1-2015

Charged Particle Dynamics in the Magnetic Field of a Long Straight Current-Carrying Wire

M. Fatuzzo
Xavier University - Cincinnati

A. Prentice

T. Toepker

The Physics Teacher

Charged Particle Dynamics in the Magnetic Field of a Long Straight Current-Carrying Wire

A. Prentice, M. Fatuzzo, and T. Toepker

Citation: The Physics Teacher **53**, 34 (2015); doi: 10.1119/1.4904240
View online: http://dx.doi.org/10.1119/1.4904240
View Table of Contents: http://scitation.aip.org/content/aapt/journal/tpt/53/1?ver=pdfcov
Published by the American Association of Physics Teachers

Charged Particle Dynamics in the Magnetic Field of a Long Straight Current-Carrying Wire

A. Prentice, M. Fatuzzo, and T. Toepker, Xavier University, Cincinnati, OH

By describing the motion of a charged particle in the well-known nonuniform field of a current-carrying long straight wire, a variety of teaching/learning opportunities are described.

1) Brief review of a standard problem
2) Vector analysis
3) Dimensionless variables
4) Coupled differential equations
5) Numerical solutions

It is standard for an introductory-level physics text[1-3] covering electromagnetism to present the trajectory of a charged particle in a *uniform* magnetic field B_0, and to demonstrate that such a particle will exhibit the following types of motion:

- Circular if the velocity is perpendicular to the field.
- Linear if the velocity is parallel to the field.
- Helical if the velocity has components both parallel and perpendicular to the field.

When the particle motion is perpendicular to the field, the resulting uniform circular motion allows Newton's second law to be expressed as

$$F = qvB_0 = m\frac{v^2}{r}, \tag{1}$$

which, along with the kinematics relation

$$v = \frac{2\pi r}{T}, \tag{2}$$

then yields the following expression for the period T of the motion:

$$T = \frac{2\pi m}{qB_0} = \frac{2\pi}{\omega}, \tag{3}$$

where $\omega = qB_0/m$ is known as the "cyclotron frequency." We will use the cyclotron frequency later on to introduce a dimensionless time, $\tau = \omega t$.

The problem

But what type of motion is possible if the field is *nonuniform*? Motivated by the desire of providing an illustrative example of such a case that is accessible to an introductory physics audience, we examine how a positive charge q moves through the magnetic field produced by a very long straight current-carrying wire,

$$B = \frac{\mu_0 I}{2\pi r}, \tag{4}$$

where I is the current through the wire and r is the distance from the wire. The magnetic field lines, which indicate the direction of the field at any point, are represented by families of circles around the wire with arrows directed in accordance to the natural curvature of our fingers when our thumb points in the direction of the current.

This paper focuses on particle trajectories that are planar, which (as shown below) is ensured by choosing initial velocities that are parallel to the wire. Figure 1 shows the adopted

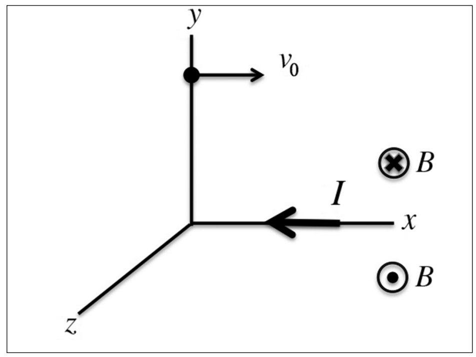


Fig. 1. Schematic of the adopted geometry, for which the positive current is directed along the negative *x*-axis, and the charged particle is launched from the *x*-*y* plane (i.e., *z* = 0) in the positive *x*-direction with an initial speed v_0. Note that in the *x*-*y* plane, the magnetic field points into the paper above the wire and out of the paper below the wire.

geometry, for which the positive current is directed along the negative x-axis, and the charged particle is launched from the x-y plane (i.e., $z = 0$) in the positive x-direction with an initial speed v_0. We note that at the location of the charge, the field is perpendicular to the x-y plane and therefore points in the negative z-direction, e.g., $\mathbf{B} = -B(r)\hat{k}$.

