_k\}_{k \geq 0}$ such that $N_k \nearrow \infty$, and for a fixed k

$$\frac{1}{N_k} \sum_{j=0}^{N_k - 1} f(T^j(x_k)) < c.$$

Consider the following sequence of measures $\{\mu_k\}_{k \geq 0}$, where for fixed k we have

$$\mu_k = \frac{1}{N_k} \sum_{j=0}^{N_k - 1} \tilde{T} \delta_{x_k}$$

where δ represents the Dirac measure. By compactness of $M(X)$, the collection of all Borel probability measures on X, there exists $\nu \in M(X)$ and increasing sequence $\{k_\ell\}_{l \geq 0} \nearrow \infty$ such that

$$\mu_{k_\ell} \xrightarrow{\text{weak}^*} \nu.$$

Recall by [W, Theorem 6.9] $\nu \in M(X,T)$; we will now show that $\int_X f\,d\nu \le c$ which will give us our contradiction. Since f is continuous we have that

$$\begin{aligned}\int_X f\,d\nu &= \lim_{\ell\to\infty}\int_X f\,d\mu_{k_\ell}\\ &= \lim_{\ell\to\infty}\frac{1}{N_\ell}\sum_{j=0}^{N_{k_\ell}-1} f(T^j(x_{k_\ell}))\\ &\le c.\end{aligned}$$

This is a contradiction, which proves our proposition. □

We use Proposition 3.3.1 and Proposition 3.3.3 in conjunction with Lemma 3.3.2 to prove Lemma 3.3.4. This lemma serves as a precursor to the key lemma and is instrumental for proving Lemma 3.3.6.

Lemma 3.3.4. *Let (X,T) be a minimal Cantor system, and let A, B be nonempty, disjoint, clopen subsets of X. If for all $\mu \in M(X,T)$, $\mu(A) < \mu(B)$, then there exists $p : A \to \mathbb{Z}^+$ such that $S : A \to B$ defined as $S(x) = T^{p(x)}(x)$ is a homeomorphism onto its image.*

Proof. Let A, B be nonempty disjoint clopen subsets of X and define $f = \mathbb{1}_B - \mathbb{1}_A$. Since both A and B are clopen it follows that $f : X \to \mathbb{Z}$ is continuous. Moreover, since $\int f d\mu > 0$ for every $\mu \in M(X,T)$ and $M(X,T)$ is compact in the weak* topology, it follows by assumption that

$$\inf\left\{\int_X f\,d\mu : \mu \in M(X,T)\right\} > 0.$$

Choose $c \in \mathbb{R}$ so that

$$\inf \left\{ \int_X f \, d\mu : \mu \in M(X,T) \right\} > c > 0$$

So by Proposition 3.3.3 find N_0 large such that for every $n \geq N_0$ and every $x \in X$ we have

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \geq c.$$

Use Proposition 2.0.9 to construct a Kakutani-Rokhlin tower partition of X such that for each i, $h_i \geq N_0$. Let the following denote our tall Kakuntani-Rokhlin tower partition:

$$\{T^j(D_i) : 1 \leq i \leq t, \, 0 \leq j < h_i\}.$$

Use Proposition 2.0.10 to refine each tower with respect to the partition

$$\{A, B, (A \cup B)^c\}$$

.

By a slight abuse of notation we will not rename our new Kakutani-Rokhlin tower partition, and with that let us look at a single column of our partition. Fix $i = 1$, and consider the column

$$\left\{T^j(D_1) : 0 \leq j < h_1\right\}.$$

Let $x \in D_1$, then as $h_1 \geq N_0$ we must have that

$$\frac{1}{h_1} \sum_{j=0}^{h_1-1} f(T^j x) \geq c > 0. \tag{3.3.1}$$

Thus, there are more B levels than A levels in this column. In other words let J and K be defined below

$$J = \{j_1, j_2, \ldots, j_m : T^{j_i}(D_1) \cap A \neq \emptyset,\ i = 1, 2, \ldots m\}$$
$$K = \{k_1, k_2, \ldots, k_r : T^{k_i}(D_1) \cap B \neq \emptyset, i = 1, 2, \ldots, r\}$$

and by (3.3.1) we have that $|J| < |K|$. Choose any injection $\Gamma : J \hookrightarrow K$.

We exploit the inherit order structure of the column to define our map p. First, we give a picture with an arbitrary injection to help the reader visualize what is going on. All A-levels in our first column are colored red and all of the B-levels in the first column are colored blue.

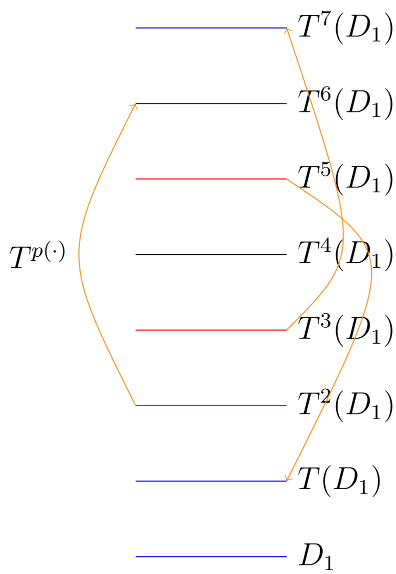

Figure 3.1: K-R Tower

We break the definition of $T^{p(\cdot)}$ into the following two cases. First fix $i \in \{1, 2, \ldots, m\}$.

Case 1: $\Gamma(j_i) > j_i$**.** In this case we can simply define $p : T^{j_i}(D_1) \to \mathbb{Z}^+$ by $p(x) = \Gamma(j_i) - j_i$. By assumption p is positive and as T is a homeomorphism we have

$$T^{p(\cdot)} = T^{\Gamma(j_i)-j_i}$$

is a homeomorphism from $T^{j_i}(D_1) \subseteq A$ to $T^{\Gamma(j_i)}(D_1) \subseteq B$. Furthermore, we see that

$$T^{p(\cdot)}(T^{j_i}(D_1)) = T^{\Gamma(j_i)-j_i}(T^{j_i}D_1) = T^{\Gamma(j_i)}(D_1) \subseteq B.$$

Note $T^{p(\cdot)}$ simply moves x up the requisite number of levels in the tower as $T^{\Gamma(j_i)}(D_1)$ lies above $T^{j_1}(D_1)$ in the column by assumption. This finishes the first case.

Case 2: $\Gamma(j_i) < j_i$. In this case we see that we must map an A level into a B level which is below it in our column. In this case we cannot move down the tower as p must be positively valued. To this end, let $T^{\lambda(\cdot)} : T^{j_i}(D_1) \to T^{j_i}(D_1)$ be the first return map where recall,

$$\lambda(x) = \inf\{n > 0 : T^n x \in T^{j_i}(D_1)\}.$$

The map λ is well defined by virtue of $T^{j_1}D_1$ being clopen and T minimal. Moreover, one can see that λ is continuous, hence λ is finitely valued as $T^{j_1}(D_1)$ is compact. Furthermore, it is well known that $T^{\lambda} : T^{j_1}(D_1) \to T^{j_1}(D_1)$ is a homeomorphism; if we let $S = T^{\Gamma(j_i)-j_i} \circ T^{\lambda}$ we have that $S : T^{j_i}(D_1) \to T^{\Gamma(j_i)}$ is a homeomorphism and the resulting p function on $T^{j_i}(D_1)$ is

$$p(x) = \lambda(x) - (\Gamma(j_i) - j_i).$$

Thus, all that is left to show is that p is a positive function. However, let

$$\lambda(T^{j_i}(D_1)) = \{t_1, t_2, \ldots, t_n\}.$$

Observe points must traverse the tower in a specified order, thus for each $\ell \in \{1, 2, \ldots, n\}$ we must have that $t_\ell \geq h_1$, hence for each ℓ, $t_\ell - (\Gamma(j_i) - j_i) > 0$. Therefore, we have found our $S : T^{j_i}(D_1) \to T^{\Gamma(j_i)}(D_1)$ of the form $S(x) = T^{p(x)}(x)$, where $p : T^{j_i}(D_1) \to \mathbb{Z}^+$ as desired.

Continuing for each i, and then for each column we see that we define p on all of A. Furthermore, it is clear that $T^{p(\cdot)}$ is a continuous surjection from A onto its image in B, as $T^{p(\cdot)}$ is a homeomorphism on each level of A. To see that $T^{p(\cdot)}$ is injective, hence a homeomorphism, observe that $T^{p(\cdot)}$ is a homeomorphism when restricted to any A level in any column in the Kakutani-Rokhlin tower partition. Moreover, the $T^{p(\cdot)}$ image of any two distinct, hence disjoint, A levels is again disjoint. Finally, as all columns of the Kakutani-Rokhlin tower partition are disjoint $T^{p(\cdot)}$ maintains its injectivity and is, therefore, a homeomorphism from A onto its image in B. □

We now immediately use Lemma 3.3.4 to prove the final lemma needed in order to prove our key lemma.

Lemma 3.3.5. *Let (X, T) be a minimal Cantor system, and let $A, B \subseteq X$ be nonempty, disjoint, clopen subsets of X with*

$$\mu(A) = \mu(B)$$

for every $\mu \in M(X, T)$. Moreover, fix $x \in A$, $y \in B$ and let $\varepsilon > 0$ be given. Then there exists clopen sets $A_1 \subseteq A$, $B_1 \subseteq B$ with the following properties:

1. *$x \in A_1$ and $y \in B_1$*
2. *$diam(A_1) < \varepsilon$, $diam(B_1) < \varepsilon$*
3. *For every $\mu \in M(X, T)$, $\mu(A_1) = \mu(B_1)$, $\mu(A_1) < \dfrac{\mu(A)}{2}$, $\mu(B_1) < \dfrac{\mu(B)}{2}$*
4. *There exists $p : A \backslash A_1 \to \mathbb{Z}^+$ such that $T^{p(\cdot)} : A \backslash A_1 \to B \backslash B_1$ is a homeomorphism.*

Proof. Let A and B be nonempty, disjoint, clopen subsets of X, and fix $x \in A$ and $y \in B$, and let $\varepsilon > 0$ be given. Recall that every measure $\mu \in M(X,T)$ is full, i.e. gives positive measure to nonempty open sets, whence $\int \mathbb{1}_A d\mu > 0$. Let

$$\alpha = \inf \left\{ \int_X \mathbb{1}_A \, d\mu : \mu \in M(X,T) \right\}.$$

Since for every $\mu \in M(X,T)$, $\mu(A) = \mu(B)$ we also have that

$$\alpha = \inf \left\{ \int_X \mathbb{1}_B \, d\mu : \mu \in M(X,T) \right\}.$$

Observe, $\mathbb{1}_A$ is continuous as A is clopen and since $M(X,T)$ is compact in the weak* topology the above infimum is achieved, whence $\alpha > 0$. By Lemma 3.3.2 there exists $\delta_\alpha > 0$ and such that for every $K \in \mathscr{B}(X)$ and for every $\mu \in M(X,T)$

$$\operatorname{diam}(K) < \delta_\alpha \Rightarrow \mu(K) < \frac{\alpha}{2}.$$

