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The Finite Embeddability Property for Some Noncommutative Knotted Varieties of RL and DRL

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The Finite Embeddability Property for some noncommutative knotted varieties of RL and DRL

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Abstract

Residuated lattices, although originally considered in the realm of algebra providing a general setting for studying ideals in ring theory [35], were later shown to form algebraic models for substructural logics; see [11] for a detailed account. The latter are non-classical logics that include intuitionistic, relevance, many-valued, and linear logic, among others. Most of the important examples of substructural logics are obtained by adding structural rules to the basic logical calculus **FL**. We denote by RL_m^n the varieties of knotted residuated lattices. Examples of these knotted rules include integrality and contraction. The extension of **FL** by the rules corresponding to these two equations is equivalent to Gentzen's original system **LJ** for intuitionism [14]. Apart from applications to logic and to abstract ring theory, residuated lattices are connected to mathematical linguistics, computer science, and quantum mechanics, among other areas. Even thought the connections to other disciplines are abundant, the current document is of purely algebraic nature.

Results in [19] establish the *finite model property* (FMP) for the implicational fragment of $\mathbf{FL_e}$ extended by some knotted rules. In [34] the *finite embeddability property* (FEP) is shown to hold for commutative RL_m^n $(xy = yx)$; the *strong finite model property* follows for the corresponding logics. Recent results by Horčík [17]

show that the word problem is undecidable for the varieties RL_m^n when $1 \leq m < n$ or $2 \leq n < m$. Therefore these varieties do not have the FEP. We refer the reader to [18] for details on how this is connected to the Burnside problems in group theory and to regularity of languages in automata theory.

In the present document, using purely algebraic methods, we prove the FEP for subvarieties of RL_m^n and DRL_m^n that satisfy properties weaker than commutativity. The proof uses the theory of residuated frames introduced in [10] and [9] .

In Chapter 1, we present the basic definitions and constructions that will be used throughout the full document. We point the reader towards Section 1.4, where we list a relevant list of varieties for which the FEP holds or not.

Chapter 2 presents a proof of the FEP for subvarieties of RL_m^n that satisfy the identity $xyx = x^2y$. The proof of this case relies on finding the free object over the class of pomonoids that satisfy the previous equality and $x^m \leq x^n$.

Chapter 3 focuses on the study of the noncommutative equation that we use to define the varieties studied in the following two chapters. This equation arises as a natural generalization of the basic equation $xyx = x^2y$.

Chapter 4 presents the FEP for RL_m^n. In the general case, the free object in the class is fairly complicated, so we identify instead an object outside the class, which is both free and structured enough to allow us to prove the result. In the last section, we extend our result to cover some other subvarieties of knotted residuated lattices. These subvarieties include the cyclic, cyclic-involutive, and representable ones.

Chapter 5 details a proof for the fully distributive case. Here we enrich the free object discovered in Chapter 4 by creating the meet semilattice generated by it.

We remark that the FEP for a variety $\mathcal{V}$ is equivalent to the condition that all finitely presented algebras in $\mathcal{V}$ are residually finite [7]. Varieties of semigroups with this property have been fully characterized in [22]. In particular, the variety of monoids axiomatized by $xyx = x^2y$ has been studied, see [30] for example, and has the FEP [22]. These results do not imply the FEP for the corresponding variety of residuated lattices, which also serves as the simplest case of our analysis.

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Chapter 1

Preliminaries

1.1 Posets and well partially ordered sets

Let P be a set. An *order* (or *partial order*) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

i. $x \leq x$ (*reflexivity*),

ii. $x \leq y$ and $y \leq x$ imply $x = y$ (*antisymmetry*), and

iii. $x \leq y$ and $y \leq z$ imply $x \leq z$ (*transitivity*).

A set P equipped with an order relation $\leq$ is said to be a *partially ordered set* (or *poset*) and we denote it as $(P, \leq)$. A poset $\mathbf{P} = (P, \leq)$ is a *chain* if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, any two elements of P are comparable). Alternative names for a chain are *linearly ordered set* and *totally ordered set*. On the other hand $\mathbf{P}$ is an *antichain* if $x \leq y$ implies $x = y$ for all $x, y \in P$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $\varphi : P \to Q$ is said to be *order-preserving* (or *monotone*) if $x \leq^{\mathbf{P}} y$ implies $\varphi(x) \leq^{\mathbf{Q}} \varphi(y)$. The direct product $\mathbf{P} \times \mathbf{Q}$ is a poset on the Cartesian product $P \times Q$, where $(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq^{\mathbf{P}} x_2$ and $y_1 \leq^{\mathbf{Q}} y_2$. The disjoint union $\mathbf{P}\dot{\cup}\mathbf{Q}$ is a poset with underlying set $P \cup Q$ and for every $p \in P, q \in Q$, p and q are incomparable.

A relation $\leq$ on a set Q which is reflexive and transitive but not necessarily anti-symmetric is called a *quasi-order* and the structure $(Q, \leq)$ is called a *quasi-ordered set*. The definitions in the previous paragraph apply to quasi-ordered sets as well.

The *dual* of a poset $\mathbf{P} = (P, \leq)$ is the poset $\mathbf{P}^{\partial} = (P, \geq)$. Let $\mathbf{P}$ be a poset and $Q \subseteq P$. Q is a *downset* if, whenever $x \in Q, y \in P$ and $y \leq x$, we have $y \in Q$. Dually, Q is an *upset* if, whenever $x \in Q, y \in P$ and $y \geq x$, we have $y \in Q$. For $X \in P$ and $x \in P$, we define

$$\downarrow X = \{y \in P : y \leq x \text{ for some } x \in X\} \text{ and } \downarrow x = \downarrow\{x\},$$

$$\uparrow X = \{y \in P : y \geq x \text{ for some } x \in X\} \text{ and } \uparrow x = \uparrow\{x\}.$$

The next definitions were introduced in [27]. Consider a poset (or quasi-ordered set) $\mathbf{P}$. An infinite sequence $p_1, p_2, \ldots$ of elements in P will we called *good* if there exist positive integers i, j such that $i < j$ and $p_i \leq p_j$. Infinite sequences that are not good are appropriately called *bad*. It is easy to see that an infinite sequence is bad if $i < j$ implies $p_i \nleq p_j$ for every i, j.

A *well partially ordered set* (wpo for short) is a poset that contains no infinite descending chains and no infinite antichains. Examples of wpo's are $(\mathbb{N}, \leq)$ and $(\mathbb{N}^k, \leq)$. The latter result is known as Dickson's lemma and it is a special case of a more general result that we will introduce later in this section. On the other hand $(\mathbb{Z}, \leq)$ and $(\mathbb{N}, |)$, where $|$ refers to the divisibility relation, are not wpo's. The first one fails because the whole poset is an infinitely descending chain. In the second one, the prime numbers form an infinite antichain.

A *well quasi-ordered set* (abbreviated wqo) is defined in an analogous manner. In the literature, wpo's have not been studied as extensively as wqo's. The theory is technically smoother if antisymmetry is not required. Furthermore, some constructions that arise do not produce a poset.

It is easy to verify that $(A, \leq)$ is a wpo (or wqo) iff every infinite sequence of elements of A is good iff there are no bad sequences in A. For a set S, we use $\mathscr{P}_{\text{fin}}(S)$ to denote the class of finite subsets of S. Given a quasi-order $(Q, \leq)$, for $A, B \in \mathscr{P}_{\text{fin}}(Q)$ we define $A \leq_{\mathscr{P}} B$ iff there exists an injection $f : A \hookrightarrow B$ such that $(\forall a \in A)(a \leq f(a))$.

The following results by Higman show ways to construct new wqo from existing ones. In particular, the first one also applies to wpo's.

Theorem 1.1.1 ([16])**.** *The products, images of order-preserving maps, and subsets of wqo's are also wqo's.*

Theorem 1.1.2 ([16])**.** *If $(Q, \leq)$ is a well quasi-ordered set, then $(\mathscr{P}_{fin}(Q), \leq_{\mathscr{P}})$ is also a wqo.*

Now, we introduce a very special kind of poset. Let $\mathbf{L} = (L, \leq)$ be a poset with the property that every pair of elements of L have a greatest lower bound (infimum) and a smallest upper bound (supremum). This poset will be called a lattice. In this case, for $a, b \in L$ we can define the operations $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. In this case, $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. The following is an equivalent characterization of a lattice.

A *lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee)$ where for all $a, b, c \in L$,

i. $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (*associativity*),

ii. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (*commutativity*),

iii. $a \vee a = a = a \wedge a$ (*idempotency*), and

iv. $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ (*absorption*).

If $\bigwedge S$ and $\bigvee S$ exist for all $S \subseteq L$, then $\mathbf{L}$ is a *complete lattice*. For a lattice to be complete, it is enough to verify that either all arbitrary meets exist, or all arbitrary joins exist. A lattice is *distributive* if for all $a, b, c \in L$ these equations are satisfied

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

To show distributivity, it suffices to verify only one of the two equations.

A *semilattice* $(A, \wedge)$ is an algebra such that the operation $\wedge$ is idempotent, associative, and commutative. The next two structures will be very important for the constructions presented in this document.

A *pomonoid* is a structure $\mathbf{P} = (P, \leq, \cdot, 1)$ such that $(P, \leq)$ is a poset, $(P, \cdot, 1)$ is a monoid, and multiplication is order-preserving. The last condition means that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$.

A *semilattice monoid* is an algebra $\mathbf{A} = (A, \wedge, \cdot, 1)$ such that $(A, \wedge)$ is a semilattice, $(A, \cdot, 1)$ is a monoid, and multiplication distributes over meet.

1.2 Concepts from Universal Algebra

A *language* (or *type* $\mathcal{F}$) is an indexed set of symbols F together with a map $\sigma : F \to \mathbb{N}$, called the *arity map*. An *operation* on a set A of *arity* n is a function from A^n to A. An algebra $\mathbf{A}$ of type $\mathcal{F}$ consists of a set A and an indexed set $(f^{\mathbf{A}})_{f \in F}$ of operations $f^{\mathbf{A}} : A^{\sigma(f)} \to A$ on A of arity $\sigma(f)$. The set A is called the *underlying set* (or the *universe* of A) and the maps $f^{\mathbf{A}}$ are called the fundamental operations of

A. We will be dealing with algebras over a finite type. Such algebras will be denoted by $\mathbf{A} = (A, f_1^{\mathbf{A}}, f_2^{\mathbf{A}}, \ldots, f_n^{\mathbf{A}})$, and most of the time we will omit the superscript $^{\mathbf{A}}$.

A *homomorphism* between two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type $\mathcal{F}$ is a map $h : A \to B$, such that for all the fundamental operations,

$$h(f^{\mathbf{A}}(a_1, a_2, \ldots, a_{\sigma(f)})) = f^{\mathbf{B}}(h(a_1), h(a_2), \ldots, h(a_{\sigma(f)})),$$

for all $a_1, a_2, \ldots, a_{\sigma(f)} \in A$ and for all $f \in F$. If h is a surjective homomorphism from A to B, then we say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$.

A *subuniverse* of an algebra $\mathbf{A}$ is a subset B of A that is closed under the operations, i.e., $f^{\mathbf{A}}(b_1, b_2, \ldots, b_{\sigma(f)}) \in B$, for all $b_1, \ldots, b_{\sigma(f)} \in B$. If B is a subuniverse of an algebra $\mathbf{A} = (A, f_1, f_2, \ldots, f_n)$, then the algebra $\mathbf{B} = (B, f_1\restriction_B, f_2\restriction_B, \ldots, f_n\restriction_B)$, where $f_i\restriction_B$ is the restriction of f_i to $B^{\sigma(f_i)}$, is called a *subalgebra* of $\mathbf{A}$.

If $\mathcal{A} = \{\mathbf{A}_i : i \in I\}$ is an indexed set of algebras of a given type $\mathcal{F}$, then the product of the algebras of $\mathcal{A}$ is the algebra $\mathbf{P} = \prod_{i\in I} \mathbf{A}_i$ with underlying set the Cartesian product of the underlying sets of the algebras in $\mathcal{A}$, type $\mathcal{F}$ and operations $f^{\mathbf{P}}, f \in F$, defined by $f^{\mathbf{P}}\left(\langle a_{i1}\rangle_{i\in I}, \ldots, \langle a_{i\sigma(f)}\rangle_{i\in I}\right) = \left\langle f^{\mathbf{A}_i}(a_{i1}, \ldots, a_{i\sigma(f)})\right\rangle_{i\in I}$, for all $\mathbf{A}_i \in \mathcal{A}$, $a_{ij} \in A_i$, $i \in I$ and $j \in \{1, \ldots, \sigma(f)\}$.

A *subdirect product* of an indexed set $\mathcal{A} = \{\mathbf{A}_i : i \in I\}$ of algebras of a given type $\mathcal{F}$, is a subalgebra $\mathbf{B}$ of the product of the algebras of $\mathcal{A}$, such that for every $i \in I$ and for every $a_i \in A_i$, there exists an element of B, whose ith coordinate is a_i. In other words, the projection to the ith coordinate map from B to A_i is surjective. An nontrivial algebra is called *subdirectly irreducible*, if it is not a subdirect product of more than one nontrivial algebra. The collection of all subdirectly irreducible members of a class of algebras $\mathcal{K}$ is denoted by $\mathcal{K}_{SI}$.

If $\mathcal{K}$ is a class of algebras, we denote by $\mathbf{S}(\mathcal{K})$, $\mathbf{H}(\mathcal{K})$, and $\mathbf{P}(\mathcal{K})$ the classes of all algebras that are subalgebras, homomorphic images, and direct products of algebras in $\mathcal{K}$, respectively. A class of algebras is called a *variety*, if it is closed under the three operators $\mathbf{S}, \mathbf{H}$, and $\mathbf{P}$. It is not hard to prove that a class $\mathcal{V}$ of algebras is a variety iff $\mathcal{V} = \mathbf{HSP}(\mathcal{V})$. Moreover, given a class of algebras $\mathcal{K}$ of the same type, the smallest variety containing $\mathcal{K}$ is $\mathbf{HSP}(\mathcal{K})$, the *variety generated* by $\mathcal{K}$.

A class $\mathcal{K}$ is called an *equational class* when it is the class of all algebraic structures of the same signature satisfying a given set of identities. The operators $\mathbf{H}, \mathbf{S}$, and $\mathbf{P}$ preserve equations. Namely, equations satisfied in $\mathcal{K}$ are also satisfied in $\mathbf{HSP}(\mathcal{K})$. Birkhoff's theorem proves that equational classes and varieties are equivalent.

A *quasi-equation* is a formula of the form $(p_1 = q_1 \& \dots \& p_n = q_n) \Rightarrow p = q$. A *quasivariety* is a class of algebras closed under $\mathbf{I}, \mathbf{S}, \mathbf{P}$, and $\mathbf{P}_U$ that contains a trivial one element algebra. $\mathbf{I}$ represents the isomorphic algebras and $\mathbf{P}_U$ stands for *ultraproducts*.

1.3 Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and the following equivalences hold:

$$xy \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x\backslash z. \tag{Res}$$

Residuation (Res) can be reformulated in equational form [2], so the class RL of residuated lattices is a variety.

Residuated lattices, although originally considered in the realm of algebra providing a general setting for studying ideals in ring theory [35], were later shown to form algebraic models for substructural logics; see [11] for a detailed account. The latter are non-classical logics that include intuitionistic, relevance, many-valued and linear logic, among others.

An *FL-algebra* is a residuated lattice with an additional arbitrary constant 0. We denote the variety of FL-algebras by FL. FL-algebras serve as algebraic models for the logical calculus **FL**, known as Full Lambek Calculus and introduced in [24] (see [11] for characterization). The subvariety of FLthat satisfies $0 = 1$ is term equivalent to RL. Most of the important examples of substructural logics are obtained by adding structural rules to **FL**. Apart from applications to logic and abstract ring theory, residuated lattices are connected to mathematical linguistics, computer science, and quantum mechanics, among other areas.

A residuated lattice is *commutative* if it has commutative multiplication. We denote the variety of all commutative residuated lattices by CRL. On the other hand, commutative FL-algebras are also called FL_e-algebras because of their correspondence with the logic $\mathbf{FL_e}$. We denote the variety of FL_e-algebras by $\mathsf{FL_e}$.

A residuated lattice is called *integral* if it has a greatest element, which is the multiplicative unit 1. In other words, it satisfies $x \leq 1$ for all x. It is called *contractive* if $x \leq x^2$, for all x. Both integrality and contraction are actually identities as they can be represented by $x \wedge 1 = x$ and $x \wedge x^2 = x$, respectively. The extension of **FL** by the rules corresponding to these two equations is equivalent to Gentzen's original system **LJ** for intuitionism [14]. These equations are members of a larger family. We denote the integral and contractive subvarieties of $\mathsf{RL}$by $\mathsf{IRL}$and KRL, respectively. The corresponding varieties of FL-algebras are denoted by $\mathsf{FL_i}$ and $\mathsf{FL_c}$.

Structural rules that involve a single variable are known as *knotted rules* [19], and their algebraic characterization are *knotted inequalities* of the form $x^m \leq x^n$, where m and n are natural numbers. We denote by RL_m^n (FL_m^n) the variety of residuated lattices (FL-algebras) that satisfy the above (in)equality; if $n = m$ we obtain the whole variety RL of residuated lattices and if $m = 0$, we obtain the trivial variety of 1-element algebras. So in this work we assume that m, n are distinct, nonnegative integers and that m is actually positive. It is clear that integrality and contraction are included among these knotted rules.

For an FL-algebra we can define two additional operations

$$\sim x = x\backslash 0, \text{ and } -x = 0/x.$$

An FL-algebra is called *involutive*, if it satisfies the identities $-\sim x = x = \sim - x$. It is *cyclic*, if it satisfies $\sim x = -x$. We denote the corresponding varieties of FL-algebras by InFL and CyFL. Note that every commutative FL-algebra is cyclic as a consequence of (Res).

Residuated lattices or FL-algebras that can be represented as subdirect products of totally ordered algebras are called *representable* and the corresponding classes are denoted by RRL and RFL. The term *semilinearity* is often used instead of representability. These classes are actually varieties as shown in [2], [21].

A residuated lattice is distributive if its lattice reduct is distributive. We use DRL(DFL) to refer to the variety of distributive residuated lattices (FL-algebras). Since products of chains are distributive, every representable algebra is distributive.

Finally, a residuated lattice is *fully distributive* if it is distributive and multiplication distributes over (both join and) meet. Distributivity of multiplication over meet is defined by the following equations. Distributivity of multiplication over join is defined in an analogous manner.

$$a \cdot (b \wedge c) = a \cdot b \wedge a \cdot c$$
$$(b \wedge c) \cdot a = b \cdot a \wedge c \cdot a.$$

All residuated lattices satisfy that multiplication distributes over join, hence full-distributivity only requires that multiplication and meet distribute over join.

We allow combinations of the preceding notations. For instance, $\mathsf{CyInDFL}$ denotes the variety of cyclic involutive distributive FL-algebras.

1.4 Finite embeddability property and decidability

A class of algebras has a *decidable* (quasi)equational theory if there is an algorithm that decides whether a (quasi)equation holds in the class or not.

B is a *partial subalgebra* of **A**, if B is a subset of A and each n-ary operation $f^{\mathbf{A}}$ on A induces a partial operation $f^{\mathbf{B}}$ on B defined as

$$f^{\mathbf{B}}(b_1, \dots, b_n) = \begin{cases} f^{\mathbf{A}}(b_1, \dots, b_n), & \text{if } f^{\mathbf{A}}(b_1, \dots, b_n) \in B, \\ \text{undefined}, & \text{if } f^{\mathbf{A}}(b_1, \dots, b_n) \notin B. \end{cases}$$

A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP), if for every algebra **A** in $\mathcal{K}$ and every finite partial subalgebra **B** of **A**, there exists a finite algebra **D** in $\mathcal{K}$ such that **B** embeds into **D**.

A class $\mathcal{K}$ has the *finite model property* (FMP), if any equation that fails in the class, fails in a finite algebra of the class. We say that $\mathcal{K}$ has the *strong finite model property* (SFMP), if every quasi-equation that fails in the class is falsified in a finite element of $\mathcal{K}$. We obtain that a variety with the FMP is generated by its finite members. Similarly, a quasivariety with the SFMP is generated by its finite members as well.

The relations between the previous properties are that every class of algebras with the FEP has the SFMP, and every class of algebras with the SFMP has the FMP. In the case of varieties of finite type, the notions of FEP and SFMP are equivalent [11]. Note that the FEP implies that if a universal first-order sentence fails in the class, then it fails in a finite algebra in the class.

One of the important consequences of the FEP is decidability. If a class $\mathcal{K}$ has the FEP, then every universal sentence that fails in $\mathcal{K}$, fails in a finite member of $\mathcal{K}$. Thus, if $\mathcal{K}$ is finitely axiomatizable, then its universal theory is decidable.

Let $\mathcal{V}$ be a variety, X a set of variables and R a set of equations over X. Starting from the free algebra in $\mathcal{V}$ over X, we factor the smallest equivalence relation containing R and we obtain an algebra that we denote by $\mathbf{A} = (X|R)$. If both X and R are finite, then we say that $\mathbf{A}$ is *finitely presented*. Two terms s, t over X represent the same element of $\mathbf{A}$ iff the quasi-equation $\mathsf{AND}\ R \Rightarrow s = t$ holds in $\mathcal{V}$. Clearly, if the quasiequational theory of a variety is decidable, then so is its word problem. Conversely, if the word problem is undecidable, then the FEP does not hold.

While $\mathsf{RL}$and FLhave the FMP [29], the word problem is undecidable [36] and they do not have the FEP [21].

Results in [19] establish the FMP for the implicational fragment of $\mathbf{FL_e}$ extended by $x^m \leq x^n$, for $m = 1, n \geq 2$, as well as for $n = 1, m \geq 2$. See [18] for details on

Table 1.1: Decidability results.

Variety	Eq. Th.	Word prob.	Univ. Th.
FL	FMP	Undec.	Undec.
RL	FMP	Undec.	Undec.
Com. RL	FMP		Undec.
Knotted RL	Varies	Undec.	Undec.
Com. Knot. RL	Dec.	Dec.	FEP
DRL		Undec.	Undec.
Com. DRL	Dec.	Undec.	Undec.
CyFL		Undec.	Undec.
CyInFL	FMP	Undec.	Undec.
IRL	Dec.	Dec.	FEP

how this is connected to the Burnside problems in group theory and to regularity of languages in automata theory. The FEP has been established for $\mathsf{IRL}$and for all its subvarieties axiomatized by equations over the language of join, multiplication and 1 [10]. Recent results by Horčík [17] show that the word problem is undecidable for the varieties RL_m^n when $1 \leq m < n$ or $2 \leq n < m$. Therefore these varieties do not have the FEP. It is clear that integrality is a strong condition. However it can be replaced by a combination of a knotted inequality and commutativity as seen in [34], where the FEP is shown to hold for commutative RL_m^n $(xy = yx)$.