Vector analysis

The ensuing particle motion is governed solely by the magnetic force,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}, \tag{5}$$

which (as illustrated by the cross product), always acts perpendicular to the particle's velocity and the magnetic field direction. As such, the force acting on a particle that is on the x-y plane (at $z = 0$) can never have a component in the z-direction, ensuring that the particle trajectories remain in the x-y plane. The particle acceleration follows directly by substituting the magnetic force into Newton's second law and can easily be written in component form as

$$a_x = \frac{q}{m}(v_y B_z - v_z B_y), \tag{6a}$$

 DOI: 10.1119/1.4904240

$$a_y = \frac{q}{m}(v_z B_x - v_x B_z), \quad (6b)$$

$$a_z = \frac{q}{m}(v_x B_y - v_y B_x), \quad (6c)$$

which, for motion confined to the *x*-*y* plane (for which B_z is the only non-zero component), simplifies to

$$a_x = \frac{q}{m} v_y B_z, \quad (7a)$$

$$a_y = -\frac{q}{m} v_x B_z, \quad (7b)$$

$$a_z = 0. \quad (7c)$$

Note that the field term B_z is negative for our particular case, which will then appear to reverse the signs in Eqs. (11a) and (11b) below.

It may be helpful for students (and teachers) to consider what type of motion is described by Eqs. (5) and (7) to get a better feel for the expected results, and to "know the answer before you calculate." (Often students obtain unreasonable results but are not aware because they did not think about or "ballpark" their answer.) Looking at Fig. 1, the initial velocity v_0 produces a positive acceleration a_y, which in turn produces a positive velocity v_y, which when then crossed with B_z will produce a negative a_x , and so on. Since the field strength varies as $1/r$, a little intuition would suggest that the radius of curvature would be larger farther from the wire and smaller closer to the wire. Let's see if our intuition is right.

Dimensionless variables

The trajectory of a positive particle can be found by solving the coupled differential Eqs. (7a) and (7b) along with the specified initial conditions. To facilitate the calculations, we rewrite the physical variables in terms of the following dimensionless variables: $\bar{x}$, $\bar{y}$, $\bar{v}_x$, $\bar{v}_y$, and τ (recall that $z = 0$ for the case considered here). Specifically, we adopt the following conventions:

1) The particle position will be expressed in terms of the initial distance to the wire y_0 (which for our case is also the distance of closest approach to the wire), so that $\bar{x} = x/y_0$ and $\bar{y} = y/y_0$.
2) The magnetic field will be expressed in terms of the field strength

$$B_0 = \frac{\mu_0 I}{2\pi y_0} \quad (8)$$

at the launch point, so that $B_z = -B_0/\bar{y}$. As noted above, the value of B_z will be negative. Note also that since the motion is restricted to the *x*-*y* plane, $r = y$ for our adopted geometry.

3) Time will be expressed in terms of the cyclotron frequency for a particle moving perpendicular to a uniform field of strength B_0, as given by Eq. (3).
4) Consistent with the above conventions, velocities will be expressed in terms of the quantity

$$\omega y_0 = \frac{q y_0 B_0}{m} = \frac{q \mu_0 I}{2\pi m}, \quad (9)$$

so that

$$\bar{v}_0 = \frac{2\pi^2 m v_0}{\mu_0 q I}. \quad (10)$$

As an exercise in using appropriate metric units, one can show that the expression in Eq. (10) is dimensionless. The concept of units is emphasized with students as a first step in verification of their mathematical expressions. Changing to dimensionless expressions simplifies equations and gives practice in dimensional analysis.