Find clopen set $A_{\frac{1}{3}} \subsetneq A$, such that

$$x \in A_{\frac{1}{3}} \text{ and } \operatorname{diam}(A_{\frac{1}{3}}) < \min\{\delta_\alpha, \varepsilon\}.$$

Let

$$\varepsilon_1 = \inf \left\{ \int_X \mathbb{1}_{A_{\frac{1}{3}}} \, d\mu : \mu \in M(X,T) \right\} > 0$$

and use Lemma 3.3.2 to obtain $\delta_1 > 0$ such that for every $K \in \mathscr{B}(X)$ and for every $\mu \in M(X,T)$

$$\operatorname{diam}(K) < \delta_1 \Rightarrow \mu(K) < \varepsilon_1.$$

Find clopen subset $B_1 \subseteq B$ such that

$$y \in B_1 \text{ and } \operatorname{diam}(B_1) < \min\{\varepsilon, \delta_1, \delta_\alpha\};$$

thus, we have for all $\mu \in M(X, T)$

$$\mu(B_1) < \varepsilon_1 < \mu(A_{\frac{1}{3}}) \Rightarrow \mu(A \backslash A_{\frac{1}{3}}) < \mu(B \backslash B_1).$$

Apply Lemma 3.3.4 to T and get $p_1 : A \backslash A_{\frac{1}{3}} \to \mathbb{Z}^+$ such that $S : A \backslash A_{\frac{1}{3}} \to B \backslash B_1$, defined by $S(x) = T^{p_1(x)}(x)$, is a homeomorphism onto its image. Then, $B \backslash S(A \backslash A_{\frac{1}{3}}) = B_1 \sqcup U_1$ where U_1 is a nonempty clopen set and B_1 and U_1 are disjoint. Furthermore, for every $\mu \in M(X, T)$ we have that

$$\mu(A_{\frac{1}{3}}) = \mu(B_1) + \mu(U_1) \tag{3.3.2}$$

We can visualize this as below.

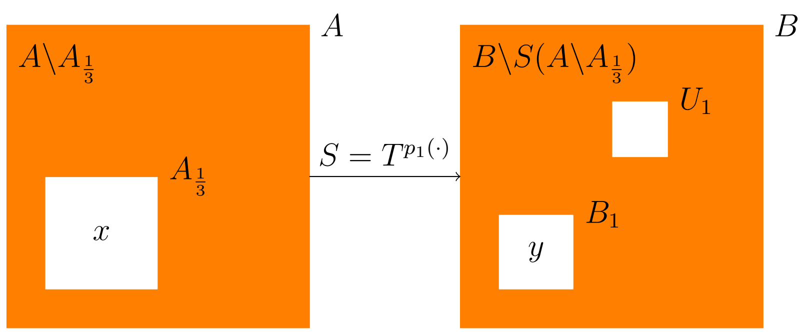

Figure 3.2: Visual of Lemma 3.3.5

Here is the intertwining nature of the proof; in order to extend to S to more of A, we apply Lemma 3.3.4 to T^{-1} with respect to the clopen sets U_1

and $A_{\frac{1}{3}}$ with a small neighborhood of x removed. By (3.3.2) above we have for every $\mu \in M(X,T)$

$$\mu(U_1) < \mu(A_{\frac{1}{3}})$$

and let

$$\varepsilon_2 = \inf\left\{\int_X (\mathbb{1}_{A_{\frac{1}{3}}} - \mathbb{1}_{U_1})d\mu : \mu \in M(X,T)\right\} > 0.$$

By Lemma 3.3.2 there exists $\delta_2 > 0$ such that for every $\mu \in M(X,T)$ and every $K \in \mathscr{B}(X)$ we have,

$$\operatorname{diam}(K) < \delta_2 \Rightarrow \mu(K) < \varepsilon_2.$$

Find clopen set $A_{\frac{2}{3}} \subsetneq A_{\frac{1}{3}}$ such that

$$x \in A_{\frac{2}{3}} \text{ and } \operatorname{diam}(A_{\frac{2}{3}}) < d_2 = \min\left\{\delta_2, \operatorname{diam}(A_{\frac{1}{3}})\right\}.$$

Thus for all $\mu \in M(X,T)$ we have that

$$\begin{aligned}\mu(A_{\frac{1}{3}} \backslash A_{\frac{2}{3}}) &= \mu(A_{\frac{1}{3}}) - \mu(A_{\frac{2}{3}}) \\ &> \mu(A_{\frac{1}{3}}) - (\mu(A_{\frac{1}{3}}) - \mu(U_1)) \\ &= \mu(U_1).\end{aligned}$$

Applying Lemma 3.3.4 to T^{-1} and U_1, recall by Proposition 3.0.6 we have $M(X,T) = M(X,T^{-1})$, we get $\hat{p}_2 : U_1 \to \mathbb{Z}^+$ such that$(T^{-1})^{\hat{p}_2(\cdot)} : U_1 \to A_{\frac{1}{3}} \backslash A_{\frac{2}{3}}$ is a homeomorphism onto its image in $A_{\frac{1}{3}} \backslash A_{\frac{2}{3}}$.

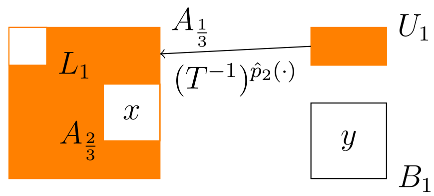

Figure 3.3: Visual of Lemma 3.3.5 (2)

We now use $\hat{p_2}$ to define $p_2 : (T^{-1})^{\hat{p_2}}(U_1) \to \mathbb{Z}^+$ by

$$p_2((T^{-1})^{\hat{p}_2(z)}(z)) = \hat{p}_2(z).$$

Observe, for any $z \in U_1$ we have

$$T^{p_2(z)}(T^{-p_2(z)}(z)) = z.$$

Similarly, the reverse composition is the identity, whence $T^{p_2(\cdot)}$ is not only a bijection but the inverse function to $(T^{-1})^{\hat{p}_2(\cdot)}$, and so is a homeomorphism itself.

This intertwining allows us to map more of A onto B using only positive powers of T and also to ensure that the diameter of B_1 is small. As was the case with p_1, we see that

$$A_{\frac{1}{3}} \backslash T^{-p_2(\cdot)}(U_1) = A_{\frac{2}{3}} \sqcup L_1$$

where L_1 is a clopen subset of $A_{\frac{1}{3}}$ with $A_{\frac{2}{3}}$ and L_1 being disjoint. Again, we have the following equality for every $\mu \in M(X,T)$

$$\mu(B_1) = \mu(A_{\frac{2}{3}}) + \mu(L_1).$$

Thus, by defining

$$A_1 = A_{\frac{2}{3}} \sqcup L_1$$

we have A_1 and B_1 as desired. □

We will use induction on our previous lemma to prove our key Lemma.

Lemma 3.3.6. *Let (X,T) be a minimal Cantor system and let A, B be nonempty disjoint, clopen subsets of X. If for all $\mu \in M(X,T)$, $\mu(A) = \mu(B)$, then there exists $p : A \to \mathbb{Z}^+$ such that $S : A \to B$, defined as $S(x) = T^{p(x)}(x)$, is a homeomorphism onto B.*

Proof. Let A and B be nonempty, disjoint, clopen subsets of X and $x \in A$. Since T is minimal there exists $n \in \mathbb{Z}^+$ such that $T^n(x) \in B$, let $y = T^n(x)$. We will use induction to find a decreasing sequences of sets $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ such that

$$\bigcap_{n\geq 0} A_n = \{x\} \text{ and } \bigcap_{n\geq 0} B_n = \{y\}$$

all while defining S on larger and larger parts of A. Let

$$\varepsilon_1 = \min\{\text{diam}(A), \text{diam}(B), 1\}.$$

Then, using Lemma 3.3.5 find clopen subsets A_1 and B_1 such that

1. $x \in A_1, y \in B_1$

2. $\operatorname{diam}(A_1) < \varepsilon_1, \operatorname{diam}(B_1) < \varepsilon_1$

3. For every $\mu \in M(X,T)$, $\mu(A_1) = \mu(B_1)$ and $\mu(A_1) < \dfrac{\mu(A)}{2}$, $\mu(B_1) < \dfrac{\mu(B)}{2}$.

4. Find $p_1 : A \backslash A_1 \to \mathbb{Z}^+$ such that

$$S_1 = T^{p_1(\cdot)} : A \backslash A_1 \to B \backslash B_1$$

is a homeomorphism.

Having defined $A_n \subseteq A_{n-1}$ and $B_n \subseteq B_{n-1}$ with $x \in A_n$, $y \in B_n$ and $\operatorname{diam}(A_n) < \varepsilon_n$, $\operatorname{diam}(B_n) < \varepsilon_n$ where

$$\varepsilon_n = \min\left\{\operatorname{diam}(A_{n-1}), \operatorname{diam}(B_{n-1}), \frac{1}{n}\right\};$$

we will use Lemma 3.3.4 to define S on more of A. Before doing so, we must check a couple of hypotheses. As a result of the construction for A_n and B_n, we also have for all $\mu \in M(X,T)$,

$$\mu(A_n) = \mu(B_n) \text{ and } \mu(A_n) < \frac{\mu(A_{n-1})}{2},\ \mu(B_{n-1}) < \frac{B_{n-1}}{2}$$

and $p_n : A_{n-1} \backslash A_n \to \mathbb{Z}^+$ such that

$$S_n : A_{n-1} \backslash A_n \to B_{n-1} \backslash B_n$$

is a homeomorphism, use Lemma 3.3.5 with

$$\varepsilon_{n+1} = \min\left\{\operatorname{diam}(A_n), \operatorname{diam}(B_n), \frac{1}{n+1}\right\}$$

to find clopen sets A_{n+1} and B_{n+1} such that

1. $x \in A_{n+1}, y \in B_{n+1}$
2. $\operatorname{diam}(A_{n+1}) < \varepsilon_{n+1}, \operatorname{diam}(B_{n+1}) < \varepsilon_{n+1}$
3. For every $\mu \in M(X,T)$, $\mu(A_{n+1}) = \mu(B_{n+1})$ and
   $\mu(A_{n+1}) < \dfrac{\mu(A_n)}{2}$, $\mu(B_{n+1}) < \dfrac{\mu(B_n)}{2}$.
4. Find $p_{n+1} : A_n \backslash A_{n+1} \to \mathbb{Z}^+$ such that

$$S_{n+1} = T^{p_{n+1}(\cdot)} : A_n \backslash A_{n+1} \to B_n \backslash B_{n+1}$$

   is a homeomorphism.

