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Black-Scholes Model: an Analysis of the Influence of Volatility

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by

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ABSTRACT

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In this thesis the influence of volatility in the Black-Scholes model is analyzed. The deduced Black-Scholes formula estimates the price of European options. Contrary to the other parameters of the formula, the future volatility of the underlying asset cannot be observed in the market. The parameter needs to be assumed in order to calculate the option price. An inaccurate assumption may lead to an erroneous volatility. It is studied how a falsely assumed volatility impacts on the option price. Empirical simulations will be carried out to get an impression of possible errors in the computations. Afterwards, those results will be discussed and linked with an evaluation of potential risks.

To my loving parents

who support me in my personal

and academic endeavors.

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Chapter 1

Introduction

During the last several decades the trade of stocks and options has experienced an increasing interest in both scientific work and everyday life. In 1973 the mathematicians Fischer Black and Myron Scholes published a paper titled "The Pricing of Options and Corporate Liabilities" (Black and Scholes 1973). In it, they illustrated how the pricing of stock options is uniquely determined by their formula. They used a method of arbitrage reasoning, which was developed by Robert Merton, to obtain their option-pricing formula. When Merton and Scholes discovered a new method to determine the value of derivatives, they were awarded the Nobel Prize in Economic Sciences in 1997. Fischer, who had collaborated (Nobelprize.org 2017) in the development of that formula, did not live to see the prize-giving, since he passed away two years prior to that. The Black-Scholes model is used to calculate the theoretical price of European put and call options, where an option is "a contract for the right to buy and sell shares at a later date or within a certain period at a particular price" (*Cambridge Online Dictionary* 2017a).

Therefore they assumed some features of the financial market, including:

European-style options: The model supposes European-style options. Those can only be exercised on the expiration date. With American-style options it is possible to exercise the option at any time during the life of the option.

Efficient markets: It is assumed that the stock's behavior is like a random walk. Meaning, at any given moment in time, the price of the underlying stock can go up or down. The future stock price is independent from the past. The market movements cannot be predicted.

No dividends: During the life of the option no dividends are paid out.

No transaction and commissions costs: It is presumed that there are no fees for buying or selling options and stocks in the Black-Scholes model. Additionally there are no barriers to trading.

Returns are lognormally distributed: As normally done in the real world, the Black-Scholes model assumes lognormal distributed profits of the stock.

Constant volatility and risk-free rate: The model's most significant assumptions are that the volatility of the underlying stock and the risk-free rate are known and constant (Bossu and Henrotte 2012). In a short term it is possible to have a relatively constant volatility, whereas it becomes variable in the long-term view. A risk-free rate is the best rate that does not involve taking a risk. In theory it supposes a rate of return without any loss. This is not feasible in reality. In real-life it is possible to use the U.S. Government Treasury Bills rate, for instance. But these treasury rate can change in times of increased volatility.

As stated above, the Black-Scholes model considers the volatility of the underlying stock to be constant. The broker assumes an estimated rate. It is believed, that it corresponds with the real market behavior. But what would happen if the assumption does not match with the development in real life? How much does a small variation in volatility affect the option price? To which consequences would those variations lead? This thesis deals with finding answers to these questions.

The outline of this thesis is as follows: Chapter 2 summarizes the Black-Scholes formula, including the stochastic argument for it (see Section 2.1) and the verification of the formula itself by a change of variables (see Section 2.2). Subsequently, the third chapter presents different simulations for both, European and digital call and put options. Errors resulting in the use of inaccurate volatility are displayed and analyzed. In a European option the payoff at maturity is the maximum of the difference of the current stock price and the striking price or zero. The payoff of a digital option is equal to one, if the underlying asset expires in the money at expiry and zero otherwise. Following this, in Chapter 4 various risk measures are considered and evaluated. In Chapter 5 the hedged errors are weighted by the probability of choosing a volatility based on historical data. Then they get evaluated with different risk measures. The final Chapter 6 provides the summary and draws a conclusion on the object of investigation.

Chapter 2

Black-Scholes Model

The Black-Scholes model is used to calculate the value of a stock option. The developers assumed some features for the financial market. The following paragraph illustrates the Black-Scholes pricing formula for European call and put options. In order to calculate the price some specific input variables are used, which are:

s - Current stock price;

t - Current time;

K - Option striking price;

r - Risk-free rate; and

$\boldsymbol{\sigma}$ **-** Standard deviation.

The Black-Scholes pricing formula for the price of a European call or put option is

$$F(t,s) = \epsilon \cdot s \cdot \Phi(\epsilon \cdot d_1(t,s)) - \epsilon \cdot Ke^{-r(T-t)} \cdot \Phi(\epsilon \cdot d_2(t,s)).$$

In this, Φ is the cumulative distribution function of $N(0,1)$,

$$\begin{aligned}
d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \cdot \left[\ln\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right], \\
d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t} \\
&= \frac{1}{\sigma\sqrt{T-t}} \cdot \left[\ln\left(\frac{s}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right], \text{ and} \\
\epsilon &= \begin{cases} +1 \text{ for a call,} \\ -1 \text{ for a put.} \end{cases}
\end{aligned}$$

The price of an option is calculated based on the current stock price s. Of course, this depends on the current time t. For this formula, the stock price is modeled by a geometric Brownian motion. The increment of a Brownian motion $X(t+s) - X(s)$ is independent of the past. Consequently, the price of a stock is not predictable. It behaves like a random walk, at any given moment in time, the price of the underlying stock can go up or down.

Additionally, the option striking price is needed. An option is a type of insurance for investors in the stock market. The investor has the right to buy or sell the underlying asset on the expiration date. Normally, a call option will only be exercised, if the striking price is below the market value. A put option will be exercised if the striking price is above the market value. The payoff at expiry for European call options is the maximum of $s - K$ and zero. An example of profit for a European call option and a striking price of 30 can be seen in Figure 1.

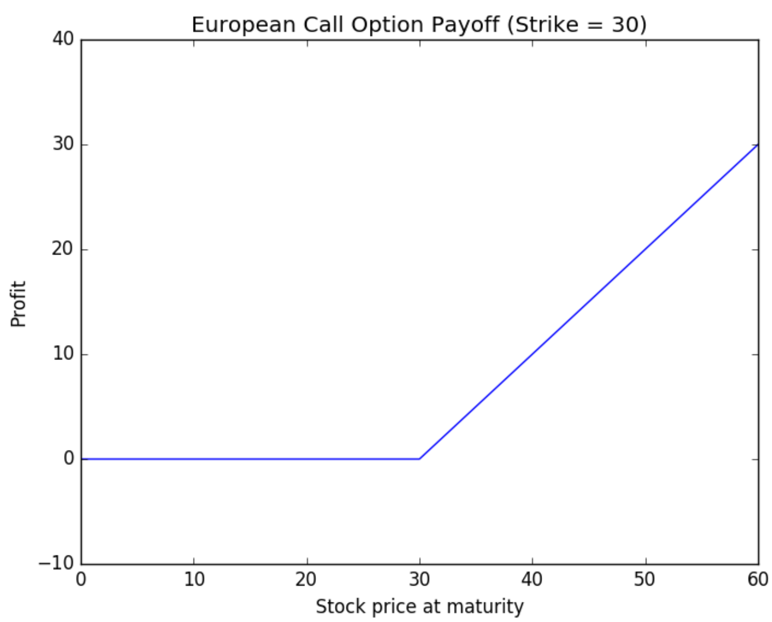


Figure 1: Profit of a European call option

If the stock price at maturity is lower than the striking price (here $K = 30$), the profit will be zero, because $\max(s - 30, 0) = 0$. As soon as the stock price at maturity is higher than the striking price, the profit will be positive by a value of $s - 30$.

Another input variable for the Black-Scholes formula is the risk-free rate. It is the best rate that does not involve taking a risk. The return of the original capital as well as the payment of interest are completely guaranteed. Usually, the risk-free rate for a given period is taken to be the return on government bonds.

Options depend on volatility, which is the annualized standard deviation of the asset's return. Intuitively, it is the amount the price swings around in a given time period. Relatively stable stocks have a lower volatility, while unstable stocks with a higher level of uncertainty are more volatile. Hence, it is more likely that the stock either has an extremely high or low value on the expiration date. Whether the stock price is massively or just slightly below the striking price of a call option is of no importance: In both cases the option will not be exercised. If the stock price is above the striking price at maturity, the option will be exercised and the payoff will be $s - K$. This means an option with high volatility will result in a higher profit, if the price rises.

The formula can be split into two parts: First, there is the expected benefit of a purchase of the underlying. Secondly, the current value of paying the exercise price is taken into consideration.

Expected benefit of purchasing the underlying completely: If the underlying stock price at maturity is above the striking price, the option will be exercised. One will get the prior specified amount of stock's units. This is worth whatever the stock price is in the market at maturity. The expected value of this is proportional to the stock price. At a time $t < T$ it can be written as $s \cdot \Phi(d_1(t, s))$. It is equal to the final stock price for $t = T$. If the final stock price is below the striking price, it will be zero.

Current value of paying the exercise price: If the option is exercised and the underlying asset is above the striking price, one will pay the striking price. The probability at time t that this is above the striking price at maturity is $\Phi(d_2(t, s))$. The expected value of paying the striking price is $K\Phi(d_2(t, s))$. This is the value of cash flow at maturity. To get the value of it at the specified date, it is necessary to discount it by the factor $e^{-r(T-t)}$. The value of the cash to buy the option is $Ke^{-r(T-t)} \cdot \Phi(d_2(t, s))$.

To get the price of an option, the second part gets subtracted from the first part. For a put option ϵ is negative. Thus, the Black-Scholes formula is

$$P(t, s) = -s\,\Phi(d_1(t, s)) + Ke^{-r(T-t)}\,\Phi(d_2(t, s)).$$

An example in options pricing can be found in Ross 2010.

2.1 Black-Scholes Equation

The Black-Scholes formula returns the price of an option. Its behavior over time is described by the Black-Scholes equation. This is a partial differential equation and will be derived in the following section.

2.1.1 Definitions

Suppose the current price of a stock is $S(0) = S_0$, and let $S(t)$ denote its price at time t. A stock price process can be approximated by a geometric Brownian motion. The change of the stock price over time is of interest. It can be written as:

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),$$

where W is a Wiener process, μ is the drift, σ is the volatility and t is the time. Additionally, the option price process $F(t, S(t))$ is necessary. The price dynamics of the derivative asset is given by applying Itô's formula:

$$\begin{aligned} dF(t,S(t)) =& \frac{\partial F}{\partial t}(t,S(t))\,dt + \frac{\partial F}{\partial s}(t,S(t))\,dS(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))\,dt \\ =& \left[\frac{\partial F}{\partial t}(t,S(t)) + \mu S(t)\frac{\partial F}{\partial s}(t,S(t))\right] dt + \sigma S(t)\frac{\partial F}{\partial s}(t,S(t))\,dW(t) \\ &+ \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))\,dt. \end{aligned}$$

To get the price of the option, it is the final goal to find $F(T, S(T))$. A portfolio based on the stock and the price function is considered.

2.1.2 Hedging Strategy

The following strategy gives a method to find the Black-Scholes partial differential equation. A portfolio based on two assets, the underlying stock and the derivative asset, is given by:

$$P(t) = \varphi(t)S(t) + F(t, S(t)),$$

where $\varphi = \varphi(t), 0 \leq t \leq T$, represents the shares of the stock and the units of the option F are equal to 1. The change of the portfolio during time can be written as

$$dP(t) = \left[\varphi(t)\, dS(t) + dF(t, S(t)) + \underbrace{S(t)\, d\varphi(t) + \varphi(t)\sigma^2 S^2(t) \frac{\partial F}{\partial s}(t, S(t))}_{\substack{\text{self-financing condition} \\ =0}} \right] dt.$$

The self-financing condition provides, that there are no exogenous infusions or withdrawals of money. The sale of an old asset should finance the purchase of a new one. This condition and the requirement that the last part of the equation is zero lead to

$$\begin{aligned} dP(t) = &\left[\mu\varphi(t)S(t) + \frac{\partial F}{\partial t}(t, S(t)) + \mu S(t) \frac{\partial F}{\partial s}(t, S(t)) \right] dt \\ &+ \sigma S(t) \left[\varphi(t) + \frac{\partial F}{\partial s}(t, S(t)) \right] dW(t) \\ &+ \frac{\sigma^2}{2} S^2(t) \frac{\partial^2 F}{\partial s^2}(t, S(t))dt. \end{aligned}$$

By buying and selling the underlying asset in the right way, one can perfectly hedge the option and eliminate risk. That is why it is desired to hold $-\frac{\partial F}{\partial s}(t, S(t))$ shares of the stock. So, $\varphi(t)$ is set to $-\frac{\partial F}{\partial s}(t, S(t))$:

$$\begin{aligned} dP(t) = &\left[\mu \left(-\frac{\partial F}{\partial s}(t, S(t)) \right) S(t) + \frac{\partial F}{\partial t}(t, S(t)) + \mu S(t) \frac{\partial F}{\partial s}(t, S(t)) \right] dt \\ &+ \sigma S(t) \left[-\frac{\partial F}{\partial s}(t, S(t)) + \frac{\partial F}{\partial s}(t, S(t)) \right] dW(t) \\ &+ \frac{\sigma^2}{2} S^2(t) \frac{\partial^2 F}{\partial s^2}(t, S(t))dt. \end{aligned}$$

That implies the following equation:

$$dP(t) = \left[\frac{\partial F}{\partial t}(t, S(t)) + \frac{\sigma^2}{2} S^2(t) \frac{\partial^2 F}{\partial s^2}(t, S(t)) \right] dt.$$

Additionally, $dP(t) = rF(t, S(t)) - rS(t)\frac{\partial F}{\partial s}(t, S(t))$ represents the bond.