Coupled differential equations

With these modifications, Eqs. (7a) and (7b) become two seemingly simple dimensionless equations

$$\frac{d\bar{v}_x}{d\tau} = -\frac{\bar{v}_y}{\bar{y}} \quad (11a)$$

$$\frac{d\bar{v}_y}{d\tau} = \frac{\bar{v}_x}{\bar{y}}, \quad (11b)$$

where

$$\frac{d\bar{x}}{d\tau} = \bar{v}_x \quad \text{and} \quad \frac{d\bar{y}}{d\tau} = \bar{v}_y. \quad (11c)$$

Three of the initial conditions are specified as $\bar{x}_0 = 0$, $\bar{y}_0 = 1$, and $\bar{v}_{y0} = 0$, leaving the fourth initial condition, $\bar{v}_{x0} = \bar{v}_0$, as the single adjustable parameter that characterizes the problem.

Numerical solutions

The dimensionless coupled equations were solved for various trajectories (as defined by the initial velocity) using the NDSolve function in Mathematica. The results for two different initial velocities are shown in Fig. 2. As predicted, the radius is smaller closer to the wire and larger away from the wire, resulting in a drift in the negative *x*-direction (the direction of the current). There are two aspects of the shape of the path that are noticeably changing as the initial velocity is varied—the size and the "tightness" of the loops—which then relate to the drift and amplitude of the trajectory.

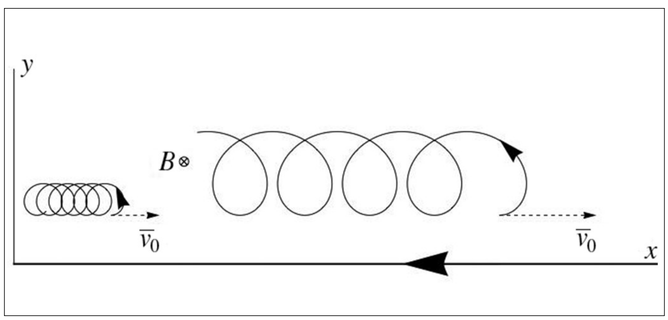


Fig. 2. Particle trajectories for two different values of initial velocity $\bar{v}_0$, where $\bar{v}_0$ = 0.25 for the tight particle trajectory (left) and $\bar{v}_0$ = 0.5 for the other particle trajectory (right).

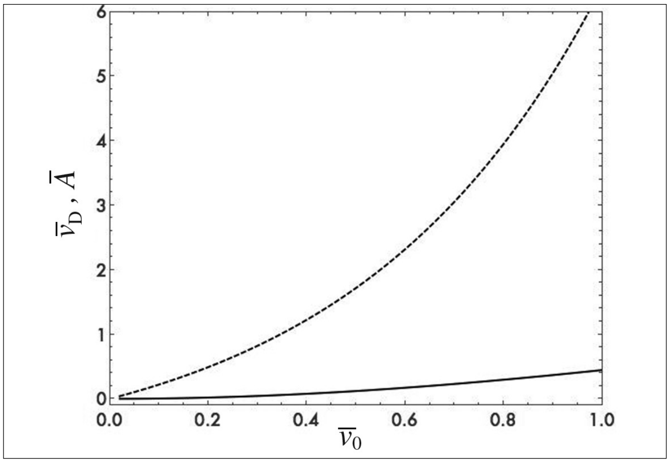


Fig. 3. Drift velocity $\bar{v}_D$ (solid curve) and orbit amplitude $\bar{A}$ (dashed curve) as a function of the initial velocity $\bar{v}_0$.

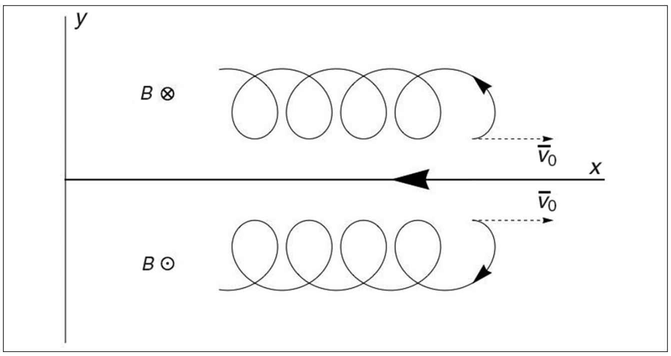


Fig. 4. Particle trajectories for two different initial positions, where $\bar{y}_0$ = 1 for the top particle trajectory and $\bar{y}_0$ = –1 for the bottom particle trajectory. In both cases, $\bar{v}_0$ = 0.5.