Therefore, by induction we have defined $p : A\backslash\{x\}$ by taking

$$p(x) = p_n(x)$$

where $x \in A_n \backslash A_{n+1}$. Moreover, we observe at this point $T^{p(\cdot)} : A\backslash\{x\} \to B\backslash\{y\}$ is a homeomorphism. We extend p to all of A by defining $p(x) = n$. Consequently, $T^{p(\cdot)}$ is a bijection on A.

All that is left to show is that $T^{p(\cdot)}$ is continuous on A. By construction $T^{p(\cdot)}$ is continuous at all points in A less our exceptional point x. Let $\varepsilon > 0$, then by the construction there exists an n such that $B_n \subseteq B_\varepsilon(y)$; thus

$T^{p(\cdot)}(A_{n+1}) \subseteq B_n$, as $T^{p(x)}(x) = y$. Hence, taking $\delta > 0$ such that the ball of radius δ about x, $B_\delta(x) \subseteq A_{n+1}$, we have that $T^{p(\cdot)}$ is continuous at x. Hence, $T^{p(\cdot)}$ is continuous on all of A. Therefore, we have defined p in such a way that the map $T^{p(\cdot)} : A \to B$ is a homeomorphism as desired. $\square$

We immediately use the above lemma and Proposition 3.0.7 to prove the following lemma.

Lemma 3.3.7. *Let (X, T) be a minimal Cantor system and $A, B \subseteq X$ nonempty clopen subsets such that $A \cap B = \emptyset$. If for all $\mu \in M(X, T)$, $\mu(A) = \mu(B)$, then for any clopen partition of A, say $A = \bigsqcup_{i=1}^{n} A_i$, there exists clopen sets $B_i \subseteq B$ with $B = \bigsqcup_{i=1}^{n} B_i$ such that for all $\mu \in M(X, T)$ and for each i we have*

$$\mu(A_i) = \mu(B_i)$$

We will use this lemma in the proof of the main theorem which is soon to follow. We will make use of the following definition due to Dahl.

Definition 3.3.8 (H. Dahl)**.** *Let $K \subseteq M(X)$, where X is a Cantor set, be a Choquet simplex consisting of non-atomic, Borel, probability measures. We say that K is a* ***dynamical simplex (D-simplex)*** *if it satisfies the following two conditions:*

1. *For clopen subsets A and B of X with $\mu(A) < \mu(B)$ for all $\mu \in K$, there exists a clopen subset $B_1 \subseteq B$ such that $\mu(A) = \mu(B_1)$ for all $\mu \in K$.*
2. *If $\nu, \mu \in \partial_e K$, $\nu \neq \mu$, then μ and ν are mutually singular.*

It is well know that condition (2) is satisfied by every $M(X,T)$ for any continuous map on a compact metric space X. Furthermore, thanks to Glasner and Weiss we have the following theorem.

Theorem 3.3.9 ([GW '95] Lemma 2.5)**.** *Let (X,T) be a minimal Cantor system and $M(X,T)$ its associated Choquet simplex of $T-$invariant measures. Then $M(X,T)$ is a $D-$simplex.*

Theorem 3.3.9 becomes useful in construction of the speedup which proves (3) $\Rightarrow$ (1). We have enough background to finish the proof of the main theorem. We recall the final portion of the main theorem we have left to prove.

Theorem 3.3.10. *Let (X_1,T_1) and (X_2,T_2) minimal Cantor systems. If there exists a homeomorphism $F : X_1 \to X_2$ such that $F_* : M(X_1,T_1) \hookrightarrow M(X_2,T_2)$ is an injection, then (X_2,T_2) is a speedup of (X_1,T_1).*

Proof. We begin with a sketch of the proof to keep in mind. The idea of the construction is to take a refining sequence of Kakutani-Rokhlin tower partitions in X_2 and copy them in X_1 using the homeomorphism F^{-1}. We observe for any fixed tower in X_2, its copy in X_1 has the property that all levels in this tower have the same measure for every T_1-invariant measure. Now using Lemma 3.3.6 we can define the speedup on all non-final levels of the tower. Then we define a set conjugacy from one tower to another. We simply iterate this process refining each previous tower. We have a great deal of freedom in this construction, enough to ensure the base and tops of the towers converge to prespecified singletons, say x and $T_1^{-1}x$, and that the sequence of towers generates the topology on X_1.

We begin by fixing $x_0 \in X_1$ and let $\{A_n\}_{n\geq 0}$ be a nested sequence of clopen sets, where

$$\bigcap_{n\geq 0} A_n = \{x_0\}.$$

For each n let $Z_n = T_1^{-1}(A_n)$ and thus,

$$\bigcap_{n\geq 0} Z_n = \{T^{-1}x_0\}.$$

We may assume with no loss of generality that $A_0 \cap Z_0 = \emptyset$. Moreover, observe for every $\mu \in M(X_1, T_1)$ and every n, $\mu(A_n) = \mu(Z_n)$. That being said, let α_0 be defined below,

$$\alpha_0 = \min\left\{\int_X \mathbb{1}_{A_0}\, d\mu : \mu \in M(X_1, T_1)\right\}.$$

Coupling the fact that $M(X_1, T_1)$ is compact in the weak* topology and both A_0 is clopen, we may conclude $\alpha_0 > 0$: let $\varepsilon_0 = \alpha_0$. Apply Theorem 2.0.11 to create $\{\mathcal{Q}(n)\}_{n\geq 0}$, a refining sequence of Kakutani-Rohklin tower partitions of X_2. Specifically,

$$\mathcal{Q}(n) = \{T_2^j(B_i(n)) : 1 \leq i \leq t(n), 0 \leq j < h_i(n)\}$$

where $t(n)$ represents the total number of columns and $h_i(n)$ represents the height of the i^{th} column in the n^{th} Kakutani-Rokhlin tower partition of X_2. Furthermore, $\{\mathcal{Q}(n)\}_{n\geq 0}$ has the following three properties:

1. $\displaystyle\bigcap_{n\in\mathbb{N}}\left(\bigcup_{1\leq i\leq t(n)} B_i(n)\right) = \{y\}$

2. For every n we have $\mathcal{Q}(n+1)$ is finer than $\mathcal{Q}(n)$.

3. $\bigcup_{n\in\mathbb{N}} \mathcal{Q}(n)$ generates the topology of X_2.

Let $\{\mathcal{P}(n)\}_{n\geq 0}$ be a sequence of finite clopen partitions which generates the topology on X_1. Use Lemma 3.3.2 with respect to ε_0 and obtain a $\delta_0 > 0$ such that for every $K \in \mathscr{B}(X_2)$ with $\operatorname{diam}(K) < \delta_0$ we have for every $\nu \in M(X_2, T_2)$, $\nu(K) < \varepsilon_0$. Since

$$\bigcap_{n\geq 0} \left(\bigcup_{1\leq i\leq t(n)} B_i(n) \right) = \{y\}$$

there exists an n_0 such that

$$\operatorname{diam} \left(\bigcup_{1\leq i\leq t(n_0)} B_i(n_0) \right) < \delta_0.$$

Thus, for every $\nu \in M(X_2, T_2)$ we have that

$$\nu \left(\bigcup_{1\leq i\leq t(n_0)} B_i(n_0) \right) < \varepsilon_0.$$

Below we give a picture of (X_2, T_2) partitioned into $\mathcal{Q}(n_0)$. We will use F^{-1} to copy $\mathcal{Q}(n_0)$ into X_1.

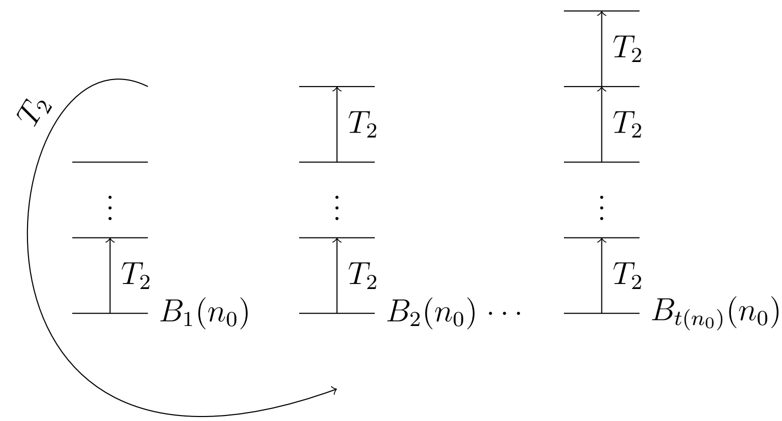

Figure 3.4: Towers to Copy

Define for $1 \leq i \leq t(n_0)$ and $0 \leq j < h_i(n_0)$

$$C'(i,j) = F^{-1}(T_2^j(B_i(n_0))).$$

We will make a series of alterations to each $C'(i,j)$ resulting in $C(i,j)$ with

$$\mu(C'(i,j)) = \mu(C(i,j))$$

for all $\mu \in M(X_1, T_1)$. Furthermore, this will be done iteratively and once completed we will have the following

$$x \in \bigcup_{i=1}^{t(n_0)} C(i,0) \subseteq A_0 \quad \text{and} \quad T_1^{-1}x \in \bigcup_{i=1}^{t(n_0)} C(i, h_i(n_0)-1) \subseteq Z_0.$$

First ensure $C(i,0) \subseteq A_0$ for each i.