Also of interest are the varieties CyInFL, and CyFL which have undecidable word problem [36]. Furthermore, any subvariety axiomatized by nontrivial lattice equations still have undecidable word problem. Table 1.1 shows some a non-exhaustive list of decidability results for varieties related to the ones we study in this document. We will prove the FEP for subvarieties of the above axiomatized by some extra conditions.

In the present document, using purely algebraic methods, we prove the FEP for subvarieties of each RL_m^n and fully distributive RL_m^n that satisfy properties weaker than commutativity. The proof uses the theory of residuated frames and their distributive counterparts introduced in [10] and [9].

1.5 Residuated frames

For posets $\mathbf{P}$ and $\mathbf{Q}$, the maps ${}^{\rhd} : P \to Q$ and ${}^{\lhd} : Q \to P$ form a *Galois connection* if for all $p \in P$ and $q \in Q$,

$$q \leq p^{\rhd} \quad \text{iff} \quad p \leq q^{\lhd}.$$

A *closure operator* on $\mathbf{P}$ is a map $\gamma : P \to P$ such that for all $x, y \in P$,

i. $x \leq \gamma(x)$ (*expansive*),

ii. $x \leq y$ implies $\gamma(x) \leq \gamma(y)$ (*monotone*), and

iii. $\gamma(\gamma(x)) = \gamma(x)$ (*idempotent*).

$\mathbf{P}_\gamma = (\gamma[P], \leq)$ denotes the poset of γ-*closed* sets.

Given a relation $R \subseteq A \times B$, for $X \subseteq A$ and $Y \subseteq B$ define

$$\begin{array}{lll} X\ R\ Y & \text{iff} & x\ R\ y \text{ for all } x \in X, y \in Y, \\ x\ R\ Y & \text{iff} & \{x\}\ R\ Y, \\ X\ R\ y & \text{iff} & X\ R\ \{y\}. \end{array}$$

Note that a pair of maps ${}^{\rhd} : \mathscr{P}(A) \to \mathscr{P}(B)$ and ${}^{\lhd} : \mathscr{P}(B) \to \mathscr{P}(A)$ forms a Galois connection iff $X^{\rhd} = \{y : X\ R\ y\}$ and $Y^{\lhd} = \{x : x\ R\ Y\}$, for some relation $R \subseteq A \times B$. In this case $({}^{\rhd}, {}^{\lhd})$ is the Galois connection *induced* by R. The closure operator $\gamma_R : \mathscr{P}(A) \to \mathscr{P}(A)$ associated with R is $\gamma_R(X) = X^{\rhd\lhd}$. For a closure operator γ on a complete lattice $\mathbf{P}$, $D \subseteq P$ is a *basis* for γ if the elements in $\gamma[P]$ are exactly the meets of elements of D. Note that the interior operator in a topological space is a closure operator under the dual order, hence this definition of basis is equivalent to the usual one in topology.

Lemma 1.5.1. *Let A and B be sets.*

1. *If R is a relation between A and B, then γ_R is a closure operator on $\mathscr{P}(A)$.*
2. *If $(^{\triangleright}, ^{\triangleleft})$ is the Galois connection induced by $R \subseteq A \times A$, then $\{\{b\}^{\triangleleft} : b \in B\}$ is a basis for γ_R.*

A *nucleus* on a pomonoid $\mathbf{G}$ is a closure operator γ on the poset reduct of $\mathbf{G}$ such that $\gamma(x)\gamma(y) \leq \gamma(xy)$ for all $x, y \in G$. The concept of a nucleus was originally defined in the context of Brouwerian algebras [33] and quantales [32].

Lemma 1.5.2 ([13])**.** *Let γ be a closure operator on a residuated lattice $\mathbf{G}$. Then γ is a nucleus if and only if $y\backslash x, x/y \in G_\gamma$ for all $x \in G_\gamma, y \in G$.*

Let $\mathbf{G} = (G, \wedge, \vee, \cdot, \backslash, /, 1)$ be a residuated lattice, γ a nucleus on $\mathbf{G}$, and for all $x, y \in G$ define $x \cdot_\gamma y = \gamma(x \cdot y)$. $\mathbf{G}_\gamma = (G_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, is called the *$\gamma$-image* or *$\gamma$-retraction* of $\mathbf{G}$.

Lemma 1.5.3 ([11], [12])**.**

1. *If $\mathbf{G}$ is a residuated lattice and γ is a nucleus on it, then the γ-retraction $\mathbf{G}_\gamma$ of $\mathbf{G}$ is a residuated lattice.*
2. *All equations and inequations involving $\{\cdot, \vee, 1\}$ that are satisfied in $\mathbf{G}$ also hold in $\mathbf{G}_\gamma$. For example, if $\mathbf{G}$ is idempotent, commutative, integral or contracting, then so is $\mathbf{G}_\gamma$.*

Nuclei can be expanded to other structures as well. Consider a monoid $\mathbf{W} = (W, \circ, \varepsilon)$. On the power set $\mathscr{P}(W)$ of W we define the operation $X \circ Y = \{z \in W : z = x \circ y \text{ for some } x \in X, y \in Y\}$. We write $x \circ y$ for the set $\{x\} \circ \{y\}$ and $x \circ Y$ for $\{x\} \circ Y$. Also, we define the sets $X/Y = \{z : \{z\} \circ Y \subseteq X\}$ and $Y\backslash X = \{z : Y \circ \{z\} \subseteq X\}$. We obtain that the algebra $\mathscr{P}(\mathbf{W}) = (\mathscr{P}(W), \cap, \cup, \circ, \backslash, /, \{\varepsilon\})$ is a residuated lattice.

For a monoid $(W, \circ, \varepsilon)$ and a set W', the next condition characterizes the relations $R \subseteq W \times W'$ for which γ_R is a nucleus on $\mathscr{P}(W, \circ)$. A relation $N \subseteq W \times W'$ is called *nuclear* on $(W, \circ, \varepsilon)$ if there exist functions $\backslash\!\!\backslash : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$ such that for all $u, v \in W, w \in W'$,

$$u \circ v \; N \; w \quad \text{iff} \quad v \; N \; u \backslash\!\!\backslash w \quad \text{iff} \quad u \; N \; w /\!\!/ v.$$

The following lemma draws the connection between nuclei and nuclear relations.

Lemma 1.5.4 ([10]). *If $(W, \circ, \varepsilon)$ is a monoid, W' is a set, and $N \subseteq W \times W'$, then γ_N is a nucleus on $\mathscr{P}(W, \circ)$ iff N is a nuclear relation.*

A *residuated frame* (as defined in [10]) is a structure of the form

$$\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, \varepsilon),$$

where $(W, \circ, \varepsilon)$ is a monoid, W' is a set, and $N \subseteq W \times W'$ is a nuclear relation on $(W, \circ, \varepsilon)$ with respect to $\backslash\!\!\backslash, /\!\!/$. Concretely, this means

- N is a binary relation from W to W', called the *Galois relation*,
- there exist functions $\backslash\!\!\backslash : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$, and
- $(u \circ v) \; N \; w$ iff $v \; N \; (u \backslash\!\!\backslash w)$ iff $u \; N \; (w /\!\!/ v)$ for all $u, v \in W$ and $w \in W'$.

As a consequence of Lemma 1.5.3 and Lemma 1.5.4, we obtain that $\mathscr{P}(W, \circ)_{\gamma_N}$ is a residuated lattice, called the *Galois algebra* of $\mathbf{W}$ and denoted by $\mathbf{W}^+$.

$\mathbf{W}^+ = (\gamma_N[\mathscr{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\varepsilon))$, where

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y), \quad X \backslash Y = \{z : X \circ z \subseteq Y\}, \quad Y/X = \{z : z \circ X \subseteq Y\},$$

$$X \circ_{\gamma_N} Y = \gamma_N(X \circ Y), \quad X \circ Y = \{z \in W : \exists x \in X, y \in Y (x \circ y = z)\}.$$

Let $\mathbf{A}$ be a residuated lattice and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$. Define $(W, \circ, 1)$ to be the submonoid of $\mathbf{A}$ generated by B. A *unary linear polynomial* of $(W, \circ, 1)$ is a map u on W of the form $u(x) = v \circ x \circ w$, for $v, w \in W$. Such polynomials are also known as *sections* and we denote the set of all sections by S_W. We use id to denote the identity polynomial ($id(x) = x$). Let $W' = S_W \times B$, and define $N \subseteq W \times W'$ by

$$x \; N \; (u, b) \Leftrightarrow u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b.$$

Given $x \in W, b \in B$, and $u \in S_W$, we define

$$x \mathbin{\backslash\!\backslash} (u, b) = \{(u(x \circ \_), b)\}, \text{ and}$$

$$(u, b) \mathbin{/\!/} y = \{(u(\_ \circ y), b)\}.$$

Then we can see that $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \{1\})$ is a residuated frame.

Let $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1\}$ be the language of residuated lattices. A *Gentzen frame* is a pair $(\mathbf{W}, \mathbf{B})$, where

i. $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \{\varepsilon\})$ is a residuated frame with binary operation $\circ$,

ii. $\mathbf{B}$ is a partial $\mathcal{L}$-algebra,

iii. $(W, \circ, \varepsilon)$ is a monoid generated by $B \subseteq W$,

Table 1.2: The theory **GN**.

$$
\begin{array}{cc}
\dfrac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ (\mathrm{CUT}) & \dfrac{}{a\ N\ a}\ (\mathrm{Id}) \\[2ex]
\dfrac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (\cdot\mathrm{L}) & \dfrac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ (\cdot\mathrm{R}) \\[2ex]
\dfrac{x\ N\ a \quad b\ N\ z}{x \circ (a\backslash b)\ N\ z}\ (\backslash\mathrm{L}) & \dfrac{a \circ x\ N\ b}{x\ N\ a\backslash b}\ (\backslash\mathrm{R}) \\[2ex]
\dfrac{x\ N\ a \quad b\ N\ z}{(b/a) \circ x\ N\ z}\ (/\mathrm{L}) & \dfrac{x \circ a\ N\ b}{x\ N\ b/a}\ (/\mathrm{R}) \\[2ex]
\dfrac{a\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}\ell) & \dfrac{b\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}r) \\[2ex]
\dfrac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (\vee\mathrm{L}) & \dfrac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ (\wedge\mathrm{R}) \\[2ex]
\dfrac{x\ N\ a}{x\ N\ a \vee b}\ (\vee\mathrm{R}\ell) & \dfrac{x\ N\ b}{x\ N\ a \vee b}\ (\vee\mathrm{R}r) \\[2ex]
\dfrac{\varepsilon\ N\ z}{1\ N\ z}\ (1\mathrm{L}) & \dfrac{}{a\ N\ a}\ (1\mathrm{R})
\end{array}
$$

iv. there is an injection of B into W' (under which we will identify B with a subset of W'), and

v. N satisfies the rules of **GN** (Table 1.2) for all $a, b \in B$, $x, y \in W$ and $z \in W'$.

A rule in **GN** is understood to hold only in case all the expressions in it make sense. For example, $(\wedge\mathrm{L}\ell)$ is read as "if $a, b, a \wedge b \in B$, $z \in W'$ and $a\ N\ z$, then $a \wedge b\ N\ z$."

Theorem 1.5.5 ([10])**.** $(\mathbf{W_{A,B}}, \mathbf{B})$ *is a Gentzen frame. Furthermore,*

1. *The map* $\{\_\}^{\lhd} : \mathbf{B} \to \mathbf{W}^+_{\mathbf{A,B}}$*, defined by* $b \mapsto \{(id, b)\}^{\lhd}$*, is an embedding of the partial subalgebra* $\mathbf{B}$ *of the residuated lattice* $\mathbf{A}$ *into* $\mathbf{W}^+_{\mathbf{A,B}}$*.*
2. *If an equation over* $\{\vee, \cdot, 1\}$ *is valid in the residuated lattice* $\mathbf{A}$*, then it is also valid in* $\mathbf{W}^+_{\mathbf{A,B}}$*, for every partial subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$*.*

1.6 Distributive frames

Given a lattice expansion $\mathbf{L} = (L, \wedge, \vee, \curlywedge)$, a map γ on L is called a *distributive* $\curlywedge$*-nucleus* if it is a nucleus with respect to $\curlywedge$ and it satisfies $\gamma(x \curlywedge y) = \gamma(x) \wedge \gamma(y)$, namely $\gamma : (L, \curlywedge) \to (L, \wedge)$ is a homomorphism.

Lemma 1.6.1 ([9]). *Let* $\mathbf{L} = (L, \wedge, \vee, \curlywedge)$ *be a lattice expansion and* γ *a distributive* $\curlywedge$*-nucleus on* $\mathbf{L}$*. Then* $\curlywedge_\gamma = \wedge$*. Furthermore,* $\mathbf{L}_\gamma$ *is distributive when* $\curlywedge$ *is a residuated operation on* $\mathbf{L}$*.*

Lemma 1.6.2 ([9]). *Let* $\mathbf{W} = (W, \curlywedge)$ *be a semilattice. Assume* γ *is a distributive* $\curlywedge$*-nucleus on* $\mathscr{P}(W)$*. Then* $\curlywedge_\gamma = \cap$ *and* $\mathscr{P}(W)_\gamma$ *is distributive.*

Given a semilattice $(W, \curlywedge)$, a relation $N \subseteq W \times W'$ is called *distributively nuclear* if it is nuclear with respect to $\curlywedge$, and it satisfies

$$\frac{x\ N\ z}{x \curlywedge y\ N\ z}\ (\curlywedge i)$$

Lemma 1.6.3 ([9]). *If* $(W, \curlywedge)$ *is a semilattice and* $N \subseteq W \times W'$*, then* γ_N *is a distributive nucleus on* $\mathscr{P}(W, \curlywedge)$ *iff* N *is a distributively nuclear relation.*

A *distributive residuated frame* is a structure of the form

$$\mathbf{W} = (W, W', N, \circ, \bbslash, /\!\!/, \curlywedge, \curlyvee\!\!\!\searrow, \swarrow\!\!\!\curlyvee, \varepsilon),$$

where $(W, \circ, \varepsilon)$ is a monoid and $(W, \curlywedge)$ is a semilattice, $N \subseteq W \times W'$ is a nuclear relation on $(W, \circ, \varepsilon)$ with respect to $\bbslash, /\!\!/$, and distributively nuclear on $(W, \curlywedge)$ with respect to $\curlyvee\!\!\!\searrow, \swarrow\!\!\!\curlyvee$. This is equivalent to

- W and W' are sets,
- N is a binary relation from W to W', called the *Galois relation*,

- there exist functions $\mathbin{\backslash\!\!\backslash} : W \times W' \to W'$ and $\mathbin{/\!\!/} : W' \times W \to W'$,
- N is $\circ$-nuclear: for all $x, y \in W$ and $z \in W'$
  - $(x \circ y) \; N \; z$ iff $y \; N \; (x \mathbin{\backslash\!\!\backslash} z)$ iff $x \; N \; (z \mathbin{/\!\!/} y)$.
- N is $\curlywedge$-nuclear: for all $x, y \in W$ and $z \in W'$
  - $(x \curlywedge y) \; N \; z$ iff $y \; N \; (x \searrow z)$ iff $x \; N \; (z \swarrow y)$.
- N is $\curlywedge$-distributive: for all $x, y, z \in W$ and $w \in W'$
  - $x \; N \; w$ implies $x \curlywedge y \; N \; w$.

Lemmas 1.5.3 and 1.5.4 provide that $\mathscr{P}(W, \circ)_{\gamma_N}$ is a distributive residuated lattice, which is the Galois algebra $\mathbf{W}^+$.

A *distributive Gentzen frame of type $\mathcal{L}$* is a pair $(\mathbf{W}, \mathbf{B})$, where

i. $\mathbf{W} = (W, W', N, \circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}, \curlywedge, \searrow, \swarrow, \varepsilon)$ is a distributive frame,

ii. $\mathbf{B}$ is a partial algebra,

iii. $(W, \circ, \curlywedge, \varepsilon)$ is a semilattice and a monoid generated by $B \subseteq W$,

iv. there is an injection of B into W' (under which we will identify B with a subset of W'), and

v. N satisfies the rules of **DGN** (Table 1.3) for all $a, b \in B, x, y \in W$ and $z \in W'$.

We consider a distributive residuated lattice $\mathbf{A}$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$. Define $\mathbf{W} = (W, \circ, \curlywedge, \varepsilon)$ to be the $\{\cdot, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B (note that we use different notation for the restriction of the operations of A on W).

We denote by S_W the set of all unary linear polynomials of $(W, \circ, \curlywedge, \varepsilon)$, namely terms containing a single variable, which appears exactly once. Let $W' = S_W \times B$, as well as the relation N from W to W', given by

$$x \; N \; (u, b) \text{ iff } u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b.$$

Table 1.3: The theory **DGN**.

$$\frac{x \; N \; a \quad a \; N \; z}{x \; N \; z} \text{ (CUT)} \qquad \frac{}{a \; N \; a} \text{ (Id)}$$

$$\frac{\varepsilon \; N \; z}{1 \; N \; z} \text{ (1L)} \qquad \frac{}{a \; N \; a} \text{ (1R)}$$

$$\frac{a \circ b \; N \; z}{a \cdot b \; N \; z} \text{ ($\cdot$L)} \qquad \frac{x \; N \; a \quad y \; N \; b}{x \circ y \; N \; a \cdot b} \text{ ($\cdot$R)}$$

$$\frac{x \; N \; a \quad b \; N \; z}{x \circ (a\backslash b) \; N \; z} \text{ ($\backslash$L)} \qquad \frac{a \circ x \; N \; b}{x \; N \; a\backslash b} \text{ ($\backslash$R)}$$

$$\frac{x \; N \; a \quad b \; N \; z}{(b/a) \circ x \; N \; z} \text{ (/L)} \qquad \frac{x \circ a \; N \; b}{x \; N \; b/a} \text{ (/R)}$$

$$\frac{a \curlywedge b \; N \; z}{a \wedge b \; N \; z} \text{ ($\wedge$L)} \qquad \frac{x \; N \; a \quad x \; N \; b}{x \; N \; a \wedge b} \text{ ($\wedge$R)}$$

$$\frac{x \; N \; a}{x \; N \; a \vee b} \text{ ($\vee$Rℓ)} \qquad \frac{x \; N \; b}{x \; N \; a \vee b} \text{ ($\vee$Rr)}$$

$$\frac{a \; N \; z \quad b \; N \; z}{a \vee b \; N \; z} \text{ ($\vee$L)} \qquad \frac{x \; N \; z}{x \curlywedge y \; N \; z} \text{ ($\curlywedge i$)}$$

Given $x \in W$, $b \in B$ and $u \in S_W$, let $x \mathbin{\backslash\!\backslash} (u, b) = \{(u(x \circ \_), b)\}$ and $(u, b) \mathbin{/\!/} y = \{(u(\_ \circ y), b)\}$. Notice that these definitions coincide with the ones presented in the previous section.

We define the relations $\curlyvee\!\!\!\!\searrow$ and $\swarrow\!\!\!\!\curlyvee$ in an analogous manner.

$$x \curlyvee\!\!\!\!\searrow (u, b) = \{(u(x \curlywedge \_), b)\} \qquad (u, b) \swarrow\!\!\!\!\curlyvee\, y = \{(u(\_ \curlywedge y), b)\}.$$

Theorem 1.6.4 ([9])**.** *The structure* $\mathbf{W_{A,B}} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \curlywedge, \curlyvee\!\!\!\!\searrow, \swarrow\!\!\!\!\curlyvee, \{\varepsilon\})$ *is a distributive frame and* $(\mathbf{W_{A,B}}, \mathbf{B})$ *is a distributive Gentzen frame. Therefore*

1. *The map* $b \mapsto \{(id, b)\}^{\triangleleft}$ *is a (partial algebra) embedding of* $\mathbf{B}$ *into* $\mathbf{W}^+_{\mathbf{A,B}}$*.*
2. *If an equation over* $\{\wedge, \vee, \cdot, 1\}$ *is valid in a fully-distributive residuated lattice* $\mathbf{A}$*, then it is also valid in* $\mathbf{W}^+_{\mathbf{A,B}}$ *for every partial subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$*. Partic-*

ularly, the Galois algebra $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ *is a fully-distributive residuated lattice (and* $\wedge_N$ *is intersection).*

Chapter 2

The FEP for subvarieties of RL_m^n axiomatized by $xyx = x^2y$

While our main goal is to prove the FEP for subvarieties of RL_m^n axiomatized by the general equation $xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}$, we will take a detour by working with the simplest noncommutative form $xyx = x^2y$. As the general case is quite involved, we deal with this simple case first to illustrate our approach, but we reuse later (without repetition) the part of the theory that we develop in this basic case.

Let $\mathcal{V}$ be the subvariety of RL_m^n axiomatized by $xyx = x^2y$. To show that $\mathcal{V}$ has the FEP, we will find the free $|B|$-generated pomonoid $\mathbf{F}$ for the corresponding pomonoid class $\mathcal{P}$ satisfying $xyx = x^2y$ and $x^m \leq x^n$ for $m \neq n$, $m \geq 1$. We start by describing its monoid reduct $\mathbf{K}$ and to achieve this objective, we study the variety $\mathcal{M}$ of monoids axiomatized by $xyx = x^2y$. Later, we consider separately the two cases $m > n$ and $m < n$, as they require slightly different arguments in the proof of finiteness.

2.1 The construction

Given a residuated lattice $\mathbf{A}$ and a subset B of it, we rely on the construction presented in Section 1.5 to obtain the algebra $\mathbf{D}$ involved in the definition of the FEP. Under the assumptions that $\mathbf{A}$ is in the appropriate variety and B is finite, we will establish the finiteness of $\mathbf{D}$.

We first consider the submonoid $\mathbf{W} = (W, \circ, 1)$ of $\mathbf{A}$ generated by B; we use $\circ$ for the restriction of the multiplication of $\mathbf{A}$ to W in order to avoid confusion. Recall the construction of the residuated frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$. S_W is the set of sections, $W' = S_W \times B$, and the binary relation N from W to W' is defined as

$$xN(u,b) \Leftrightarrow u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b.$$

Furthermore, for $X \subseteq W$ and $Z \subseteq W'$ we have

$$X^{\rhd} = \{z \in W' : x \; N \; z, \text{ for all } x \in X\} \quad \text{and}$$
$$Z^{\lhd} = \{x \in W : x \; N \; z, \text{ for all } z \in Z\}.$$

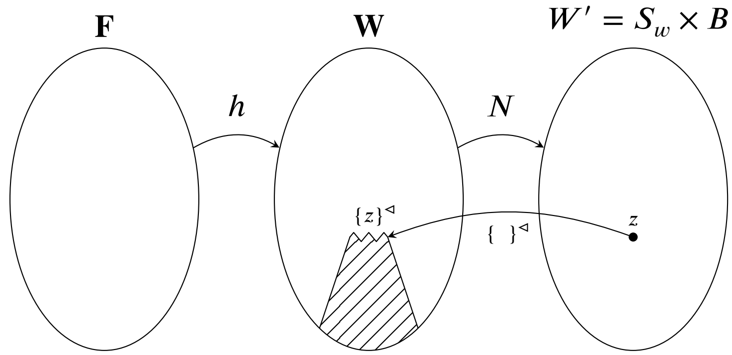
By Theorem 1.5.5, the algebra

$$\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}} = (\gamma[\mathscr{P}(W)], \cap, \cup_\gamma, \cdot_\gamma, \backslash, /, \gamma(\{1\}))$$

is a residuated lattice and the map $b \mapsto \{(id, b)\}^{\lhd}$ is an embedding of the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$. The closed sets $\{z\}^{\lhd}$ for $z \in W'$ form a basis for $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$, namely each closed set is an intersection of basic closed sets, and $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ belongs to all varieties of residuated lattices axiomatized over $\{\vee, \cdot, 1\}$ that contain $\mathbf{A}$.

