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Fr. Joseph Wagner S.J.
Xavier University - Cincinnati

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What the Integral Does: Physics Students' Efforts at Making Sense of Integration

Joseph F. Wagner

Xavier University, Department of Mathematics

3800 Victory Parkway, Cincinnati, Ohio, 45207

Abstract. Students use a variety of resources to make sense of integration, and interpreting the definite integral as a sum of infinitesimal products (rooted in the concept of a Riemann sum) is particularly useful in many physical contexts. This study of beginning and upper-level undergraduate physics students examines some obstacles students encounter when trying to make sense of integration, as well as some discomforts and skepticism some students maintain even after constructing useful conceptions of the integral. In particular, many students attempt to explain what integration does by trying to interpret the algebraic manipulations and computations involved in finding antiderivatives. This tendency, perhaps arising from their past experience of making sense of algebraic expressions and equations, suggests a reluctance to use their understanding of "what a Riemann sum does" to interpret "what an integral does."

Keywords: integration, Riemann sums, student understanding

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I. INTRODUCTION

Researchers have argued that the Riemann sum-based interpretation of the definite integral is perhaps the most valuable interpretation for making sense of integration in applied contexts, particularly physics [1,2]. Generally, a "Riemann sum-based interpretation" refers to imagining the definite integral as a sum of products, in which one of the factors is an infinitesimal or a "very small amount." Despite the utility of Riemann sum-based interpretations, many students do not develop such reasoning in their calculus courses, despite having studied Riemann sums [3]. It is likely that at least part of the reason for this situation is that Riemann sum-based interpretations are not generally emphasized in traditional calculus classrooms, where procedural methods and "area under the curve" ideas dominate [3]. I argue here, however, that there are also other factors at work that can interfere with students' ability to adopt Riemann sum-based reasoning, and that even among students who do adopt such reasoning, these factors continue to lead students to doubt its legitimacy. In particular, the central thesis of this paper is that the algebraic solution process for finding an area as a limit of a Riemann sum is inherently different from the solution process for finding an area through the computation of an antiderivative, and that this difference can cause varying levels of confusion and puzzlement to students.

II. WHAT DO INTEGRALS DO?

Although students and experts alike commonly use an "area under a curve" interpretation of the definite integral, it is important to note that this is really a means of interpreting what the *value* of the integral might represent. Riemann sums, however, can be used as part of a *process* through which that value is found. Of interest to this research is how students understand the process of using Riemann sums, and how they understand the process of integration. That is, from a student's perspective, what do these two processes *do* and what do they have to do with each other?

Under standard definitions of the embedded symbols, the definite integral can be expressed (or defined) as a Riemann sum:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

As a mathematical, algebraic statement, the Riemann sum representation of an area as a combination of rectangular areas has explanatory sense built into it, in that a meaningful algebraic and geometric process for finding area can be mapped onto the symbols of the expression, and the process can be directly modeled and investigated. The area of each rectangle is computed by multiplying its height and width, and is represented by $f(x_i)\Delta x$. The summation indicates that all the rectangles are to be added together and combined into a total area. The limiting process mathematizes the notion of letting the width of intervals get "very small" (or, equivalently, letting the number of rectangles increase without bound). Furthermore, the fact that the limiting process has a completion point, the limit itself, supports the notion of the rectangles themselves becoming "infinitesimally thin." In short, the Riemann sum process algebraically does what it says it does. There is a clear way in which algebraic and geometric meaning can be mapped on to the algebraic syntax.