Recall that for every $\mu \in M(X_1, T_1)$

$$
\begin{aligned}
\mu\left(\bigsqcup_{i=1}^{t(n_0)} C'(i,0)\right) &= \sum_{i=1}^{t(n_0)} \mu(C'(i,0)) \\
&= \sum_{i=1}^{t(n_0)} \mu(F^{-1}B_i(n_0)) \\
&= \sum_{i=1}^{t(n_0)} \nu_\mu(B_i(n_0)) < \varepsilon_0 \leq \mu(A_0).
\end{aligned}
$$

In particular, for every $\mu \in M(X_1, T_1)$ we have

$$\mu(A_0) - \sum_{i=1}^{t(n_0)} \mu(C'(i,0)) > 0. \tag{3.3.3}$$

Define

$$D_0(i) = C'(i,0) \cap A_0 \text{ and } D_0'(i) = C'(i,0) \cap A_0^c$$

and by the above we have for every $\mu \in M(X_1, T_1)$

$$\sum_{i=1}^{t(n_0)} \mu(D_0'(i)) < \mu\left(A_0 \backslash \bigsqcup_{i=1}^{t(n_0)} D_0(i)\right). \tag{3.3.4}$$

Fix $i = 1$. It may be the case that $D_0'(1) \neq \emptyset$ and in this case we wish to amend this, and to do it in a way which preserves all the measures of each clopen set $C'(i,j)$. We know from (3.3.4) above that for $\mu \in M(X_1, T_1)$

$$\mu(D_0'(1)) < \mu\left(A_0 \backslash \bigsqcup_{i=1}^{t(n_0)} D_0(i)\right).$$

Thus, as $M(X_1, T_1)$ is a D-simplex, there exists $C_1 \subseteq A_0 \backslash \bigsqcup_{i=1}^{t(n_0)} D_0(i)$ clopen such that for every $\mu \in M(X_1, T_1)$, $\mu(C_1) = \mu(D_0'(1))$. Note that C_1 is partitioned by

$$\bigsqcup_{i=1}^{t(n_0)} \bigsqcup_{j=0}^{h_i(n_0)-1} C'(i,j)$$

into

$$C_1 = \bigsqcup_{k=1}^{m} C_{i_k,j_k}(1),$$

where $C_{i_k,j_k}(1) \subseteq C'(i_k, j_k)$. Hence, by Lemma 3.3.7 there exists a partition of $D_0'(1)$,

$$D_0'(1) = \bigsqcup_{k=1}^{m} D_{i_k,j_k}(1)$$

where for all $\mu \in M(X_1, T_1)$ and each $k = 1, 2, \ldots, m$

$$\mu(C_{i_k,j_k}(1)) = \mu(D_{i_k,j_k}(1)).$$

Define

$$C(1,0) = D_0(1) \sqcup C_1$$

and for each $k = 1, 2, \ldots, m$

$$C''(i_k, j_k) = C'(i_k, j_k) \backslash C_{i_k,j_k}(1) \sqcup D_{i_k,j_k}(1).$$

Observe, for every $\mu \in M(X_1, T_1)$ and $k = 1, 2, \ldots, m$ we have

$$\mu(C(1,0)) = \mu(C'(1,0)) \quad \mu(C''(i_k, j_k)) = \mu(C'(i_k, j_k)),$$

and of course all the measure of the unaffected $C'(i,j)$ still have the same measure for each $\mu \in M(X_1, T_1)$. Combining (3.3.3) and (3.3.4) from above reveals

$$\sum_{i=2}^{t(n_0)} \mu(D_0'(i)) < \mu\left(A_0 \setminus \bigsqcup_{i=2}^{t(n_0)} D_0(i) \sqcup C(1,0) \right).$$

We now simply repeat the above argument. Inequalities (3.3.3) and (3.3.4) allow us to do this construction for each $i = 1, 2, \ldots, t(n_0)$ defining $C(i,0)$ for $i = 1, 2, \ldots, t(n_0)$. Furthermore, by construction we have the following two properties

1. For every $\mu \in M(X_1, T_1)$ and every i, $\mu(C(i,0)) = \mu(C'(i,0))$.

2. For every $i \neq j$, $C(i,0) \cap C(j,0) = \emptyset$.

Second, ensure $x \in \bigcup_{i=1}^{t(n_0)} C(i,0)$**.**

In adjusting to construct $C(i,0)$, $i = 1, 2, \ldots, t(n_0)$, we may not have captured x. If not, then

$$x \in A_0 \setminus \bigsqcup_{i=1}^{n_0} C(i,0).$$

Use Proposition 3.3.2 and find a small clopen subset of $A_0 \setminus \bigsqcup_{i=1}^{n_0} C(i,0)$ containing x and exchange it with part of $C(1,0)$.

Third, repeat steps one and two to obtain $C(i, h_i(n_0) - 1)$, $i = 1, 2, \ldots, t(n_0)$**.**

Notice that $\varepsilon_0 = \min\{\alpha_0, \zeta_0\}$, so we can repeat the above two steps using the same sort of calculations and construction to obtain clopen sets

$$C(i, h_i(n_0) - 1),\ i = 1, 2, \ldots, t(n_0)$$

such that

$$T_1^{-1}x \in \bigsqcup_{i=1}^{t(n_0)} C(i, h_i(n_0) - 1) \subseteq Z_0.$$

Furthermore, for every $\mu \in M(X_1, T_1)$

$$\mu(C'(i, h_i(n_0) - 1)) = \mu(C(i, h_i(n_0) - 1)).$$

Having defined $C(i, 0)$ and $C(i, h_i(n_0) - 1)$ for $i = 1, 2, \ldots, m$ we wish to keep consistent notation and thus rename any $C'(i, j)$ to simply $C(i, j)$. Whence, we have the following:

- $x \in \bigsqcup_{i=1}^{t(n_0)} C(i, 0) \subseteq A_0$
- $T_1^{-1}x \in \bigsqcup_{i=1}^{t(n_0)} C(i, h_i(n_0) - 1) \subseteq Z_0$
- For every $\mu \in M(X_1, T_1)$ and fixed $i = 1, 2, \ldots, t(n_0)$ we have

$$\mu(C(i, 0)) = \mu(C(i, 1)) = \cdots = \mu(C(i, h_i(n_0) - 1)).$$

Now, we use repeated applications of our key lemma, Lemma 3.3.6, to define our speedup on nearly all of X_1. Specifically,

$$S(C(i,j)) = C(i,j+1)$$

for each $1 \le i \le t(n_0)$ and $0 \le j < h_i(n_0) - 2$. Thus S is defined on $X_1 \backslash \left(\bigsqcup_{i=1}^{t(n_0)} C(i, h_i(n_0) - 2) \right)$.

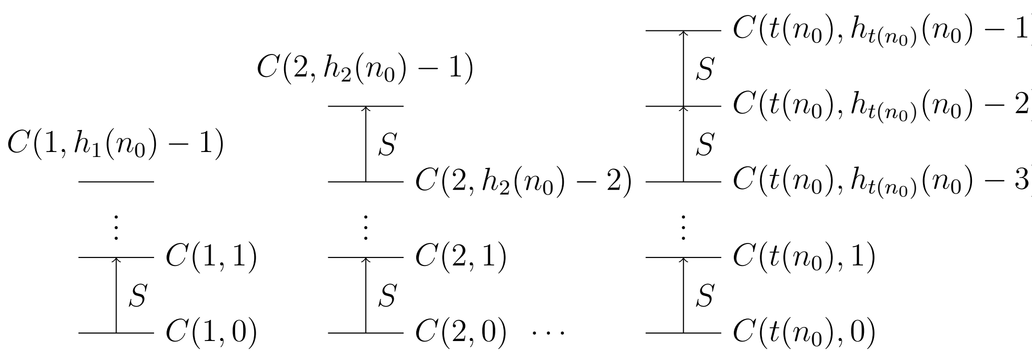

Figure 3.5: Defining S Towers

Formally, let

$$\mathscr{P}'(0) = \{S^j(C(i,0)) : 1 \le i \le t(n_0),\ 0 \le j < h_i(n_0) - 1\}$$

where $S^j(C(i,0)) = C(i,j)$. Refine $\mathscr{P}'(0)$ with respect to each clopen set in $\mathcal{P}$ as in Proposition 2.0.10, thus preserving the tower structure, and call the result $\mathscr{P}(0)$. So X_1 now looks like

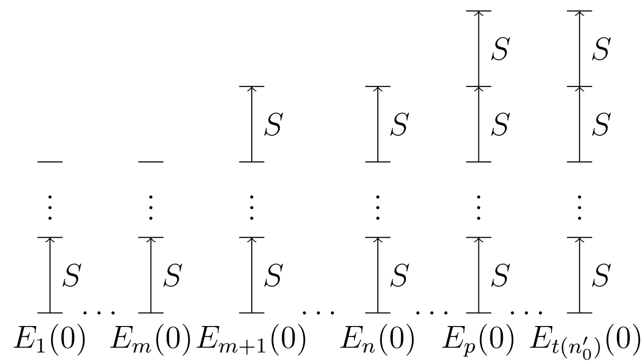

Figure 3.6: Split S Towers

where for each i

$$C(i,0) = \bigsqcup_{j=k_i}^{k_{i+1}-1} E_j(0) \text{ and}$$

$$\mathscr{P}(0) = \{S^j E_i(0) : 1 \le i \le t'(n_0), 0 \le j < h_i'(n_0) - 1\}$$

where $t'(n_0)$ is the new number of base levels and $h_i'(n_0)$ gives the height of the respective column. Because $\mu(C(i,0)) = \mu(C'(i,0))$ for all $\mu \in M(X_1, T_1)$, $F : X_1 \to X_2$ is a homeomorphism and through the use of Lemma 3.3.7 we can refine $\mathscr{Q}(n_0)$, our tower partition in X_2 to look exactly like $\mathscr{P}(0)$. That is, there are sets $B_j'(0)$ such that

$$\mu \circ F^{-1}(B_j'(0)) = \mu(E_j(0)) \text{ and } B_i(n_0) = \bigsqcup_{\ell=k_i}^{k_{i+1}-1} B_\ell'(0)$$

and set

$$\mathscr{Q}'(n_0) = \{T_2^j B_\ell'(0) : 1 \le \ell \le t'(n_0), 0 \le j < h_i'(n_0)\}.$$

Hence, X_2 looks like Figure 3.7.

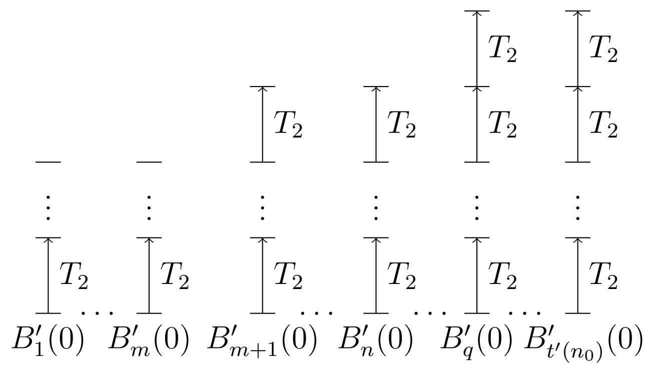

Figure 3.7: Splitting T_2 Towers

Define a map on the level of sets, which in the limit will give us our conjugacy. Define $\Phi_0 : \mathscr{P}(0) \to \mathcal{Q}'(n_0)$ by

$$\Phi_0(S^j(E_i(0))) = T_2^j(B'_i(0)).$$

We have now completed the first step of our construction!