We will take $\mathbf{D}$ to be the algebra $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$. Clearly, to prove that $\mathbf{D}$ is finite, it suffices to prove that there are only finitely many basic closed sets. Note that these sets $\{z\}^{\lhd} = \{x \in W : x \; N \; z\}$, for $z \in W'$, are actually downsets in $(W, \leq^{\mathbf{A}})$.

Figure 2.1: A residuated frame construction.



To prove finiteness we will be considering a relatively free pomonoid **F** and an order-preserving surjective homomorphism $h : \mathbf{F} \to \mathbf{W}$. It will be important to show that the underlying poset of **F** is dually well partially ordered. Figure 2.1 gives a pictorial representation of the situation.

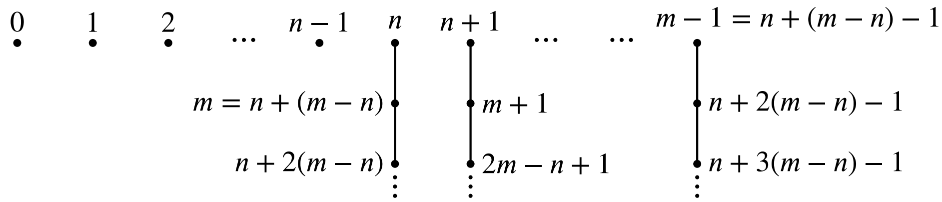
To guarantee the existence of the pomonoid map h in the above diagram, the auxiliary pomonoid **F** will be chosen to be free over a class of pomonoids that includes $(W, \leq^{\mathbf{A}}, \circ, 1)$. This class will satisfy the identities $x^m \leq x^n$ and $xyx = x^2y$. As W is generated by B as a monoid, we will take **F** to be $|B|$-generated.

For illustrative purposes we first consider the 1-generated case. In this case multiplication is commutative. As it turns out, the 1-generated free pomonoid satisfying $x^m \leq x^n$ is easy to describe. Nevertheless, this structure will provide important insights about the consequences of the knotted inequality.

2.2 The 1-generated pomonoid satisfying $x^m \leq x^n$

For the 1-generated case, the monoid reduct is clearly $(\mathbb{N}, +, 0)$. We will start by considering the case $m > n$. The order, which we denote by $\leq^m_n$, is the one used

Figure 2.2: The order $\leq_n^m$ for $m > n$.



in [34]. Clearly, $m \leq_n^m n$, as this is the translation of $x^m \leq x^n$ in additive notation, where the generator of $\mathbb{N}$ is the number 1. The order needs to capture all (1-variable) consequences of this identity. Motivated by these consequences (see Lemma 2.2.1), for $m > n$ we define the relation $\leq_n^m$ on the nonnegative integers $\mathbb{N}$ by

$$u \leq_n^m v \text{ if and only if } u = v, \text{ or } n \leq v < u \text{ and } u \equiv v \pmod{m - n}.$$

The corresponding Hasse diagram is shown in Figure 2.2.

It is easy to see that this relation is a partial order on $\mathbb{N}$ that is compatible with $+$ in the sense that $u \leq_n^m v$ implies $u + w \leq_n^m v + w$ for all $u, v, w \in \mathbb{N}$. Consequently, for all $u \in \mathbb{N}, mu \leq_n^m nu$.

Lemma 2.2.1. *Assume that a pomonoid $(P, \preccurlyeq, \cdot, 1)$ satisfies $x^m \preccurlyeq x^n$, for all $x \in P$ and fixed $m > n \in \mathbb{N}$. Then $x^u \preccurlyeq x^v$ whenever $u \leq_n^m v$.*

Proof. For $j > i \in \mathbb{N}$,

$$\begin{aligned} x^m = x^{n+(m-n)} &\preccurlyeq x^n \\ \Rightarrow x^{n+(i+1)(m-n)} &\preccurlyeq x^{n+i(m-n)} \\ \Rightarrow \quad x^{n+j(m-n)} &\preccurlyeq x^{n+i(m-n)} \end{aligned}$$

If $u \leq_n^m v$, we have two possibilities. The first, $u = v$, implies that $x^u \preccurlyeq x^v$. Otherwise, $n \leq v < u$ and $u \equiv v \pmod{m-n}$. Then, for some $j \in \mathbb{N}, u = v + j(m-n) = n + (v-n) + j(m-n)$.

From the previous result $x^{n+j(m-n)} \preccurlyeq x^{n+0(m-n)}$. Multiplying both sides by x^{v-n}, we obtain the result $x^u \preccurlyeq x^v$. □

Using the lemma, it is easy to see (and we prove it more generally in Lemma 2.3.2) that $(\mathbb{N}, \leq_n^m, +, 0)$ is the free 1-generated pomonoid satisfying $x^m \leq x^n$, when $m > n$; when $m < n$ we obtain the dual order.

Under the assumption of commutativity the free k-generated pomonoid satisfying $x^m \leq x^n$ is simply the kth power of the 1-generated one. This is exploited in [34] to establish the FEP for the commutative case.

2.3 Finitely generated free algebras for $\mathcal{M}$ and $\mathcal{P}$

We first identify and construct the free algebra $\mathbf{K}$ over k generators for the variety of monoids $\mathcal{M}$ axiomatized by $xyx = x^2y$; later we will take k to be $|B|$ and $\mathbf{K}$ will serve as the monoid reduct of $\mathbf{F}$. Clearly, $\mathbf{K}$ is a homomorphic image of the free k-generated monoid of words in a k-element alphabet $\{z_1, \dots, z_k\}$, and the elements of $\mathbf{K}$ can be viewed as equivalence classes of such words.

Note that the defining equation $xyx = x^2y$ implies that $x^i y x^j = x^{i+j} y$, for $i, j \geq 1$. Hence, each word is equivalent to one where all occurrences of each generator are consecutive. Thus every element of $\mathbf{K}$ can be identified by such a word. Even though viewing the elements of $\mathbf{K}$ as such representative words is useful (and we will need to consider it later as a set that we call H in the general case), here we opt for an equivalent description directly. Note that each such word is completely specified by the order in which the generators appear and the list of their exponents.

For instance, if we have 5 generators $\{z_1, z_2, z_3, z_4, z_5\}$, then

$$z_5^3 z_1^4 z_3^2$$

can be encoded by the list of the exponents of the generators $(4, 0, 2, 0, 3)$ and the order in which the generators appear 513. In the order 513 we only include the indices of the generators that have exponents greater than 0. With this encoding, $z_5^3 z_1^4 z_3^2$ would be represented as $((4, 0, 2, 0, 3), 513)$.

To capture the above order of generators, we consider the set S of all words over $\{1, 2 \ldots, k\}$ where each element appears at most once (*linear* words). For $s \in S$, let $|s|$ denote the length of s. When $i > 0$, we define s_i to represent the ith position in s, where $i \leq |s|$. We also define the *content* of a word

$$\operatorname{cont}(s) = \{s_i : i \leq |s|\},$$

or set of elements that appear in s. Finally, we define the *support* of a vector

$$\operatorname{supp}(x_1, \ldots, x_j) = \{i : 1 \leq i \leq j \text{ and } x_i \neq 0\}.$$

It is clear, then, that the underlying set of $\mathbf{K}$ can be taken to be

$$F = \{(\vec{x}, s) \in \mathbb{N}^k \times S : \operatorname{cont}(s) = \operatorname{supp}(\vec{x})\}.$$

The benefit of this alternative description of the elements is that now the operation in $\mathbf{K}$ is easier to explain. Note that when we multiply two words in $\mathbf{K}$, the exponents of the corresponding generators are added.

On the other hand, the order of the generators is consolidated following the rule $xyx = xy$, as it is visible in the following calculation

$$\begin{aligned}(z_5^3 z_1^4 z_3^2)(z_1 z_2^3 z_4 z_3^2) &= z_5^3(z_1^4 z_3^2 z_1) z_2^3 z_4 z_3^2 \\ &= z_5^3 z_1^5 (z_3^2 (z_2^3 z_4) z_3^2) \\ &= z_5^3 z_1^5 z_3^4 z_2^3 z_4,\end{aligned}$$

which translates to

$$((4,0,2,0,3), 513)((1,3,2,1,0), 1243) = ((5,3,4,1,3), 51324).$$

Therefore, we endow S with the operation of concatenation followed by deletion for each generator of all occurrences except the leftmost one. It is clear then that $\mathbf{S} = (S, \cdot, \varepsilon)$ is the free idempotent monoid on k generators $\{1, 2 \ldots, k\}$ that satisfies the equation $xyx = xy$. Clearly S is finite; in fact,

$$|S| = \sum_{i=0}^{k} \binom{k}{i} i!$$

Let $\cdot^{\mathbf{F}}$ be the binary operation in the direct product of the monoids $(\mathbb{N}^k, +, 0)$ and $\mathbf{S}$, both of them in $\mathcal{M}$. We define the algebra $\mathbf{K} = (F, \cdot^{\mathbf{F}}, 0)$, which is a submonoid of the direct product $(\mathbb{N}^k, +, 0) \times \mathbf{S}$, and prove that $\mathbf{K}$ is the free object in $\mathcal{M}$. The notation for the operation in $\mathbf{K}$ is chosen in anticipation of the fact that $\mathbf{K}$ will be the monoid reduct of $\mathbf{F}$.

Lemma 2.3.1. $\mathbf{K} = (F, \cdot^{\mathbf{F}}, 0)$ *is the free algebra in* $\mathcal{M}$ *on* k *generators. Moreover, the set* F *is generated by the vectors:* $z_1 = ((1, 0, \ldots, 0), 1), z_2 = ((0, 1, \ldots, 0), 2)$, $\ldots$, $z_k = ((0, 0, \ldots, 1), k)$. *Also, the identity element is* $0 = (\vec{0}, \varepsilon)$.

Proof. First, we show that F is generated by the aforementioned vectors. For $x = (\vec{x}, s) \in F$, we have that $x_i = 0$ when $i \notin \mathrm{cont}(s)$, since $\mathrm{cont}(s) = \mathrm{supp}(\vec{x})$. Note that $z_{s_i}^{x_{s_i}} = ((0, \dots, x_{s_i}, \dots, 0), s_i)$, where x_{s_i} appears in the s_ith position and since $s = s_1 \dots s_{|s|}$, we have

$$(\vec{x}, s) = z_{s_1}^{x_{s_1}} \cdots^{\mathbf{F}} z_{s_{|s|}}^{x_{s_{|s|}}}.$$

We will show that $\mathbf{K}$ has the universal mapping property. Consider a monoid $\mathbf{M} \in \mathcal{M}$, the set of generators $Z = \{z_i : 1 \leq i \leq k\}$, the map inc given by $inc(z_j) = z_j$, and a map h_1, as in the following diagram.

$$\begin{tikzcd}
F \arrow[r, dashed, "h"] & M \\
Z \arrow[u, hook, "inc"] \arrow[ur, "h_1"'] &
\end{tikzcd}$$

We write m_i for $h_1(z_i)$ and define the function $h : F \to M$ by

$$h((\vec{0}, \varepsilon)) = u_M, \text{ the unit of } \mathbf{M},$$
$$h((x_1, \dots, x_k), s) = m_{s_1}^{x_{s_1}} \cdots^{\mathbf{M}} m_{s_{|s|}}^{x_{s_{|s|}}}.$$

Clearly,

$$h(z_j) = h_1(z_j) \text{ for all } j \in \{1, \dots, k\}.$$

To show that h is a homomorphism, take $x = (\vec{x}, s), y = (\vec{y}, t) \in F$. Note that $x_i = 0$ when $i \notin \mathrm{cont}(s)$ and $y_i = 0$ when $i \notin \mathrm{cont}(t)$. Let $p = |s|$ and $q = |t|$.

$$\begin{aligned}
h(x) \cdot^{\mathbf{M}} h(y) &= m_{s_1}^{x_{s_1}} \cdots^{\mathbf{M}} m_{s_p}^{x_{s_p}} \cdot^{\mathbf{M}} m_{t_1}^{y_{t_1}} \cdots^{\mathbf{M}} m_{t_q}^{y_{t_q}} \\
&= m_{(s\cdot t)_1}^{x_{(s\cdot t)_1} + y_{(s\cdot t)_1}} \cdots^{\mathbf{M}} m_{(s\cdot t)_{|s\cdot t|}}^{x_{(s\cdot t)_{|s\cdot t|}} + y_{(s\cdot t)_{|s\cdot t|}}} \quad (\text{since } \mathbf{M} \in \mathcal{M}) \\
&= h((\vec{x} + \vec{y}, s \cdot t)) \\
&= h(x \cdot^{\mathbf{F}} y).
\end{aligned}$$

The uniqueness of h follows from the fact that F is generated by Z. $\square$

We consider the case $x^m \leq x^n$ where $m > n$. Let $\mathbf{F}$ be the subpomonoid of the direct product $(\mathbb{N}, \leq_n^m, +, 0)^k \times (S, =, \cdot, \varepsilon)$ with underlying set F. More explicitly, for $x, y \in F$, we have $x \leq^{\mathbf{F}} y$ iff $\vec{x} \leq_n^m \vec{y}$ and $s_x = s_y$.

Lemma 2.3.2. $(F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0)$ *is the free k-generated object in* $\mathcal{P}$ *when* $m > n$.

Proof. Consider $\mathbf{M} \in \mathcal{P}$ with the order $\leq^{\mathbf{M}}$. The monoid reduct of $\mathbf{M}$ belongs to $\mathcal{M}$. We will verify that the knotted inequality is satisfied in $\mathbf{F}$. Let $x = (\vec{x}, s) \in F$, then $x^n = (n \cdot \vec{x}, s)$ and $x^m = (m \cdot \vec{x}, s)$. Recall that $\mathbf{S}$ is idempotent, so $s = s^n = s^m$. Furthermore, $x_i \leq_n^m x_i$ implies that $mx_i \leq_n^m nx_i$ for all $i \in \{1, \dots, k\}$. Therefore $x^m \leq^{\mathbf{A}} x^n$.

Consider the map h from the universal mapping property for $\mathbf{K}$. We only need to show that this map is order-preserving.

For $x, y \in F$ such that $x \leq^{\mathbf{F}} y$, we set $s = s_x = s_y$ and $\ell = |s|$. By construction,

$$h(x) = m_{s_1}^{x_{s_1}} \cdots^{\mathbf{M}} m_{s_\ell}^{x_{s_\ell}}.$$
$$h(y) = m_{s_1}^{y_{s_1}} \cdots^{\mathbf{M}} m_{s_\ell}^{y_{s_\ell}}.$$

For every $i \in \mathrm{cont}(s)$, $x_i \leq_n^m y_i$, which implies that $m_i^{x_i} \leq^{\mathbf{M}} m_i^{y_i}$ by Lemma 2.2.1. Multiplying these inequalities in the order determined by s, we obtain that $h(x) \leq^{\mathbf{M}} h(y)$. □

In the next section, we will make use of the fact that $\mathbf{F}$ is actually dually well partially ordered to prove finiteness of the algebra $\mathbf{D}$. Recall that a partially ordered set is said to be well partially ordered if it has no infinite antichains and no infinite descending chains.

Note that $(\mathbb{N}, \leq_n^m)$ is the disjoint union of n one-element chains and $(m-n)$ chains isomorphic to $(\mathbb{Z}^-, \leq)$. Thus, it is a dually well partially ordered set. This order can be extended to the direct product $(\mathbb{N}, \leq_n^m)^k$, which is dually well partially ordered as well. Also, $(S, =)$ is dually well partially ordered since it is finite. Then, it follows that $(F, \leq^{\mathbf{F}})$ is dually well partially ordered.

2.4 The FEP for $\mathcal{V}$ when $m > n$

Let $B = \{b_1, \ldots, b_k\}$ and extend the assignment $h_1 : Z \to W$ that sends $z_i \mapsto b_i$ for each $i = 1, \ldots, k$ to a pomonoid homomorphism $h : (F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0) \to (W, \leq^{\mathbf{A}}, \circ, 1)$ by the universal mapping property. The map h is surjective because B generates $(W, \circ, 1)$.

To show that $\mathbf{D} = \mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $\{z\}^{\lhd} = \{x \in W : xNz\}$, for $z \in W'$. Now, for each $b \in B$, we define $C_b = \{\{(u, b)\}^{\lhd} : u \in S_W\}$. We will show that the growth of the poset $(C_b, \supseteq)$ is bounded in every possible direction.

Recall that a poset is well partially ordered if and only if it has no bad sequences, where a sequence $(p_i)_{i\in\mathbb{N}}$ is bad when $i < j$ implies that $p_i \nleq p_j$. Obviously, a poset is dually well partially ordered if and only if the dual order has no bad sequences.

Lemma 2.4.1. *For each $b \in B$, $(C_b, \supseteq)$ is a dually well partially ordered set.*

Proof. It suffices to show that $(C_b, \supseteq)$ is a homomorphic image of $(F^2, \leq^{\mathbf{F}})$. Define $\varphi : F^2 \to C_b$ by $\varphi(y, w) = \{(h(y) \circ \_ \circ h(w), b)\}^{\lhd}$. The map φ is necessarily surjective because h is surjective.

Now, we only need to show that φ is order-preserving. Let $(y_1, w_1), (y_2, w_2) \in F^2$ be such that $(y_1, w_1) \leq^{\mathbf{F}} (y_2, w_2)$. We have that for all $x \in F$,

$$y_1 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_1 \leq^{\mathbf{F}} y_2 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_2$$
$$\Rightarrow h(y_1) \circ h(x) \circ h(w_1) \leq^{\mathbf{A}} h(y_2) \circ h(x) \circ h(w_2).$$

If $x \in \varphi(y_2, w_2)$, then $h(y_2) \circ h(x) \circ h(w_2) \leq^{\mathbf{A}} b$. Thus $h(y_1) \circ h(x) \circ h(w_1) \leq^{\mathbf{A}} b$, which implies that $x \in \varphi(y_1, w_1)$. So $\varphi(y_1, w_1) \supseteq \varphi(y_2, w_2)$. This proves that $(C_b, \supseteq)$ is a dually well partially ordered set. □

The previous lemma shows that $(C_b, \supseteq)$ has no infinite antichains or infinite ascending chains.

Lemma 2.4.2. $(C_b, \supseteq)$ *has no infinite descending chains.*

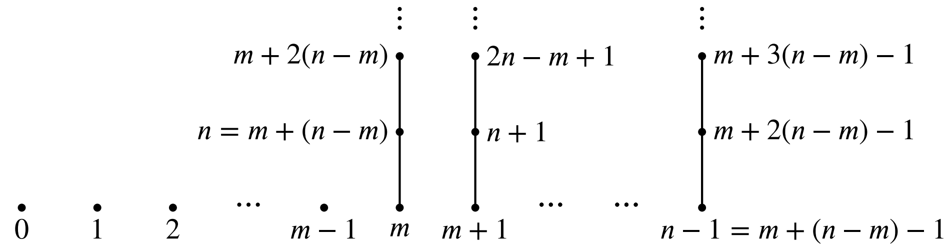
Proof. Suppose that there exists an infinite descending chain

$$\{(u_1, b)\}^{\triangleleft} \subset \{(u_2, b)\}^{\triangleleft} \subset \dots$$

in C_b. For each $i \in \mathbb{Z}^+$, choose $w_i \in \{(u_i, b)\}^{\triangleleft} \setminus \{(u_{i-1}, b)\}^{\triangleleft}$. Note that for $i < j$, we cannot have have $w_i \geq^{\mathbf{A}} w_j$, because $\{(u_i, b)\}^{\triangleleft}$ is a downset. Therefore, $(w_i)_{i \in \mathbb{N}}$ is a bad sequence in the dual of $(W, \leq^{\mathbf{A}})$, which is a contradiction as the latter is a homomorphic image of the dually well partially ordered set $(F, \leq^{\mathbf{F}})$. □

Since $(C_b, \supseteq)$ has no infinite ascending chains, no infinite descending chains and no infinite antichains, C_b is finite for every $b \in B$. Thus, there are finitely many sets of the form $\{(u, b)\}^{\triangleleft}$ because B is finite.

Figure 2.3: The order $\leq_n^m$ for $m < n$.



2.5 The FEP for $\mathcal{V}$ when $m < n$

For $m < n$, we define the relation $\leq_n^m$ (notice that the bigger value is the index n in this order) on $\mathbb{N}$ by:

$$u \leq_n^m v \text{ if and only if } u = v, \text{ or } m \leq u < v \text{ and } v \equiv u \pmod{n - m}.$$

The corresponding Hasse diagram is in Figure 2.3.

This order is the dual of the order $\leq_m^n$, i.e., $\leq_n^m = \geq_m^n$ and $(\mathbb{N}, \leq_n^m)^k$ is a well partially ordered set.

As before, we construct the free pomonoid $(F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0)$ for this class. In this case, $(F, \leq^{\mathbf{F}})$ is a well partially ordered set. By the universal mapping property, there exists a surjective homomorphism $h : (F, \cdot^{\mathbf{F}}, 0, \leq^{\mathbf{F}}) \to (W, \circ, 1, \leq^{\mathbf{A}})$. Following the previous construction we conclude that it is sufficient to show that there are only finitely many closed sets of the form $\{(u, b)\}^{\triangleleft}$ with $u \in S_W, b \in B$.

Once again, we define $C_b = \{\{(u, b)\}^{\triangleleft} : u \in S_W\}$. From Lemma 2.4, we have that for each $b \in B, (C_b, \supseteq)$ is a well partially ordered set (note that F carries the dual order of the last section), so it has no infinite antichains or descending chains.

Lemma 2.5.1. $(C_b, \supseteq)$ *has no infinite ascending chains.*

Proof. Suppose that there exists an infinite ascending chain

$$\{(u_1, b)\}^{\lhd} \supset \{(u_2, b)\}^{\lhd} \supset \dots$$

in C_b. For each $i \in \mathbb{Z}^+$, choose $w_i \in \{(u_i, b)\}^{\lhd} \setminus \{(u_{i+1}, b)\}^{\lhd}$. Suppose that for some $i < j$, we have $w_i \leq^{\mathbf{A}} w_j$. Then $w_i \in \{(u_j, b)\}^{\lhd}$ because $\{(u_j, b)\}^{\lhd}$ is a downset. From $\{(u_{i+1}, b)\}^{\lhd} \supseteq \{(u_j, b)\}^{\lhd}$, it follows that $w_i \in \{(u_{i+1}, b)\}^{\lhd}$. However, this contradicts the fact that $w_i \in \{(u_i, b)\}^{\lhd} \setminus \{(u_{i+1}, b)\}^{\lhd}$. We conclude that $i < j \Rightarrow w_i \nleq^{\mathbf{A}} w_j$ and $(w_i)_{i \in \mathbb{N}}$ is a bad sequence in W. Since $(W, \leq^{\mathbf{A}})$ is a homomorphic image of $(F, \leq^{\mathbf{F}})$, it follows that the latter is not a well partially ordered set, which is a contradiction. Therefore, we cannot have an infinite ascending chain. □

Since $(C_b, \supseteq)$ has no infinite ascending chains, infinite descending chains and no infinite antichains, C_b is finite for every $b \in B$. Thus, there are finitely many sets of the form $\{(u, b)\}^{\lhd}$ because B is finite.