As one can see, there is no randomness in the equation and with the notion of an arbitrage-free market the equation needs to be equal to the bond. In an arbitrage-free there market are no differences in profit depending on the asset. Whereas, arbitrage is the purchase and sale of an asset at the same time to benefit from the differences in price. With this assumptions and substitutions the Black-Scholes equation is:

$$\frac{\partial F}{\partial t}(t,s) + \frac{\sigma^2}{2}s^2\frac{\partial^2 F}{\partial s^2}(t,s) + \underbrace{rs\frac{\partial F}{\partial s}(t,s) - rF(t,s)}_{\text{bond}} \equiv 0, \qquad \forall\, 0 < t < T\,, 0 < s \tag{1}$$

$$F(T,s) = C(T,s) \qquad \forall\, s > 0.$$

Consequently, there is only one right price for an option.

2.2 Verification of the Black-Scholes Formula

The right price for the option is returned by the Black-Scholes formula, which will be verified in the following section. It is necessary to solve (1). The following changes of variables are considered:

$$s = Ke^x, \tag{2a}$$

$$F(t,s) = K \cdot f(\tau, x), \text{ and} \tag{2b}$$

$$\tau = \frac{(T-t)\,\sigma^2}{2}. \tag{2c}$$

The Black-Scholes equation is a linear parabolic equation of the form

$$\frac{\partial f}{\partial \tau}(\tau,x) = \frac{\partial^2 f}{\partial x^2}(\tau,x) + a\frac{\partial f}{\partial x}(\tau,x) + bf(\tau,x). \tag{3}$$

It can be reduced to a diffusion equation:

$$\frac{\partial h}{\partial \tau}(\tau,x) = \frac{\partial^2 h}{\partial x^2}(\tau,x). \tag{4}$$

2.2.1 Reduction to a Diffusion Equation

In the Black-Scholes model the interest rate r and the volatility σ are constant. With the change of variables in (2) it is possible to rewrite the Black-Scholes equation in terms of f. The partial derivatives of $F(t,s)$ are:

$$\frac{\partial F}{\partial t}(t,s) \overset{2b}{=} K\frac{\partial f}{\partial \tau}(\tau,x)\frac{\partial \tau}{\partial t} \overset{2c}{=} -K\frac{\sigma^2}{2}\frac{\partial f}{\partial \tau}(\tau,x), \tag{5a}$$

$$\frac{\partial F}{\partial s}(t,s) \overset{2b}{=} K\frac{\partial f}{\partial x}(\tau,x)\frac{\partial x}{\partial s} \overset{2a}{=} K\frac{\partial f}{\partial x}(\tau,x)\frac{\partial}{\partial s}\cdot \ln\left(\frac{s}{K}\right) = \frac{K}{s}\frac{\partial f}{\partial x}(\tau,x) = e^{-x}\frac{\partial f}{\partial x}(\tau,x), \quad \text{and} \tag{5b}$$

$$\begin{aligned}\frac{\partial^2 F}{\partial s^2}(t,s) &\overset{5b}{=} \frac{\partial}{\partial s}\frac{K}{s}\frac{\partial f}{\partial x}(\tau,x) = -\frac{K}{s^2}\frac{\partial f}{\partial x}(\tau,x) + \frac{K}{s}\frac{\partial}{s}\frac{\partial f}{\partial x}(\tau,x)\\ &= -\frac{K}{s^2}\frac{\partial f}{\partial x}(\tau,x) + \frac{K}{s}\left(\frac{\partial x}{\partial s}\frac{\partial}{\partial x}\right)\frac{\partial f}{\partial x}(\tau,x) = -e^{-2x}\frac{\partial f}{\partial x}(\tau,x) + e^{-2x}\frac{\partial^2 f}{\partial x^2}(\tau,x).\end{aligned} \tag{5c}$$

Inserting the partial derivatives of $F(t,s)$ (see (5)) into the Black-Scholes equation (1) leads to:

$$\frac{\partial f}{\partial \tau}(\tau,x) = \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial f}{\partial x}(\tau,x) + \frac{\partial^2 f}{\partial x^2}(\tau,x) - \frac{2r}{\sigma^2}\cdot f(\tau,x), \tag{6}$$

and so $f(\tau,x)$ satisfies (3) with

$$a = \frac{2r}{\sigma^2} - 1, \qquad \text{and} \qquad b = -\frac{2r}{\sigma^2} = -(1+a). \tag{7}$$

The solution of an equation like (6) is of the form

$$f(\tau,x) = w(\tau)\cdot g(x)\cdot h(\tau,x). \tag{8}$$

To get $f(\tau,x)$ it is necessary to calculate the partial derivatives of $f(\tau,x)$. They are given by:

$$\frac{\partial f}{\partial \tau}(\tau,x) = \frac{\partial w}{\partial \tau}(\tau)\cdot g(x)h(\tau,x) + w(\tau)g(x)\cdot\frac{\partial h}{\partial \tau}(\tau,x), \tag{9a}$$

$$\frac{\partial f}{\partial x}(\tau,x) = w(\tau)\cdot\frac{\partial g}{\partial x}(x)\cdot h(\tau,x) + w(\tau)g(x)\cdot\frac{\partial h}{\partial x}(\tau,x), \quad \text{and} \tag{9b}$$

$$\frac{\partial^2 f}{\partial x^2}(\tau,x) = w(\tau)\cdot\frac{\partial^2 g}{\partial x^2}(x)\cdot h(\tau,x) + 2w(\tau)\frac{\partial g}{\partial x}(x)\frac{\partial h}{\partial x}(\tau,x) + w(\tau)g(x)\cdot\frac{\partial^2 h}{\partial x^2}(\tau,x). \tag{9c}$$

Substituting (9) into (6) results in:

$$\begin{aligned}\frac{\partial w}{\partial \tau}(\tau)g(x)h(\tau,x)+w(\tau)g(x)\frac{\partial h}{\partial \tau}(\tau,x) =& w(\tau)\cdot\frac{\partial^2 g}{\partial x^2}(x)\cdot h(\tau,x)+2w(\tau)\frac{\partial g}{\partial x}(x)\frac{\partial h}{\partial x}(\tau,x)\\ &+w(\tau)g(x)\cdot\frac{\partial^2 h}{\partial x^2}(\tau,x)\\ &+a\left[w(\tau)\cdot\frac{\partial g}{\partial x}(x)\cdot h(\tau,x)+w(\tau)g(x)\cdot\frac{\partial h}{\partial x}(\tau,x)\right]\\ &+bw(\tau)g(x)h(\tau,x),\end{aligned} \tag{10}$$

which can be satisfied, if w and g are of the form

$$w(\tau)=c_1\,exp(\hat{w}(\tau)),\text{ and} \tag{11a}$$

$$g(x)=c_2\,exp(\hat{g}(x)), \tag{11b}$$

in which c_1, c_2 $\in\mathbb{R}$ are constant. They have the following derivatives:

$$\frac{\partial w}{\partial \tau}(\tau)=w(\tau)\frac{\partial \hat{w}}{\partial \tau}(\tau), \tag{12a}$$

$$\frac{\partial g}{\partial x}(x)=g(x)\frac{\partial \hat{g}}{\partial x}(x),\text{ and} \tag{12b}$$

$$\frac{\partial^2 g}{\partial x^2}(x)=g(x)\frac{\partial^2 \hat{g}}{\partial x^2}(x)+\left(\frac{\partial \hat{g}}{\partial x}(x)\right)^2. \tag{12c}$$

Substituting (11) and (12) into (10) leads to:

$$\begin{aligned}\frac{\partial h}{\partial \tau}(\tau,x)=&\frac{\partial^2 h}{\partial x^2}(\tau,x)+\frac{\partial h}{\partial x}(\tau,x)\left[2\frac{\partial \hat{g}}{\partial x}(x)+a\right]\\ &+h(\tau,x)\left[-\frac{\partial \hat{w}}{\partial \tau}(\tau)+\frac{\partial^2 \hat{g}}{\partial x^2}(x)+\left(\frac{\partial \hat{g}}{\partial x}(x)\right)^2+a\frac{\partial \hat{g}}{\partial x}(x)+b\right].\end{aligned} \tag{13}$$

In order for (13) to be of the form of the diffusion equation (4), it is required that $2\frac{\partial \hat{g}}{\partial x}(x)+a=0$, which implies $\hat{g}(x) = \frac{-ax}{2} + c_1$ for some $c_1 \in \mathbb{R}$ and also that

$$\begin{aligned}
& -\frac{\partial \hat{w}}{\partial \tau}(\tau) + \frac{\partial^2 \hat{g}}{\partial x^2}(x) + \left(\frac{\partial \hat{g}}{\partial x}(x)\right)^2 + a\frac{\partial \hat{g}}{\partial x}(x) + b = 0 \\
\Leftrightarrow & -\frac{\partial \hat{w}}{\partial \tau}(\tau) + \frac{a^2}{4} - \frac{a^2}{2} + b = 0 \\
\Leftrightarrow & \frac{\partial \hat{w}}{\partial \tau}(\tau) = -\frac{a^2}{4} + b \\
\Rightarrow & \hat{w}(\tau) = -\left(\frac{4+4a+a^2}{4}\right)\tau + c_2
\end{aligned}$$

for some $c_2 \in \mathbb{R}$.

Putting that back into the solution in (8) and expressing $\hat{w}(\tau)$ in terms of a of (7) one gets:

$$f(\tau, x) = ce^{-\left(\frac{a^2}{4}+a+1\right)\tau} \cdot e^{-\frac{a}{2}x} \cdot h(\tau, x), \; c \in \mathbb{R}.$$

In summary, the Black-Scholes equation has been reduced to a diffusion equation

$$\frac{\partial h}{\partial \tau}(\tau, x) = \frac{\partial^2 h}{\partial x^2}(\tau, x), \qquad x \in \mathbb{R}, \, \tau \in \left[0, \sigma\frac{T}{2}\right]$$

by the following change of variables:

$$S = Ke^x, \qquad \text{and} \qquad \tau = (T-t) \cdot \sigma^2/2.$$

The price of an option is

$$F(\tau, x) = K \cdot ce^{-\left(\frac{a^2}{4}+a+1\right)\tau} \cdot e^{-\frac{a}{2}x} \cdot h(\tau, x), \text{ in which } a = 2r/\sigma^2 - 1. \tag{14}$$

2.2.2 Solution of the Diffusion Equation

The next subsection deals with the solution of the diffusion equation with $h(\tau, x)$ (14). It can be solved by using Fourier transforms. For a function f the Fourier transform with respect to variable x is defined by

$$\mathcal{F}(f(x))(k) = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x)\, dx,$$

and the Fourier transform of its derivatives is given by

$$\mathcal{F}(f^{(n)}(x))(k) = \tilde{f}(k) = (ik)^n \cdot \mathcal{F}(f(x))(k).$$

The Fourier transform of the diffusion equation can be written as follows:

$$\frac{\partial \tilde{h}}{\partial \tau} = -k^2 \tilde{h},$$

which results in

$$\tilde{h}(\tau, k) = \tilde{h}(0, k) \cdot e^{-k^2 \tau}. \tag{15}$$

$\tilde{h}(0, x)$ is the Fourier transform of the initial condition of h, which corresponds to the terminal condition at expiry $t = T$ of the option, since $\tau = \frac{(T-t)\sigma^2}{2}$.