One can characterize the drift velocity $\bar{v}_D$ by dividing the distance moved along the x-axis during one cycle by the time required to complete the cycle, and the orbit amplitude $\bar{A}$ by taking the difference between the maximum and minimum values of $\bar{y}$. These values are shown in Fig. 3 as a function of the initial velocity $\bar{v}_0$. Not surprising, both quantities increase with $\bar{v}_0$.

As an exercise, one can ask what happens if the initial velocity is reversed, or if the magnetic field is reversed, or both. Figure 4, similar to Fig. 1, shows two particles moving parallel to the wire, but with one moving in the $y > 0$ region and the other moving in the $y < 0$ region. The mirror symmetry becomes obvious when one realizes that the problem has rotational symmetry about the x-axis. *Any* initial velocity vector parallel to the wire will yield the same general trajectory.

A student should consider whether the velocity vector changes over the trajectory (yes, since the direction is changing) and whether the speed (or kinetic energy) changes over the trajectory. A little vector analysis in the appendix demonstrates that the speed does not change. Since the magnitude of the velocity vector does not change, one could start the motion at any point on the trajectory with the velocity tangent to the curve. Mathematica is used by students primarily in research projects. This paper is part of a more extensive student project by A. Prentice, who examined more general trajectories as shown in Fig. 5.

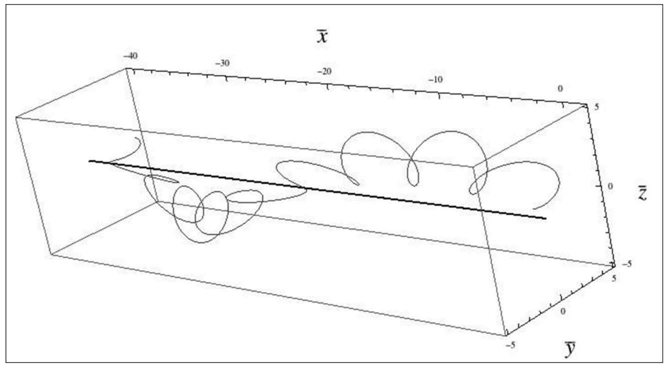


Fig. 5. Particle trajectory for a particle injected at $\bar{x}_0$ = 0, $\bar{y}_0$ = 1, $\bar{z}_0$ = 0 with an initial velocity $\bar{v}_{0x}$ = 0.75, $\bar{v}_{0z}$ = 0.2. The basic drift motion is still observed, with a helical-like structure in its path around the wire. Note that the current is in the negative *x*-direction.

Conclusion

The goal of this paper was to examine the trajectory of a charged particle in a relatively simple non-uniform magnetic field. The complexity that arises by adopting a well-known non-uniform field may not have been anticipated! Further interesting (and complex) considerations arise by removing the constraint of two-dimensional motion, as demonstrated by the illustrative example shown in Fig. 5. The analysis of these more complex trajectories is left for future work.

More advanced discussions of the trajectories of charged particles in magnetic fields are available. [4,5]

References

1. D. C. Giancoli, *Physics*, 7th ed. (Pearson, Glenview, IL, 2004), Chap. 20.
2. R. D. Knight, *Physics,* 3rd ed. (Pearson/Addison-Wesley, San Francisco, CA, 2008), Chap. 33.
3. D. J. Griffiths, *Introduction To Electrodynamics*, 3rd ed. (Prentice Hall, New Jersey, 1999), Chap. 5.
4. M. Kaan Öztürk,"Trajectories of charged particles trapped in Earth's magnetic field," *Am. J. Phys*. **80**, 420–428 (May 2012).
5. I. H. Hutchinson, *Introduction to Plasma Physics*, silas.psfc.mit.edu/introplasma.