From the last two sections we obtain the following theorem.

Theorem 2.5.2. *The variety axiomatized relative to* RL *by* $x^m \leq x^n$*, for some natural numbers* $m \neq n$*,* $m \geq 1$*, and by* $xyx = x^2y$ *has the finite embeddability property.*

Chapter 3

A noncommutative monoid equation

Our basic equation $xyx = x^2y$ can be generalized to a larger set of variables as follows. Given a vector $a = (a_0, a_1, \dots, a_r) \in \mathbb{N}^{r+1}$, where $\sum_{i=0}^{r} a_i = r+1$ and $\prod_{i=0}^{r} a_i = 0$, (namely an additive, nontrivial, decomposition of the number $r+1$), we consider the equation

$$xy_1xy_2\cdots y_rx = x^{a_0}y_1x^{a_1}y_2\cdots y_rx^{a_r}. \tag{a}$$

The condition on the product of the a_i's being equal to 0 is equivalent to some exponent being equal to 0.

In order to study the FEP for varieties axiomatized by (a), we need to describe the monoid reduct. We define $\mathcal{K}(a)$ to be the variety of monoids axiomatized by (a). For these purpose we will prove certain properties that this equation implies.

We already studied the simplest noncommutative version in Chapter 2 when $a = (2, 0)$. In this chapter, for an arbitrary a, we will study the equation (a) in full generality. We will obtain a result that resembles commutativity for very specific subwords of certain words that we describe explicitly. However, we are going to find that some pieces of our words will not be affected by the equation. These pieces will be referred to as *walls*.

The main theorem in this chapter will take advantage of the fact that some of the elements between the walls can be moved. The purpose of this theorem is to produce a normal form for some elements of monoids in the variety $\mathcal{K}(a)$. In the next chapter, we will use these results to characterize a relatively free algebra $\mathbf{F}$.

3.1 The variety $\mathcal{K}(a)$

We fix a countable set of variables $Y = \{x, y_1, y_2, y_3, \dots\}$. As usual Y^* denotes the set of all words (finite sequences) over Y. For $s \in Y^*$, we define $s^x_{i,j}$ when $i \leq j$ to be the subword of s between the ith and the jth occurrences of x, inclusively.

$$s = \cdots \underbrace{\overset{\substack{i\text{th}\\ \downarrow}}{x} \cdots \overset{\substack{j\text{th}\\ \downarrow}}{x}}_{s^x_{i,j}} \cdots$$

We even allow it to be defined for $j = r + 1$, where r is equal to the number of occurrences of x in s, so as to include the (final) part of s to the right of the last occurrence of x.

For convenience, we consider the infinite word $w = xy_1xy_2xy_3\cdots$, and extend the above notation to $w^x_{i,j} = xy_i \cdots y_{j-1}x$, for $i \geq 1, j \geq 0$. In the following we abbreviate $w^x_{i,j}$ to $w_{i,j}$. Also, for $i > j$, $w_{i,j} = 1$. Throughout the paper we will be working modulo the theory of monoids and we will identify terms in the language of monoids with words; for example, we will simply write xyx for $(xy)x$ or $x(yx)$ following standard practice of omitting parentheses. Also, we will allow substitution instances of the equations we will be considering.

Note that the left-hand side of (a) is simply $w_{1,r+1}$. The position of the x's with respect to the y_i's will change when moving from the left-hand side to the right-hand side of the equation. For example, we can represent the equation $w_{1,8} = y_1x^3y_2y_3y_4xy_5y_6x^3y_7x$ in the following manner

$$\begin{array}{ccccccccccccccc}
x & y_1 & x & y_2 & x & y_3 & x & y_4 & x & y_5 & x & y_6 & x & y_7 & x \\
\searrow & & \downarrow & & \swarrow & & \searrow & & \searrow & & \searrow & & \downarrow & & \downarrow \\
y_1 & x & x & x & y_2 & y_3 & y_4 & x & y_5 & y_6 & x & x & x & y_7 & x
\end{array}$$

From this diagram, we can see that some x's remain in the same position relative to the y_i's. We say that these variables *drop down*. On the other hand, the rest will shift left or right. In the previous case the second, seventh, and eighth occurrences of x drop down, the third *shifts left*, and the rest *shift right*. Note that a right shift can not be immediately followed by a left shift. In the rest of the document, we use the word 'shift' to refer to either a right or a left shift. Note that a variable, such as the x between y_4 and y_5 above, may shift (to the right) by more than one position.

For a given word $t \in Y^*$, we define $\mathrm{del}_x(t)$ to be the result of erasing all occurrences of the variable x in the word t. We also define $|t|_x$ to be the number of occurrences of x in the word t. For example

$$\mathrm{del}_x(w_{i,j}) = y_iy_{i-1}\cdots y_{j+1} \text{ and}$$

$$|w_{i,j}|_x = j - i + 1.$$

We consider initial and final subwords of the left-hand side of (a) that start and end with an x and all occurrences of x inside them drop down. A *wall* (initial or final) will be such a subword (possibly empty) of maximal length. For instance, in the case of the equation $w_{1,8} = y_1x^3y_2y_3y_4xy_5y_6x^3y_7x$, the front (initial) wall is empty, and the back (final) wall is $w_{7,8} = xy_7x$.

We now define two different types of walls. A front wall is *attractive* if the first occurrence of x after the wall shifts left. By symmetry, a back wall is attractive if the last occurrence of x before the wall shifts right. A wall is *repulsive* if it is not attractive. So a front wall is repulsive if the closest x variable on the right of it shifts right. For example, in $w_{1,8} = y_1 x^3 y_2 y_3 y_4 x y_5 y_6 x^3 y_7 x$ the (empty) front wall is repulsive and the back wall is attractive. Note that attractive walls are never empty.

We define some additional convenient notation. For $s \in Y^*$, we define

$$\begin{aligned} s^x_{\underset{\rightarrow}{i}} &= s^x_{0,i} \\ s^x_{\underset{\leftarrow}{j}} &= s^x_{|s|_x - j+1, |s|_x+1}. \end{aligned}$$

So, $s^x_{\underset{\rightarrow}{i}}$ is the initial segment of s ending in x and containing the first i-many x's in s; similarly $s^x_{\underset{\leftarrow}{j}}$ is the final segment of s starting with x and containing the last j-many x's.

Consider (a) and let p and q be the number of x's in the front and back wall, respectively. In other words, $(w_{1,r+1})^x_{\underset{\rightarrow}{p}}$ and $(w_{1,r+1})^x_{\underset{\leftarrow}{q}}$ are the front and back wall, respectively. For $u = w^x_{1,r}$ we define u^x_{mid} to be the part of u strictly between $u^x_{\underset{\rightarrow}{p}}$ and $u^x_{\underset{\leftarrow}{q}}$. Hence

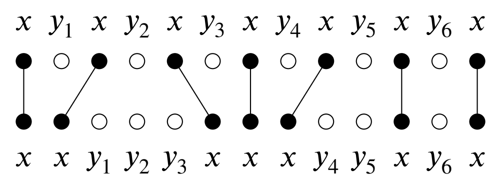
$$u = u^x_{\underset{\rightarrow}{p}} u^x_{\mathrm{mid}} u^x_{\underset{\leftarrow}{q}}.$$

If the front wall is attractive, then (a) yields

$$u = u^x_{\underset{\rightarrow}{p}} x u' u^x_{\underset{\leftarrow}{q}}, \tag{3.1.1}$$

where $|u'|_x = |u^x_{\mathrm{mid}}|_x - 1$ and $\mathrm{del}_x(u') = y_p y_{p+1} \cdots y_{r-q+1}$. Attractive back walls behave similarly. The final effect of the original equation is to attract one or more x's next to the attractive wall.

Figure 3.1: Graphical representation of an instance of equation (a).



On the other hand, if a wall is repulsive, x's are pushed away from it. This becomes clear when we look at the previous setup with a repulsive front wall. In this case (a) yields

$$u = \underset{\rightarrow}{u^x_p}\, y_p y_{p+1} u' \underset{\leftarrow}{u^x_q}\,, \tag{3.1.2}$$

where $|u'|_x = |u^x_{\text{mid}}|_x$ and $\text{del}_x(u') = y_{p+2} y_{p+3} \cdots y_{r-q+1}$.

3.2 Examples

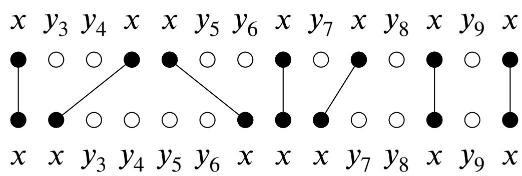
We will focus our attention on some specific examples to show how (a) can be used to change the position of some of the occurrences of x. All the substitution instances below will fix x.

Consider the equation

$$w_{1,7} = x^2 y_1 y_2 y_3 x^3 y_4 y_5 x y_6 x. \tag{3.2.1}$$

We can represent it by a diagram. Figure 3.1 represents the equation graphically. In this figure black circles denote occurrences of x and white circles denote y_i's. In this example, the front is x and the back wall is xy_6x. The front wall is attractive, while the back wall is repulsive.

Note that substitutions that fix the variable x allow us to treat consecutive (possibly empty) sequences of y_i's as if they were a single y_i in the application of (a). In other words, maximal (possibly empty) blocks of consecutive white dots between black dots can be treated as a single white dot. The following diagram is an example of the latter fact as such an instance of (a).



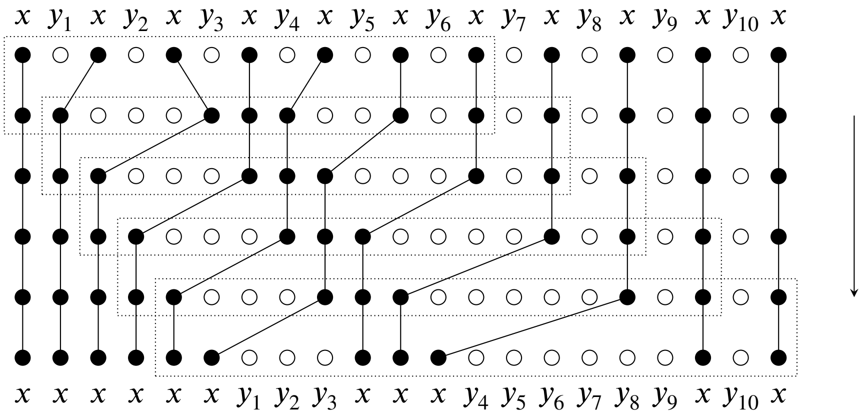
Notice that the previous substitution instance treats y_3y_4 as a single y_i and the corresponding y_j between the second and the third occurrence of x does not appear. The only requirement for being able to apply (3.2.1) is to select a portion that contains the correct number of x's, which is seven in this case. This determines the values assigned to the y_i's.

Given equation (a), we will prove that for sufficiently large values of ℓ (for example $\ell \geq 2r$), a large number of occurrences of x in the word $w_{1,\ell}$ can be collected to adjacent positions.

We will use the diagram notation to exemplify the effect of equation (3.2.1). We start with a general expression where x appears 11 times.

In the next diagram, the dotted boxes indicate the places where substitution instances of (3.2.1) have been used. For instance, the first dotted box represents the following transformation

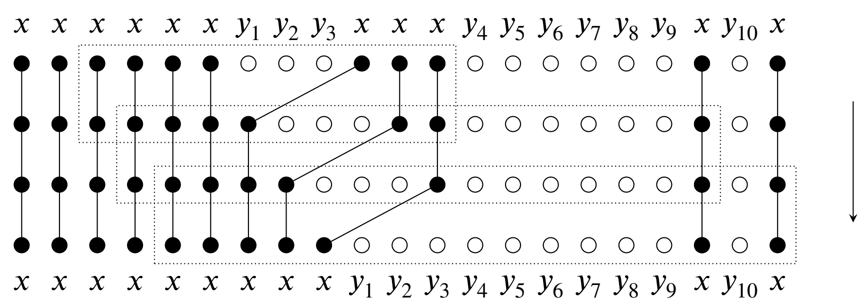
$$(xy_1xy_2xy_3xy_4xy_5xy_6x)y_7xy_8xy_9xy_{10}x = (xxy_1y_2y_3x^3y_4y_5xy_6x)y_7xy_8xy_9xy_{10}x.$$



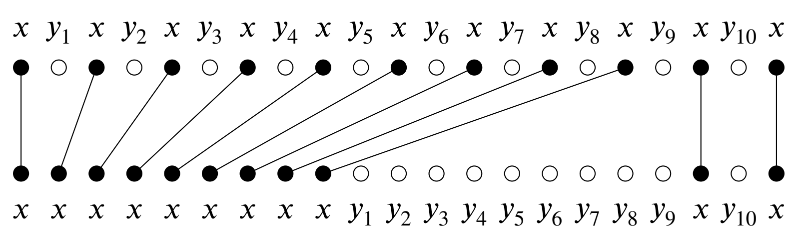
The resulting equality is

$$xy_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9xy_{10}x = x^6y_1y_2y_3x^3y_4y_5y_6y_7y_8y_9xy_{10}x.$$

Notice that this process moves many of the x's next to the attractive wall. Next, we move the remaining occurrences of x between the walls. For the next argument, we use the repulsivity of the back wall.



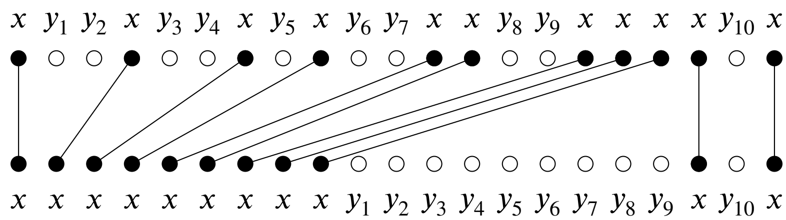
Finally, we conclude that



Equivalently,

$$xy_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9xy_{10}x = xx^8y_1y_2y_3y_4y_5y_6y_7y_8y_9xy_{10}x.$$

Furthermore, we can use substitution instances of the last equation such as the following



Hence

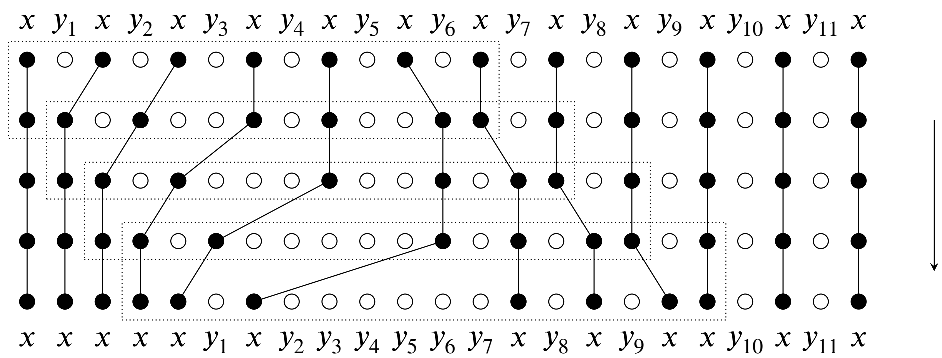
$$\begin{aligned}\underline{x}y_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9\underline{xy_{10}x} &= \underline{x}x^8y_1y_2y_3y_4y_5y_6y_7y_8y_9\underline{xy_{10}x}\\ &= \underline{x}y_1y_2xy_3y_4xy_5xy_6y_7x^2y_8y_9x^3\underline{xy_{10}x}.\end{aligned}$$

Notice that the walls remained intact. Also, the order of the occurrences of x between the walls is not important. In the next section, we will show that this is a general fact.

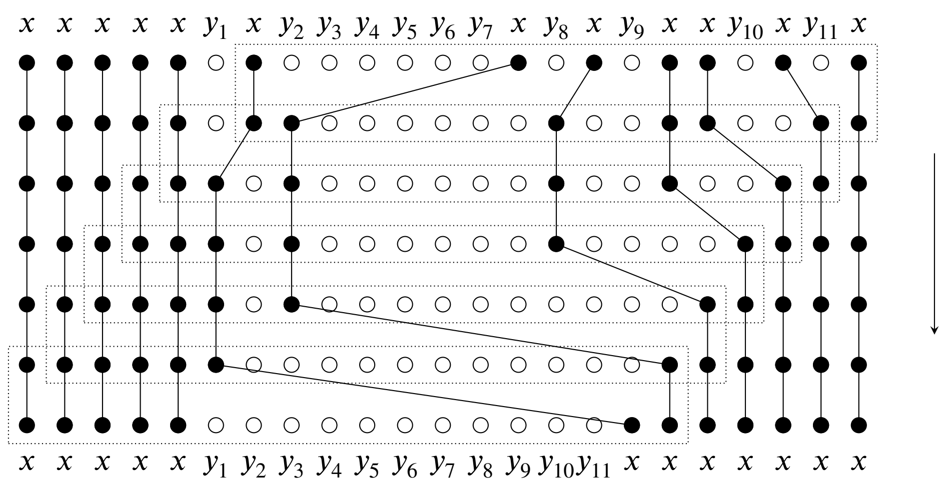
Another possibility is to have two attractive walls. Consider the equality

$$xy_1xy_2xy_3xy_4xy_5xy_6x = x^2y_1xy_2y_3xy_4xy_5y_6x^2. \tag{3.2.2}$$

Our goal is to use the previous equality to gather more variables together, as in the last example. We start by considering a longer expression and gathering x's closer to the front wall.



Now, we gather the rest of the x's on the right.



The additional step in the end illustrates the method that will be used in the general case.

The final equation is

$$\underline{x}y_1xy_2xy_3xy_4xy_5xy_6xy_7xy_8xy_9xy_{10}xy_{11}\underline{x} = \underline{x}x^4y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10}y_{11}x^6\underline{x}.$$

3.3 The equation $xy_1xy_2\cdots y_rx = x^{a_0}y_1x^{a_1}y_2\cdots y_rx^{a_r}$

We will use substitution instances of (a) that fix the variable x and for every i assign to y_i a (possibly empty) word over $\{y_1, y_2, \ldots\}$. Such a substitution instance of the left hand side of (a) is determined by words that contain exactly $r+1$ occurrences of x.

Lemma 3.3.1. *Assume equation* (a)*,* $xy_1xy_2\cdots y_rx = x^{a_0}y_1x^{a_1}y_2\cdots y_rx^{a_r}$*, has a front wall that contains p-many x's, and a back wall that contains q-many x's. Let* $\ell \geq r+1$ *and* $0 \leq k \leq \ell - r$*.*

If the front wall is attractive, then

$$w_{1,\ell} = w_{1,p}x^k t w_{\ell-q+1,\ell}. \tag{3.3.1}$$

Similarly, if the back wall is attractive, then

$$w_{1,\ell} = w_{1,p}t x^k w_{\ell-q+1,\ell}. \tag{3.3.2}$$

In both cases, $t \in Y^*$ *is such that* $\mathrm{del}_x(t) = y_py_{p+1}\cdots y_{\ell-q}$ *and* $|t|_x = \ell-p-q-k$*.*

Proof. We consider the case where there is an attractive front wall. A symmetric argument proves the result for an attractive back wall. We prove the result by induction on $\ell \geq r+1$.

The base case is $\ell = r+1$, which implies that $k \leq 1$. If $k = 0$, the result is trivial. When $k = 1$, the equality follows immediately from (3.1.1).

$$w_{1,r+1} = w_{1,p}xtw_{r-q+2,r+1},$$

where t satisfies $\mathrm{del}_x(t) = y_py_{p+1}\cdots y_{r-q+1}$ and $|t|_x = r-p-q$. Now assume that the result is true for some $\ell \geq r+1$.

We will show the result holds for $\ell+1$. Consider $1 \leq k \leq \ell+1-r$ (the case $k = 0$ is trivial). Notice that an attractive wall cannot be empty, hence $p \geq 1$. If $p > 1$, we have

$$\begin{aligned} w_{1,\ell+1} &= (w_{1,\ell})y_\ell x \\ &= (w_{1,p}xt'w_{\ell+1-q,\ell})y_\ell x \quad \text{(by Induction Hypothesis and } 1 \leq \ell - r) \\ &= xy_1(w_{2,p}xt'w_{\ell+1-q,\ell+1}) \\ &= xy_1w_{2,p}xx^{k-1}tw_{\ell+2-q,\ell+1} \quad \text{(by IH and } k-1 \leq \ell - r) \\ &= w_{1,p}x^k tw_{\ell+2-q,\ell+1} \end{aligned}$$

where $|t'|_x = \ell - p - q - 1$ and $\mathrm{del}_x(t') = y_p y_{p+1} \cdots y_{\ell-q}$. Thus $|t|_x = \ell + 1 - p - q - k$ and $\mathrm{del}_x(t) = y_p y_{p+1} \cdots y_{\ell+1-q}$. If $p = 1$, the argument is essentially the same, with the only difference that we erase y_1 and $w_{2,p}$ from the previous equalities. This concludes the induction. $\square$

The next lemma deals with the case of repulsive walls.

Lemma 3.3.2. *Assume* (a) *has a front wall that contains p-many x's, and a back wall that contains q-many x's. Let* $s_1, s_2, s_3 \in Y^*$, $\ell \geq r$, $|s_1|_x = p$, $|s_2|_x \geq 1$, *and* $|s_3|_x = q$.

If the front wall is repulsive, then

$$s_1 s_2 x^\ell = s_1 s_2' x^{\ell+1}. \tag{3.3.3}$$

Similarly, if the back wall is repulsive, then

$$x^\ell s_2 s_3 = x^{\ell+1} s_2' s_3. \tag{3.3.4}$$

In both cases, $s_2' \in Y^*$ *is such that* $\mathrm{del}_x(s_2') = \mathrm{del}_x(s_2)$ *and* $|s_2'|_x = |s_2|_x - 1$.