To find the solution for $h(\tau, x)$ it is required to apply the inverse Fourier transform. By defining

$$\mathcal{F}(h_1) = \tilde{h_1} = e^{-k^2 \tau}, \qquad \text{and} \qquad \mathcal{F}(h_2) = \tilde{h_2} = \tilde{h}(0, k) \tag{16}$$

Equation (15) gives

$$\tilde{h}(\tau, k) = \tilde{h_1}(\tau, k) * \tilde{h_2}(\tau, k). \tag{17}$$

A useful property of the Fourier transform is the convolution theorem: A Fourier transform of a convolution product of two functions f and g is equal to the product of the Fourier transforms of both:

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

So, applying the convolution theorem to the inverse Fourier transform of (17) leads to

$$h(\tau, x) = (h_1 * h_2)(\tau, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_1(\tau, x - \xi) h_2(\tau, \xi)\, d\xi. \tag{18}$$

The inverse Fourier transforms of (16) are:

$$\mathcal{F}^{-1}(\tilde{h_1}) = h_1 = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{4\tau}}, \qquad \text{and} \qquad \mathcal{F}^{-1}(\tilde{h_2}) = h_2 = h(0,x). \tag{19}$$

Inserting (19) into (18) gives:

$$h(\tau, x) = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4\tau}} h(0,\xi)\, d\xi. \tag{20}$$

The general solution in (20) satisfies the diffusion equation (4) with initial condition $h(0,x)$. That implies the following partial derivatives:

$$\begin{aligned}
\frac{\partial h}{\partial \tau}(\tau,x) &= -\frac{1}{2\tau} h(\tau,x) + \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} \frac{(x-\xi)^2}{4\tau^2} e^{-\frac{(x-\xi)^2}{4\tau}} h_0(\xi)\, d\xi, \\
\frac{\partial h}{\partial x}(\tau,x) &= \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} \frac{-(x-\xi)}{2\tau} e^{-\frac{(x-\xi)^2}{4\tau}} h_0(\xi)\, d\xi, \quad \text{and} \\
\frac{\partial^2 h}{\partial x^2}(\tau,x) &= -\frac{1}{2\tau} h(\tau,x) + \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} \frac{(x-\xi)^2}{4\tau^2} e^{-\frac{(x-\xi)^2}{4\tau}} h_0(\xi)\, d\xi.
\end{aligned}$$

These equations satisfy the diffusion equation (4):

$$\lim_{\tau\to 0} h(\tau,x) = \lim_{\tau \to 0} \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{4\tau}} h(\tau,\xi) d\xi \; = \; h(0,x).$$

2.2.3 Application to European Options

Correspondent to the change of variables the payoff condition in (14) at $\tau = 0$ is equivalent to the payoff at expiry $t = T$. The payoff at expiry for European options is given by:

$$F(T,s) = \max[\epsilon(s-K), 0], \tag{21}$$

where $\epsilon = 1$ for a call option and $\epsilon = -1$ for a put option. Equation (21) can be expressed in terms of the new variables from (2) as:

$$\begin{aligned}
h(0,x) &= \frac{1}{K} e^{\frac{a}{2}x} F(T, Ke^x) \\
&= \frac{1}{K} e^{\frac{a}{2}x} \max[\epsilon\,(Ke^x - K)\,, 0] \\
&= \max\left[\epsilon \left(e^{\left(\frac{a}{2}+1\right)x} - e^{\frac{a}{2}x}\right), 0\right].
\end{aligned} \tag{22}$$

Substituting (22) into the general solution (20) gives:

$$h(\tau,x)=\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}e^{-\frac{(x-\xi)^2}{4\tau}}\max\left[\epsilon\left(e^{\left(\frac{a}{2}+1\right)x}-e^{\frac{a}{2}x}\right),0\right]d\xi.$$

Since $e^{\left(\frac{\alpha}{2}+1\right)x}-e^{\frac{\alpha}{2}x}>0$ if and only if $x>0$ for all $\alpha\in\mathbb{R}$ the integration domain can be rewritten as $[0,\infty)$. A change in the variables to $\eta=\xi/\epsilon$ and $d\eta=d\xi/\epsilon$ leads to:

$$\begin{aligned}h(\tau,x)&=\epsilon\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}e^{-\frac{(x-\epsilon\eta)^2}{4\tau}}\left[e^{\epsilon\left(\frac{a}{2}+1\right)\eta}-e^{\epsilon\frac{a}{2}\eta}\right]d\eta\\&=\epsilon\frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty}e^{-\frac{(x-\epsilon\eta)^2}{4\tau}}\cdot e^{\epsilon\frac{a}{2}\eta+\epsilon\eta}\,d\eta-\int_0^{\infty}e^{-\frac{(x-\epsilon\eta)^2}{4\tau}}\cdot e^{\epsilon\frac{a}{2}\eta}\,d\eta\\&=\epsilon e^{\left(\frac{a}{2}+1\right)\left(x+a\frac{\tau}{2}+\tau\right)}\Phi\left(\epsilon\frac{x+a\tau+2\tau}{\sqrt{2\tau}}\right)-\epsilon e^{\frac{a}{2}\left(x+a\frac{\tau}{2}\right)}\Phi\left(\epsilon\frac{x+a\tau}{\sqrt{2\tau}}\right),\end{aligned}$$

where Φ denotes the cumulative standard normal distribution function. After changing the variables back to the initial ones (see Section 2.2.4), one will get the Black-Scholes formula.

2.2.4 Reversing the Change of Variables

Going back to the initial variables with (14) leads to

$$\begin{aligned}F(t,s)=&Ke^{-\left(\frac{a^2}{2}+a+1\right)\tau}e^{-\frac{a}{2}x}*\left[e^{\left(\frac{a}{2}+1\right)\left(x+a\frac{\tau}{2}+\tau\right)}\Phi\left(\epsilon\frac{x+a\tau+2\tau}{\sqrt{2\tau}}\right)-e^{\frac{a}{2}\left(x+a\frac{\tau}{2}\right)}\Phi\left(\epsilon\frac{x+a\tau}{\sqrt{2\tau}}\right)\right]\\=&\epsilon Ke^{x}\Phi\left(\epsilon\frac{x+a\tau+2\tau}{\sqrt{2\tau}}\right)-\epsilon Ke^{-a\tau-\tau}\Phi\left(\epsilon\frac{x+a\tau}{\sqrt{2\tau}}\right)\\=&\epsilon s\Phi\left(\epsilon\underbrace{\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)\right]}_{\equiv d_1(t,s)}\right)\\&-\epsilon Ke^{-r(T-t)}\Phi\left(\epsilon\underbrace{\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)\right]}_{\equiv d_2(t,s)}\right)\\=&\epsilon s\Phi(\epsilon d_1(t,s))-\epsilon Ke^{-r(T-t)}\Phi(\epsilon d_2(t,s)),\end{aligned}$$

which is the Black-Scholes formula.

Chapter 3

Effects of Incorrect Volatility

In this chapter some simulations for European and digital call and put options are presented. The Black-Scholes model depends on volatility, which is assumed to be constant and should correspond with the real behavior. Consequently, the same applies for the option price $F(t, s)$. So, it is reasonable to ask what impact a wrong assumed volatility might have on the option. This question is analyzed within this chapter.

When the Black-Scholes model is correct, the option price $F(t, s)$ satisfies

$$\begin{aligned}F(T, S(T)) = {} & \underbrace{F(0, S_0)}_{\text{option value at } t=0} - \underbrace{\varphi(0)S_0}_{\text{shares of stock with value } S_0 \text{ at } t=0} + \underbrace{\varphi(T)S(T)}_{\text{shares of stock with value } S(T) \text{ at } t=T} \\ & + \int_0^T \left[\frac{\partial F}{\partial t}(t, S(t)) + rS(t)\frac{\partial F}{\partial s}(t, S(t)) - rF(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 F}{\partial s^2}(t, S(t)) \right] dt \\ & + \underbrace{\int_0^T \sigma S(t) \left(\frac{\partial F}{\partial s}(t, S(t)) + \varphi(t) \right) dW(t)}_{=0}.\end{aligned}$$

By the choice of $\varphi(t) = -\frac{\partial F}{\partial s}(t, S(t))$, the integrand in the stochastic integral is zero and the regular integral is zero as well, since F satisfies the Black-Scholes equation (1). Thus, the value of the portfolio at the final time equals the value of the option.

A question naturally arises about the effect of using the wrong volatility parameter σ. In the following analysis, let σ_* denote the true volatility so the stock price process S satisfies the stochastic differential equation $dS(t) = \mu S(t)\, dt + \sigma_* S(t)\, dW(t)$. Also let σ denote the volatility parameter used to determine the function F. Now Itô's formula gives:

$$
\begin{aligned}
F(T,S(T)) = {} & F(0,S_0) - \varphi(0)S_0 + \varphi(T)S(T) \\
& + \int_0^T \left[\frac{\partial F}{\partial t}(t,S(t)) + rS(t)\frac{\partial F}{\partial s}(t,S(t)) - rF(t,S(t)) + \frac{1}{2}\sigma_*^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))\right] dt \\
& + \underbrace{\int_0^T \sigma_* S(t)\left(\frac{\partial F}{\partial s}(t,S(t)) + \varphi(t)\right) dW(t)}_{=0} \\
= {} & F(0,S_0) - \varphi(0)S_0 + \varphi(T)S(T) \\
& + \underbrace{\int_0^T \left[\frac{\partial F}{\partial t}(t,S(t)) + rS(t)\frac{\partial F}{\partial s}(t,S(t)) - rF(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))\right] dt}_{=0} \\
& + \int_0^T \frac{1}{2}(\sigma_*^2 - \sigma^2) S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))\, dt \\
& + \underbrace{\int_0^T \sigma_* S(t)\left(\frac{\partial F}{\partial s}(t,S(t)) + \varphi(t)\right) dW(t)}_{=0}. \qquad (23)
\end{aligned}
$$

With this formula the hedging portfolio errors are calculated. By analyzing the effects of an incorrect volatility in calculating option prices one might think about applying the Greeks Vega to F. Vega measures how sensitive the option price is in respect to the volatility parameter σ. It is given as

$$
\begin{aligned}
\mathcal{V} = \frac{\partial F}{\partial \sigma} &= s\sqrt{T-t}\,\Phi'(d_1) \\
&= s\sqrt{T-t}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{d_1^2}{2}}.
\end{aligned}
$$

Vega is always positive and the option price increases as σ increases. In reality the volatility of a stock price changes over time. “Vega is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset” (Hull 2000).

In the Black-Scholes model the volatility is assumed to be constant. There is no change over time and a trader estimates the used parameter σ. While Vega analyzes the sensitivity of the option price in respect to σ this thesis deals with the question of what impact an incorrect volatility might have on the option price. It observes the accuracy of the hedging portfolio. These are two different ways of analyzing.

In all following simulations a risk-free rate of zero and a time interval from 0 to 2 with 200 timesteps are assumed. Moreover, the starting stock price S_0 is 20 and the striking price K is 16. For a meaningful result 100 different exemplary stocks are used. Those are the same for all simulations. In the simulations σ_* and σ are equal to $[0.1, 0.2, ..., 1.0]$.

In this chapter, the general approach for the simulations is discussed. Section 3.1 introduces the formulas to calculate the option price for European options. It is followed by Section 3.2 which covers the same calculations for digital options. Afterwards, the results of these simulations for call and put options are displayed and analyzed.

3.1 European Option

The first simulations dissemble the errors of using σ_* and σ for the option price of European call and put options. Its payoff at expiry for call options is $\max(s-K,0)$. The Black-Scholes formula, which was verified in Section 2.2, is used for these simulations:

$$F(t,s) = \epsilon \cdot s \cdot \Phi(\epsilon \cdot d_1(t,s)) - \epsilon \cdot Ke^{-r(T-t)} \cdot \Phi(\epsilon \cdot d_2(t,s)).$$

The integral to calculate the option value is

$$\int_0^T \frac{\partial F}{\partial t}(t,S(t)) + rS(t)\frac{\partial F}{\partial s}(t,S(t)) - rF(t,S(t)) + \frac{1}{2}\sigma_*^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))dt,$$

with

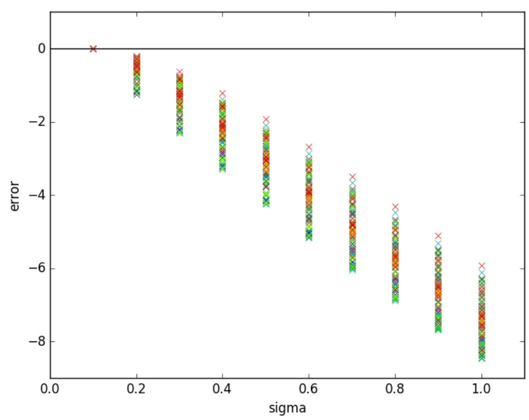
$$\frac{\partial F}{\partial t}(t,s) = \frac{se^{-\frac{d_1^2(t,s)}{2}}}{\sqrt{2\pi}}\left[\frac{d_1(t,s)}{2(T-t)} - \frac{1}{\sigma\sqrt{T-t}}\left(r+\frac{\sigma^2}{2}\right)\right] - \epsilon Kre^{-r(T-t)}\Phi(\epsilon d_2(t,s))$$
$$-\frac{Ke^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\left[\frac{d_2(t,s)}{2(T-t)} - \frac{1}{\sigma\sqrt{T-t}}\left(r-\frac{\sigma^2}{2}\right)\right],$$

$$\frac{\partial F}{\partial s}(t,s) = \epsilon\Phi(\epsilon d_1(t,s)) + \frac{e^{-\frac{d_1^2(t,s)}{2}}}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}} - \frac{Ke^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\frac{1}{s\sigma\sqrt{T-t}}, \text{ and}$$