Appendix

Does the kinetic energy change over the trajectory?

$$K=\frac{1}{2}mv^2=\frac{1}{2}m\,\mathbf{v}\cdot\mathbf{v} \tag{A1}$$

$$\frac{dK}{dt}=\frac{1}{2}m\left(\frac{d\,\mathbf{v}}{dt}\cdot\mathbf{v}+\mathbf{v}\cdot\frac{d\,\mathbf{v}}{dt}\right). \tag{A2}$$

But the dot product commutes, so

$$\frac{dK}{dt}=m\left(\frac{d\,\mathbf{v}}{dt}\cdot\mathbf{v}\right). \tag{A3}$$

Looking at the force equation,

$$\mathbf{F}=m\frac{d\,\mathbf{v}}{dt}=q(\,\mathbf{v}\times\mathbf{B}\,). \tag{A4}$$

Taking the dot product of (A4) with **v**:

$$m\left(\mathbf{v}\cdot\frac{d\,\mathbf{v}}{dt}\right)=q\,\mathbf{v}\cdot(\,\mathbf{v}\times\mathbf{B}\,)=0, \tag{A5}$$

where the right-hand side is zero since $\mathbf{v}$ is perpendicular to $\mathbf{v}\times\mathbf{B}$. Thus,

$$m\left(\mathbf{v}\cdot\frac{d\,\mathbf{v}}{dt}\right)=0. \tag{A6}$$

But from above, it follows that the time derivative of K is also zero. Thus, the kinetic energy is constant over the trajectory and the magnitude of the velocity also remains constant.

Xavier University, Cincinnati, OH 45207; toepker@xavier.edu

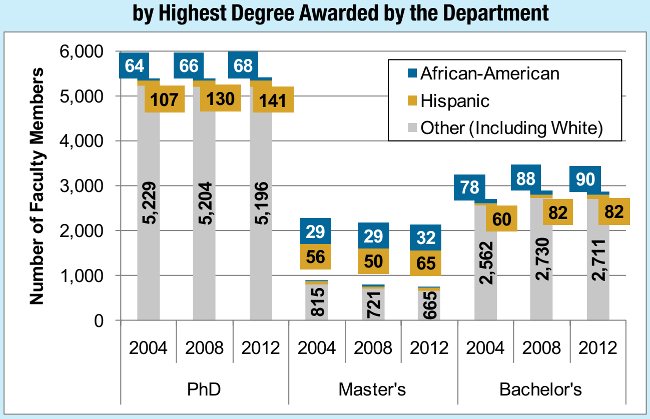
And the Survey Says ...

Susan C. White
American Institute of Physics
Statistical Research Center
College Park, MD 20740; swhite@aip.org

Underrepresented minorities among physics faculty members

The United States is becoming more and more diverse, but the representation of some minority groups in physics and astronomy lags behind. Although 13% of the U.S. population is African-American or black, and 17% is Hispanic (U.S. Census), the representation of these two groups in physics is much lower. For this reason, African-Americans and Hispanics are considered underrepresented minorities (URMs) in physics. Furthermore, the representation of Native Americans in physics is so low that data often cannot be reported. While the percentage of Hispanic physics faculty members has increased from 2.7% in 2004 to 3.2% in 2012, the representation of African-Americans has stayed relatively constant over this period at about 2%. Among the more than 9,000 faculty members in physics departments in 2012, there were 288 Hispanics and 190 African-Americans.

Number of African-American and Hispanic Physics Faculty by Highest Degree Awarded by the Department



www.aip.org/statistics

Next month we will take a closer look at where the minority faculty members work. If you have any questions or comments, please contact Susan White at the Statistical Research Center of the American Institute of Physics (swhite@aip.org).

DOI: 10.1119/1.4904241