Inductive step

We now move onto the second (inductive) step of our construction. Let $\varepsilon_1 = \min\{\alpha_1, \rho_0\}$ where

$$\alpha_1 = \min\left\{\int_X \mathbb{1}_{A_1}\, d\mu : \mu \in M(X_1, T_1)\right\} > 0$$
$$\rho_0 = \min_{1\le i\le t'(n_0)}\left\{\int_X \mathbb{1}_{E_i(0)}\, d\mu : \mu \in M(X_1, T_1)\right\} > 0$$

and find $n_1 > n_0$ large enough such that the following are true:

1. For every $\nu \in M(X_2, T_2)$

$$\nu\left(\bigcup_{i=1}^{t(n_1)} B_i(n_1)\right) < \varepsilon_1.$$

2. $\mathcal{Q}(n_1)$ refines $\mathcal{Q}'(n_0)$ i.e. $\mathcal{Q}(n_1) \geq \mathcal{Q}'(n_0)$.

Now as $\mathcal{Q}(n_1) \geq \mathcal{Q}'(n_0)$ we see that each column in $\mathcal{Q}(n_1)$ is simply made up of stacking towers from $\mathcal{Q}'(n_0)$ upon one another. So we view $\mathcal{Q}(n_1)$ not only as a space time partition, but also as a labeled or tagged partition by the previous tower construction, in this case tagged by the towers of $\mathcal{Q}'(n_0)$. We give a picture as an illustrative example of the tagging or labeling of the towers.

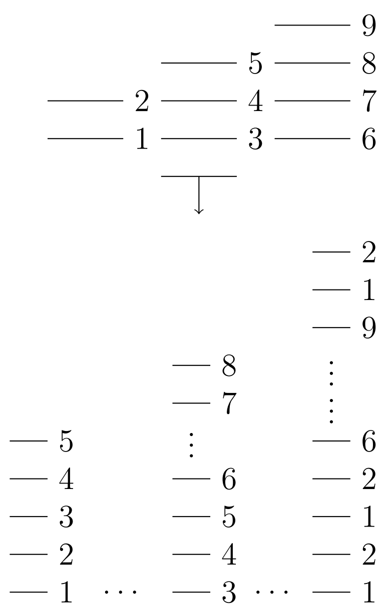

Figure 3.8: Example of Tower Labeling

As a consequence of $\mathcal{Q}(n_1) \geq \mathcal{Q}'(n_0)$, we see that for $1 \leq i \leq t'(n_0)$ and $0 \leq j < h_i'(n_0)$ we have

$$T_2^j(B_i'(0)) = \bigsqcup_{a=1}^{m} T_2^{j_a}(B_{i_a}(n_1))$$

and thus it follows that

$$F^{-1}(T_2^j(B_i'(0))) = \bigsqcup_{a=1}^{m} F^{-1}(T_2^{j_a}(B_{i_a}(n_1))).$$

As a result of Lemma 3.3.7 we can write

$$S^j(E_i(0)) = \bigsqcup_{a=1}^{m} E_{(i,a)}(1).$$

Using Lemma 3.3.7 on each copied tower of X_2 in X_1, we can copy $\mathcal{Q}(n_1)$ in X in a way which refines our $\mathscr{P}(0)$: call this collection $\mathscr{P}'(1)$. Recall, we have already defined S on a large portion of X_1 and we do not need, nor want, to be redefining S on this portion of the space. Following the tagging from $\mathcal{Q}(n_1)$, extend S on any and all previous undefined pieces, save for the top levels of each column. As before, using Proposition 2.0.10 refine $\mathscr{P}'(1)$ with respect to each clopen set in $\mathcal{P}(1)$ and call $\mathscr{P}(1)$ the result of this refinement. Specifically,

$$\mathscr{P}(1) = \{S^j(E_i(1)) : 1 \leq i \leq t'(n_1), 0 \leq j < h_i'(n_1) - 1\}.$$

Use Lemma 3.3.7 and F to push this refinement onto $\mathcal{Q}(n_1)$, resulting in

$$\mathcal{Q}'(n_1) = \{T_2^j(B_i'(1)) : 1 \leq i \leq t'(n_1), 0 \leq j < h_i'(n_1) - 1\}.$$

As before we define $\Phi_1 : \mathscr{P}(1) \to \mathcal{Q}'(n_1)$ by

$$\Phi_1(S^j(E_i(1))) = T_2^j(B_i'(1)),$$

and by construction Φ_1 extends Φ_0. We continue this process and thus by induction we see that we will have defined $S : X_1 \backslash \{T_1^{-1}x\} \to X_1 \backslash \{x\}$. By construction, S is a homeomorphism and so by defining $p(T_1^{-1}x) = 1$ we see that S now lifts to a homeomorphism on all of X_1. Furthermore, $\{\Phi_n\}_{n \geq 0}$ induces, by way of intersection, a point map $\varphi : X_1 \to X_2$, which is our conjugacy from (X_1, S) onto (X_2, T_2). The fact φ is well defined and a homeomorphism is due to both $\{\mathscr{P}(k)\}_{k \geq 0}$ and $\{\mathcal{Q}'(n_k)\}_{k \geq 0}$ being generating for the topology of X_1 and X_2 respectively. Moreover, φ conjugates S and T_2 is built into the definition of each Φ_n and each Φ_{n+1} extends the previous Φ_n. Therefore, our theorem as been proved. □

3.4 Speedup Equivalence

We wish to view speedups as a relation; and, to that end, it will be helpful to introduce some notation. Let (X_i, T_i), $i = 1, 2$ be minimal Cantor systems and write $T_1 \rightsquigarrow T_2$ to mean that (X_2, T_2) is a speedup of (X_1, T_1). Moreover, define (X_1, T_1) and (X_2, T_2) to be ***speedup equivalent***, written $T_1 \leftrightsquigarrow T_2$, if and only if $T_1 \rightsquigarrow T_2$ and $T_2 \rightsquigarrow T_1$. It is straight forward to verify that speedup equivalence is indeed an equivalence relation. Combining [GPS '95, Thm 2.2] with our main theorem we obtain the following corollary.

Corollary 1. *Let* (X_1, T_1) *and* (X_2, T_2) *be minimal Cantor systems. If* (X_1, T_1) *and* (X_2, T_2) *are orbit equivalent, then* (X_1, T_1) *and* (X_2, T_2) *are speedup equivalent (i.e.* $T_1 \leftrightsquigarrow T_2$*).*

Rephrasing Corollary 1 above, as equivalence relations, orbit equivalence is contained in speedup equivalence. This leads us to a fundamental question: are orbit equivalence and speedup equivalence the same equivalence relation? At this time, we only have the partial answer in the form of the Theorem 3.4.2. However, before we can prove the aforementioned theorem, we need a proposition for which the proof is straight forward and hence omitted.

Proposition 3.4.1. *Let* (X_i, T_i) *be minimal Cantor systems and* $\varphi : X_1 \to X_2$ *be a homeomorphism. If* $\varphi_* : M(X_1, T_1) \hookrightarrow M(X_2, T_2)$ *is an injection, then* φ_* *preserves pairs of mutually singular measures.*

We now use Proposition 3.4.1 to prove the following theorem.

Theorem 3.4.2. *Let* (X_i, T_i)*,* $i = 1, 2$*, be minimal Cantor systems each with finitely many ergodic measures. If* (X_1, T_1) *and* (X_2, T_2) *are speedup equivalent, then* (X_1, T_1) *and* (X_2, T_2) *are orbit equivalent.*

Proof. Since $T_1 \leftrightsquigarrow T_2$, combining part (3) of Theorem 3.0.8 and Proposition 3.4.1 it follows immediately that

$$|\partial_e(M(X_1, T_1))| = |\partial_e(M(X_2, T_2))|$$

and without loss of generality we may assume $|\partial_e(M(X_1, T_1))| = n$ for some $n \in \mathbb{Z}^+$. Every measure μ in $M(X_2, T_2)$ is a convex combination of ergodic measures in a unique way. With this in mind, for $\mu \in M(X_2, T_2)$ let $E(\mu)$ denote the collection of all ergodic measures of $M(X_2, T_2)$ which have a positive coefficient in the unique ergodic decomposition of μ. Observe if $\mu_1, \mu_2 \in M(X_2, T_2)$ with $\mu_1 \neq \mu_2$ and $\mu_1 \perp \mu_2$, then

$$E(\mu_1) \cap E(\mu_2) = \emptyset.$$

Now as $T_1 \rightsquigarrow T_2$ there exists $\varphi : X_1 \to X_2$, a homeomorphism, such that

$$\varphi_* : M(X_1, T_1) \hookrightarrow M(X_2, T_2)$$

is an injection. Since $|\partial_e(M(X_1, T_1))| = |\partial_e(M(X_2, T_2))| = n$ and φ_* is injective, we see that $\{E(\varphi_*(\nu_i))\}_{i=1}^n$, where $\{\nu_i\}_{i=1}^n = \partial_e(M(X_1, T_1))$, is a collection of n pairwise disjoint sets, as distinct ergodic measures are mutually singular. It follows that for each $i = 1, 2, \ldots, n$, $E(\nu_i)$ is a distinct singleton, and therefore $\varphi_*(\partial_e(M(X_1, T_1))) = \partial_e(M(X_2, T_2))$. Coupling the facts that φ_* is an affine map and a bijection on extreme points, we may conclude that φ_* is a bijection, and hence is an affine homeomorphism between $M(X_1, T_1)$ and $M(X_2, T_2)$ arising from a space homeomorphism. Therefore, by [GPS '95, Thm. 2.2] (X_1, T_1) and (X_2, T_2) are orbit equivalent. $\square$

There are two obstacles which arise when trying to extend Theorem 3.4.2 to the infinite dimensional case. The first is whether or not it is always true that

$$\varphi_*(\partial_e(M(X_1, T_1))) \subseteq \partial_e(M(X_2, T_2))$$

whenever $T_1 \rightsquigarrow T_2$. The second is whether the Schröder-Bernstein Theorem holds in the category of simple dimension groups with our morphisms. The Schröder-Bernstein Theorem for dimension groups and simple dimension groups was addressed in the Glasner and Weiss paper [GW '95], which we discuss below.

We must remark that speedup equivalence looks quite similar to weak orbit equivalence, especially in terms of weakly isomorphic dimension groups. Observe that we have surjective homomorphisms, and the key difference is that we require our homomorphism to exhaust the positive cone in the image space. We mention this here because one avenue to try to answer the speedup equivalence question would be to show that given two dimension groups $(G_1, G_1^+, \mathbf{1})$ and $(G_2, G_2^+, \mathbf{1})$ with surjective group homomorphisms φ_1, φ_2 satisfying

$$\varphi_1 : G_1 \to G_2 \text{ and } \varphi_1(G_1^+) = G_2^+,\ \varphi_1(\mathbf{1}) = \mathbf{1}$$
$$\varphi_2 : G_2 \to G_1 \text{ and } \varphi_2(G_2^+) = G_1^+,\ \varphi_2(\mathbf{1}) = \mathbf{1},$$

then in fact $(G_1, G_1^+, \mathbf{1}) \cong (G_2, G_2^+, \mathbf{1})$. However, Glasner and Weiss, in [GW '95], gave a beautiful counter example, [GW '95, Example 4.2], which shows that even if $Inf\, G = 0$ the Schröeder-Bernstein Theorem fails for simple dimension groups. Unfortunately, their example fails to exhaust the positive cone. Since this cannot happen with speedups, this example would need some modification to apply.