Proof. We first look at the repulsive front wall case. We start by establishing the following

$$\begin{aligned} w_{1,p}y_pxy_{p+1}x^{\ell} &= (w_{1,p}y_pxy_{p+1}x^{r-p})x^{\ell-r+p} \\ &= (w_{1,p}y_py_{p+1}x^{r+1-p})x^{\ell-r+p} \quad \text{(by (3.1.2))} \\ &= w_{1,p}y_py_{p+1}x^{\ell+1}. \end{aligned}$$

Notice that in the last equation, we did not modify the front wall. Hence, every part of the equation before y_p remained the same. So, if we consider an $s \in Y^*$ such that $|s|_x = p$, we obtain

$$sxyx^{\ell} = syx^{\ell+1}.$$

We conclude that if $|s_2|_x \geq 1$, $|s_1|_x = p$ and $\ell \geq r$, then we have

$$s_1s_2x^{\ell} = s_1s_2'x^{\ell+1},$$

where $\mathrm{del}_x(s_2) = \mathrm{del}_x(s_2')$ and $|s_2'|_x = |s_2|_x - 1$. So (3.3.3) holds. For the repulsive back wall the proof is analogous. $\square$

As mentioned before, if there are enough occurrences of x in a word, the x's between the walls can be moved anywhere between the walls. The next theorem proves this fact. This is the main result in this chapter and it will allow us to find a nice representation for monoids that satisfy (a).

Theorem 3.3.3. *Assume* (a) *holds and let* p *and* q *be the number of* x*'s in the front and back wall, respectively. We set* $w_1 = w_{1,p}$ *and* $w_2 = w_{2r-q+1,2r}$*. For all* $w, w' \in Y^*$ *such that* $\mathrm{del}_x(w) = \mathrm{del}_x(w') = y_py_{p+1}\cdots y_{2r-q}$ *and* $|w|_x = |w'|_x = 2r-p-q$*, we have* $w_1ww_2 = w_1w'w_2$.

Proof. We separate the proof into three cases.

Case 1. One wall is attractive and the other is repulsive.

We begin with the case of an attractive front wall. Consider $k \geq r$ and words $w_f, w_b, s \in Y^*$ such that $|w_f|_x = p, |w_b|_x = q$, w_f ends with an x, and w_b starts with an x. We will prove by induction on $n = |s|_x \geq 0$ that

$$w_f x^k s w_b = w_f x^{k+n} \mathrm{del}_x(s) w_b. \tag{3.3.5}$$

The base case $n = 0$ follows trivially because $s = \mathrm{del}_x(s)$.

Assume that the result is true for $|s|_x = n$. We want to show that it holds for $|s|_x = n + 1$. Since $|w_b|_x = q, k \geq r$ and $|s|_x \geq 1$, we have that

$$\begin{aligned} w_f x^k s w_b &= w_f (x^k s w_b) \\ &= w_f (x^{k+1} s' w_b), \quad \text{(by (3.3.4))} \end{aligned}$$

where $\mathrm{del}_x(s') = \mathrm{del}_x(s)$ and $|s'|_x = |s|_x - 1 = n$. Since $k + 1 \geq r$ and $|s'|_x = |s|_x - 1 = n$, we obtain

$$\begin{aligned} w_f x^k s w_b &= w_f x^{k+1} s' w_b \\ &= w_f x^{k+n+1} \mathrm{del}_x(s') w_b \quad \text{(by Induction Hypothesis)} \\ &= w_f x^{k+n+1} \mathrm{del}_x(s) w_b, \end{aligned}$$

as desired.

Now we are ready to prove the lemma for this case. Using (3.3.1) for $\ell = 2r$ and $k = r$, we have

$$w_{1,2r} = w_{1,p} x^r t w_{2r-q+1,2r},$$

where t is such that $\mathrm{del}_x(t) = y_p y_{p+1} \cdots y_{2r-q}$ and $|t|_x = r-p-q$. Since $|w_{2r-q+1,2r}|_x = q$ and $|w_{1,p}|_x = p$, we obtain

$$\begin{aligned} w_{1,2r} &= w_{1,p} x^r t w_{2r-q+1,2r} \\ &= w_{1,p} x^{2r-p-q} \mathrm{del}_x(t) w_{2r-q+1,2r} \quad \text{(by (3.3.5))} \\ &= w_1 x^{2r-p-q} y_p \cdots y_{2r-q} w_2. \end{aligned}$$

Consider w, w' as defined in the hypothesis. We can use substitution instances of the last equation to show

$$\begin{aligned} w_1 w w_2 &= w_1 x^{2r-p-q} \mathrm{del}_x(w) w_2, \text{ and} \\ w_1 w' w_2 &= w_1 x^{2r-p-q} \mathrm{del}_x(w') w_2. \end{aligned}$$

Therefore $w_1 w w_2 = w_1 w' w_2$ and the theorem follows. A symmetric argument takes care of the case where the back wall is attractive and the front wall is repulsive.

Case 2. Both walls are attractive.

In (a), let j be the number of consecutive occurrences of x after the front wall that are the input for a left shift or a drop-down. In other words, to determine j we stop counting when we find the leftmost variable that shifts right. In (a) the x after y_{p+j} shifts right.

Using (3.3.1) for $\ell = 2r$ and $k = j$, we obtain

$$w_{1,2r} = w_{1,p} x^j t w_{2r-q+1,2r} \tag{3.3.6}$$

where t is such that $\mathrm{del}_x(t) = y_p y_{p+1} \cdots y_{2r-q}$. Notice that further applications of (a) do not change the word x^j next to the wall. This happens because in such substitution instances the x's of x^j will correspond to a drop-down or a left shift ,which means that they do not drift to the right.

If we apply (3.3.2) to (3.3.6), for $\ell = 2r$ and $k = r$, we have

$$\begin{aligned} w_{1,2r} &= w_{1,p} x^j t w_{2r-q+1,2r} \\ &= w_{1,p} x^j t' x^r w_{2r-q+1,2r}, \end{aligned}$$

where $|t'|_x = |t|_x - r$ and $\text{del}_x(t') = \text{del}_x(t)$.

Now, we will show that we can move all the x's of t' next to x^r. Let $k \geq r$, $w_f, w_b, s \in Y^*$ be such that $|w_f|_x = p$, $|w_b|_x = q$, $|s|_x = n$, w_f has an x at the last position, and w_b has an x at the first position.

Using induction on $n \geq 0$, we will prove that

$$w_f x^j s x^k w_b = w_f x^j \text{del}_x(s) x^{k+n} w_b.$$

The case $n = 0$ follows from $s = \text{del}_x(s)$. Recall that in (a) the variable x after y_{p+j} shifts right. Using this fact, we prove the following intermediate result

$$\begin{aligned} w_{1,p} x^j y_p x y_{p+1} x^r &= (w_{1,p} x^j y_p x y_{p+1} x^{r-p-j}) x^{p+j} \\ &= (w_{1,p} x^j y_p y_{p+1} x^{r-p-j+1}) x^{p+j} \quad \text{(by } (a)\text{)} \\ &= w_{1,p} x^j y_p y_{p+1} x^{r+1}. \end{aligned}$$

We have that for $n = |s|_x = 1$,

$$\begin{aligned} w_f x^j s x^k w_b &= (w_f x^j s x^r) x^{k-r} w_b \\ &= w_f x^j \text{del}_x(s) x^{k+1} w_b. \end{aligned}$$

Assume that the result is true for $|s|_x = n \geq 1$. We will prove it for $|s|_x = n + 1$.

We first define $t \in Y^*$ by $s = ts^x_{\underset{\leftarrow}{1}}$ and $s' = ty$, where $y \in Y \setminus \mathrm{cont}(w_f t w_b)$. Since $|ty|_x = n$, we have

$$\begin{aligned} w_f x^j s' x^k w_b &= w_f x^j t y x^k w_b \\ &= w_f x^j \mathrm{del}_x(ty) x^{k+n} w_b \quad \text{(by Induction Hypothesis)} \\ &= w_f x^j \mathrm{del}_x(t) y x^{k+n} w_b. \end{aligned}$$

By applying the substitution that assigns $s^x_{\underset{\leftarrow}{1}}$ to y and fixes all other variables in Y, we have

$$\begin{aligned} w_f x^j s x^k w_b &= w_f x^j \mathrm{del}_x(t) s^x_{\underset{\leftarrow}{1}} x^{k+n} w_b \\ &= w_f x^j \mathrm{del}_x(ts^x_{\underset{\leftarrow}{1}}) x^{k+n+1} w_b \quad \text{(case } n = 1\text{)} \\ &= w_f x^j \mathrm{del}_x(s) x^{k+n+1} w_b. \end{aligned}$$

So the word x^r acts like an attractor for the x's between the walls that are not part of x^j. Therefore

$$\begin{aligned} w_{1,2r} &= w_{1,p} x^j y_p y_{p+1} \cdots y_{2r-q} x^{2r-p-q-j} w_{2r-q+1,2r} \\ &= w_1 x^j y_p y_{p+1} \cdots y_{2r-q} x^{2r-p-q-j} w_2. \end{aligned}$$

Consider w and w' as defined in the hypothesis. We can use substitution instances of the last equation to show

$$\begin{aligned} w_1 w w_2 &= w_1 x^j \mathrm{del}_x(w) x^{2r-p-q-j} w_2, \text{ and} \\ w_1 w' w_2 &= w_1 x^j \mathrm{del}_x(w') x^{2r-p-q-j} w_2. \end{aligned}$$

Thus $w_1 w w_2 = w_1 w' w_2$.

Case 3. Both walls are repulsive.

Considering the sequence of shifts and drop-downs, let j be the index such that the variable x after y_j is the input of a left shift and it is the leftmost such x. We notice that all the x's before that particular x are either drop-downs or right shifts. We know that the variable x before y_j is a drop down because a right shift can not be followed by a left shift. In other words (a) has a portion between the walls that looks like this,

$$\begin{array}{ccccccccccc} x & y_1 & x & \cdots & x & y_j & x & \cdots & x & y_r & x \\ \downarrow & & \downarrow & & \downarrow & \swarrow & & & \downarrow & & \downarrow \\ \multicolumn{5}{c}{\underbrace{x \quad y_1 \quad x \quad \cdots \quad x}_{\substack{\text{drop-downs}\\ \text{or right shifts}}}} & x & & \cdots & x & y_r & x \end{array}$$

We use substitution instances of (a) to create a word x^r in the middle. We start by proving the following intermediate result, for all $k \geq 0$,

$$w_{1,r+k} = w_{1,p} t x^{k+1} s w_{r+k+1-q,r+k}, \tag{3.3.7}$$

where $\mathrm{del}_x(t) = y_p y_{p+1} \cdots y_{j-1}$, $\mathrm{del}_x(s) = y_j y_{j+1} \cdots y_{r+k-q}$, $|t|_x = j - p - 1$, and $|s|_x = r - q - j$.

We prove this by induction on k. The base case $k = 0$ is trivial. Assume the result holds for k and consider the case $k + 1$.

$$\begin{aligned} w_{1,r+k+1} &= (w_{1,r+k}) y_{r+k} x \\ &= (w_{1,p} t' x^{k+1} s' w_{r+k+1-q,r+k}) y_{r+k} x \quad \text{(by Induction Hypothesis)} \end{aligned}$$

where $\mathrm{del}_x(t') = y_p y_{p+1} \cdots y_{j-1}$, $\mathrm{del}_x(s') = y_j y_{j+1} \cdots y_{r+k-q}$, $|t'|_x = j - p - 1$, and $|s'|_x = r - q - j$.

Now, let $u, v \in Y^*$ be such that $uv = w_{1,p}t'x^{k+1}$ and $|v|_x = j$.

$$\begin{aligned}
w_{1,r+k+1} &= w_{1,p}t'x^{k+1}s'w_{r+k+1-q,r+k}y_{r+k}x \\
&= u(vs'xy_{r+k+1-q}w_{r+k+2-q,r+k+1}) \\
&= u(v'xsw_{r+k+2-q,r+k+1}) \quad \text{(by } (a))
\end{aligned}$$

where $\mathrm{del}_x(v') = \mathrm{del}_x(v)$, $|v'|_x = |v|_x$, $|s|_x = |s'|_x$, and $\mathrm{del}_x(s) = \mathrm{del}_x(s')y_{r+k+1-q}$. Notice that during the application of (a), the first j-many occurrences of x either drop-down or shift right, by definition of j. Hence $w_{1,p}t'x^{k+1} = uv = uv' = w_{1,p}tx^{k+1}$, where $\mathrm{del}_x(t) = \mathrm{del}_x(t')$ and $|t|_x = |t'|_x$.

This happens because all the occurrences of x in v stayed in the same position or shifted right, which means that x's in the word x^{k+1} remained together. Furthermore, the elements of the wall $w_{1,p}$ also remain unchanged. Therefore, we have

$$\begin{aligned}
w_{1,r+k+1} &= uv'xsw_{r+k+2-q,r+k} \\
&= w_{1,p}tx^{(k+1)+1}sw_{r+(k+1)+1-q,r+(k+1)}.
\end{aligned}$$

So (3.3.7) is proven.

Since the walls are repulsive, the results of lemma 3.3.2 apply here. The lemma implies that x^r acts like an attractor. By a routine induction, the equations (3.3.3) and (3.3.4) imply the following result

$$\begin{aligned}
w_{1,2r} &= w_{1,p}tx^{r+1}sw_{2r+1-q,2r} \quad \text{(by (3.3.7))} \\
&= w_{1,p}y_py_{p+1}\cdots y_{j-1}x^{2r-p-q}y_j\cdots y_{2r-q}w_{2r+1-q,2r} \quad \text{(by lemma 3.3.2)}
\end{aligned}$$

To finish the proof, we will show that we can shift the word x^{2r-p-q} next to a wall. An instance of (a) produces

$$w_{1,p}y_px^{r-p-q}y_{p+1}xw_{p+2,p+q+1} = w_{1,p}y_px^{r-p-q+1}y_{p+1}w_{p+2,p+q+1}.$$

After flipping the equation we have

$$w_{1,p}y_px^{r-p-q+1}y_{p+1}w_{p+2,p+q+1} = w_{1,p}y_px^{r-p-q}y_{p+1}xw_{p+2,p+q+1}.$$

Notice that in the last equation, we do not modify the front wall. Furthermore, every part of it before y_p remains unchanged. Similarly, the back wall is not modified. If we consider $s, t \in Y^*$ such that $|s|_x \geq p, |t|_x \geq q$, and t starts with x, we obtain

$$sx^{r-p-q+1}yt = sx^{r-p-q}yxt. \tag{3.3.8}$$

Now consider $s, t \in Y^*$, $k \in \mathbb{N}$, $y \in Y \setminus \{x\}$ such that $|s|_x \geq p, |t|_x = q$, and t starts with x. We will show by induction on k that

$$sx^{r-p-q+k}yt = sx^{r-p-q}yx^kt. \tag{3.3.9}$$

The base case $k = 0$ is trivial. Assume that the result is true for k. We will prove that it holds for $k + 1$.

$$\begin{aligned} sx^{r-p-q+k+1}yt &= (sx)x^{r-p-q+k}yt \\ &= (sx)x^{r-p-q}yx^kt \quad \text{(by Induction Hypothesis)} \\ &= sx^{r-p-q+1}yx^kt \\ &= sx^{r-p-q}yxx^kt \quad \text{(by (3.3.8))} \end{aligned}$$

as desired.

We finish the proof as follows.

$$\begin{aligned}
w_{1,2r} &= w_{1,p}y_p \cdots y_{j-1}x^{2r-p-q}y_j \cdots y_{2r-q}w_{2r+1-q,2r} \\
&= w_{1,p}y_p \cdots y_{j-1}x^{r-p-q}y_j \cdots y_{2r-q}x^r w_{2r-q+1,2r} \quad \text{(by (3.3.9))} \\
&= w_{1,p}y_p \cdots y_{2r-q}x^{2r-p-q}w_{2r-q+1,2r} \quad \text{(by lemma 3.3.2)}
\end{aligned}$$

We conclude that

$$w_{1,2r} = w_1 y_p \cdots y_{2r-q}x^{2r-p-q}w_2.$$

This is the same result we obtained for the attractive back wall in the first case. Thus, the theorem is proven. □

Chapter 4

The FEP for subvarieties of RL_m^n axiomatized by (a)

For an appropriate $a \in \mathbb{N}^{r+1}$, we define $\mathcal{V}_m^n(a)$ as the subvariety of RL_m^n such that their monoid reducts belong to $\mathcal{K}(a)$. This variety is defined by equations strong enough to allow for a description of a relatively free algebra.

In the present chapter, we will show that these varieties enjoy the FEP. The results from Section 4.3 are presented in [5]. Finally, we expand the basic results to other related subvarieties.

4.1 The varieties $\mathcal{K}(\ell, p, q)$

For $a \in \mathbb{N}^{r+1}$, such that $\sum_{i=0}^{r} a_i = r + 1$ and $\prod_{i=0}^{r} a_i = 0$, we previously defined $\mathcal{K}(a)$ as the variety of monoids axiomatized by

$$w_{1,r} = xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}.$$

Similarly, for $p + q < \ell$, we define $\mathcal{K}(\ell, p, q)$ as the variety of monoids axiomatized by the equations of the form

$$w_1 w w_2 = w_1 w' w_2, \tag{Σ}$$

for all w, w' such that $\mathrm{del}_x(w) = \mathrm{del}_x(w') = y_p y_{p+1} \cdots y_{\ell-q}$, $|w|_x = |w'|_x = \ell - p - q$, with $w_1 = w_{1,p}$ and $w_2 = w_{\ell-q+1,\ell}$. Note that the above axiomatization is finite.

For a given a, we define

$$p_a = \max\{j : (\forall i < j)(a_i = 1)\}.$$

$$q_a = \max\{j : (\forall i > r - j)(a_i = 1)\}.$$

Theorem 3.3.3 implies that $\mathcal{K}(a)$ is a subvariety of $\mathcal{K}(2r, p_a, q_a)$. Notice that the family of equations (Σ), for $|w|_x = |w'|_x = \ell - p - q$, implies the same equations for $|w|_x = |w'|_x \geq \ell - p - q$. The result is summarized in the next lemma.

Lemma 4.1.1. *For $\ell, p, q, \ell', p', q' \in \mathbb{N}$, $\ell > p + q$, and $\ell' > p' + q'$, if $\ell \leq \ell'$, $p \leq p'$, $q \leq q'$, then*

$$\mathcal{K}(\ell, p, q) \subseteq \mathcal{K}(\ell', p', q').$$

Proof. Let (Σ) be the set of equations that axiomatize $\mathcal{K}(\ell, p, q)$ and (Σ') the set of equations that axiomatize $\mathcal{K}(\ell', p', q')$. It is enough to show that the equations in (Σ') can be deduced from (Σ). That would mean that every monoid in $\mathcal{K}(\ell, p, q)$ satisfies (Σ') and it belongs to $\mathcal{K}(\ell', p', q')$ as well.

We start by showing that

$$\mathcal{K}(\ell, p, q) \subseteq \mathcal{K}(\ell + 1, p, q).$$

Notice that (Σ) implies that for every w, $w_1 w w_2 = w_1 x^{\ell-p-q} y_p y_{p+1} \cdots y_{\ell+1-q} w_2$. Consider a $u \in Y^*$ such that $\mathrm{del}_x(u) = y_p y_{p+1} \cdots y_{\ell+1-q}$ and $|u|_x = \ell + 1 - p - q$.

We define $v \in Y^*$ such that $u = vu^x_{\underset{\leftarrow}{1}}$. Then

$$\begin{aligned}
w_{1,p}uw_{\ell+2-q,\ell+1} &= (w_{1,p}vu^x_{\underset{\leftarrow}{1}}\, w_{\ell+2-q,\ell})y_\ell x \\
&= w_{1,p}x^{\ell-p-q}\mathrm{del}_x(v)u^x_{\underset{\leftarrow}{1}}\, w_{\ell+2-q,\ell+1} \qquad \text{(by } (\Sigma)\text{)} \\
&= xy_1(w_{2,p}xx^{\ell-1-p-q}\mathrm{del}_x(v)u^x_{\underset{\leftarrow}{1}}\, w_{\ell+2-q,\ell+1}) \\
&= w_{1,p}xx^{\ell-p-q}\mathrm{del}_x(v)\mathrm{del}_x(u^x_{\underset{\leftarrow}{1}})w_{\ell+2-q,\ell+1} \qquad \text{(by } (\Sigma)\text{)} \\
&= w_{1,p}x^{\ell+1-p-q}\mathrm{del}_x(u)w_{\ell+2-q,\ell+1}.
\end{aligned}$$

The last equation is enough to obtain all the defining equations of $\mathcal{K}(\ell+1,p,q)$, which proves the result. Using this result, an easy induction proves that for any $i \in \mathbb{N}$,

$$\mathcal{K}(\ell,p,q) \subseteq \mathcal{K}(\ell+i,p,q).$$

We also have that for $i, j \in \mathbb{N}$, such that $p+q+i+j < \ell$,

$$\mathcal{K}(\ell,p,q) \subseteq \mathcal{K}(\ell,p+i,q+j).$$

This follows from the fact that the defining equations of $\mathcal{K}(\ell,p+i,q+j)$ are a subset of (Σ). The lemma follows from the previous two results.

$$\begin{aligned}
\mathcal{K}(\ell,p,q) &\subseteq \mathcal{K}(\ell',p,q) \\
&\subseteq \mathcal{K}(\ell',p',q').
\end{aligned}$$

□

For the rest of the section, we will work with $\mathcal{K}(\ell,p,q)$ where $p > 0$. The requirement that p be positive will be explained later.

For a set C, $s \in C^*$ and $c \in C$, recall the definitions

$$s^c_{\underset{\rightarrow}{i}} = s^c_{0,i} \qquad\qquad s^c_{\underset{\leftarrow}{j}} = s^c_{|s|_c-j+1,|s|_c+1}.$$

For $p > 0, q \geq 0, \ell > p + q$, we defined s^c_{mid} to be the part of s strictly between $s^c_{\underset{\rightarrow}{p}}$ and $s^c_{\underset{\leftarrow}{q}}$. Hence

$$s = s^c_{\underset{\rightarrow}{p}} s^c_{\text{mid}} s^c_{\underset{\leftarrow}{q}}.$$

Now, for $\square \in \{D, N\}$, we consider the functions $\alpha^c_\square : C^* \to C^*$ defined by

$$\alpha^c_\square(s) = \begin{cases} s^c_{\underset{\rightarrow}{p}} c^{\hat{s}-p-q} \mathrm{del}_c(s^c_{\text{mid}}) s^c_{\underset{\leftarrow}{q}} & \text{if } |s|_c \geq \ell, \\ s & \text{otherwise,} \end{cases}$$

where $\hat{s} = |s|_c$ or ℓ. For $\hat{s} = |s|_c$ we obtain the function α^c_N; for $\hat{s} = \ell$ we obtain the function α^c_D. We will illustrate this by considering $C = \{a, b, c\}$, $p = 2, q = 1, \ell = 5$, and $s = cabcac^2bac^4acb \in C^*$. Then we have

$$\alpha^b_N(s) = \alpha^b_D(s) = s$$

$$\alpha^c_N(s) = \underline{cabc}c^6aba^2\underline{cb}$$

$$\alpha^c_D(s) = \underline{cabc}c^2aba^2\underline{cb},$$

where the underlined words represent $s^c_{\underset{\rightarrow}{2}}$ and $s^c_{\underset{\leftarrow}{1}}$.