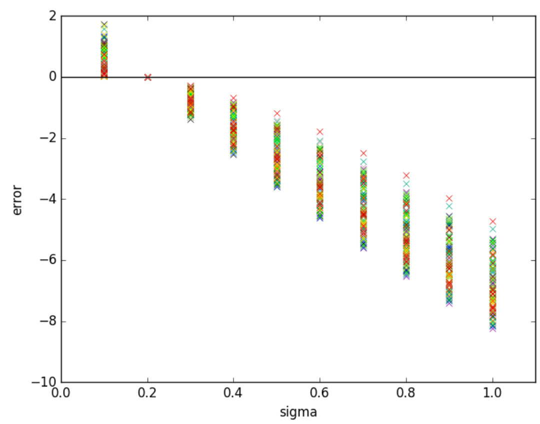
$$\frac{\partial^2 F}{\partial s^2}(t,s) = \frac{2}{s^2\sigma^2}\left[\frac{se^{-\frac{d_1^2(t,s)}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} - \frac{d_1(t,s)}{2(T-t)}\right)\right.$$
$$\left. + \frac{Ke^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} + \frac{d_2(t,s)}{2(T-t)}\right)\right].$$

3.1.1 Call Option

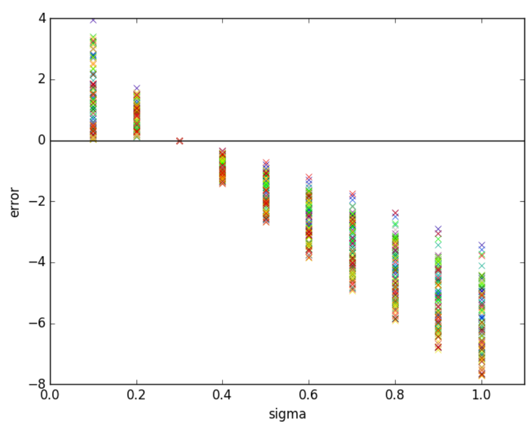
In Figure 2 (on two pages) one can see all errors for each σ_* and σ for a European call option. If the assumed volatility coincides with the real volatility ($\sigma = \sigma_*$), then the error is zero as expected. For that case the same formula, as a broker utilizes, is used. But if σ is smaller than σ_*, the errors are greater than zero. Those imply a shortfall, meaning that the option exceeds the amount of cash that is available. If σ is greater than σ_*, then the errors are smaller than zero. Those errors do not imply a shortfall but imply a surplus. As one can see in (23), deviations from zero result from $\int_0^T \frac{1}{2}(\sigma_*^2 - \sigma^2)S^2(t)\frac{\partial^2 F}{\partial s^2}(t, S(t))\,dt$. For European call options $\frac{\partial^2 F}{\partial s^2}(t, S(t))$ is positive. Hence, for σ_* greater than σ, $\sigma_* - \sigma$ is positive, and the integral as well is greater than zero. On the other hand, for σ_* less than σ, $\sigma_* - \sigma$ is negative, and the integral is, too.



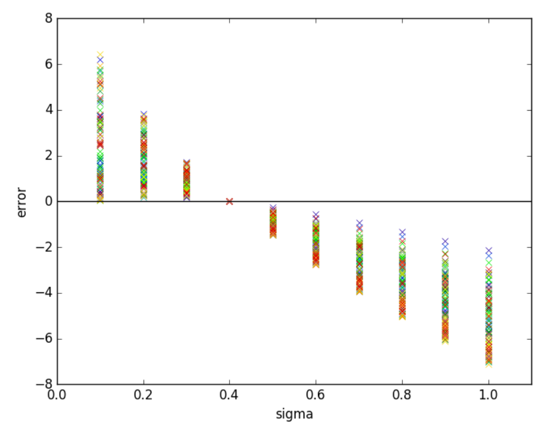
(a) $\sigma_* = 0.1$



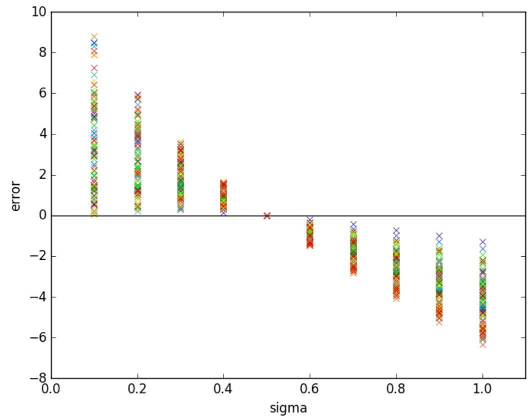
(b) $\sigma_* = 0.2$



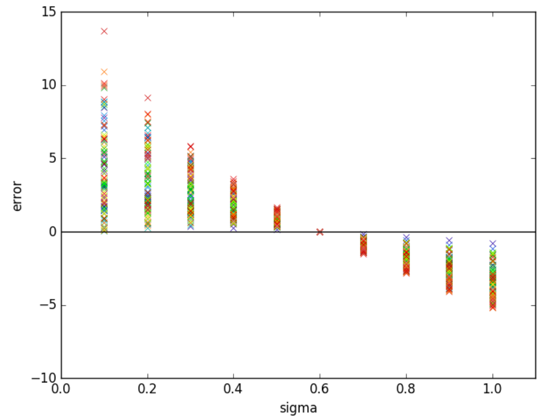
(c) $\sigma_* = 0.3$



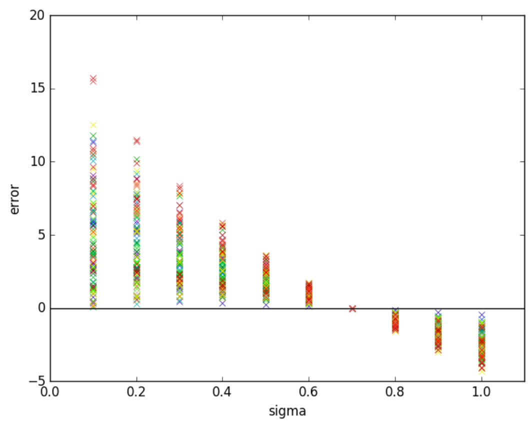
(d) $\sigma_* = 0.4$



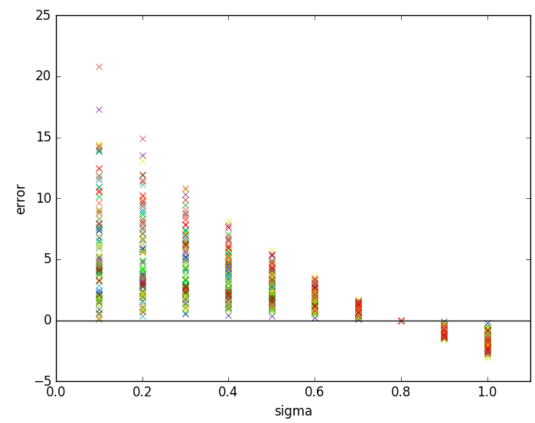
(e) $\sigma_* = 0.5$



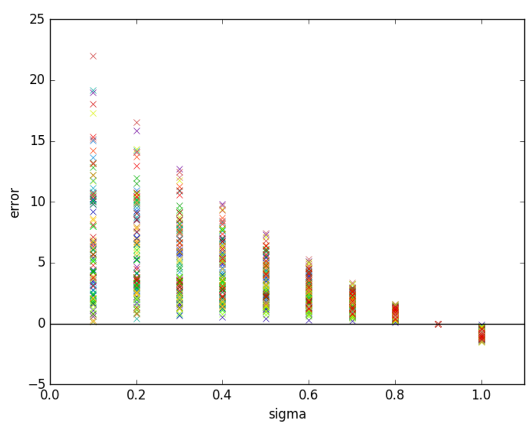
(f) $\sigma_* = 0.6$



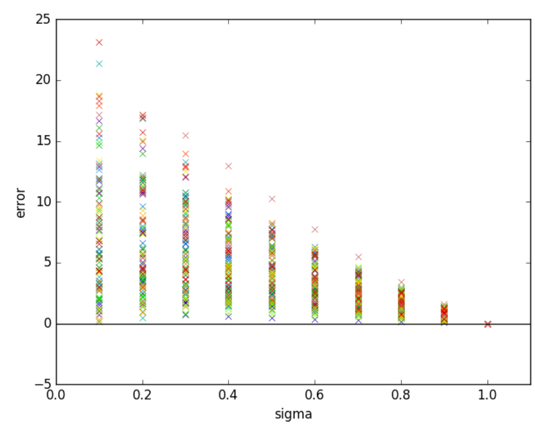
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$



(i) $\sigma_* = 0.9$



(j) $\sigma_* = 1.0$

Figure 2: Hedged portfolio errors per σ for a European call option with different σ_*

The range of the errors for various σ_* differs. In Figure 3 all errors for each σ_* are presented. As a result of the error behavior for σ smaller than σ_* and greater than σ_*, one can see a positive trend of the errors. Additionally, the bigger the real volatility gets, the greater is the range of the errors. For the real volatility $\sigma_* = 0.1$ the errors lie within the interval $[-8.446, 0.0]$, for $\sigma_* = 0.5$ they belong to $[-6.325, 8.799]$, thus, a 1.8 time larger interval, and for $\sigma_* = 1.0$ the errors lie within $[0.0, 23.151]$. This is 2.7 times as big as the first one.

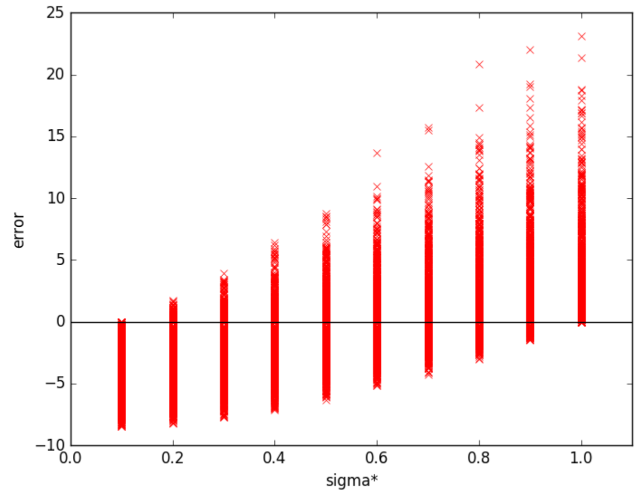


Figure 3: Range of hedged errors per σ_* for a European call option

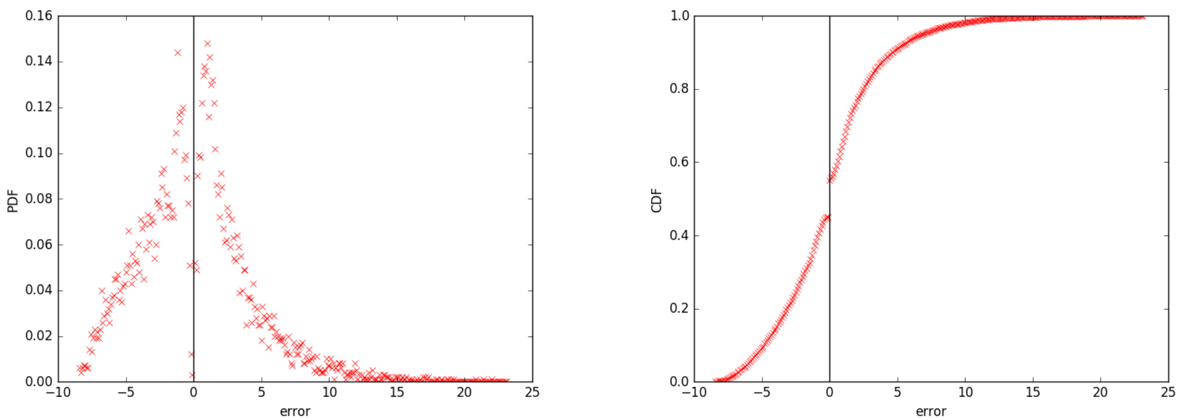
In the table below the minimal and maximal errors for each σ_* and σ are displayed. For $\sigma_* = \sigma$ these values are zero, as expected. The bigger the difference between σ and σ_* is, the bigger the errors are.