The situation with regard to the definite integral, however, is quite different from the perspective of a student who considers a definite integral to be the means of calculating an area using antidifferentiation. Although the symbols used in expressing a definite integral lend themselves to being interpreted as the sum of the products of lengths and widths of rectangles, the actual process of computing the definite integral is entirely different. Consider a simple example:

$$\int_0^2 x^3 dx = \frac{1}{4}x^4 \Big|_0^2 = \frac{16}{4} - 0 = 4$$

In this case, what the integral "does" is transform the integrand, x^3, into its antiderivative, $\frac{1}{4}x^4$, through a process that cannot be subjected to algebraic sense-making. The power rule for finding such an antiderivative can be readily proven, of course, but the computation takes advantage of a known pattern associated with antiderivatives of polynomials, and the algebraic manipulation is immune to any sort of geometric explanation or metaphorical algebraic interpretation, except perhaps in the simplest cases. In addition, no actual "summation" of any sort takes place, nor does anything "get very small," and the differential *dx* appears simply to be extraneous and to evaporate in the solution process. (Indeed, a number of the beginning students wondered aloud why the differential was used at all.) Riemann sum-based reasoning may fit the syntax of the original integral expression, but it cannot be used to explain or extract meaning from the computational process.

At the very least, nothing would suggest that importing a Riemann sum-based explanation into the process of integration via antidifferentiation ought to be automatic or natural for students, at least, not for students who are accustomed to trying to make sense of their mathematical activities. Nothing they *do* when computing a definite integral is at all related to Riemann sum-based ideas. I am not aware of any other circumstances in typical mathematics curricula prior to the study of calculus that asks students to use the semantics of one mathematical process to interpret the syntax of another. The purpose of this research is to document how this conflict between syntax and meaning is manifested in the reasoning used by undergraduate physics students.

III. METHODS

Data for this study are taken from extensive interviews between individual undergraduate physics students and the author (who was not their instructor) at a large public university. Eight beginning students were selected from among volunteers in an introductory calculus-based physics course, focusing primarily on classical mechanics. These students, representing a variety of majors, included first- and second-year students who had all completed at least one academic quarter each of differential and integral calculus. Seven upper-level students were also selected from a third-year cohort of undergraduate physics majors, each of whom had completed two quarters of multivariable (vector) calculus. Students were interviewed using open-ended interview methods about every other week during the course of an eight-week term, with each interview lasting about 45-60 minutes. All students completed at least two interviews, and most completed four. The questions presented to students involved primarily problems and concepts of calculus, both in abstract mathematical form and in applied problem contexts. The interviewer asked students to answer questions and solve problems, sometimes using mechanical devices, thinking aloud as best they could. Additional questioning continued until the interviewer believed he understood the reasoning the students were using, but, in general, the interviewer refrained from evaluating the students' ideas or offering instruction. All interviews were audio- and video-recorded for later analysis. The portions of the interviews relevant to the present research all involved students' use and interpretation of definite integrals in both abstract and applied contexts.

Analysis of the student interviews used qualitative methods consistent the methods of *knowledge analysis* [6], an emergent analytical method suited to a knowledge-in-pieces epistemology. Careful attention was given to students' reasoning strategies and their patterns of use, particular use of language and gesture, the use of intuitive and naive knowledge, and changes and patterns of reasoning across contexts.

IV. SEARCHING FOR MEANING IN INTEGRATION

A thorough presentation of the data requires a careful and extended analysis of a large number of lengthy interview transcripts. In the limited space allotted here, I offer only summaries of the central findings of this work, absent the detailed transcripts that support these findings. A more comprehensive paper is in preparation.

At the time of the interviews, none of the beginning students demonstrated an ability to use Riemann sum-based reasoning to interpret the definite integral. All of these students, however, had studied both Riemann sums and definite integration. When asked about the relationship between the two topics, many students indicated an awareness of some relationship, but none could articulate it. About half identified them as two different ways of finding the same thing, an area. The other half focused on Riemann sums as a means of *approximating* an area, while a definite integral could find it *exactly*. As such, integrals were preferred, and Riemann sums were invoked only as a last resort when an integral could not be directly computed. When asked *why* solving a definite integral through antidifferentiation found an area, none of the beginning students (and only one of the upper-level students) could answer the question.

Most of the beginning students, either spontaneously or under direct questioning, indicated that their knowledge of

Riemann sums did not in any way inform their understanding of the definite integral. Throughout the interviews, however, both beginning and upper-level students sometimes showed evidence of searching for meaning in the procedures used to compute definite integrals through antidifferentiation and the Fundamental Theorem of Calculus. This search for meaning arose in at least three different aspects of the solution process.