Notice that $\alpha^c_D \circ \alpha^c_N = \alpha^c_D$. Basically the operations α^c_N, α^c_D change s when it contains sufficiently many c's. α^c_N moves all the copies of c except for the first p many and the last q many. The copies in between are placed right after the pth copy of c. α^c_D works similarly with the only difference that it truncates the number of c's in the output. It this setting it is important that $p > 0$ in order to have a fixed position where to place $c^{\hat{s}-p-q}$.

Furthermore, for $c_i, c_j \in C$, $\alpha^{c_i}_\square\big(\alpha^{c_j}_\square(s)\big) = \alpha^{c_j}_\square\big(\alpha^{c_i}_\square(s)\big)$. This follows from the fact that every $\alpha^c_\square$ (possibly) moves only c's to positions occupied already by c's, leaving the rest unchanged, and does so based solely on the relative positions of the occurrences of the c's, ignoring the rest of the word.

Lemma 4.1.2. *For $c \in C$ and $\square \in \{D, N\}$, $s, t \in C^*$,*

$$\alpha^c_\square(st) = \alpha^c_\square\left(\alpha^c_\square(s)\alpha^c_\square(t)\right).$$

Proof. First we show that $\widehat{st} = \widehat{\alpha^c_\square(s)\alpha^c_\square(t)}$. If $\hat{s} = \ell$, then $\widehat{\alpha^c_\square(s)\alpha^c_\square(t)} = \ell = \widehat{st}$. When $\hat{s} = |s|_c$, we have $|st|_c = |\alpha^c_N(s)\alpha^c_N(t)|_c$ since $|u|_c = |\alpha^c_N(u)|_c$ for all $u \in C^*$.

We also have that $(st)^c_{\underrightarrow{p}} = \left(\alpha^c_\square(s)\alpha^c_\square(t)\right)^c_{\underrightarrow{p}}$ and $(st)^c_{\underleftarrow{q}} = \left(\alpha^c_\square(s)\alpha^c_\square(t)\right)^c_{\underleftarrow{q}}$. Both results follow from the fact that $\alpha^c_\square$ does not modify the initial or final segments of a word.

To prove the lemma, we only need to prove the following intermediate result

$$\mathrm{del}_c\left((st)^c_{\mathrm{mid}}\right) = \mathrm{del}_c\left((\alpha^c_\square(s)\alpha^c_\square(t))^c_{\mathrm{mid}}\right).$$

We divide the analysis into the following cases.

Case $|s|_c, |t|_c < \ell$. Here $\alpha^c_\square$ does not modify s or t, so the result is trivial.

Case $|s|_c, |t|_c \geq \ell$.

$$\begin{aligned}
\mathrm{del}_c\left((\alpha^c_\square(s)\alpha^c_\square(t))^c_{\mathrm{mid}}\right) &= \mathrm{del}_c\left(\left(s^c_{\underrightarrow{p}}\, c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{\underleftarrow{q}}\, t^c_{\underrightarrow{p}}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})t^c_{\underleftarrow{q}}\right)^c_{\mathrm{mid}}\right) \\
&= \mathrm{del}_c\left(c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{\underleftarrow{q}}\, t^c_{\underrightarrow{p}}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})\right) \\
&= \mathrm{del}_c\left(s^c_{\mathrm{mid}}s^c_{\underleftarrow{q}}\, t^c_{\underrightarrow{p}}\, t^c_{\mathrm{mid}}\right) = \mathrm{del}_c\left((st)^c_{\mathrm{mid}}\right).
\end{aligned}$$

Case $|s|_c \geq \ell, q \leq |t|_c < \ell$.

$$\begin{aligned}
\mathrm{del}_c\left((\alpha^c_\square(s)\alpha^c_\square(t))^c_{\mathrm{mid}}\right) &= \mathrm{del}_c\left(\left(s^c_{\underrightarrow{p}}\, c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{\underleftarrow{q}}\, t\right)^c_{\mathrm{mid}}\right) \\
&= \mathrm{del}_c\left(c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{\underleftarrow{q}}\, t_{\underrightarrow{|t|_c-q+1}}\right) \\
&= \mathrm{del}_c\left(s^c_{\mathrm{mid}}s^c_{\underleftarrow{q}}\, t_{\underrightarrow{|t|_c-q+1}}\right) = \mathrm{del}_c\left((st)^c_{\mathrm{mid}}\right).
\end{aligned}$$

Case $p \le |s|_c < \ell, |t|_c \ge \ell$.

$$\begin{aligned}
\mathrm{del}_c\big((\alpha^c_\Box(s)\alpha^c_\Box(t))^c_{\mathrm{mid}}\big) &= \mathrm{del}_c\left(\left(st^c_{\underrightarrow{p}}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})t^c_{\underleftarrow{q}}\right)^c_{\mathrm{mid}}\right) \\
&= \mathrm{del}_c\left(s^c_{\underleftarrow{|s|_c-p+1}}t^c_{\underrightarrow{p}}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})\right) \\
&= \mathrm{del}_c\left(s^c_{\underleftarrow{|s|_c-p+1}}t^c_{\underrightarrow{p}}\, t^c_{\mathrm{mid}}\right) = \mathrm{del}_c\big((st)^c_{\mathrm{mid}}\big).
\end{aligned}$$

Case $|s|_c \ge \ell, |t|_c < q$.

$$\begin{aligned}
\mathrm{del}_c\big((\alpha^c_\Box(s)\alpha^c_\Box(t))^c_{\mathrm{mid}}\big) &= \mathrm{del}_c\left(\left(s^c_{\underrightarrow{p}}\, c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{\underleftarrow{q}}\, t\right)^c_{\mathrm{mid}}\right) \\
&= \mathrm{del}_c\big(c^{\hat{s}-p-q}\mathrm{del}_c(s^c_{\mathrm{mid}})s^c_{|s|_c-q+1,|s|_c+|t|_c-q+1}\big) \\
&= \mathrm{del}_c\big(s^c_{\mathrm{mid}}s^c_{|s|_c-q+1,|s|_c+|t|_c-q+1}\big) = \mathrm{del}_c\big((st)^c_{\mathrm{mid}}\big).
\end{aligned}$$

Case $|s|_c < p, |t|_c \ge \ell$.

$$\begin{aligned}
\mathrm{del}_c\big((\alpha^c_\Box(s)\alpha^c_\Box(t))^c_{\mathrm{mid}}\big) &= \mathrm{del}_c\left(\left(st^c_{\underrightarrow{p}}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})t^c_{\underleftarrow{q}}\right)^c_{\mathrm{mid}}\right) \\
&= \mathrm{del}_c\big(t^c_{p-|s|_c,p}\, c^{\hat{t}-p-q}\mathrm{del}_c(t^c_{\mathrm{mid}})\big) \\
&= \mathrm{del}_c\big(t^c_{p-|s|_c,p}\, t^c_{\mathrm{mid}}\big) = \mathrm{del}_c\big((st)^c_{\mathrm{mid}}\big).
\end{aligned}$$

Using the previous results, we have

$$\begin{aligned}
\alpha^c_\Box\big(\alpha^c_\Box(s)\alpha^c_\Box(t)\big) &= \big(\alpha^c_\Box(s)\alpha^c_\Box(t)\big)^c_{\underrightarrow{p}}\, c^{\widehat{st}-p-q}\mathrm{del}_c\big((\alpha^c_\Box(s)\alpha^c_\Box(t))^c_{\mathrm{mid}}\big)\big(\alpha^c_\Box(s)\alpha^c_\Box(t)\big)^c_{\underleftarrow{q}} \\
&= (st)^c_{\underrightarrow{p}}\, c^{\widehat{st}-p-q}\mathrm{del}_c\big((st)^c_{\mathrm{mid}}\big)(st)^c_{\underleftarrow{q}} \\
&= \alpha^c_\Box(st). \qquad \Box
\end{aligned}$$

For $s \in C^*$, $E = \{e_1, \ldots, e_k\} \subseteq C$, and $\Box \in \{D, N\}$, let $\alpha^E_\Box(s) = \alpha^{e_1}_\Box \circ \alpha^{e_2}_\Box \circ \cdots \circ \alpha^{e_k}_\Box(s)$. As noted previously, the order of the e_i's does not affect the result of the composition. Recall that for $s \in C^*$, $\mathrm{cont}(s)$ represents the set of elements of C that appear in s.

Now, for $s \in C^*$, we define

$$\alpha_\square(s) = \alpha_\square^{\mathrm{cont}(s)}(s).$$

Note that if $\mathrm{cont}(s) \subseteq E$, we have $\alpha_\square^{\mathrm{cont}(s)}(s) = \alpha_\square^E(s)$. The following is a consequence of the last lemma.

Lemma 4.1.3. *For $s, t \in C^*$ and $\square \in \{D, N\}$, $\alpha_\square(\alpha_\square(s)\alpha_\square(t)) = \alpha_\square(st)$.*

Proof. Let $E = \mathrm{cont}(s) \cup \mathrm{cont}(t)$. We will show that

$$\alpha_\square^E\left(\alpha_\square^E(s)\alpha_\square^E(t)\right) = \alpha_\square^E(st).$$

We prove the result by induction on $|E|$. The base case $|E| = 1$ follows from Lemma 4.1.2. For the induction hypothesis, assume that the result is true for $|E| = k$. We will prove it for $|E| = k + 1$. Assume that $e \in E$ and $E' = E \setminus \{e\}$. We have

$$\begin{aligned}
\alpha_\square^E(\alpha_\square^E(s)\alpha_\square^E(t)) &= \alpha_\square^{E'} \circ \alpha_\square^e \left(\alpha_\square^e(\alpha_\square^{E'}(s))\alpha_\square^e(\alpha_\square^{E'}(t))\right) \\
&= \alpha_\square^{E'} \circ \alpha_\square^e \left(\alpha_\square^{E'}(s)\alpha_\square^{E'}(t)\right) \quad \text{(by Lemma 4.1.2)} \\
&= \alpha_\square^e \circ \alpha_\square^{E'} \left(\alpha_\square^{E'}(s)\alpha_\square^{E'}(t)\right) \\
&= \alpha_\square^e \circ \alpha_\square^{E'}(st) \quad \text{(by Induction Hypothesis)} \\
&= \alpha_\square^E(st)
\end{aligned}$$

□

Remark 4.1.4. Let $C = \{c_1, \dots, c_k\}$. These results provide the following characterization of $\alpha_\square(s)$. $\alpha_N(s)$ is the element of C^* obtained from s by moving next to the pth occurrence of c_i the $(p+1)$th, the $(p+2)$th, and up to the $(|s|_c - q)$th occurrence of c_i, simultaneously for each c_i with more than ℓ-many occurrences in s. Thus, by collecting all these consecutive occurrences next to the pth occurrence of c_i, we obtain a power of c_i. If we further truncate this exponent to be at most $\ell - p - q$, for each c_i, then we obtain the element $\alpha_D(s)$.

We define the infinite set $X = \{x_1, x_2, \dots\}$. For all $z \in X^*$, $x \in X$

$$(\Sigma) + (\mathrm{Mon}) \vdash \alpha_N^x(z) = z,$$

where (Mon) represents an axiomatization of the class of all monoids. Here we follow standard practice, when working in the theory of monoids, and consider terms modulo the monoid axioms, even thought the two sides of $\alpha_N^x(z) = z$ are not terms, but elements of the free monoid. Since $(\Sigma) + (\mathrm{Mon})$ axiomatize $\mathcal{K}(\ell, p, q)$, we have that for $\mathbf{M} \in \mathcal{K}(\ell, p, q)$, $\mathbf{M} \vDash \alpha_N^x(z) = z$. By definition of α_N, we have that

$$\mathbf{M} \vDash \alpha_N(z) = z. \tag{4.1.1}$$

Thus for all $z \in X^*$ and for any homomorphism $h : X^* \to \mathbf{M}$, we have $h(\alpha_N(z)) = h(z)$.

4.2 Finitely generated free monoids for the varieties $\mathcal{K}(\ell, p, q)$, where $p > 0$

For $X_k = \{x_1, x_2, \dots x_k\}$, we define $H = \alpha_N[X_k^*]$. We define the multiplication as $s \cdot^{\mathbf{H}} t = \alpha_N(st)$ for $s, t \in H$. We will show that $\mathbf{H} = (H, \cdot^{\mathbf{H}}, \varepsilon)$ is a monoid by proving that α_N is a homomorphism. For $s, t \in X_k^*$, Lemma 4.1.3 implies

$$\begin{aligned} \alpha_N(st) &= \alpha_N\left(\alpha_N(s)\alpha_N(t)\right) \\ &= \alpha_N(s) \cdot^{\mathbf{H}} \alpha_N(t). \end{aligned}$$

Figure 4.1: A free algebra in $\mathcal{K}(\ell, p, q)$ on k generators.

$$
\begin{tikzcd}
H \arrow[r, dashed, "h\restriction"] & M \\
X_k^* \arrow[u, "\alpha_N"] \arrow[ur, "h"] & \\
X_k \arrow[u, hook, "i"] \arrow[uur, "h_1"'] &
\end{tikzcd}
$$

The result implies that $\mathbf{H}$ is a homomorphic image of X_k^* and hence a monoid. Furthermore, for every element $s \in H$, the equation $s = \alpha_N(s)$ holds. Hence $\mathbf{H}$ satisfies (Σ) and $\mathbf{H} \in \mathcal{K}(\ell, p, q)$.

The multiplication defined is the least restrictive operation that makes α_N a homomorphism. We claim that $\mathbf{H}$ is the free monoid in k generators over $\mathcal{K}(\ell, p, q)$.

Lemma 4.2.1. $\mathbf{H} = (H, \cdot^{\mathbf{H}}, \varepsilon)$ *is the free algebra in* $\mathcal{K}(\ell, p, q)$ *on* k *generators. Moreover, the set* H *is generated by* X_k.

Proof. By definition, H is generated by $X_k = \{x_1, \ldots, x_k\}$. We will show that $\mathbf{H}$ has the universal mapping property. Consider a monoid $\mathbf{M} \in \mathcal{K}(\ell, p, q)$, the map i given by $i(x_j) = x_j$, and a map $h_1 : X_k \to M$.

Since X_k^* is the free k-generated monoid, there exists a homomorphism $h : X_k^* \to M$. We claim that the restriction $h\restriction := h\restriction_H$ is a homomorphism. Figure 4.1 depicts the situation.

Notice that

$$h\restriction(\varepsilon) = h(\varepsilon) = u_M, \text{ the unit of } M, \text{ and}$$

$$h\restriction(x_j) = h(x_j) = h_1(x_j), \text{ for all } j \in \{1, \ldots, k\}.$$

Let $x, y \in H$, then

$$\begin{aligned} h\restriction(x \cdot^{\mathbf{H}} y) &= h\restriction(\alpha_N(xy)) \\ &= h(\alpha_N(xy)) \\ &= h(xy) \qquad \text{(by (4.1.1))} \\ &= h(x) \cdot^{\mathbf{M}} h(y) \\ &= h\restriction(x) \cdot^{\mathbf{M}} h\restriction(y). \end{aligned}$$

Thus $h\restriction$ is a homomorphism. The uniqueness of $h\restriction$ follows from the fact that X generates H . □

Let $S = \alpha_D[X_k^*] = \alpha_D[H]$. Consider the function $\varphi : H \to \mathbb{N}^k \times S$ defined by $\varphi(s) = (|s|_{x_1}, \dots, |s|_{x_k}, \alpha_D(s))$ for $s \in H$ and define $F = \varphi[H] \subseteq \mathbb{N}^k \times S$. It is easy to see that S is a finite set because there are only finitely many x_i's and each one appears at most ℓ times. We define multiplication on S by $s \cdot t = \alpha_D(st)$, for $s, t \in S$. By Lemma 4.1.3, α_D is a homomorphism, so $\mathbf{S} = (S, \cdot, \varepsilon)$ is a monoid.

We define a binary operation $\cdot^{\mathbf{F}}$ for $(x, s), (y, t) \in F$ as follows

$$(x, s) \cdot^{\mathbf{F}} (y, t) = (x + y, s \cdot t).$$

Let $\mathbf{K} = (F, \cdot^{\mathbf{F}}, 0)$, where $0 = (0, \dots, 0, \varepsilon)$.

Lemma 4.2.2. $\varphi : \mathbf{H} \to \mathbf{K}$ *is an isomorphism .*

Proof. We first will show that φ is injective. Assume that $\varphi(s) = \varphi(t)$. Then, $|s|_{x_i} = |t|_{x_i}$ for every i and $\alpha_D(s) = \alpha_D(t)$. Given that α_D only truncates the exponents in s, t, and nothing else, we conclude that $\alpha_N(s) = \alpha_N(t)$. Therefore $s = t$ because both of them are in H.

We now show that φ is a homomorphism. Let $s, t \in H$.

$$
\begin{aligned}
\varphi(s \cdot^{\mathbf{H}} t) &= \varphi(\alpha_N(st)) \\
&= \big(|\alpha_N(st)|_{x_1}, \dots, |\alpha_N(st)|_{x_k}, \alpha_D(\alpha_N(st))\big) \\
&= \big(|st|_{x_1}, \dots, |st|_{x_k}, \alpha_D(st)\big) \\
&= \big(|s|_{x_1} + |t|_{x_1}, \dots, |s|_{x_k} + |t|_{x_k}, \alpha_D(\alpha_D(s)\alpha_D(t))\big) \quad \text{(by Lemma 4.1.3)} \\
&= \big(|s|_{x_1} + |t|_{x_1}, \dots, |s|_{x_k} + |t|_{x_k}, \alpha_D(s) \cdot \alpha_D(t)\big) \\
&= \big(|s|_{x_1}, \dots, |s|_{x_k}, \alpha_D(s)\big) \cdot^{\mathbf{F}} \big(|t|_{x_1}, \dots, |t|_{x_k}, \alpha_D(t)\big) \\
&= \varphi(s) \cdot^{\mathbf{F}} \varphi(t).
\end{aligned}
$$

Hence φ is an injective homomorphism onto its image, which means that it is an isomorphism. □

Since X_k generates H, we have that $\varphi[X_k]$ generates F. Therefore F is generated by $z_1 = (1, 0, \dots, 0, x_1), \dots, z_k = (0, 0, \dots, 1, x_k)$. By Lemmas 4.2.1 and 4.2.2, **K** has the universal mapping property for $\mathcal{K}(\ell, p, q)$. Let $Z = \{z_1, \dots, z_k\}$ and $\mathbf{M} \in \mathcal{K}(\ell, p, q)$.

$$
\begin{tikzcd}
F \arrow[r, dashed, "h"] & M \\
Z \arrow[u, hook, "i"] \arrow[ur, "h_1"'] &
\end{tikzcd}
$$

Corollary 4.2.3. $\mathbf{K} = (F, \cdot^{\mathbf{F}}, 0)$ *is the free k-generated object in* $\mathcal{K}(\ell, p, q)$.

4.3 The FEP for $\mathcal{V}_m^n(a)$

The monoid reduct of $\mathbf{A} \in \mathcal{V}_m^n(a)$ belongs to $\mathcal{K}(a)$ by definition. As shown in Section 4.1, define $p > 0, q$, and ℓ such that

$$\mathcal{K}(a) \subseteq \mathcal{K}(\ell, p, q).$$

Let $B = \{b_1, \ldots, b_k\}$. Since $\mathbf{W}$ belongs to $\mathcal{K}(\ell, p, q)$, we can extend the assignment $h_1 : Z \to W$ that sends $z_i \mapsto b_i$, for each $i = 1, \ldots, k$, to a monoid homomorphism $h : (F, \cdot^{\mathbf{F}}, 0) \to (W, \circ, 1)$, by the universal mapping property. The map h is surjective because B generates $(W, \circ, 1)$.

Now, we will extend $\mathbf{K}$ to a pomonoid $\mathbf{F}$. Recall the order $(\mathbb{N}, \leq_n^m)$, defined on Section 2.2. We prove the next lemma for that order.

Lemma 4.3.1. *Let $m > n$ and r be natural numbers. If $x \leq_{n+r}^{m+r} y$ and $r \leq x, y$, then $x - r \leq_n^m y - r$.*

Proof. If $x = y$, then $x - r = y - r \geq 0$. This implies that $x - r \leq_n^m y - r$. Otherwise $n + r \leq y < x$ and $x \equiv y \pmod{m + r - n - r}$. We obtain that $n \leq y - r < x - r$ and $x - r \equiv y - r \pmod{m - n}$, which means that $x - r \leq_n^m y - r$ by definition. $\square$

Let $m' = m+p+q$ and $n' = n+p+q$. Recall from the last section that $F \subseteq \mathbb{N}^k \times S$. Also, $(\mathbb{N}, \leq_{n'}^{m'})^k$ and $(S, =)$ are dually well partially ordered sets.

For $m > n$ (if $m < n$ we use the dual order), we extend $\mathbf{K}$ to $\mathbf{F} = (F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0)$, which is a subpomonoid of the direct product $(\mathbb{N}, \leq_{n'}^{m'}, +, 0)^k \times (S, =, \cdot, \varepsilon)$. For $x = (\vec{x}, s_x), y = (\vec{y}, s_y) \in F$, $x \leq^{\mathbf{F}} y$ iff $\vec{x} \leq_{n'}^{m'} \vec{y}$ and $s_x = s_y$.

We will show that $h : (F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0) \to (W, \leq^{\mathbf{A}}, \circ, 1)$ is order-preserving. Consider $w, y \in F$ such that $w \leq^{\mathbf{F}} y$. Let $w = (w_1, \ldots, w_k, s)$ and $y = (y_1, \ldots, y_k, s)$.

Since both share the same signature s, we only have to make sure that the variables raised to their exponents are comparable and in the correct order. We know that every $x_i \in \mathrm{cont}(s)$ is truncated or not. If it is not truncated, then $h(w)$ and $h(y)$ contain the variables in the same order. If it is truncated, then we have that $w_i \leq_{n'}^{m'} y_i$. By Lemma 4.3.1, we obtain $w_i - p - q \leq_n^m y_i - p - q$ and by Lemma 2.2.1, $b_i^{w_i - p - q} \leq^{\mathbf{A}} b_i^{y_i - p - q}$. Multiplying these inequalities in the order determined by s, we obtain that $h(w) \leq^{\mathbf{A}} h(y)$.

We know that $(F, \leq^{\mathbf{F}})$ is a (dually) well partially ordered set for $m < n$ ($m > n$, respectively) and that $h : (F, \leq^{\mathbf{F}}, \cdot^{\mathbf{F}}, 0) \to (W, \leq^{\mathbf{A}}, \circ, 1)$ is a surjective monotone homomorphism. As in Sections 2.4 and 2.5, it follows that there are only finitely many closed sets of the form $\{(u, b)\}^{\triangleleft}$, where $u \in S_W, b \in B$.

Notice that we can extend the construction for residuated lattices to cover FL-algebras. We only need to add the constant 0 to our set B.

Theorem 4.3.2 ([5])**.** *The varieties axiomatized relative to* RL *and* FL *by* $x^m \leq x^n$*, for some natural numbers* $m \neq n$*,* $m \geq 1$*, and by one of the equations* (a)*, have the FEP. Furthermore, every subvariety axiomatized by equations over the language of join, multiplication, and identity also has the FEP.*

Since commutativity implies that (a) holds for any a, we obtain the next consequence.