σ	0.1		0.2		0.3		0.4		0.5	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−0.0	0.0	−1.26317	−0.20272	−2.29893	−0.62572	−3.29365	−1.21195	−4.24101	−1.91469
0.2	0.01725	1.74193	−0.0	0.0	−1.37948	−0.2881	−2.54286	−0.6728	−3.5895	−1.17719
0.3	0.03103	3.95978	0.10827	1.7183	−0.0	0.0	−1.41563	−0.32582	−2.68168	−0.70953
0.4	0.04576	6.43765	0.16839	3.82998	0.18028	1.71909	−0.0	0.0	−1.46822	−0.2521
0.5	0.06153	8.79896	0.21586	5.95566	0.28417	3.63902	0.15769	1.68743	−0.0	0.0
0.6	0.0786	13.68867	0.2632	9.16158	0.37344	5.87134	0.261	3.62161	0.13984	1.69377
0.7	0.09748	15.72432	0.3141	11.48889	0.46387	8.36547	0.34761	5.81608	0.2392	3.66835
0.8	0.11874	20.84193	0.36935	14.90892	0.55814	10.85935	0.43068	8.14752	0.32273	5.76965
0.9	0.14294	22.02865	0.42886	16.52301	0.65563	12.68412	0.51477	9.81151	0.4007	7.41496
1.0	0.17058	23.15056	0.49293	17.17745	0.75632	15.44842	0.60077	12.94376	0.47764	10.27263

σ	0.6		0.7		0.8		0.9		1.0	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−5.16345	−2.6832	−6.04396	−3.48492	−6.88422	−4.29916	−7.68423	−5.11172	−8.44617	−5.91205
0.2	−4.61004	−1.78796	−5.5942	−2.47496	−6.52925	−3.20922	−7.40969	−3.96793	−8.2379	−4.73422
0.3	−3.84038	−1.17984	−4.92047	−1.73908	−5.92255	−2.36188	−6.85537	−2.89681	−7.72461	−3.42139
0.4	−2.77036	−0.56958	−3.94844	−0.93235	−5.05927	−1.32116	−6.12589	−1.72149	−7.11762	−2.12297
0.5	−1.47025	−0.19815	−2.82984	−0.43597	−4.08846	−0.70478	−5.2506	−0.9936	−6.32478	−1.29238
0.6	−0.0	0.0	−1.4998	−0.16272	−2.84257	−0.34831	−4.05612	−0.55281	−5.16187	−0.77021
0.7	0.12488	1.74405	−0.0	0.0	−1.57478	−0.13715	−3.00006	−0.28642	−4.29473	−0.44578
0.8	0.21878	3.63057	0.11191	1.71712	−0.0	0.0	−1.56533	−0.11748	−3.04521	−0.24018
0.9	0.29856	5.27866	0.19966	3.3392	0.10044	1.59774	−0.0	0.0	−1.51064	−0.10165
1.0	0.37215	7.75293	0.27515	5.47548	0.18177	3.44154	0.09016	1.62599	−0.0	0.0

Table 1: Minimum and maximum hedged errors for a European call option

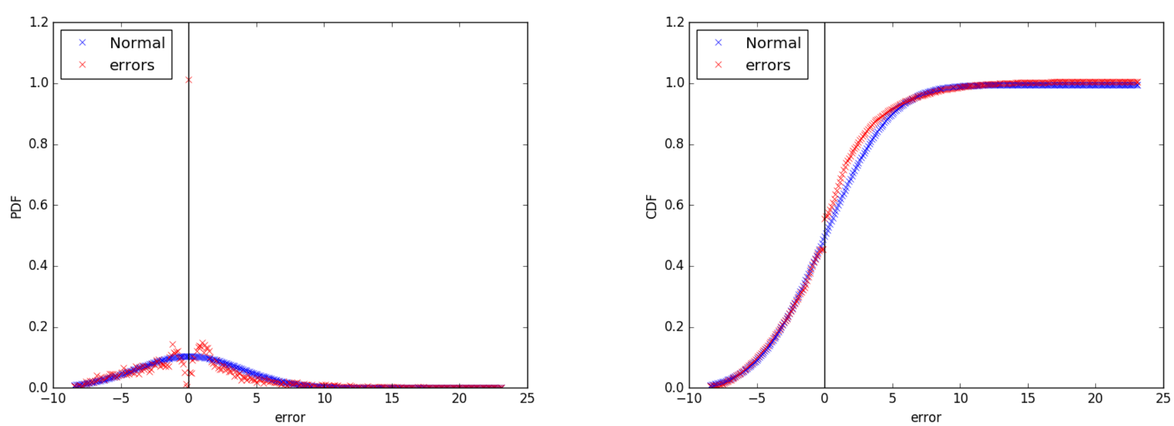
Figure 4 displays two probability plots. In Figure 4(a) one can see the approximated probability density function. Each point in the plots denotes the relative probability, that an error occurs in the interval x_i and x_{i+1}, where Δx is 0.1. To get a better impression of all values, $f(0)$ is not plotted. With $f(0) = 1.012$ it is much higher than all other values. Furthermore, in Figure 4(b), the approximated cumulative distribution function is shown. One can see the huge jump at zero, resulting from the high probability for an error equals zero.



(a) Approximated probability density function **(b)** Approximated cumulative distribution function

Figure 4: Approximated probability density and cumulative distribution function for a European call option

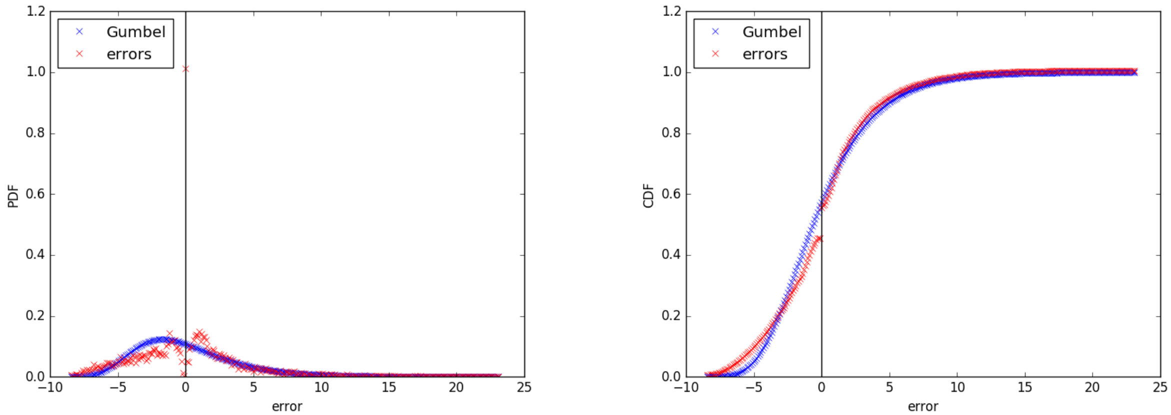
By examining these plots, one could assume, that the errors might be normally distributed. In Figure 5 the approximated probability plots of the calculated errors are compared to a normal distribution with estimated parameters. There is a high coverage.



(a) PDF **(b)** CDF

Figure 5: Comparison of errors of a European call option to an estimated normal distribution

Another possible distribution might be the Gumbel distribution (see comparisons in Figure 6).



(a) Gumbel distribution

(b) Normal distribution

Figure 6: Comparison of errors of a European call option to an estimated Gumbel distribution

As seen for the Normal distribution, a Gumbel distribution with estimated parameters covers the data, also. Usually a Gumbel distribution is used, for instance, in meteorology for weather predictions. It is a typical distribution for annual scenarios.

Definition 1. *A continuous random variable X is distributed by a Gumbel distribution with scale parameter $\beta > 0$ and shape parameter $\mu \in \mathbb{R}$, if it has the probability density of*

$$f(x) = \frac{1}{\beta} e^{-\frac{1}{\beta}(x-\mu)} e^{-e^{-\frac{1}{\beta}(x-\mu)}}, \; x \in \mathbb{R},$$

and probability distribution function of

$$F(x) = e^{e^{-\frac{1}{\beta}(x-\mu)}}, x \in \mathbb{R}.$$

Letting $\gamma = 0.5772$ be the Euler–Mascheroni constant, the parameters β and μ satisfy

$$\mathbb{E}[X] = \mu + \beta\gamma, \qquad \text{and} \qquad Var[X] = \frac{(\pi\beta)^2}{6}$$

A further analysis of possible distributions might bring clarity to these assumptions. Maybe, it is possible to find an appropriate one for those errors.

3.1.2 Put Option

The hedged errors for European put options with a risk-free rate $\boldsymbol{r = 0}$ should be the same as those for European call options. The errors are results of

$$\int_0^T \frac{\partial F}{\partial t}(t,S(t)) + \underbrace{rS(t)\frac{\partial F}{\partial s}(t,S(t))}_{=0} - \underbrace{rF(t,S(t))}_{=0} + \frac{1}{2}\sigma_*^2 S^2(t)\frac{\partial^2 F}{\partial s^2}(t,S(t))dt,$$

with

$$\frac{\partial F}{\partial t}(t,s) = \frac{se^{-\frac{d_1^2(t,s)}{2}}}{\sqrt{2\pi}}\left[\frac{d_1(t,s)}{2(T-t)} - \frac{1}{\sigma\sqrt{T-t}}\left(r+\frac{\sigma^2}{2}\right)\right] - \underbrace{\epsilon Kre^{-r(T-t)}\Phi(\epsilon d_2(t,s))}_{=0},$$

$$- \frac{Ke^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\left[\frac{d_2(t,s)}{2(T-t)} - \frac{1}{\sigma\sqrt{T-t}}\left(r-\frac{\sigma^2}{2}\right)\right], \quad \text{and}$$

$$\frac{\partial^2 F}{\partial s^2}(t,s) = \frac{2}{s^2\sigma^2}\left[\frac{se^{-\frac{d_1^2(t,s)}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} - \frac{d_1(t,s)}{2(T-t)}\right)\right.$$

$$\left. + \frac{Ke^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}} + \frac{d_2(t,s)}{2(T-t)}\right)\right].$$

In $\frac{\partial^2 F}{\partial s^2}(t,S(t))$ is no ϵ in place. Hence, the integral for a call option and the integral for a put option are the same and there is no change in sign. The probability functions, shown in Figure 16 are covered by the normal and the Gumbel distribution as well (see Figure 17 and Figure 18). The minimal and maximal errors do not differ from those of a call option. The tables can be seen in Appendix B.1.

3.2 Digital Option

As a further example, a digital option is considered. Its payoff at expiry is equal to 1 if the underlying asset is in the money and zero otherwise. Let $F(t, s)$ denote the price of a digital option at time t when the underlying stock has value s. The terminal condition therefore is

$$F(T, s) = \begin{cases} \Theta(s - K) \text{ for a call,} \\ 1 - \Theta(s - K) \text{ for a put,} \end{cases}$$

where $\Theta(\xi)$ is the Heaviside distribution, which is defined as follows:

$$\Theta(\xi) = \begin{cases} 0 \text{ for } \xi \leq 0, \\ 1 \text{ for } \xi > 0. \end{cases}$$

With this change in the terminal condition the Black-Scholes formula can be written as

$$F(t, s) = e^{-r(T-t)}\Phi(\epsilon d_2(t, s)),$$

where $\epsilon = 1$ for a call option and $\epsilon = -1$ for a put option. $F(t, s)$ results in the discount risk that the stock price s is above or below K at time T. The integrand for calculating the option value is

$$\int_0^T \frac{\partial F}{\partial t}(t, S(t)) + rS(t)\frac{\partial F}{\partial S}(t, S(t)) - rF(t, S(t)) + \frac{1}{2}\sigma_*^2 S^2(t)\frac{\partial^2 F}{\partial S^2}(t, S(t))dt,$$

with

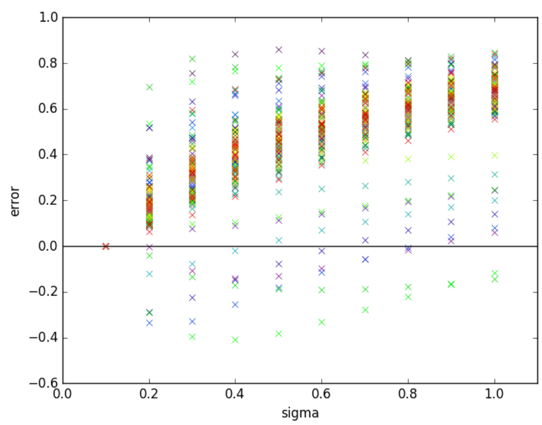
$$\frac{\partial F}{\partial t}(t, s) = re^{-r(T-t)}\Phi(\epsilon d_2(t, s)) + e^{-r(T-t)}\frac{\epsilon e^{-\frac{d_2^2(t,s)}{2}}}{\sqrt{2\pi}}\left[\frac{d_2(t, s)}{2(T-t)} - \frac{1}{\sigma\sqrt{T-t}}\left(r + \frac{\sigma^2}{2}\right)\right],$$

$$\frac{\partial F}{\partial s}(t, s) = \frac{\epsilon e^{-r(T-t)}}{\sqrt{T-t}}\frac{e^{-\frac{d_2^2(t,s)}{2}}}{s\sigma\sqrt{T-t}}, \text{ and}$$