A. Searching for meaning in symbol manipulations

I described above how the algebra of Riemann sums directly supports algebraic and geometric sense-making for finding areas. Several students in this study showed evidence of looking for or expecting similar sense in the symbolic manipulations of the antidifferentiation process. These students, for example, questioned why the power rule for finding the antiderivative of a polynomial should result in a function that gives area, or why that same rule used in an applied context resulted in a change of units between the original function and its antiderivative. They specifically noted an expectation of meaning to be found in the symbolic manipulations. One beginning student, for example, attempted to interpret the symbolic manipulations of the power rule:

> I don't know why, like, bringing up a constant in the exponent, or whatever you have to do to solve it … I don't know why that means that it's now revolutions instead of revolutions per minute, if I was integrating revolutions per minute.

Even upper-level students appeared to expect to find sense in the symbolic manipulations. One upper-level student, also discussing the power rule, suggested that there should be a geometric explanation for it:

> I know it's the power rule, but I guess they never showed me why behind the power rule, or like, the visual, a connection between the graph.

The data lead me to conclude that some students are treating the symbolic manipulations of antidifferentiation as though they are algebraic manipulations, and should be subject to algebraic interpretation and sense-making. Both physics and mathematics educators alike have emphasized the importance of developing conceptual understanding to underlie algebraic skills [4,5], but we have not yet directed attention to the possibility of students' subsequent search for such meaning in the context of integration and antidifferentiation techniques where it cannot be found in the same way.

B. Searching for meaning in substitution procedures

Some students showed evidence of finding significance in the substitution procedures using the limits of integration at the end of the integral evaluation process. The appeal to the substitution procedures typically arose as students were trying to explain why the units of an integrand changed in the process of integration; for example, why the antiderivative with respect to time of a function with units of revolutions per second was a function with units of revolutions. A student using Riemann sum-based reasoning could appeal to a cancellation of units between the integrand and the differential. Without such Riemann sum-based reasoning skills, these students appealed, instead, to the substitution process as the source of the change in units. For these students, the antiderivative, in itself, essentially maintained the units of the original integrand. They reasoned, however, that during the substitution procedure, the units of the limit were inserted into the antiderivative, thereby resulting in a change of units. One beginning student gave an extensive explanation for this and maintained his reasoning for the final units of "revolutions" under repeated questioning: "Because you're adding in the time component. You're substituting in." He argued that *only* during the substitution process the units of the original integrand cancelled with the units of the integral's limits.

C. Searching for meaning in the geometry of the antiderivative

In their attempt to explain why a definite integral could be interpreted as an area, some students sought geometric structure within the algebraic form of the antiderivative. They reasoned that, since finding an antiderivative is necessary for finding the area under the curve, there ought to be a way to uncover the area calculations within the algebraic structure of the antiderivative. This is, in a sense, an attempt to construct a direct parallel to finding the underlying geometric structure within a Riemann sum, where heights, widths, and sums of rectangles are all represented in the algebra.

One beginning student made a considerable effort to deconstruct an antiderivative algebraically in an attempt to match its algebraic structure to her graph of the area she knew it was used to find. In the end, she exhibited some satisfaction in mapping her calculations to two area regions, the difference of which gave her the final answer. She could not, of course, explain why the algebra produced the correct areas, and she quickly realized that she still could not explain why antiderivative process should be used at all. At that point, she returned to a written expression of the power rule, appeared to try to make sense of why it should yield and area, and soon gave up: "I don't know."

Recall that none of these students ever invoked Riemann sum-based reasoning to interpret an antiderivative, and most gave convincing evidence that they could not do so. In these circumstances, attempts to explain why integration does what it does led students to bring out the only other tools they had at their disposal: algebraic and geometric reasoning tools that serve well in other circumstances. What was lacking was an awareness that antidifferentiation procedures are not subject to algebraic reasoning.