Corollary 4.3.3 ([34])**.** *The varieties of commutative knotted residuated lattices have the FEP.*

4.4 The FEP for related subvarieties

In Theorem 4.3.2, we obtained the FEP for subvarieties axiomatized by equations over the language of join, multiplication, and identity. In this section, we will extend the result to subvarieties that are axiomatized by equations outside of the language $(\wedge, \cdot, 1)$.

We will begin by considering the subvarieties of cyclic FL-algebras and cyclic involutive FL-algebras.

Lemma 4.4.1. *The subvarieties of* CyFL_m^n *axiomatized by one of the equations* (a) *have the FEP.*

Proof. We will actually prove that our construction preserves cyclicity. Consider a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ in the variety. Let $B_0 = B \cup \{0\}$. We construct the finite algebra $\mathbf{D}$ into which $\mathbf{B}_0$ is embedded. By Theorem 1.5.5, $0_D = \{(id, 0)\}^{\triangleleft}$ is the zero element of $\mathbf{D}$. Let $X \in D$ and $z \in \sim X = X \backslash 0_D$. Hence $z \in \{y : X \circ y \subseteq 0_D\}$.

We have that $X \circ z \subseteq 0_D$ iff for all $x \in X$, $x \circ z \leq^{\mathbf{A}} 0$. Cyclicity implies that for all $y \in A$, $y \backslash 0 = 0/y$. Residuation implies $xy \leq^{\mathbf{A}} 0 \Leftrightarrow x \leq^{\mathbf{A}} 0/y \Leftrightarrow x \leq^{\mathbf{A}} y \backslash 0 \Leftrightarrow yx \leq^{\mathbf{A}} 0$.

Hence,

$$
\begin{aligned}
& X \circ z \subseteq 0_D \\
\Leftrightarrow \quad & \forall x \in X, x \circ z \leq^{\mathbf{A}} 0 \\
\Leftrightarrow \quad & \forall x \in X, z \circ x \leq^{\mathbf{A}} 0 \\
\Leftrightarrow \quad & z \circ X \subseteq 0_D.
\end{aligned}
$$

The last line is equivalent to $z \in -X = 0_D/X$. Therefore, for all $X \in D$, $\sim X = -X$. We conclude that **D** is cyclic. □

Lemma 4.4.2. *The subvarieties of* CyInFL^n_m *axiomatized by one of the equations* (a) *have the FEP.*

Proof. Let **A** be an algebra in the variety. Consider a partial subalgebra **B**. We define $B^\sim = \{\sim b : b \in B\}$ and $B^\star = B \cup B^\sim \cup \{0\}$. As before, we construct the finite algebra **D** where $\mathbf{B}^\star$ embeds into. From the previous theorem we know that **D** is cyclic. We just need to verify that it is involutive, i.e., for all $X \in D$, $-\sim X = X$.

Let $z \in -\sim X$. Recalling the definitions of $-$ and $\sim$, we have

$$-\sim X = 0_D/\sim X = \{y : \{y\} \circ \sim X \subseteq 0_D\},$$
$$\sim X = X \backslash 0_D = \{y : X \circ \{y\} \subseteq 0_D\}.$$

Then $z \in -\sim X$ is equivalent to the condition:

for all $a \in W$, $X \circ \{a\} \subseteq 0_D$ implies $\{z \circ a\} \subseteq 0_D$, which implies $z \circ a \leq^{\mathbf{A}} 0$.

Let $X = \bigcap_{i \in I} \{(u_i, c_i)\}^{\triangleleft}$, where $u_i \in S_W, c_i \in B^\star$ and I is a finite indexing set. Let $u_i = r_i \circ \_ \circ s_i$ for $r_i, s_i \in W$. Take an arbitrary $x \in X$. For all $i \in I$,

$$\begin{aligned} & x \in \{(u_i, c_i)\}^{\triangleleft} \\ \Leftrightarrow \quad & r_i \circ x \circ s_i \leq^{\mathbf{A}} c_i \\ \Leftrightarrow \quad & r_i \circ x \circ s_i \circ (\sim c_i) \leq^{\mathbf{A}} c_i \circ (\sim c_i) \leq^{\mathbf{A}} 0 \\ \Leftrightarrow \quad & x \circ s_i \circ (\sim c_i) \circ r_i \leq^{\mathbf{A}} 0. \quad \text{(By cyclicity)} \end{aligned}$$

Note that $\sim c_i \in B^\star$, then $s_i \circ (\sim c_i) \circ r_i \in W$. Since x was arbitrary, we obtain $X \circ \{s_i \circ (\sim c_i) \circ r_i\} \subseteq 0_D$ and by the condition stated above we have $z \circ s_i \circ (\sim c_i) \circ r_i \leq^{\mathbf{A}} 0$.

Notice that the steps above are reversible, hence

$$z \circ s_i \circ (\sim c_i) \circ r_i \leq^{\mathbf{A}} 0 \quad \Leftrightarrow \quad r_i \circ z \circ s_i \leq^{\mathbf{A}} c_i.$$

Therefore $z \in \{(u_i, c_i)\}^{\lhd}$, for every $i \in I$ and $z \in X$, as desired. This shows that $X \subseteq -{\sim}X$. The other inclusion holds trivially because for all $z \in X$, $a \in W$, $X \circ \{a\} \subseteq 0_D$ implies $\{z \circ a\} \subseteq 0_D$. □

Next, we study the subvarieties of representable residuated lattices and representable FL-algebras, RRL and RFL, respectively. Recall that algebras in these varieties can be represented as subdirect products of totally ordered algebras. We will utilize the following condition for the FEP (modified from [1]).

Lemma 4.4.3. *For a variety $\mathcal{V}$, if every finite partial subalgebra* $\mathbf{B}$ *of an algebra* $\mathbf{A} \in \mathcal{V}_{\mathrm{SI}}$ *embeds into a finite* $\mathbf{D} \in \mathcal{V}$*, then $\mathcal{V}$ has the FEP.*

Proof. Let $\mathbf{A} \in \mathcal{V}$ have a finite partial subalgebra $\mathbf{B}$. Take a subdirect representation $\prod_{i \in I} \mathbf{A}_i$ of $\mathbf{A}$. Let $\mathbf{B}_i$ be the projection of $\mathbf{B}$ on the coordinate i. Clearly each $\mathbf{B}_i$ is a finite partial subalgebra of $\mathbf{A}_i \in \mathcal{V}_{\mathrm{SI}}$ and thus it is embeddable into a finite algebra $\mathbf{D}_i \in \mathcal{V}$ (note that is not necessary for this algebra to be subdirectly irreducible). Since $\mathbf{B}$ is finite, finitely many coordinates suffice to separate the elements of B. Let J be the index set of those finitely many coordinates. We obtain that $\mathbf{B}$ embeds into $\prod_{j \in J} \mathbf{D}_j \in \mathcal{V}$. □

Lemma 4.4.4. *The subvarieties of* RFL_m^n *and* RRL_m^n *axiomatized by one of the equations* (a) *have the FEP.*

Proof. By the previous lemma, it suffices to show that the FEP holds for the set of subdirectly irreducible members of RRL_m^n. Any algebra $\mathbf{A}$ in that class is a chain, as well as any finite partial algebra $\mathbf{B}$. Let $\mathbf{D}$ be the finite algebra generated by our construction. We only need to show that this residuated lattice is representable. Notice that the pomonoid $\mathbf{W}$ generated by B will be a chain under the order inherited from $\mathbf{A}$. Therefore, all the elements $\mathbf{D}$ are downsets on a chain, which means that they form a chain themselves. Thus, $\mathbf{D}$ is a finite chain, which means that it is representable (and subdirectly irreducible). □

The identities $x \wedge {\sim}x \leq 0$ and $x \wedge -x \leq 0$ are known as *pseudo-complementation*.

Lemma 4.4.5. *The subvarieties of* FL_m^n *axiomatized by pseudo-complementation and one of the equations* (a) *have the FEP.*

Proof. We only need to show that pseudo-complementation is preserved in our construction for FL-algebras. Assume $z \in X \cap {\sim}X$, then $z \in X$ and $X \circ \{z\} \subseteq 0_D$. Particularly $z^2 \leq^{\mathbf{A}} 0$, which implies that $z \leq^{\mathbf{A}} {\sim}z$. We have that

$$z = z \wedge z \leq^{\mathbf{A}} z \wedge {\sim}z \leq^{\mathbf{A}} 0.$$

Thus $X \cap {\sim}X \subseteq 0_D$ as desired. The other inequality is proven similarly. □

The property of being zero-bounded ($0 \leq x$) is preserved as well. The verification is straightforward. The proofs of the previous lemmas rely on the same construction of $\mathbf{D}$, therefore they can be combined freely. The following theorem extends results from [23] and [11].

Theorem 4.4.6. *Let* $\mathcal{V}$ *be a subvariety of* FL_m^n *axiomatized by an equation of the form* (a) *and any combination of the following identities:*

1. *representability,*
2. *cyclicity,*

3. *cyclicity plus involution,*

4. *pseudo-complementation,*

5. $0 = 1$*,*

6. $0 \leq x$*,*

7. *any identity over the language of* $\{\vee, \cdot, 1\}$.

Then $\mathcal{V}$ *has the FEP and its universal theory is decidable.*

Distributivity is another interesting property to study. We know that all representable residuated lattices are distributive, however our construction is not powerful enough to capture distributivity without representability. In the next chapter, we study subvarieties of distributive residuated lattices. In particular, we focus our attention to the fully distributive ones.

Chapter 5

The FEP for some fully distributive residuated lattices

In [9] it is shown, by developing a theory for a distributive version of residuated frames, that in the presence of integrality we can obtain the FEP for all varieties of distributive residuated lattices axiomatized over the language $\{\wedge, \vee, \cdot, 1\}$. For instance, the FEP is established for all integral and fully distributive residuated lattices. In all residuated lattices multiplication distributes over join, but if we further know that both multiplication and join distribute over meet, we call the residuated lattice fully distributive. Algebras such as lattice-ordered groups, Heyting algebras, and all semilinear residuated lattices (including MV-algebras and BL-algebras) are fully distributive residuated lattices. Furthermore, fully distributive residuated lattices admit a nice representation theorem [8].

In this chapter, we relax the integrality condition with a combination of a knotted inequality and a noncommutative equation. We consider a variety $\mathcal{D}_m^n(a)$ of fully distributive residuated lattices axiomatized by a knotted rule $x^m \leq x^n$, $m \neq n$, $m \geq 1$, and an equation of the form (a). (We may also assume that the axiomatization of the variety contains further equations over the language $\{\wedge, \vee, \cdot, 1\}$.) We will show that $\mathcal{D}_m^n(a)$ has the FEP. Thus, we obtain infinitely many varieties of fully distributive residuated lattices with the FEP, outside the setting of integrality or commutativity.

5.1 Construction of a finite D

Given a fully distributive residuated lattice $\mathbf{A} \in \mathcal{D}_m^n(a)$ and a subset B of it, we rely on the construction presented in Section 1.6 to obtain the algebra $\mathbf{D}$ involved in the definition of the FEP. Under the assumptions that $\mathbf{A}$ is in the appropriate variety and B is finite, we will establish the finiteness of $\mathbf{D}$.

Define $\mathbf{W} = (W, \circ, \wedge, \varepsilon)$ to be the $\{\cdot, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B. (Note that we use different notation for the restriction of the operations of $\mathbf{A}$ on the set W.) Since $\mathbf{A}$ is fully distributive, $\mathbf{W}$ is a semilattice monoid. Observe that the linear polynomials over $(W, \circ, \wedge, \varepsilon)$, containing a single variable x, must look like $u(x) = (y \circ x \circ z) \wedge w$ for $y, z \in W$ and $y \in W \cup \{\top\}$. Here we write $y \circ x \circ z \wedge \top$ for $y \circ x \circ z$, in order to have uniform notation, where $\top$ is a new symbol used only for this purpose. Since multiplication distributes over meet, we can even assume that y and z do not have $\wedge$ in them. Recall from Section 1.6 that the set of all such polynomials is S_W and the set W' equals $S_W \times B$. Also, the relation N from W to W' is given by

$$x N(u, b) \quad \Leftrightarrow \quad u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b.$$

By Theorem 1.6.4, the algebra

$$\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}} = (\gamma[\mathscr{P}(W)], \cap, \cup_\gamma, \cdot_\gamma, \backslash, /, \gamma\{1\})$$

is a distributive residuated lattice and the map $b \mapsto \{(id, b)\}^{\lhd}$ is an embedding of the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ into $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$. The closed sets $\{z\}^{\lhd}$ for $z \in W'$ form a basis for $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ and this algebra belongs to all varieties of fully distributive lattices axiomatized over $\{\wedge, \vee, \cdot, 1\}$ that contain $\mathbf{A}$.

We will take $\mathbf{D}$ to be the algebra $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$. Clearly, to prove that $\mathbf{D}$ is finite, it suffices to prove that there are only finitely many basic closed sets. Note that these sets $\{z\}^{\lhd} = \{x \in W : x \ N \ z\}$, for $z \in W'$ are downsets in $(W, \leq^{\mathbf{A}})$.

To prove finiteness, we will construct a relatively free semilattice monoid $\mathbf{F}$ and a surjective homomorphism $h : \mathbf{F} \to \mathbf{W}$. As in the nondistributive case, it will be important to show that the underlying poset of $\mathbf{F}$ is dually well partially ordered.

To guarantee the existence of the map h in the above diagram, the auxiliary algebra $\mathbf{F}$ will be chosen to be free over a class of semilattice monoids that includes $\mathbf{W}$. This class will satisfy the identity (a). $\mathbf{F}$ will also satisfy $x^m \leq x^n$ for selected elements. Therefore, it is not the free algebra in any variety containing $\mathbf{W}$, but it will be free enough for our purposes.

We begin by describing the construction of a semilattice from a poset and extending it to the construction of a semilattice monoid from a pomonoid.

5.2 The semilattice construction $\mathscr{M}$

Given a poset $\mathbf{Q}$, we define $\mathscr{M}(Q)$ as the set of all nonempty finitely generated upsets of $\mathbf{Q}$. If A is a nonempty finitely generated upset of $\mathbf{Q}$, then the set mA of its minimal elements is nonempty and $A = {\uparrow} mA$. Also, the union of two nonempty finitely generated upsets A and B is finitely generated by $m(A \cup B) \subseteq mA \cup mB$.

Clearly $\mathscr{M}(Q)$ supports a meet semilattice under the operation $A \wedge B = A \cup B$, for $A, B \in \mathscr{M}(Q)$. If $\mathbf{Q}$ is a pomonoid, then we can further define

$$A \bullet B = \uparrow(AB), \text{ where } AB = \{ab : a \in A, b \in B\}.$$

Lemma 5.2.1. *Let* $\mathbf{Q}$ *be a pomonoid. For* $A, B \in \mathscr{M}(Q)$*, we have* $A \bullet B \in \mathscr{M}(Q)$*. Specifically,*

$$A \bullet B = \uparrow[(mA)(mB)].$$

Proof. Using order-preservation, it is easy to see that for all $C, D \subseteq Q$,

$$(\uparrow C)(\uparrow D) \subseteq \uparrow[CD].$$

Then for $A, B \in \mathscr{M}(Q)$ we have $AB = (\uparrow mA)(\uparrow mB) \subseteq \uparrow[(mA)(mB)] \subseteq \uparrow[AB]$. We have that

$$A \bullet B = \uparrow(AB) \subseteq \uparrow\uparrow[(mA)(mB)] \subseteq \uparrow\uparrow[AB] = \uparrow[AB],$$

namely $A \bullet B = \uparrow[(mA)(mB)]$. Hence it is finitely generated by $(mA)(mB)$ and thus in $\mathscr{M}(Q)$. □

We can now define the semilattice extension $\mathscr{M}(\mathbf{Q})$ of a pomonoid $\mathbf{Q}$ as

$$\mathscr{M}(\mathbf{Q}) = (\mathscr{M}(Q), \wedge, \bullet, \uparrow\{1\}).$$

Lemma 5.2.2. *If* $\mathbf{Q}$ *is a pomonoid, then* $\mathscr{M}(\mathbf{Q})$ *is a semilattice monoid under the operations defined above.*

Proof. It is clear that $\uparrow\{1\}$ is the identity for $\bullet$. Multiplication is associative as shown below. Let $A, B, C \in \mathscr{M}(Q)$.

$$
\begin{aligned}
A \bullet (B \bullet C) &= {\uparrow}\{ad : a \in A, d \in B \bullet C\} \\
&= {\uparrow}\{ad : a \in A, d \in {\uparrow}\{bc : b \in B, c \in C\}\} \\
&= {\uparrow}\{a(bc) : a \in A, b \in B, c \in C\} \quad \text{(by order-preservation)} \\
&= {\uparrow}\{(ab)c : a \in A, b \in B, c \in C\} \\
&= {\uparrow}\{dc : d \in {\uparrow}\{ab : a \in A, b \in B\}, c \in C\} \\
&= (A \bullet B) \bullet C.
\end{aligned}
$$

Now, we show that multiplication distributes over meet. Let $A, B, C \in \mathscr{M}(Q)$.

$$
\begin{aligned}
A \bullet (B \wedge C) &= A \bullet (B \cup C) \\
&= {\uparrow}\{ad : a \in A, d \in B \cup C\} \\
&= \bigcup_{\substack{a \in A \\ d \in B \cup C}} {\uparrow}\{ad\} \\
&= \Big(\bigcup_{\substack{a \in A \\ b \in B}} {\uparrow}\{ab\}\Big) \cup \Big(\bigcup_{\substack{a \in A \\ c \in C}} {\uparrow}\{ac\}\Big) \\
&= A \bullet B \wedge A \bullet C.
\end{aligned}
$$

The other equality $(B \wedge C) \bullet A = B \bullet A \wedge C \bullet A$ can be proven using a symmetric argument. □

The next lemma shows that we can extend pomonoid homomorphisms (order-preserving monoid homomorphisms) to semilattice homomorphism on the semilattice extensions created by $\mathscr{M}$.

Lemma 5.2.3. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are pomonoids and* $f : \mathbf{P} \to \mathbf{Q}$ *is a (surjective) pomonoid homomorphism then* $\mathscr{M}f : \mathscr{M}(\mathbf{P}) \to \mathscr{M}(\mathbf{Q})$ *is a (surjective) semilattice monoid homomorphism, where* $\mathscr{M}f(A) = \uparrow\{f(a) : a \in mA\}$.

Proof. We have that $\mathscr{M}f(A)$ is in $\mathscr{M}(Q)$ because it is an upset and the set mA is finite. Hence $\mathscr{M}f(A)$ is a finitely generated upset of $\mathscr{M}(\mathbf{Q})$. Let $A, B \in \mathscr{M}(P)$. Since f is order-preserving, we obtain that for any $E \subseteq P$ such that $mA \subseteq E \subseteq A$

$$\uparrow\{f(a) : a \in mA\} = \uparrow\{f(a) : a \in A\} = \uparrow\{f(a) : a \in E\}.$$

We use the previous result to show that $\mathscr{M}f$ is a homomorphism.

$$\begin{aligned}
\mathscr{M}f(A \bullet B) &= \uparrow\{f(d) : d \in m(\uparrow[(mA)(mB)])\} \\
&= \uparrow\{f(d) : d \in m((mA)(mB))\} \\
&= \uparrow\{f(d) : d \in (mA)(mB)\} \\
&= \uparrow\{f(a)f(b) : a \in (mA), b \in (mB)\} \\
&= \uparrow\big(\{f(a) : a \in mA\} \cdot \{f(b) : b \in mB\}\big) \\
&= \uparrow\{f(a) : a \in mA\} \bullet \uparrow\{f(b) : b \in mB\} \\
&= \mathscr{M}f(A) \bullet \mathscr{M}f(B). \\
\mathscr{M}f(A \wedge B) &= \uparrow\{f(d) : d \in m(A \cup B)\} \\
&= \uparrow\{f(d) : d \in mA \cup mB)\} \\
&= \uparrow\{f(d) : d \in mA\} \cup \uparrow\{f(d) : d \in mB\} \\
&= \mathscr{M}f(A) \wedge \mathscr{M}f(B).
\end{aligned}$$

Let $C \in \mathscr{M}(Q)$ and $mC = \{c_1, \dots, c_j\}$. If f is surjective, then for every c_i there exists a $b_i \in P$ such that $f(b_i) = c_i$. Let $B = \uparrow\{b_i : 1 \le i \le j\}$. It is clear that $B \in \mathscr{M}(P)$ and $\mathscr{M}f(B) = C$. □

Remark 5.2.4. *$\mathscr{M}$ is a functor from the category of pomonoids to the category of semilattice monoids.*

Finally, we provide a connection between a semilattice monoid and its underlying pomonoid. We know that the monoid reduct of a semilattice monoid is actually a pomonoid because $a(b \wedge c) = ab \wedge ac$ implies $b \leq c \Rightarrow ab \leq ac$ (right distributivity of multiplication over meet is proven similarly).

Lemma 5.2.5. *Let $\mathbf{S} = (S, \wedge, \cdot, 1)$ be a semilattice monoid and $\mathbf{S}_p = (S_p, \leq, \cdot, 1)$ be its corresponding pomonoid reduct. There exists a surjective homomorphism $\psi : \mathscr{M}(\mathbf{S}_p) \to \mathbf{S}$ defined by $\psi(A) = \bigwedge_{a \in A} a$.*

Proof. Note that $\psi(\uparrow\{a\}) = \bigwedge_{b \geq a} b = a = \psi(\{a\})$. Furthermore, $\psi(\uparrow A) = \psi(A)$. Let $A, B \in \mathscr{M}(S_p)$,

$$
\begin{aligned}
\psi(A \wedge B) &= \psi(A \cup B) \\
&= \bigwedge_{c \in A \cup B} c \\
&= \bigwedge_{c \in A} c \wedge \bigwedge_{c \in B} c \quad \text{(idempotency of meet)} \\
&= \psi(A) \wedge \psi(B).
\end{aligned}
$$

$$
\begin{aligned}
\psi(A \bullet B) &= \psi(\uparrow\{ab : a \in A, b \in B\}) \\
&= \psi(\{ab : a \in A, b \in B\}) \\
&= \bigwedge_{\substack{a \in A \\ b \in B}} ab \\
&= \Big(\bigwedge_{a \in A} a\Big) \cdot \Big(\bigwedge_{b \in B} b\Big) \quad \text{(multiplication distributes over meet)} \\
&= \psi(A) \cdot \psi(B).
\end{aligned}
$$

Every element in $x \in S$ can be written as $x = \bigwedge_{i=1}^{k} a_i$, where $a_i \in S_p$. Since $A = \uparrow\{a_1, \dots, a_k\} \in \mathscr{M}(S_p)$, we have that $\psi(A) = x$ and ψ is surjective. $\square$

5.3 The FEP for $\mathcal{D}_m^n(a)$ when $m > n$

We consider the case when the algebra $\mathbf{A} \in \mathcal{D}_m^n(a)$ and $m > n$. Let $B = \{b_1, \dots, b_k\}$ and $X_k = \{x_1, \dots, x_k\}$. In Section 4.3 we obtained the following result.