3.5 Example

We will now use the main theorem, Theorem 3.0.8, to show the aforementioned claim: speedups can leave the conjugacy class and even the orbit equivalence class of the original system. This will be demonstrated by showing that the simplex of invariant Borel probability measure can grow: see Proposition 3.0.7. To do so, we will use Theorem 3.0.8 in conjunction with Theorem 2.3.2, which recall says that states and invariant measures are in bijective correspondence.

Let (X, T) be the dyadic odometer. It is well known that the dimension group associated to this system is $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, \mathbf{1})$. Since (X, T) is uniquely ergodic, by Proposition 2.3.2 it follows that $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, \mathbf{1})$ has only one state. Our goal is to construct a simple dimension group with two states, such that it factors onto $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, \mathbf{1})$ in the sense of the main theorem. One can show that the following is a dimension group

$$(\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$$

where

$$\mathbb{Z}[\tfrac{1}{2}]^{++} = \{x \in \mathbb{Z}[\tfrac{1}{2}] : x > 0\}.$$

Note, the Riesz interpolation property is satisfied as $\mathbb{Z}[\frac{1}{2}]$ is a totally ordered set. To see that $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$ is a simple dimension group we use the following lemma from [G].

Lemma 3.5.1 ([G] Lemma 14.1)**.** *Let G be a nonzero directed Abelian group. Then G is simple if and only if every nonzero element of G^+ is an order-unit in G.*

An ordered group G is *directed* if all elements of G have the form $x - y$ for some $x, y \in G^+$ and so any dimension group is directed. One can use Lemma 3.5.1 to show that $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$ is indeed a simple dimension group. Now by Theorem 2.3.1 there exists minimal Cantor system (X_2, T_2) such that

$$(K^0(X_2,T_2), K^0(X_2,T_2)^+, \mathbf{1}) \cong (\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1)).$$

In addition, one can verify that

$$\pi_1 : (\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1)) \to (\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^+, 1)$$

satisfies condition (2) of the main theorem, Theorem 3.0.8, whence (X_2, T_2) is a speedup of (X, T) the dyadic odometer.

We will now show (X_2, T_2) and (X, T) are not conjugate to one another, hence speedups can leave their conjugacy classes. Furthermore, we will actually show that (X_2, T_2) and (X, T) cannot even be orbit equivalent. To see this, it suffices to show, by [GPS '95, Theorem 2.2], that their respective dimension groups modulo infinitesimals are not isomorphic as dimension groups. To accomplish this, we will use states and show $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$ and $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, 1)$ have different state spaces. Recall all states can be realized as integration against invariant probability measures, hence as (X, T) is uniquely ergodic it has exactly one state, namely the identity map. Thus, it suffices to show that (X_2, T_2) has more than one invariant measure, or more to the point, that $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$ has more than one state.

Before we begin we must deal with one technical aspect, that is, we know that $(K^0(X_2,T_2), K^0(X_2,T_2)^+, \mathbf{1})$ is isomorphic as a dimension group to $(\mathbb{Z}[\frac{1}{2}] \oplus$

$\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$, so we must show that this group has only trivial infinitesimals. Recall infinitesimals evaluate to 0 for every state on the dimension group, and

$$\pi_i : (\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1)) \to \mathbb{R}$$

$i = 1, 2$ are states. From this we can deduce that the only infinitesimal of $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$ is $(0,0)$, hence

$$(K^0(X_2,T_2)/Inf(K^0(X_2,T_2)), K^0(X_2,T_2)^+/Inf(K^0(X_2,T_2)), \mathbf{1}) \cong$$
$$(\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$$

as dimension groups. Furthermore, as $\pi_1 \neq \pi_2$, on $(\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^{++} \oplus \mathbb{Z}[\frac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$, we have

$$(\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{+}, 1) \ncong (\mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}[\tfrac{1}{2}]^{++} \oplus \mathbb{Z}[\tfrac{1}{2}]^{++} \cup \{(0,0)\}, (1,1))$$

and so (X,T) and (X_2,T_2) are not orbit equivalent, hence not conjugate.

Chapter 4

Bounded Speedups and Entropy

4.1 Bounded Speedups

In this chapter we will look at the properties of speedups, $T^{p(\cdot)}$, when p is bounded function. We begin with a definition.

Definition 4.1.1. *Let (X_1, T_1) and (X_2, T_2) be minimal Cantor systems. We say (X_2, T_2) is a* **bounded speedup** *of (X_1, T_1) if (X_2, T_2) is conjugate to (X_1, S) where S is a minimal homeomorphism of X defined by*

$$S(x) = T_1^{p(x)}(x)$$

where $p : X \to \mathbb{Z}^+$ is a bounded function.

We will show, in the case of the bounded speedups, that p has a tremendous amount of structure.

Proposition 4.1.2. *Let (X,T) be a minimal Cantor system and (X,S) a bounded speedup of (X,T) with $p : X \to \mathbb{Z}^+$. Then p is bounded if and only if p is continuous.*

Proof. The converse is obvious and thus we only show the necessary condition. Suppose p is bounded, thus

$$p(X) = \{z_1, \ldots, z_n\}$$

for some $z_1, \ldots, z_n \in \mathbb{Z}^+$. For each $i = 1, \ldots, n$, $p^{-1}(\{z_i\})$ is closed, from Proposition 3.0.4. We will show that each preimage is clopen, hence making p continuous. Fix $i = 1, \ldots, n$ and please observe,

$$(p^{-1}(\{z_i\}))^c = \bigsqcup_{j \neq i} p^{-1}(\{z_j\})$$

is a closed set. Thus for every $i = 1, \ldots, n$, $p^{-1}(\{z_i\})$ is clopen, and therefore p is continuous. □

Now, that we see, in the case of bounded speedups, that p is a continuous function. We now show two structural theorems about p.

Lemma 4.1.3. *Let (X,T) be a minimal Cantor system, and (X,S) a bounded speedup of (X,T) with $p : X \to \mathbb{Z}^+$. There exists $k \in \mathbb{Z}^+$ such that for every $x \in X$*

$$\mathcal{O}_T(x) = \bigsqcup_{j=1}^{k} \mathcal{O}_S(x_j).$$

We will call k the ***orbit number*** *for $T \rightsquigarrow S$.*

Proof. Let $x \in X$ be given and consider the T-orbit block of length $N = \max p$:

$$\mathcal{O}_T(x, N) = \{x, Tx, \dots T^{N-1}x\}.$$

By construction, $S(x) \in \mathcal{O}_T(x, N+1)$ thus

$$\mathcal{O}_T(x, N+1) \subset \bigsqcup_{i=1}^{k} \mathcal{O}_S(x_i)$$

for appropriately chosen x_i in $\mathcal{O}_T(x, N+1)$ and for some $k \leq N$. Further, for any $x' \in \mathcal{O}_T(x)$

$$\mathcal{O}_S(x') \cap \mathcal{O}_T(x, N+1) \neq \emptyset$$

as S cannot skip over this block, thus

$$\mathcal{O}_T(x) = \bigsqcup_{i=1}^{k} \mathcal{O}_S(x_i).$$

We will now show the T-orbit of every point in X decomposes into k many S-suborbits. Let $y \in X$ and let $\delta > 0$ be the Lebesgue number for the clopen partition $p^{-1}(\mathbb{Z}^+)$. As T is uniformly continuous there is $\delta_T > 0$ such that for any $z_1, z_2 \in X$ with $d(z_1, z_2) < \delta_T$ we have

$$\max_{0 \leq i \leq N} \{d(T^i z_1, T^i z_2)\} < \delta.$$

By the minimality of T there exists $m \in \mathbb{Z}^+$ such that $T^m x \in B_{\delta_T}(y)$. Then, by construction $\mathcal{O}_T(y, N+1)$ decomposes into the same k S-suborbits as $\mathcal{O}_T(T^m x, N+1)$, and by the previous argument for x we are done. $\square$

Before we can show the last structure theorem for p we will need the following theorem due to Gottschalk and Hedlund.

Theorem 4.1.4 (Gottschalk & Hedlund). *Let T be a minimal transformation of the compact metric space X, and $g \in C(X)$. The following are equivalent:*

1. *$g = f - f \circ T$, for some $f \in C(X)$.*
2. *There exists $x_0 \in X$ for which*

$$\sup_n \left| \sum_{j=0}^{n-1} g \circ T^j(x_0) \right| < \infty.$$

Finally, we will use the above lemma in conjunction with Lemma 4.1.3 to show that p "almost" a constant function.

Lemma 4.1.5. *Let (X, T) be a minimal Cantor system and let (X, S) be a bounded speedup of (X, T) with $p : X \to \mathbb{Z}^+$. Let k be the orbit number for $T \rightsquigarrow S$, then*

$$p(x) = k + (f(x) - (f \circ T)(x))$$

for some $f \in C(X, \mathbb{Z})$.

Proof. To show that $p(x) = k + (f(x) - (f \circ T)(x))$ we will use Theorem 4.1.4 and show there exists $x \in X$ such that

$$\sup_n \left| \sum_{i=0}^{n} (p - k) \circ T^j(x) \right| < \infty.$$

Let $x \in X$ be given, suppose $n > N = \max p$, and let k be such that $\mathcal{O}_T(x) = \sqcup_{i=1}^{k} \mathcal{O}_S(x_i)$. Define

$$j_i = \min \left\{ m \in \mathbb{Z}^+ : \sum_{j=0}^{m-1} p(S^j(x_i)) > n \right\},$$

That is, how long x stays in the S-orbit of x_i for the first n, T iterates of x. We will show, by cases, for each $i = 1, \ldots, k$

$$\left| \sum_{j=0}^{j_i - 1} p(S^j x_i) - n \right| \leq N.$$

$\underline{\sum_{j=0}^{j_i-1} p(S^j x_i) - n > N}$: If this is the case then

$$\sum_{j=0}^{j_i-2} p(S^j x_i) \geq \sum_{j=0}^{j_i-1} p(S^j x_i) - \max p > n$$

contradicting the fact that j_i is the minimum of all such numbers.