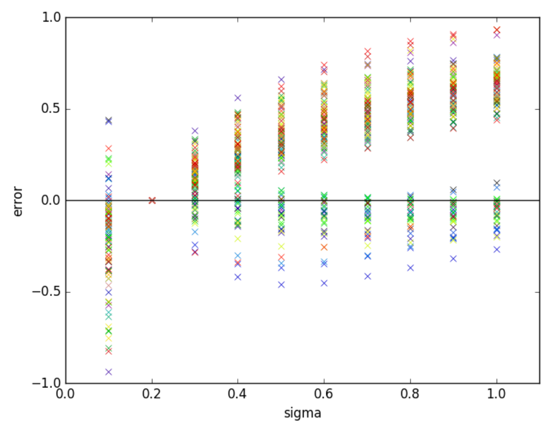
$$\frac{\partial^2 F}{\partial s^2}(t, s) = -\frac{2}{s^2\sigma^2}\frac{\epsilon e^{-r(T-t)}e^{-\frac{d_2^2(t,s)}{2}}}{2\sqrt{2\pi}}\left[\frac{\sigma}{\sqrt{T-t}} + \frac{d_2(t, s)}{T-t}\right].$$

3.2.1 Call Option

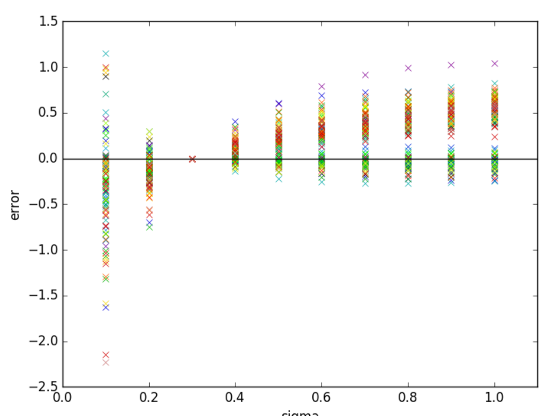
The subfigures of Figure 7 (on two pages) show the computed errors for a digital call option. As for European call options, the errors are zero for $\sigma = \sigma_*$. In difference to it, one cannot say, that all errors are positive for σ smaller than σ_* or negative for σ greater than σ_*. For both cases the errors are positive and negative. For digital call options $\frac{\partial^2 F}{\partial s^2}(t, s)$ is not always positive. Hence, the sign of the hedged errors does not only depend on the sign of $\sigma_* - \sigma$. The greater the difference between σ and σ_* is, the bigger are the absolute errors. More precisely, for all σ and σ_* occur positive errors and for all combinations of σ and σ_* except of $\sigma = \sigma_*$ it is possible to have a shortfall. That leads to an excess in the amount of cash that is available. Whether there occurs a shortfall or not, depends on the chosen stock, or in this case, on the chosen seed, with which the stock is simulated.



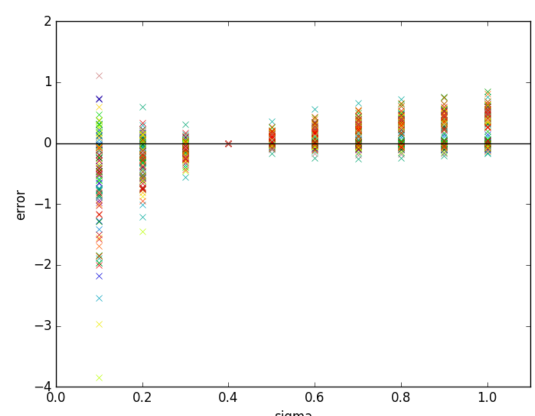
(a) $\sigma_* = 0.1$



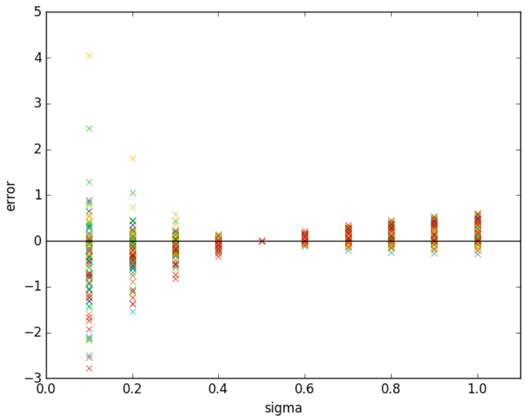
(b) $\sigma_* = 0.2$



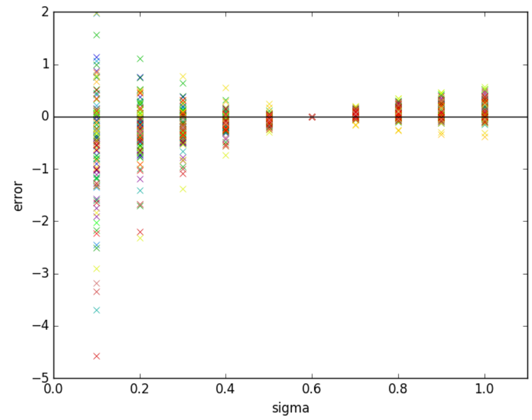
(c) $\sigma_* = 0.3$



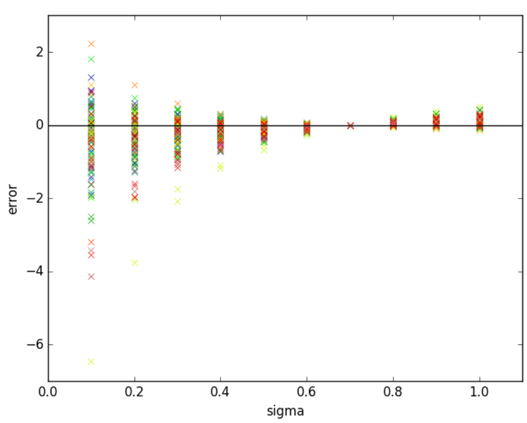
(d) $\sigma_* = 0.4$



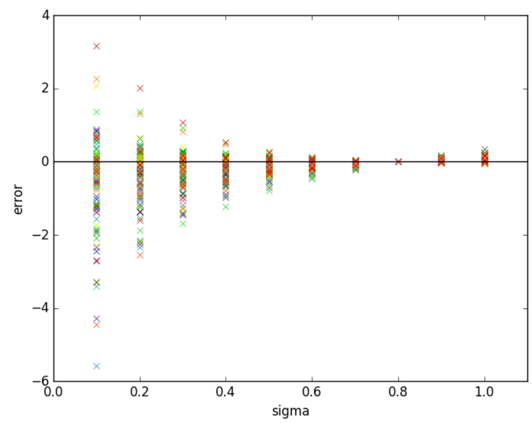
(e) $\sigma_* = 0.5$



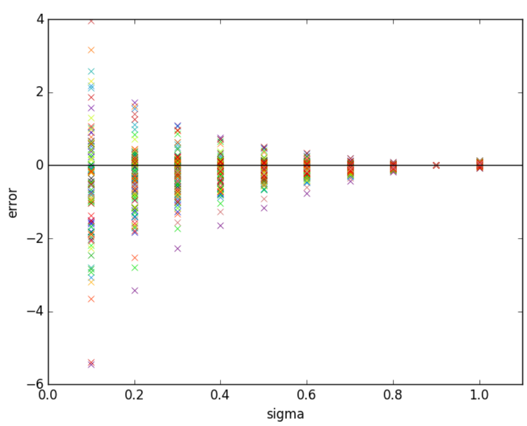
(f) $\sigma_* = 0.6$



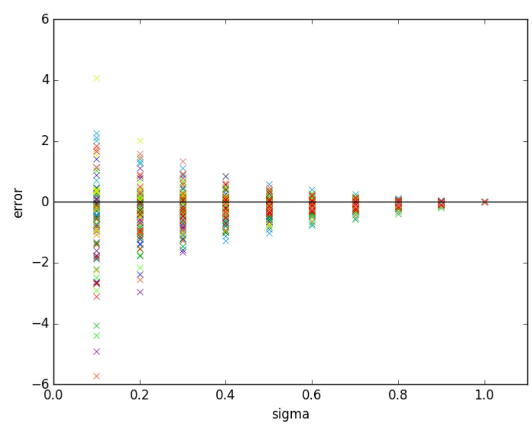
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$



(i) $\sigma_* = 0.9$



(j) $\sigma_* = 1.0$

Figure 7: Hedged portfolio errors per σ for a digital call option with different σ_*

The range of the errors for different σ_* differs. Figure 8 displays all errors for each σ_*. Furthermore, the range of the errors gets bigger for rising σ_*. For $\sigma_* = 0.1$ the errors are within $[-0.409, 0.861]$, for $\sigma_* = 0.5$ they belong to $[-2.783, 4.053]$, and for $\sigma_* = 1.0$ the errors are in the interval $[-5.712, 4.064]$. So, for $\sigma_* = 1.0$ the interval is 7.8 times larger as for $\sigma = 0.1$. The larger σ_* is, the more unpredictable are the errors.

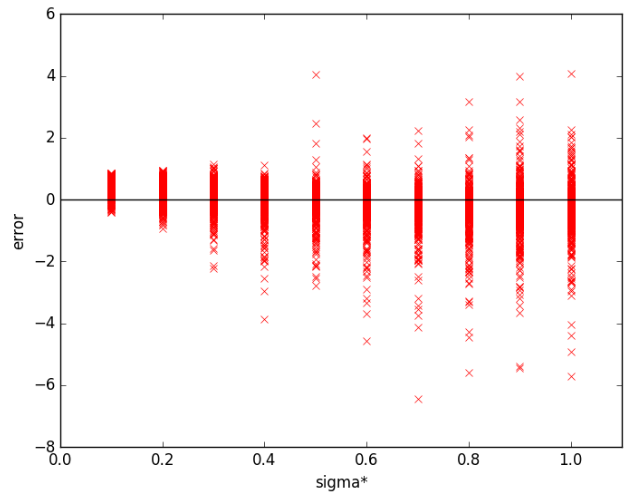


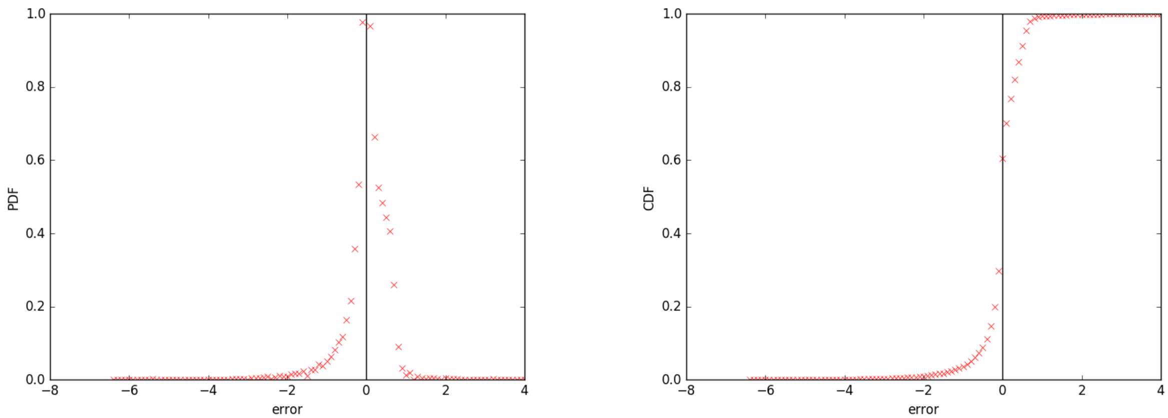
Figure 8: Range of hedged errors per σ_* for a Digital call option

As expected, the minimum and maximum errors, which are shown in table 2, are exactly zero for $\sigma = \sigma_*$. The table lists the minimum and maximum values for all combinations of σ and σ_*.