Lemma 5.3.1. *Let* $\mathbf{M} = (M, \leq^{\mathbf{M}}, \cdot^{\mathbf{M}}, 1)$ *be a pomonoid that satisfies* (a) *and* $x^m \leq x^n$. *There exists a dually well partially ordered pomonoid* $\mathbf{H} = (H, \leq^{\mathbf{H}}, \cdot, 1)$ *such that every map* $g_1 : X_k \to M$ *extends to an order-preserving monoid homomorphism* $g : \mathbf{H} \to \mathbf{M}$.

We define $\mathbf{F} = \mathscr{M}(\mathbf{H})$. By the k-freeness of $\mathbf{H}$ we obtain the following.

Lemma 5.3.2. *There is a surjective semilattice monoid homomorphism* $h : \mathbf{F} \to \mathbf{W}$.

Proof. Let $g_1(x_i) = b_i$ for $i \in \{1, \dots, k\}$. The following diagram is obtained by combining the results from Lemmas 5.2.3, 5.2.5, and 5.3.1.

$$\begin{tikzcd}
 & W \\
F \arrow[ur, "h"] \arrow[r, two heads, "\mathscr{M}g"'] & \mathscr{M}(W_p) \arrow[u, two heads, "\psi"'] \\
H \arrow[u, hook] \arrow[r, two heads, "g"] & W_p \arrow[u, hook] \\
X_k \arrow[u, hook] \arrow[ur, "g_1"'] &
\end{tikzcd}$$

Where $F = \mathscr{M}(H)$. Since B generates $\mathbf{W}_p$, we have that g is surjective. Hence $\mathscr{M}g$ and h are surjective. $\square$

The only part that we are missing in our setup is that $\mathbf{F}$ is dually well partially ordered (it has no infinite antichains and no infinite ascending chains).

Lemma 5.3.3. *If the pomonoid* $\mathbf{Q}$ *is dually well partially ordered, then so is* $\mathscr{M}(\mathbf{Q})$*.*

Proof. Let $A, B \in \mathscr{M}(Q)$. The order in $\mathscr{M}(\mathbf{Q})$ is given by $A \leq_{\mathscr{M}} B$ iff $A = A \wedge B = A \cup B$. So the elements of $\mathscr{M}(Q)$ are ordered under reverse inclusion. If $\mathbf{Q}$ is dually well partially ordered then all finitely generated upsets are actually finite. So we cannot have any infinite ascending chains in $\mathscr{M}(\mathbf{Q})$.

To show that there are no infinite antichains, we will prove that every antichain in $\mathscr{M}(\mathbf{Q})$ would produce an antichain in a well known wqo. Let $\leq := \geq^{\mathbf{Q}}$ be the dual order of $\mathbf{Q}$. Hence $(Q, \leq)$ is a wpo (and wqo). To show that there are no infinite antichains, we use the order $\leq_{\mathscr{P}}$ presented in Section 1.1. For $A, B \in \mathscr{P}_{\text{fin}}(Q)$, $A \leq_{\mathscr{P}} B$ iff there exists an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$. By Lemma 1.1.2, $(\mathscr{P}_{\text{fin}}(Q), \leq_{\mathscr{P}})$ is also a wqo.

We will show that for $A, B \in \mathscr{M}(Q) \subseteq \mathscr{P}_{\text{fin}}(Q)$, if $A \leq_{\mathscr{P}} B$, then $A \subseteq B$. (Note that the converse is trivially true.) Recall that $A = \uparrow mA$. $A \leq_{\mathscr{P}} B$ implies that there exists an injective $f : A \to B$ such that $a \leq f(a)$. Hence $f(a) \leq^{\mathbf{Q}} a$ for all $a \in A$. Particularly, $f(a) \leq^{\mathbf{Q}} a$ for $a \in mA$. Since B is an upset, we obtain

$$\begin{aligned} A &= \bigcup_{a \in mA} \uparrow\{a\} \\ &\subseteq \bigcup_{a \in mA} \uparrow\{f(a)\} \\ &\subseteq \bigcup_{b \in B} \uparrow\{b\} = B. \end{aligned}$$

The contrapositive of $A \leq_{\mathscr{P}} B \Rightarrow A \subseteq B$ implies that if $A \nsubseteq B$ and $B \nsubseteq A$, then A and B are incomparable in $(\mathscr{P}_{\text{fin}}(Q), \leq_{\mathscr{P}})$. We conclude that every antichain in $(\mathscr{M}(Q), \leq_{\mathscr{M}})$ is an antichain in $(\mathscr{P}_{\text{fin}}(Q), \leq_{\mathscr{P}})$. This implies that $(\mathscr{M}(Q), \leq_{\mathscr{M}})$ has no infinite antichains and it is dually well partially ordered. $\square$

Lemma 5.3.4. *For each $b \in B$, $(C_b, \supseteq)$ is a (dually) well partially ordered set. Recall $C_b = \{\{(u, b)\}^{\triangleleft} : u \in S_W\}$.*

Proof. We defined $u = (y \circ \_ \circ z) \wedge w$ for $y, z \in W$ and $y \in W \cup \{\top\}$, where $(y \circ \_ \circ z) \wedge \top = y \circ \_ \circ z$. Based on this, we define $F_\top = F \cup \{\top\}$. We extend the order in $\leq^{\mathbf{F}}$ to include $\top$ by defining $x \leq^{\mathbf{F}} \top$, for all $x \in F_\top$.

It suffices to show that $(C_b, \supseteq)$ is a homomorphic image of $(F^2 \times F_\top, \leq^{\mathbf{F}})$. Define $\varphi : F^2 \times F_\top \to C_b$ by $\varphi(y, w, z) = \{(h(y) \circ \_ \circ h(w) \wedge h(z), b)\}^{\triangleleft}$, where we take $h(\top) = \top$ and rely on the convention presented above. The notation $\top$ is included to make the calculations simpler. We remark that $h(y) \circ \_ \circ h(w) \wedge h(z) \in S_w$ even though h was extended. The map φ is necessarily surjective because h is surjective.

Let $(y, w, z) \in F^2 \times F_\top$ and $x \in F$. We have that

$$h(y \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w \wedge z) = h(y) \circ h(x) \circ h(w) \wedge h(z).$$

When $z \neq \top$, this equality is a consequence of h being a homomorphism. If $z = \top$, then it becomes $h(y \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w) = h(y) \circ h(x) \circ h(w)$, which also holds.

Let $(y_1, w_1, z_1), (y_2, w_2, z_2) \in F^2 \times F_\top$ be such that $(y_1, w_1, z_1) \leq^{\mathbf{F}} (y_2, w_2, z_2)$. We have that for all $x \in F$,

$$y_1 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_1 \wedge z_1 \leq^{\mathbf{F}} y_2 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_2 \wedge z_2.$$

Then, from h being a homomorphism, we obtain

$$h(y_1 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_1 \wedge z_1) \leq^{\mathbf{A}} h(y_2 \cdot^{\mathbf{F}} x \cdot^{\mathbf{F}} w_2 \wedge z_2)$$
$$\Rightarrow h(y_1) \circ h(x) \circ h(w_1) \wedge h(z_1) \leq^{\mathbf{A}} h(y_2) \circ h(x) \circ h(w_2) \wedge h(z_2).$$

If $x \in \varphi(y_2, w_2, z_2)$, then $h(y_2) \circ h(x) \circ h(w_2) \wedge h(z_2) \leq^{\mathbf{A}} b$. Thus, $h(y_1) \circ h(x) \circ h(w_1) \wedge h(z_1) \leq^{\mathbf{A}} b$, which implies that $x \in \varphi(y_1, w_1, z_1)$. So $\varphi(y_1, w_1, z_1) \supseteq \varphi(y_2, w_2, z_2)$. This proves that $(C_b, \supseteq)$ is a (dually) well partially ordered set. $\square$

Lemmas 2.4.2 and 5.3.4 imply that C_b is finite for every $b \in B$. Thus $\mathbf{D}$ is finite and the FEP holds in this case.

5.4 The FEP for $\mathcal{D}^n_m(a)$ when $m < n$

In the dual case, we have that the construction of Lemma 5.3.2 still applies. On the other hand, the question of whether $\mathbf{F}$ is well partially ordered is more involved. The proof found in the previous section relies on an order that does not capture the behavior of $\mathbf{F}$. We will examine the *order structure* of $\mathbf{H}$, defined in Chapter 4, and provide a characterization of its finitely generated upsets.

Recall the construction of the pomonoid $\mathbf{H}$ on generators $X_k = \{x_1, \dots, x_k\}$, presented in chapter 4, for a fixed vector a. First, we defined $p, q, \ell \in \mathbb{N}, p > 0$ such that $\mathcal{K}(a) \subseteq \mathcal{K}(\ell, p, q)$ and used those values to define $H = \alpha_N[X_k^*]$ and $S = \alpha_D[X_k^*]$.

Second, we defined $\varphi : H \to \mathbb{N}^k \times S$ by $\varphi(y) = (|y|_{x_1}, \dots, |y|_{x_k}, \alpha_D(y))$.

Finally, we used φ to realize $\mathbf{H}$ as a subpomonoid of $(\mathbb{N}, \leq^{m'}_{n'}, +, 0)^k \times (S, =, \cdot, \varepsilon)$, where $m' = m + p + q$, $n' = n + p + q$, and the order $\leq^{m'}_{n'}$ is the one defined in Section 2.5. We will use the notation $\mathbb{N}_\ell = \{a \in \mathbb{N} : a \geq \ell\}$.

Our immediate goal is to describe the order and the elements of $\varphi[H]$. Take $s \in S$ and define H_s as the elements $y \in H$ with signature s ($s = \alpha_D(y)$). Therefore,

$$H = \bigcup_{s \in S} H_s \times \{s\}.$$

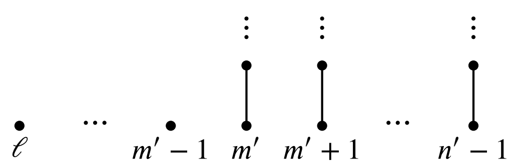
For a specific $s \in S$, we describe the order of H_s. For instance, let $y \in H_s$ and $s \in S$ be such that $|s|_{x_i} = \ell$ only for $i = 1, 3$. If i is not 1 or 3, then $|y|_{x_i} = |s|_{x_i}$ by definition of α_D. On the other hand, $|y|_{x_1}, |y|_{x_3} \geq \ell$. Hence

$$\varphi[H_s] = \mathbb{N}_\ell \times \{|s|_{x_2}\} \times \mathbb{N}_\ell \times \{|s|_{x_4}\} \times \cdots \times \{|s|_{x_k}\} \times \{s\}.$$

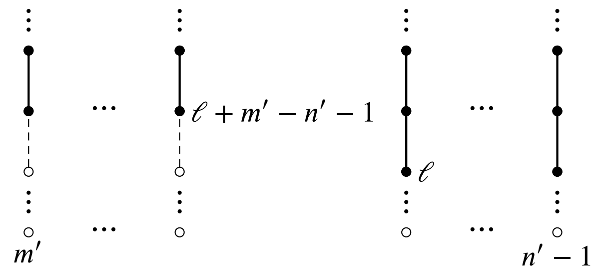
Since the order in singletons is trivial, we obtain that $(H_s, \leq^{\mathbf{H}})$ is order isomorphic to $(\mathbb{N}_\ell, \leq_{n'}^{m'})^2$ in this case.

In general, we fix a signature $s \in S$. Let $y \in H_s$, then $\varphi(y) = (|y|_{x_1}, \ldots, |y|_{x_k}, s)$. For each i, if $|s|_{x_i} < \ell$, then $|y|_{x_i} = |s|_{x_i}$ by the definition of α_D. Otherwise, $|s|_{x_i} = \ell$ and $|y|_{x_i} \geq \ell$. Therefore, $\varphi(H_s)$ is equal to product where each factor is either $\mathbb{N}_\ell$ or a singleton of the form $\{|s|_{x_i}\}$. Let j_s be equal to number of i's for which $|s|_{x_i} = \ell$. We obtain that $(H_s, \leq^{\mathbf{H}})$ is order isomorphic to $(\mathbb{N}_\ell, \leq_{n'}^{m'})^{j_s}$ by simply dropping the singletons.

The Hasse diagram of the order $(\mathbb{N}_\ell, \leq_{n'}^{m'})$ is one of the following. If $\ell < m'$, we obtain



When $\ell \geq m'$, it looks like



Thus, $\mathbb{N}_\ell$ is the union of disjoint chains in every case. Some of these chains can have a single element, while the rest are order isomorphic to the naturals under the usual order $(\mathbb{N}, \leq)$. Direct products of disjoint union of chains naturally split as the disjoint union of connected pieces. These pieces are product of chains.[1]

Let c be the number of disjoint chains in $(\mathbb{N}_\ell, \leq_{n'}^{m'})$. Since $(\mathbb{N}_\ell, \leq_{n'}^{m'})^{j_s}$ is the direct product of disjoint unions of c many chains, it will produce a disjoint union of c^{j_s} products of chains. These products are all order isomorphic to $(\mathbb{N}, \leq)^e$ for various values of e with $1 \leq e \leq j_s$. We have that

$$(H, \leq^{\mathbf{H}}) = \dot{\bigcup_{s \in S}} (H_s, \leq^{\mathbf{H}}).$$

This means that $(H, \leq^{\mathbf{H}})$ is the finite disjoint union of finite products of chains. These products of chains are order isomorphic to $(\mathbb{N}, \leq)^e$ for $1 \leq e \leq k$, where $\leq$ represents the usual order in $\mathbb{N}$.

The latter is the characterization that we were looking for. For a fixed e, we will analyze the order of the set of finitely generated upsets on $\mathbf{Q} = (\mathbb{N}, \leq)^e$. Recall that the construction $\mathscr{M}$ orders upsets by reverse inclusion. For $A, B \in \mathscr{M}(Q)$, $A \supseteq B$ is equivalent to the condition that for all $y \in B$, there exists $x \in A$ such that $x \leq y$.

[1]For posets $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, $\mathbf{R} \times (\mathbf{P} \dot{\cup} \mathbf{Q}) \simeq (\mathbf{R} \times \mathbf{P}) \dot{\cup} (\mathbf{R} \times \mathbf{Q})$. Hence $\dot{\bigcup_{i \in I}} \mathbf{C}_i \times \dot{\bigcup_{j \in J}} \mathbf{D}_j \simeq \dot{\bigcup_{i \in I, j \in J}} (\mathbf{C}_i \times \mathbf{D}_j)$.

Given a poset $\mathbf{P}$, we define an order $\preceq^{\forall}_{\exists}$ on $\mathscr{P}(P)$ by

$$A \preceq^{\forall}_{\exists} B \Leftrightarrow (\forall y \in B)(\exists x \in A)[x \leq y] \Leftrightarrow \uparrow A \supseteq B.$$

The subsets of P can be infinite. In particular, we notice that this order coincides with the one in $\mathscr{M}(P)$ when we restrict our attention to finitely generated upsets.

To characterize the conditions under which $(\mathscr{P}(P), \preceq^{\forall}_{\exists})$ is a wqo, we will utilize the concept of *better quasi-ordered set* (bqo). The definition of a bqo is rather technical and involved. We will omit the definition since it is beyond the scope of this document and we will simply rely on its properties.

Better quasi-ordered sets were introduced by Nash-Williams [28] as a generalization of wqo's. The following properties are known about them.

Lemma 5.4.1 ([28], [26])**.**

1. *Every bqo is a wqo.*
2. *Finite partial orders and well ordered chains are bqo.*
3. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are bqo's, then their direct product* $\mathbf{P} \times \mathbf{Q}$*, disjoint union* $\mathbf{P} \dot\cup \mathbf{Q}$*, and ordinal sum* $\mathbf{P} \oplus \mathbf{Q}$ *are also bqo's.*

This lemma implies that for all $e \in \mathbb{N}$, $(\mathbb{N}, \leq)^e$ is a bqo because $(\mathbb{N}, \leq)$ is a well ordered chain (it has no infinite descending chains). Furthermore, the lemma implies that every $(H_s, \leq^{\mathbf{H}})$ is a bqo because it is the disjoint union of finitely (c^{j_s}) many bqo's.

The next theorem provides the connection between bqo's and the order $\preceq^{\forall}_{\exists}$.

Theorem 5.4.2 ([25])**.** $(A, \leq)$ *is a bqo iff* $(\mathscr{P}(A), \preceq^{\forall}_{\exists})$ *is a bqo.*

The previous theorem implies that if $(A, \leq)$ is a bqo, then $(\mathscr{M}(A), \supseteq)$ is a bqo and a wqo. We also know that sets ordered under (reverse) inclusion form a poset.

Corollary 5.4.3. *$(H, \leq^{\mathbf{H}})$ is a bqo, which implies that $(\mathscr{M}(H), \supseteq)$ is a well partially ordered set.*

Remark 5.4.4. It is not sufficient that $(H, \leq^{\mathbf{H}})$ is a wpo. Rado discovered a well partially ordered set for which its finite subsets ordered by $\leq_{\exists}^{\forall}$ have an infinite antichain (see [31] and [20]). Here the *Rado structure* is $(\{(i,j) \in \mathbb{N}^2 : i < j\}, \leq_r)$ where

$$(i_1, j_1) \leq_r (i_2, j_2) \Leftrightarrow (i_1 = i_2 \text{ and } j_1 < j_2) \text{ or } (j_1 < i_2).$$

By Lemma 5.3.4, $(C_b, \supseteq)$ is a well partially ordered set. Lemma 2.5.1 implies that it has no infinite ascending chains. Therefore, for every $b \in B$, C_b is a finite set, which implies that $\mathbf{D}$ is finite.

We combine the result of the last two sections in the next theorem. Furthermore, the constructions presented in Section 4.4 that did not include the involutive subvarieties still work in this case. We obtain

Theorem 5.4.5 ([4])**.** *Let $\mathcal{V}$ be a subvariety of fully distributive FL_m^n axiomatized by some equation of the form (a) and any combination of the following identities:*

1. *cyclicity,*
2. *pseudo complementation,*
3. $0 = 1$*,*
4. $0 \leq x$*,*
5. *any identity over the language of $\{\wedge, \vee, \cdot, 1\}$.*

Then $\mathcal{V}$ has the FEP and its universal theory is decidable.

Bibliography

[1] W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *Algebra Universalis*, 43(2-3):233–257, 2000.

[2] K. Blount and C. Tsinakis. The structure of residuated lattices. *Internat. J. Algebra Comput.*, 13(4):437–461, 2003.

[3] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.

[4] R. Cardona and N. Galatos. The FEP for some varieties of fully distributive residuated lattices. Manuscript.

[5] R. Cardona and N. Galatos. The finite embeddability property for noncommutative knotted extensions of RL. *Internat. J. Algebra Comput.*, 25(3):349–379, 2015.

[6] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[7] T. Evans. Some connections between residual finiteness, finite embeddability and the word problem. *J. London Math. Soc. (2)*, 1:399–403, 1969.

[8] N. Galatos and R. Horčík. Cayley's and Holland's theorems for idempotent semirings and their applications to residuated lattices. *Semigroup Forum*, 87(3):569–589, 2013.

[9] N. Galatos and P. Jipsen. Distributive residuated frames. Manuscript.

[10] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Trans. Amer. Math. Soc.*, 365(3):1219–1249, 2013.

[11] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.

[12] N. Galatos and H. Ono. Cut elimination and strong separation for substructural logics: an algebraic approach. *Ann. Pure Appl. Logic*, 161(9):1097–1133, 2010.

[13] N. Galatos and C. Tsinakis. Generalized MV-algebras. *J. Algebra*, 283(1):254–291, 2005.

[14] G. Gentzen. Untersuchungen über das logische Schließen. I. *Mathematische Zeitschrift*, 39(1):176–210, 1935.

[15] G. Grätzer. *Universal algebra*. Springer, New York, second edition, 2008. With appendices by Grätzer, Bjarni Jónsson, Walter Taylor, Robert W. Quackenbush, Günter H. Wenzel, and Grätzer and W. A. Lampe.

[16] G. Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, s3-2(1):326–336, 1952.

[17] R. Horčík. Word problem for knotted residuated lattices. *J. Pure Appl. Algebra*, 219(5):1548–1563, 2015.

[18] R. Horčík. Residuated lattices, regular languages, and burnside problem. Talk presented at TACL, `http://www2.cs.cas.cz/~horcik/confs/tacl2013.pdf`.

[19] R. Hori, H. Ono, and H. Schellinx. Extending intuitionistic linear logic with knotted structural rules. *Notre Dame J. Formal Logic*, 35(2):219–242, 1994.

[20] P. Jančar. A note on well quasi-orderings for powersets. *Inform. Process. Lett.*, 72(5-6):155–160, 1999.

[21] P. Jipsen and C. Tsinakis. A survey of residuated lattices. In *Ordered algebraic structures*, volume 7 of *Dev. Math.*, pages 19–56. Kluwer Acad. Publ., Dordrecht, 2002.

[22] O. G. Kharlampovich and M. V. Sapir. Algorithmic problems in varieties. *Internat. J. Algebra Comput.*, 5(4-5):379–602, 1995.

[23] T. Kowalski and H. Ono. Fuzzy logics from substructural perspective. *Fuzzy Sets and Systems*, 161(3):301–310, 2010.

[24] J. Lambek. The mathematics of sentence structure. *Amer. Math. Monthly*, 65:154–170, 1958.

[25] A. Marcone. Fine analysis of the quasi-orderings on the power set. *Order*, 18(4):339–347 (2002), 2001.

[26] E. C. Milner. Basic wqo- and bqo-theory. In *Graphs and order (Banff, Alta., 1984)*, volume 147 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 487–502. Reidel, Dordrecht, 1985.

[27] C. S. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Philos. Soc.*, 59:833–835, 1963.

[28] C. S. J. A. Nash-Williams. On well-quasi-ordering infinite trees. *Proc. Cambridge Philos. Soc.*, 61:697–720, 1965.

[29] M. Okada and K. Terui. The finite model property for various fragments of intuitionistic linear logic. *J. Symbolic Logic*, 64(2):790–802, 1999.

[30] G. Pollák. On hereditarily finitely based varieties of semigroups. *Acta Sci. Math. (Szeged)*, 37(3-4):339–348, 1975.

[31] R. Rado. Partial well-ordering of sets of vectors. *Mathematika*, 1:89–95, 1954.

[32] K. I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990.

[33] J. Schmidt and C. Tsinakis. Relative pseudo-complements, join-extensions, and meet-retractions. *Math. Z.*, 157(3):271–284, 1977.

[34] C. J. van Alten. The finite model property for knotted extensions of propositional linear logic. *J. Symbolic Logic*, 70(1):84–98, 2005.

[35] M. Ward and R. P. Dilworth. Residuated lattices. *Trans. Amer. Math. Soc.*, 45(3):335–354, 1939.

[36] A. M. Wille. The word problem for involutive residuated lattices and related structures. *Arch. Math. (Basel)*, 87(6):546–553, 2006.