$\underline{n - \sum_{j=0}^{j_i-1} p(S^j x_i) > N}$: If this is the case then

$$n > \sum_{j=0}^{j_i-1} p(S^j x_i) + N \geq \sum_{j=0}^{j_i} p(S^j x_i)$$

again contradicting the definition of j_i. Therefore,

$$\left| \sum_{j=0}^{j_i - 1} p(S^j x_i) - n \right| \leq N = \max p$$

as desired. Armed with this bound we make the follow estimate,

$$\begin{aligned}\left|\sum_{i=0}^{n-1}(p-k)\circ T^i(x)\right| &= \left|\sum_{i=0}^{n-1}p(T^ix)-kn\right| \\ &\leq \left|\sum_{i=0}^{j_1-1}p(S^ix_1)-n\right| + \cdots + \left|\sum_{i=0}^{j_k-1}p(S^ix_k)-n\right| \\ &\leq kN.\end{aligned}$$

Therefore, by Theorem 4.1.4 there exists $f \in C(X,\mathbb{R})$ such that,

$$p(x)-k = f(x)-(f\circ T)(x)$$

as desired. Observe, as $p = k + (f - f\circ T)$ and p takes only values in $\mathbb{Z}^+$ we may assume that $f \in C(X,\mathbb{Z})$. □

4.1.1 Introducing new invariant measures.

It was shown in Chapter 3 that in general speedups can introduce measure which are invariant for the speedup, yet not invariant for the original system. We will show that this can happen even with bounded speedups. In particular, our example will show that p can be taken to be a constant. First, we will need a few definitions and lemmas.

We begin with a brief introduction to eigenvalues associated to minimal Cantor systems.

Definition 4.1.6. *Let (X,T) be a minimal Cantor system. A complex number λ is a* **continuous eigenvalue** *of (X,T) if there exists a continuous function*

$f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a **continuous eigenfunction** *(associated to λ).*

Next we define the *periodic spectrum* for a minimal Cantor system.

Definition 4.1.7. *Let (X, T) be a minimal Cantor system. The* **periodic spectrum** *of T is the collection of positive integers p such that $e^{2\pi i/p}$ is an eigenvalue of the linear operator $f \mapsto f \circ T$ acting on the space of continuous functions from X to the unit circle.*

Please note that if a minimal Cantor system (X, T) is topologically weak mixing (i.e. has no non-constant continuous eigenfunctions), then the periodic spectrum of T is trivial, i.e. is $\{1\}$.

In the following lemma we see how continuous eigenvalues play a role in the minimality of the square of a minimal transformation.

Lemma 4.1.8. *Let (X, T) be a minimal Cantor system. The following are equivalent:*

1. *(X, T^2) is not a minimal Cantor system*
2. *There exists a nonempty, clopen set E such that $E \cap TE = \emptyset$ and $E \cup TE = X$*
3. *-1 is a continuous eigenvalue for T.*

Proof. (1) $\Rightarrow$ (2); Then there exists a nonempty, closed set E such that $T^2 E = E$. Note, $TE \cap E = \emptyset$ otherwise $TE \cap E$ is a closed, invariant set. In addition, by minimality it follows that $E \cup TE = X$. All that is left to show is that E is open, but this follows immediately from $(TE)^c = E$.

(2) $\Rightarrow$ (3) : Define $f(x) = \mathbb{1}_E - \mathbb{1}_{TE}$, clearly f is continuous. Moreover, f is an eigenfunction with eigenvalue $\lambda = -1$ as $f \circ T = -f$.

(3) $\Rightarrow$ (1) : Let f be an continuous eigenfunction associated to $\lambda = -1$ and let $x \in X$ be given with $f(x) \neq 0$. We will show that f takes exactly 2 values, namely $f(x)$ and $-f(x)$. Let $x_0 \in X$ be given, by minimality there exists an increasing sequence $\{n_k\}$ such that

$$T^{n_k}(x) \to x_0 \quad \text{and thus} \quad f(T^{n_k}x) \to f(x_0).$$

Since for every k, $f(T^{n_k}x) = \pm f(x)$, it follows that $f(x_0) = f(x)$ or $f(x_0) = -f(x)$. Therefore, f takes two distinct values. However, this implies T^2 cannot be minimal as $f \circ T^2 = f$ and f is a non-constant, continuous function. □

Lastly, we will describe the discrete part of the spectrum of a measurable dynamical system.

Definition 4.1.9. *Let (Y, S, ν) be an ergodic measurable dynamical system. The discrete part of the spectrum of the unitary linear operator acting on $L^2(Y, \nu)$ by $f \mapsto f \circ T$ is referred to as the* **discrete spectrum of** *T. The pth root of unity $e^{2\pi i/p}$ is in the discrete spectrum of T if and only if there exists a measurable set B with $\nu(S^j B \cap B) = 0$ for $0 < j < p$ and $\nu(S^p B \cap B) = \nu(B) = 1/p$.*

Please recall the following theorem.

Theorem 1.0.3 (Jewett-Krieger). *Let $(Y, \mathscr{S}, S, \nu)$ be an ergodic automorphism of a non-atomic Lebesgue probability space. There exists a uniquely ergodic, minimal Cantor system (X, T), with a unique invariant Borel probability measure μ, such that $(Y, \mathscr{S}, S, \nu)$ is measurably conjugate to $(X, \mathscr{B}(X), T, \mu)$.*

Furthermore, we will need to make use of a realization theorem of Ormes. First, recall

Definition 1.0.6: *Let* (X_1, T_1) *and* (X_2, T_2) *be minimal Cantor systems. We say* (X_1, T_1) *and* (X_2, T_2) *are* **orbit equivalent** *if there exists a homeomorphism* $F : X_1 \to X_2$ *which preserves orbits, i.e. there exists functions* $m, n : X_1 \to \mathbb{Z}$ *such that*

$$(F \circ T_1^{n(x)})(x) = (T_2 \circ F)(x) \quad \text{and} \quad (F \circ T_1)(x) = (T_2^{m(x)} \circ F)(x).$$

Further, (X_1, T_1) *and* (X_2, T_2) *are* **strongly orbit equivalent** *if* m *and* n *each have at most one point of discontinuity.*

Please recall a theorem of Ormes.

Theorem 4.1.10 (SORT [O '97])**.** *Let* T *be a minimal homeomorphism of the Cantor set* X *and let* μ *be an ergodic* T*-invariant Borel probability measure. Let* S *be an ergodic automorphism of a non-atomic Lebesgue space* (Y, ν)*. The following are equivalent:*

1. $e^{2\pi i/p}$ *is in the discrete spectrum of* S *for all* $p \in \mathbb{N}$ *such that* p *is in the periodic spectrum of* S*.*

2. *There exists a minimal homeomorphism* T' *of* X *strongly orbit equivalent to* T *and an ergodic* T'*-invariant Borel probability measure* μ' *such that* (T', μ') *is measurably conjugate to* (S, ν)*. Furthermore, we may choose* (T', u') *such that*

   (a) *The identity map is a strong orbit equivalence between* T *and* T'

   (b) $\mu' = \mu$

Equipped with the two previous theorems, we now show that bounded speedups can introduce new invariant measures.

Theorem 4.1.11. *Bounded speedups need not preserve the simplex of invariant measures.*

Proof. Let (X, T) be a topologically weak mixing, uniquely ergodic, minimal Cantor system, such systems exists by Theorem 1.0.3. Moreover let (Y, S, μ) be an ergodic measure preserving automorphism of a non-atomic Lebesgue probability space, such that S^2 is not ergodic, note the measurable dyadic odometer is such a system. Observe, since (X, T) is topologically weak mixing its spectrum, including periodic spectrum, is trivial and thus is contained in the discrete spectrum of S. Hence, by Theorem 4.1.10 there exists $T' : X \to X$ a minimal homeomorphism, strongly orbit equivalent to (X, T) and $(X, \mathscr{B}(X), T', \mu)$ is measurable conjugate to $(Y, \mathscr{S}, S, \nu)$. Since strong orbit equivalence implies orbit equivalence it follows that μ is the unique T'-invariant measure. In addition, by Lemma 4.1.8 (X, T^2) is minimal. Since (X, T') is strongly orbit equivalent to (X, T), it follows that $(X, (T')^2)$ is minimal and, hence, a speedup of (X, T'). Observe, $(X, \mathscr{B}(X), (T')^2, \mu)$ is not ergodic as it is measurably conjugate to $(Y, \mathscr{S}, S^2, \nu)$. Finally, $M(X, (T')^2)$ is a simplex which has at least two extreme points, as μ is not ergodic for $(T')^2$. Therefore, $M(X, T') \subsetneq M(X, (T')^2)$ as desired. □

From the proof of the previous theorem, we obtain the following corollary about bounded speedups.

Corollary 2. *Let (X, T) be a minimal Cantor system and (X, S) a speedup of (X, T). The homeomorphism $\varphi : X_1 \to X_2$, such that $\varphi_* : M(X, T) \hookrightarrow$*

$M(X,S)$, as guaranteed by Theorem 3.0.8, need not preserve ergodic measures. That is, if $\mu \in \partial_e(M(X,T))$, then $\varphi_*(\mu)$ may not be ergodic in $M(X,S)$.

4.2 Entropy

The study of entropy is a fundamental theory of dynamical systems. Entropy, briefly, is the exponential growth rate of the number of distinguishable orbits of length n. We recall the definition of topological entropy for minimal Cantor systems. Let (X,T) be a Cantor system and let α and β be a finite clopen partitions of X.

Definition 4.2.1. *If α and β are clopen partitions of X, their* **join** *denoted $\alpha \vee \beta$ is the clopen partition of X by all sets of the form $A \cap B$ where $A \in \alpha$, $B \in \beta$. Similarly, we can define the join $\vee_{i=1}^{n} \alpha_i$ of any finite collection of open covers of X.*

If we are going to assign a number to a dynamical system, we should count the number of partition elements.

Definition 4.2.2. *If α is a clopen partition of X, let $N(\alpha)$ denote the number of nonempty elements of α. We define the* **entropy of the partition** *α to be $H(\alpha) = \log(N(\alpha))$.*

Since we are finding the growth rates of orbits, we should see how the partition of α changes when we apply our transformation T. Hence, the following definition.

Definition 4.2.3. *If α is a clopen partition of X and $T : X \to X$ is a homeomorphism, then the* **entropy of T relative to** α *is given by:*

$$h(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

Remark 4.2.4. *It is rather remarkable that the above limit exists. The standard proof relies on the fact, for a given clopen partition α, that the sequence $a_n = H(\vee_{i=0}^{n-1} T^{-i}\alpha)$ is a subadditive, that is for every $m, n \geq 1$ $a_{m+n} \leq a_m + a_n$. It is known that, for any subadditive sequence, the new sequence $b_n = (1/n)a_n$ not only converges, but it converges to $\inf_n b_n$.*

Finally, we want to be able to disregard partitions which do not reveal much information about the growth rates of orbits. For example, taking $\alpha = X$ cannot distinguish any orbits, hence does not yield any new or even useful information about the growth rates of the orbits of T. Therefore, the topological entropy of T, is then defined as

$$h(T) = \sup_{\alpha} h(T, \alpha)$$

where α ranges over all clopen partitions of X. It will be useful to be able to use T instead of T^{-1} when computing entropy. The following theorem guarantees that we will compute the same number.