σ	0.1		0.2		0.3		0.4		0.5	
σ_*	min	max	min	max	min	max	min	max	min	max
0.1	−0.0	0.0	−0.33553	0.69596	−0.39598	0.81909	−0.4089	0.84093	−0.38057	0.86144
0.2	−0.93597	0.43874	−0.0	0.0	−0.28152	0.38146	−0.41608	0.56269	−0.4583	0.66098
0.3	−2.23362	1.14688	−0.75207	0.29879	−0.0	0.0	−0.14206	0.40826	−0.21797	0.60346
0.4	−3.85218	1.1118	−1.45442	0.60089	−0.56028	0.30849	−0.0	0.0	−0.17327	0.3554
0.5	−2.78322	4.05266	−1.53309	1.81307	−0.81913	0.58599	−0.32932	0.14907	−0.0	0.0
0.6	−4.57163	1.9838	−2.31187	1.11357	−1.38601	0.77661	−0.73243	0.55312	−0.29455	0.24524
0.7	−6.44623	2.22838	−3.7527	1.10072	−2.06799	0.60759	−1.18136	0.31489	−0.67626	0.17748
0.8	−5.57665	3.16849	−2.53266	2.02159	−1.68064	1.08427	−1.21509	0.5402	−0.79377	0.27258
0.9	−5.44333	3.97245	−3.4115	1.73042	−2.27686	1.10118	−1.64622	0.75955	−1.16224	0.5092
1.0	−5.7122	4.06376	−2.94918	2.02218	−1.63871	1.33377	−1.2812	0.86331	−1.03195	0.58233
σ	0.6		0.7		0.8		0.9		1.0	
σ_*	min	max	min	max	min	max	min	max	min	max
0.1	−0.33226	0.85372	−0.27671	0.83539	−0.22049	0.8148	−0.1666	0.82985	−0.14307	0.84713
0.2	−0.44978	0.74035	−0.41522	0.81845	−0.36862	0.87075	−0.31761	0.90729	−0.26625	0.93459
0.3	−0.25453	0.79246	−0.26901	0.91565	−0.27012	0.98766	−0.26181	1.02661	−0.24641	1.04503
0.4	−0.24387	0.554	−0.25682	0.6606	−0.24026	0.71692	−0.20948	0.75958	−0.17255	0.84994
0.5	−0.11137	0.21036	−0.19352	0.35326	−0.24473	0.45884	−0.27144	0.54089	−0.28106	0.60576
0.6	−0.0	0.0	−0.1606	0.20292	−0.26673	0.34652	−0.34257	0.46836	−0.38583	0.568
0.7	−0.28913	0.0741	−0.0	0.0	−0.05413	0.21727	−0.10043	0.38315	−0.13708	0.51204
0.8	−0.45694	0.12523	−0.21292	0.04726	−0.0	0.0	−0.03256	0.18678	−0.05582	0.34621
0.9	−0.75885	0.33829	−0.43591	0.19917	−0.18751	0.08753	−0.0	0.0	−0.06713	0.14186
1.0	−0.76856	0.39577	−0.57006	0.25057	−0.38601	0.13962	−0.18987	0.05787	−0.0	0.0

Table 2: Minimum and maximum hedged errors for a Digital call option

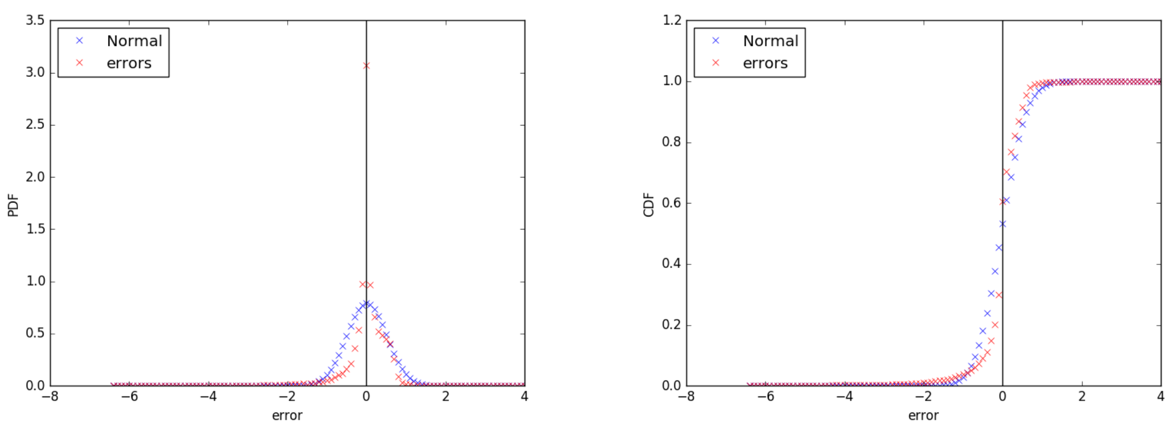
In Figure 9 the approximated probability density function and the approximated cumulative distribution function are posed. Each point in Figure 9(a) denotes the relative probability, that an error occurs in the interval x_i and x_{i+1}, where Δx is 0.1. To get a better impression of all errors, $f(0) = 3.068$ is not plotted. This value is much higher than all other probabilities, as shown by the huge jump at 0 in Figure 9(b).



(a) Approximated probability density function **(b)** Approximated cumulative distribution function

Figure 9: Approximated probability density and cumulative distribution function for a digital call option

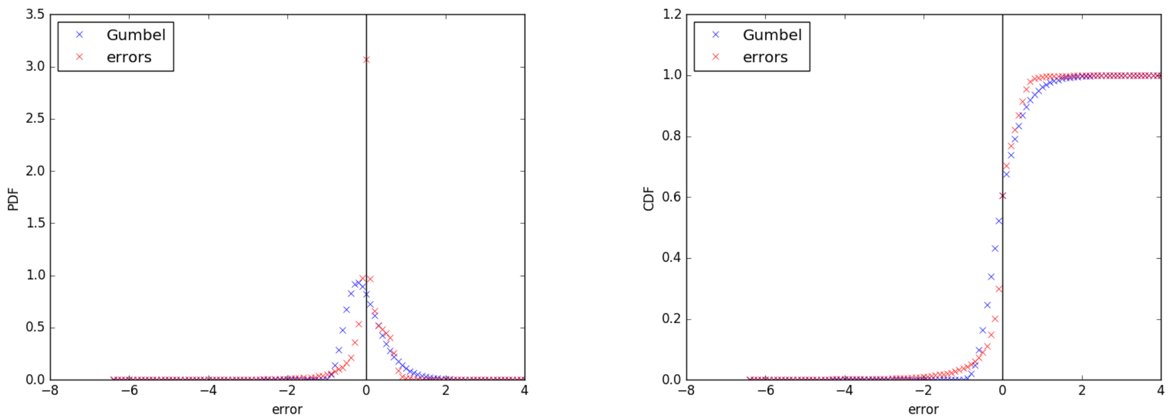
As the function for European call options, those functions lead to the suspicion, that the errors might be normal or Gumbel distributed. In Figure 10 the plots from above are compared to the probability density function and the cumulative distribution function of a normal distribution with estimated parameters. There is a high coverage.



(a) PDF **(b)** CDF

Figure 10: Comparison of errors of a digital call option to normal distribution

This can be observed by comparing the plots to a Gumbel distribution, too.



(a) PDF **(b)** CDF

Figure 11: Comparison of errors of a digital call option to Gumbel distribution

Similarly, through further analyses it might be possible, to find an appropriate distribution for those errors.

3.2.2 Put Option

Unlike the European options, the errors for digital put options are mirrored at the x-axis compared to those of digital call options. There is no sign changing ϵ in the integral for European options. For this calculations, however, there is an ϵ in the $\frac{\partial^2 F}{\partial s^2}(t, s)$-term. So, the errors change their sign. This can be seen in the plots in Appendix A.2. The errors behave similar to those of the call option. Consequently, the probability functions shown in Figure 21 are covered by the normal and the Gumbel distribution with estimated parameters as well (see Figure 22 and Figure 23).

The minimal and maximal errors per σ and σ_* do not differ essential from those of the digital call option, except of the sign, resulting from the above mentioned cause. The tables can be seen in Appendix B.2.

Chapter 4

Evaluation of Risks

In this chapter different ways of measuring risks are introduced. Risk is defined as a "hazard, a chance of bad consequences, loss or exposure to mischance" (*Concise Oxford English Dictionary* 2011). It is the volatility of unexpected outcomes. There are different types of risks defined. This thesis is dealing with model risks. They are associated with using a mis-specified model. Particularly, wrong inputs and feeds are a potential problem in modeling. Traders assume parameters to calculate the option price. In this case the volatility of the stock is a potential problem for the model. The value cannot be observed. Wrong assumed input parameters may lead to wrong models (Cruz 2008). In financial markets it is important to know the risk of a shortfall, which is "an amount that is less than the level that was expected or needed" (*Cambridge Online Dictionary* 2017b). For measuring shortfalls downside risks should be minimized. In this case positive variations of the first integral in (23) denote shortfalls of an option. For calculating the risk of a shortfall different risk measures were defined. Such a risk measure has three desirable properties:

Definition 2. *A risk measure ρ is defined to have the following properties for any two random variables X and Y:*

1. *Translation invariance: If $\alpha \in \mathbb{R}$, then $\rho(X + \alpha) = \rho(X) + \alpha$*
2. *Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$*
3. *Positive homogeneity: If $\alpha \in \mathbb{R}$, then $\rho(\alpha X) = \alpha\rho(X)$*

In this chapter three risk measures are introduced, Value at Risk (see Section 4.1), conditional Value at Risk (see Section 4.2) and Lower Partial Moments (see Section 4.3). All are evaluated for all hedged errors and for every single σ itself (see Section 4.4).

4.1 Value at Risk - VaR

In the end of the last century one of the most popular ways for measuring risks has been Value at Risk (VaR) (Jorion 1997). With that the risk of all the trading positions of a financial institution were quantified. In regard to Choudhry and Wong VaR "is the maximum loss which can occur with $\alpha\%$ confidence over a holding period of t days."

Definition 3. *Assume the stochastic loss X. Suppose, X follows the distribution F_X. Given a confidence level $\alpha \in (0,1)$, $VaR_\alpha[X]$ is defined as the α-th quantile of F_X:*

$$
\begin{aligned}
VaR_\alpha[X] &= \inf\{x \in \mathbb{R} : \mathbb{P}(X > x) \leq 1 - \alpha\} \\
&= \inf\{x \in \mathbb{R} : 1 - F_X(x) \leq 1 - \alpha\} \\
&= \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.
\end{aligned}
$$

For the simulations the risk of a shortfall is of interest. The value of the option exceeds the amount in the portfolio when the error is positive. Hence, the confidence level should be great. Here $\alpha = 0.95$ is chosen. The VaR with $\alpha = 0.95$ for all simulations are:

European call option: $VaR_{0.95} = 6.8$ European put option: $VaR_{0.95} = 6.8$

digital call option: $VaR_{0.95} = 0.5$ digital put option: $VaR_{0.95} = 0.7$

According to these values, the maximum loss with confidence level $\alpha = 0.95$ for all assumed σ is 6.8 for European options, 0.5 for digital call options and 0.7 for digital put options.

4.2 Conditional Value at Risk - CVaR

VaR is a pretty fair risk measure that is really popular in financial institutions. However, in 1999 Artzner et al. criticized VaR in their article "Coherent Measures of Risk". They delineated the definitions of a coherent measure of risk. VaR does not satisfy a condition of subadditivity (see Definition 4). Thus, using VaR could lead to excess risk taking, for example by suggesting to decrease the diversification to reduce risk. That contradicts empirical tests and fundamental financial theory.

Additionally, VaR at a specified probability level α does not provide any information about the fatness of the distribution's upper tail. New ways of risk measuring have emerged. A very important type are coherent risk measures (Acerbi and Tasche 2001). The properties further include subadditivity, which is defined as follows:

Definition 4. *A coherent risk measure ρ is defined to have the following property additional to those in Definition 2 for any two random variables X and Y:*

4. *Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

One coherent risk measure is the conditional Value at Risk (CVaR), also known as expected shortfall (ES) or Tail Expected Loss. While VaR looks on what is likely to happen, CVaR considers how bad things will go in the most pessimistic prediction. This is done by calculating the expected loss given that the loss exceeds $VaR_\alpha[X]$.

Definition 5. *Assume the stochastic loss X. Suppose, X follows the distribution F_X. Given a confidence level $\alpha \in (0, 1)$, $CVaR_\alpha[X]$ is defined as:*

$$\begin{aligned} CVaR_\alpha[X] &= \mathbb{E}[X|X \geq VaR_\alpha[X]\,] \\ &= \frac{1}{1-\alpha} \int_\alpha^1 VaR_p[X]\, dp. \end{aligned}$$

To get the CVaR for a shortfall, a confidence level of $\alpha = 0.95$ is used. Following are the CVaRs for all simulations given:

European call option: $CVaR_{0.95} = 9.776$ European put option: $CVaR_{0.95} = 9.776$

digital call option: $CVaR_{0.95} = 0.768$ digital put option: $CVaR_{0.95} = 1.393$

The expected loss given that the loss exceeds $VaR_\alpha[X]$ for all assumed σ is 9.776 for European options, 0.768 for digital call options and 1.393 for digital put options.

4.3 Lower Partial Moments - LPM

Lower Partial Moments (LPM) are downside-risk measures which refer only to a part of the probability density. They gather just the negative deviations from a boundary b. Thereby, they take all information of the probability distribution into account. The boundary b could be the expected value $\mathbb{E}[X]$ or an arbitrary target amount.