V. SKEPTICISM OF RIEMANN SUM REASONING

In contrast to the beginning students, all of the upper-level students demonstrated competence in using Riemann sum-based reasoning to interpret definite integrals. They all, in fact, used it quite well for both abstract mathematics problems and contextually rooted physics problems. Nonetheless, a number of the students clearly and repeatedly expressed skepticism in the validity of using such reasoning.

One upper-level student interrupted his otherwise clear Riemann sum-based explanation for an integral he set up with a humorous expression of embarrassment:

> Yeah, I do it. I don't-, I'm not proud of it, but I hope there's some way to justify it.

Asked to explain his comment, the student indicated that such reasoning seemed to him to be "kind of a trick," but that he could not justify it mathematically, and he did not know if it *could* be justified mathematically. He seemed particularly troubled by interpreting a differential as an infinitesimal, calling such an identification "hokey."

Another upper-level student, equally skilled at demonstrating Riemann sum-based reasoning when asked to do so, went out of his way to *avoid* using such reasoning. His explicit reason was that the Riemann sum explanation did not reflect what integration actually does. He observed, correctly, that integration via antidifferentiation involved a function transformation, but that this transformation took place through an entirely mysterious process that was not subject to sense making:

> Like, it's impossible to actually accurately explain what this integral is conceptually. It's impossible to do it…. It's not possible … to talk about an infinitesimal volume and an infinitesimal density. That doesn't make sense.

This student's case is particularly striking because, unlike the beginning students, he understood that the antidifferentiation process could not be subjected to algebraic or geometric interpretation, but he equally rejected a Riemann sum-based explanation because he could not accept that one could reason sensibly about infinitesimals. What all of these students have in common, however, is an inability to reconcile reasoning about Riemann sums with the actual computational process of calculating a definite integral through antidifferentiation.

VI. CONCLUSION

A number of researchers in both physics and mathematics education have observed that many students, even those with a strong calculus background, fail to use Riemann sum-based interpretations of the definite integral, despite its unique value to supporting sense-making in many applied contexts. There can be little doubt that part of the reason for students' unfamiliarity with such reasoning process is that they are not given much emphasis in traditional calculus curricula. What I hope this research demonstrates, however, is that addressing this situation is more complex than it may first appear.

I am arguing that there are additional explanations for why students do not quickly pick up Riemann sum-based reasoning, and why such reasoning may seem puzzling or suspect to them even when they have been taught to use it. I hope it is clear that this paper should not be interpreted as another exposition of "student deficits." To the contrary, the heart of the argument is that most of the students who took part in this research were actively trying to make sense of the mathematical activities that make up the integration process. The problems they ran into, however, exist because of the peculiar marriage that must take place between the reasoning of Riemann sums and their limits, and the algebraic symbols and symbolic manipulations that represent the process of integration by means of antidifferentiation. At face value, there is no obvious reason that students can find for using Riemann sums to interpret antidifferentiation procedures that not only appear to be, but actually *are*, algebraically unrelated to the complex limit and summation procedures they have learned for Riemann sums. The algebra of Riemann sums readily supports reasoning about area computations; the procedures on which the Fundamental Theorem of Calculus is based do not. We should not be surprised that students question the validity of using the reasoning for one to interpret the computational results of the other. If they are to be successful, increased attempts to introduce students to the use of Riemann sum-based reasoning will need to accommodate these peculiar hurdles that students will encounter.

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REFERENCES

[1] V. Sealey, J. Math. Behav. **33**, 230 (2014).
[2] L. Doughty, E. McLoughlin, and P. van Kampen, Am. J. Phys. **82**, 1093 (2014).
[3] S. R. Jones, Int. J. Math. Ed. Sci. Tech. **46**, 721 (2015).
[4] C. Kieran, in *Second handbook of research on mathematics teaching and learning*, edited by F. Lester (Information Age, Charlotte, 2007).
[5] B. L. Sherin, Cogn. Instr. **19**, 479 (2001).
[6] A. A. diSessa, B. Sherin, and M. Levin, in *Knowledge and interaction: A synthetic agenda for the learning sciences*, edited by A. A. diSessa et al. (Routledge, New York, in press).