Theorem 4.2.5. *If (X, T) is a minimal Cantor system, then $h(T) = h(T^{-1})$.*

Topological entropy draws its inspiration from measure theoretic entropy. Briefly, given a minimal Cantor system (X, T) and an invariant, Borel probability measure μ, we let $h_\mu(T)$ be the measure theoretic entropy of T with respect to the measure μ. What follows is a theorem describing the relationship between measure theoretic entropy and topological entropy.

Theorem 4.2.6 (The Variational Principle)**.** *Let* (X, T) *be a minimal Cantor system, then*

$$h(T) = \sup\{h_\mu(T) : \mu \in M(X, T)\}.$$

We remark that one can simply take the supremum over all ergodic measures of T.

Since we are interested in computing, or bounding, topological entropy of speedups, it is helpful to look at how entropy interacts with orbit equivalence. Please observe the following theorem of Boyle and Handelman [BH '94] which addresses this very issue.

Theorem 4.2.7 ([BH '94])**.** *Suppose* $0 < \log(\alpha) < \infty$*. There exists a homeomorphism* S *strongly orbit equivalent to the dyadic odometer such that* $h(S) = \log(\alpha)$*.*

Combining the above result with the characterization of strong orbit equivalence and orbit equivalence in [GPS '95] and the main result in Chapter 3, we see that there is no hope to say anything generally about the entropy of speedups. However, in the case of bounded speedups, we provide upper and lower bounds for the entropy. The following is a Theorem of Neveu which address the entropy of speedups in the measurable category.

Theorem 4.2.8 ([N2 '69])**.** *Suppose* $(X, \mathscr{B}, \mu, T)$ *is an ergodic automorphism and* $(X, \mathscr{B}, \mu, S)$ *is an aperiodic automorphism of the form*

$$S(x) = T^{p(x)}(x)$$

where $p : X \to \mathbb{Z}^+$. *Then,* $h_\mu(S) = (\int p\, d\mu) h_\mu(T)$ *whenever* $\int p\, d\mu$ *is finite.*

We present the full theorem of the entropy of bounded speedups below.

Theorem 4.2.9. *Let* (X, T) *be a minimal Cantor system and* (X, S) *be a bounded speedup of* (X, T) *with* $p : X \to \mathbb{Z}^+$ *bounded. The entropy of* S *lies within the following interval*

$$kh(T) \leq h(S) \leq \left(\int p\, d\mu_1 \right) h(T)$$

where k *is the orbit number for* $T \rightsquigarrow S$ *and* $\displaystyle\int p\, d\mu_1 = \sup_{\mu \in M(X,S)} \int p\, d\mu.$

Using the above theorem of Neveu's we are easily able to provide a lower bound for the entropy of bounded speedups.

Proposition 4.2.10. *Let* (X, T) *be a minimal Cantor system and* (X, S) *a bounded speedup of* (X, T) *with* $p : X \to \mathbb{Z}^+$. *The entropy* $h(S) \geq kh(T)$ *where* k *is the orbit number for* $T \rightsquigarrow S$.

Proof. Let $k \in \mathbb{Z}^+$ be the orbit number for $T \rightsquigarrow S$ and consider the following calculation

$$\begin{aligned}
h(S) &= \sup_{\mu \in M(X,S)} h_\mu(S) \\
&\geq \sup_{\nu \in M(X,T)} h_\nu(S) \\
&\geq \sup_{\nu \in \partial_E(M(X,T))} \left(\int p \, d\nu \right) h_\nu(T) \\
&= \sup_{\nu \in \partial_E(M(X,T))} \left(\int (k + (f - f \circ T)) \, d\nu \right) h_\nu(T) \text{ by Lemma 4.1.5} \\
&= \sup_{\nu \in \partial_E(M(X,T))} k h_\nu(T) \\
&= k \left(\sup_{\nu \in \partial_E(M(X,T))} h_\nu(T) \right) \\
&= k h(T).
\end{aligned}$$

□

Notice, that by using [N2 '69] we can obtain a sharper lower bound than one might expect as:

$$\inf_{\mu \in M(X,S)} \int p \, d\mu \leq k.$$

Before proceeding with a lemma needed to prove the upper bound we recall a theorem about construction invariant measures.

Theorem 4.2.11 ([W] Theorem 6.9)**.** *Let $T : X \to X$ be continuous. If $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence in $M(X)$ and we form the new sequence $\{\mu_n\}_{n=1}^{\infty}$ by*

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\tilde{T}^i\sigma_n$$

then any limit point μ of $\{\mu_n\}$ is a member of $M(X,T)$.

We will use Theorem 4.2.11 to prove the following lemma.

Lemma 4.2.12. *Let (X,T) be minimal Cantor system and let (X,S) be a bounded speedup of (X,T) with bounded $p : X \to \mathbb{Z}^+$. For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for every $x \in X$*

$$\frac{1}{n}\sum_{i=0}^{n-1} p(S^i(x)) < \left(\sup_{\mu\in M(X,S)}\int p\,d\mu\right) + \varepsilon.$$

Proof. Let

$$\int p\,d\mu_1 = \sup_{\mu\in M(X,S)}\int p\,d\mu$$

and assume the conclusion is false. Thus, there exists an $\varepsilon > 0$ and an increasing sequence of positive integers $\{n_k\}$ and corresponding sequence of points $\{x_{n_k}\}$ which have the property that

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1} p(S^i(x_{n_k})) \geq \int p\,d\mu_1 + \varepsilon.$$

Define

$$\nu_{n_k} = \frac{1}{n_k}\sum_{i=0}^{n_k-1}\tilde{S}^i(\delta_{x_{n_k}})$$

where $\delta_{x_{n_k}}$ represents the Dirac point-mass measure of x_{n_k}. As $M(X)$ is compact in the weak* topology there exists $\nu \in M(X)$ and a subsequence $\{n_\ell\} \subset \{n_k\}$ for which

$$\nu_{n_\ell} \xrightarrow{\text{weak}^*} \nu.$$

By Theorem 4.2.11 $\nu \in M(X,S)$. Since p is continuous we can make the following estimate

$$\begin{aligned}\int p\,d\nu &= \lim_{\ell\to\infty} \int p\,d\nu_{n_\ell} \\ &= \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{i=0}^{n_\ell-1} p(S^i x_{n_\ell}) \\ &\geq \int p\,d\mu_1 + \varepsilon\end{aligned}$$

which yields our contradiction as $\int p\,d\mu_1 = \sup_{\mu\in M(X,S)} \int p\,d\mu$. □

Now we use Lemma 4.2.12 to prove our upper bound on the entropy of a bounded speedup.

Proposition 4.2.13. *Let (X,T) be a minimal Cantor system and let (X,S) be a bounded speedup of (X,T) with $p : X \to \mathbb{Z}^+$, then we have the following inequality*

$$h(S) \leq \left(\int p\,d\mu_1\right) h(T).$$

Proof. Fix $\varepsilon > 0$ and fix α a finite clopen partition of X such that $p^{-1}(\mathbb{Z}^+) \leq \alpha$. By Lemma 4.2.12 there exists a $N \in \mathbb{N}$ such that for every $n \geq N$ and $x \in X$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} p(S^i x) < \int p\,d\mu_1 + \varepsilon.$$

It immediately follows then that for every $n > N$

$$\bigvee_{i=0}^{n} S^i(\alpha) \leq \bigvee_{i=0}^{\lceil n(\int p\, d\mu_1 + \varepsilon)\rceil} T^i(\alpha).$$

From this we can immediately deduce that

$$\begin{aligned} h(S,\alpha) &\leq \lim_{n\to} \frac{1}{n+1} H\left(\bigvee_{i=0}^{\lceil n(\int p\, d\mu_1 + \varepsilon)\rceil} T^i(\alpha)\right) \\ &= \lim_{n\to\infty} \frac{\lceil n(\int p\, d\mu_1 + \varepsilon)\rceil}{n+1} \cdot \frac{1}{\lceil n(\int p\, d\mu_1 + \varepsilon)\rceil} H\left(\bigvee_{i=0}^{\lceil n(\int p\, d\mu_1 + \varepsilon)\rceil} T^i(\alpha)\right) \\ &= \left(\int p\, d\mu_1 + \varepsilon\right) h(T). \end{aligned}$$

Whence, it follows that

$$h(S) \leq \left(\int p\, d\mu_1 + \varepsilon\right) h(T).$$

Since ε was arbitrarily given we may conclude that

$$h(S) \leq \left(\int p\, d\mu_1\right) h(T)$$

as desired. $\square$

Putting together Proposition 4.2.10 and Proposition 4.2.13 we obtain the main theorem of this section.

Theorem 4.2.14. *Let (X,T) be a minimal Cantor system and let (X,S) be a bounded speedup of (X,T) with $p : X \to \mathbb{Z}^+$. The entropy of S is bounded* *as follows:*

$$kh(T) \leq h(S) \leq \left(\int p\, d\mu_1\right) h(T)$$

where k is the orbit number for $T \rightsquigarrow S$, and $\int p\, d\mu_1 = \sup\limits_{\mu \in M(X,S)} \int p\, d\mu$.

An immediate corollary of the above describes the entropy of S when $M(X,T) = M(X,S)$. Observe, in this case the two systems are orbit equivalent.

Corollary 3. *Let (X,T) be a minimal Cantor system and (X,S) a bounded speedup of (X,T) with $p : X \to \mathbb{Z}^+$. If $M(X,T) = M(X,S)$, then*

$$h(S) = kh(T)$$

where k is the orbit number for $T \rightsquigarrow S$.

Corollary 4. *Let (X,T) be a minimal Cantor system and let (X,S) be a bounded speedup of (X,T), if $h(T) = 0$, then $h(S) = 0$.*

Corollary 5. *Let (X,T) be a minimal Cantor system and let (X,S) be a bounded speedup of (X,T), if $h(T) > 0$, then $h(S) > 0$, in fact $h(S) \geq h(T)$, with a strict inequality for a non-trivial p function, that is, $p \not\equiv 1$.*

Remark 4.2.15. *We would like to point out two observations about the entropy in general about topological speedups. First, from Theorem 4.2.14 the entropy of a bounded speedup can only increase, whereas in the unbounded case entropy can decrease.*

Second, the only instances where a bounded speedup of a minimal Cantor system could be conjugate to the original system is if the original system has entropy 0 or ∞.

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