Definition 6. *The general function for computing a LPM of order m, $m \in \mathbb{R}^+$ and boundary b is:*

$$LPM_m(b, X) = \mathbb{E}[\max(b - X, 0)^m].$$

The order m determines in which way the bound is scored. If the risk aversion of the investor is high, then the order m should be high, too.

In this thesis, a shortfall will arise, if the hedged error is positive. For the simulations instead of an LPM an Upper Partial Moment (UPM) is developed. It calculates the positive deviations from a boundary b. The definition for such an UPM is:

Definition 7. *The general function for computing a UPM of order m, $m \in \mathbb{R}^+$ and boundary b is:*

$$UPM_m(b, X) = \mathbb{E}[\max(X - b, 0)^m].$$

The order m determines in which way the bound is scored.

For $m = 0$ the UPM gives the probability, that the border b gets exceeded. This results in the following values for all simulations:

European call option: $UPM_0 = 0.9999$ European put option: $UPM_0 = 0.9999$

digital call option: $UPM_0 = 0.9998$ digital put option: $UPM_0 = 0.9999$

The theoretical result of UPM_0 is 1. Due to truncation errors the practical results are deviant. The expected deviation from b ($m = 1$) for all simulations is:

European call option: $UPM_1 = 1.42613$ European put option: $UPM_1 = 1.42613$

digital call option: $UPM_1 = 0.14493$ digital put option: $UPM_1 = 0.13698$

For $m = 2$ the expected squared deviation from b is computed:

European call option: $UPM_2 = 8.52$ European put option: $UPM_2 = 8.52$

digital call option: $UPM_2 = 0.084$ digital put option: $UPM_2 = 0.166$

4.4 Risks for Each σ

In the preceding sections different risk measures were introduced and evaluated for all hedged errors. In real life a trader does not have knowledge about the volatility of the stock, instead it gets estimated. For the purpose of usage the risks for each assumed σ is necessary. Giving that the risks for all estimations are known, a broker can assess how high the risk of a shortfall for the estimated volatility might be.

The hedged errors for all different parameters σ can be contemplated in Appendix C.1. Below are the VaRs for the hedged errors which occur for $\sigma = 0.1$, for which VaR is the highest.

European call option: $VaR_{0.95} = 11.92$ European put option: $VaR_{0.95} = 11.92$

digital call option: $VaR_{0.95} = 0.92$ digital put option: $VaR_{0.95} = 2.33$

The expected loss given that the loss exceeds $VaR_{\alpha}[X]$ for each σ is calculated by the conditional value at risk. Below are the CVaRs for the errors which occur for $\sigma = 0.1$, for which CVaR is the highest.

European call option: $CVaR_{0.95} = 15.46$ European put option: $CVaR_{0.95} = 15.46$

digital call option: $CVaR_{0.95} = 1.753$ digital put option: $CVaR_{0.95} = 3.401$

For all other σ the values for VaR are in Appendix C.2.1 and for CVaR they can be considered in Appendix C.2.2. For European options and $\sigma = 1.0$ VaR and CVaR is negative. In this case σ_* is smaller than or equal to σ. Hence, there does not exist a shortfall.

The upper partial moments for an assumed volatility $\sigma = 0.1$ are:

$m = 0$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$

digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$m = 1$

European call option: $UPM_1 = 3.73277$ European put option: $UPM_1 = 3.73277$

digital call option: $UPM_1 = 0.15507$ digital put option: $UPM_1 = 0.56593$

$m = 2$

European call option: $UPM_2 = 30.235$ European put option: $UPM_2 = 30.235$

digital call option: $UPM_2 = 0.21$ digital put option: $UPM_2 = 1.062$

The UPM for all other σ decreases. They can be seen in Appendix C.2.3, C.2.4 and C.2.5. For European options and $\sigma = 1.0$, UPM_1 and UPM_2 are zero. As seen before, σ_* is smaller than or equal to σ in this case and there does not exist a shortfall. Hence, $\max(X - b, 0)^m = 0^m = 0$.

Chapter 5

Risks for Weighted Errors

In the previous chapters one could see the hedged portfolio errors for European and digital call and put options. Different risk measures were contemplated. For all options exists a risk of a shortfall. It was assumed that there is no knowledge about a distribution of the volatility of the stocks. But the stock price process behaves like a random walk. Future values are independent from the past. So, it might be possible to estimate the used volatility for pricing an option based on historical data. A first step for rating σ is presented in this chapter.

As seen before, the considered price process S satisfies

$$dS(t) = \mu S(t)\,dt + \sigma_* S(t)\,dW(t), \qquad S(0) = S_0,$$

and it can be shown that S is

$$S(t) = S_0 e^{(\mu-\sigma^2/2)t+\sigma_* W(t)}, \qquad t \geq 0.$$

Let X denote the logarithm of the ratios:

$$X(t) := \ln\left(\frac{S(t+1)}{S(t)}\right) = \mu - \frac{\sigma_*^2}{2} + \sigma_*(W(t+1) - W(t)). \tag{25}$$

This random variable is normally distributed with mean $\mu - \frac{\sigma_*^2}{2}$ and variance σ_*^2.

Considering the stock price random variables at times $t = 0, 1, \ldots, m, m+1, m+2, \ldots, m+n+1$ and using the logarithm of the ratios as in (25) one can break this into two collections:

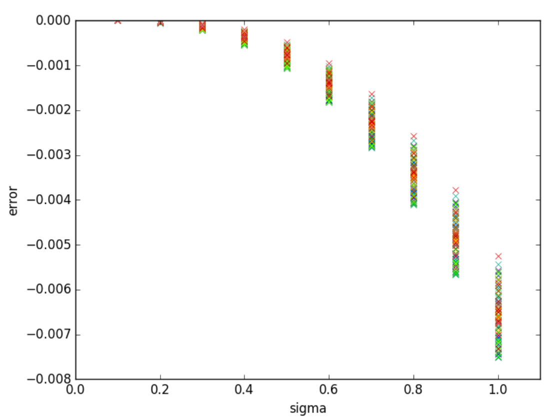
$$\{X(0), X(1), \ldots, X(m-1)\} \qquad \text{and} \qquad \{X(m), X(m+1), \ldots, X(m+n)\}.$$

These collections are independent because the increments of the Brownian motion are independent. The first collection can be thought of as historical values and the second as the future. One could use the first collection to estimate the value of σ_*^2 using the sample variance $\hat{\sigma}^2$. This gives an unbiased estimator. It turns out, that $\hat{\sigma}^2$ is χ-squared distributed:

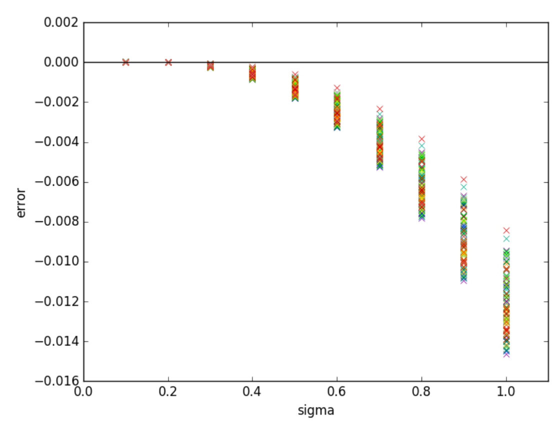
$$\hat{\sigma}^2 \sim \frac{\sigma_*^2}{n-1}\,\chi_{n-1}^2.$$

Based on historical data a trader or broker might estimate the volatility for pricing an option. This leads to an additional interpretation of the hedged errors. Resting on the previous observations the distribution for the volatility can be calculated. The hedged portfolio errors from Chapter 3 can be weighted by the appropriate probability of using a particular σ. In this thesis it is assumed, that eleven historical stock values ($n = 11$) are known.

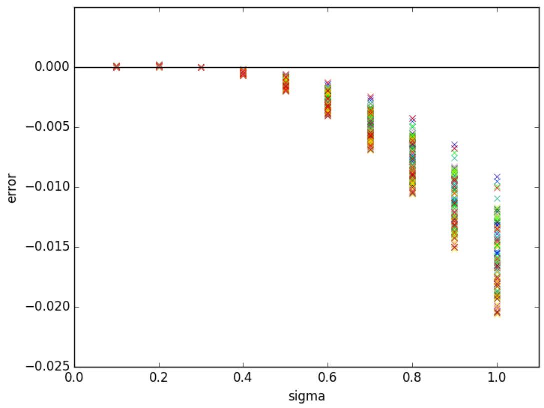
In Figure 12 (on three pages) one can see the weighted errors for European call options and those the other options are presented in Appendix D. The probability for a specific volatility is a value between 0 and 1. The weighted errors get multiplied by the probability. So, they are smaller than the hedged errors in Chapter 3. As seen there, the errors for σ less than σ_* are positive and for σ greater than σ_* they are negative. While a negative divergence from zero implies a surplus, positive discrepancies imply a shortfall. The risk for a shortfall is analyzed afterwards.



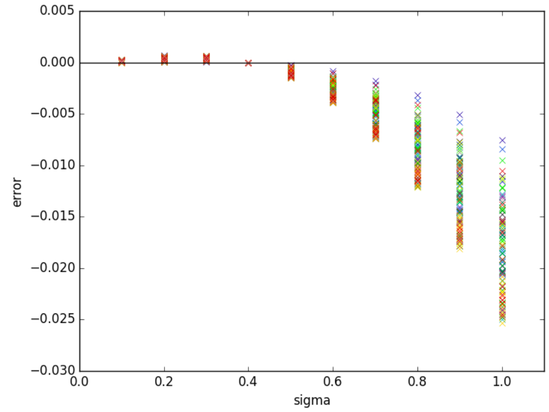
(a) $\sigma_* = 0.1$



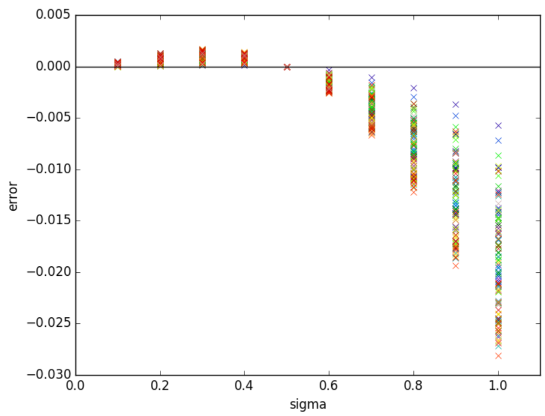
(b) $\sigma_* = 0.2$



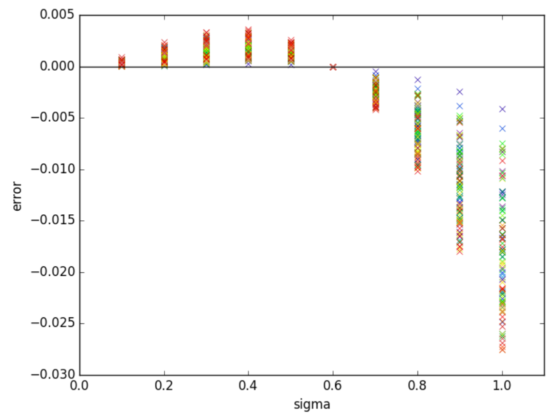
(c) $\sigma_* = 0.3$



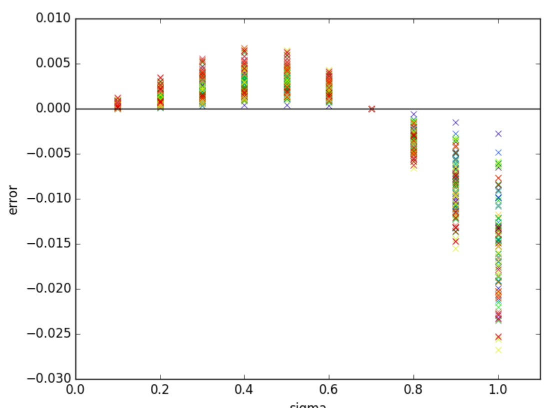
(d) $\sigma_* = 0.4$



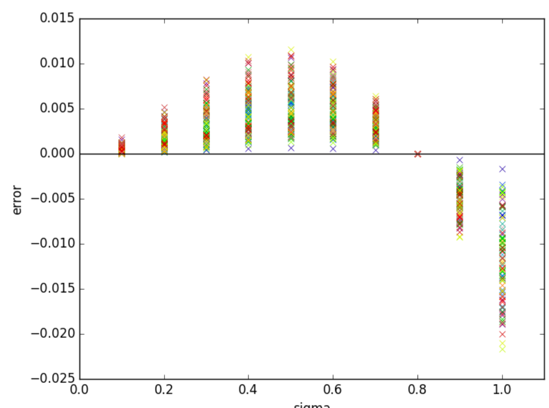
(e) $\sigma_* = 0.5$



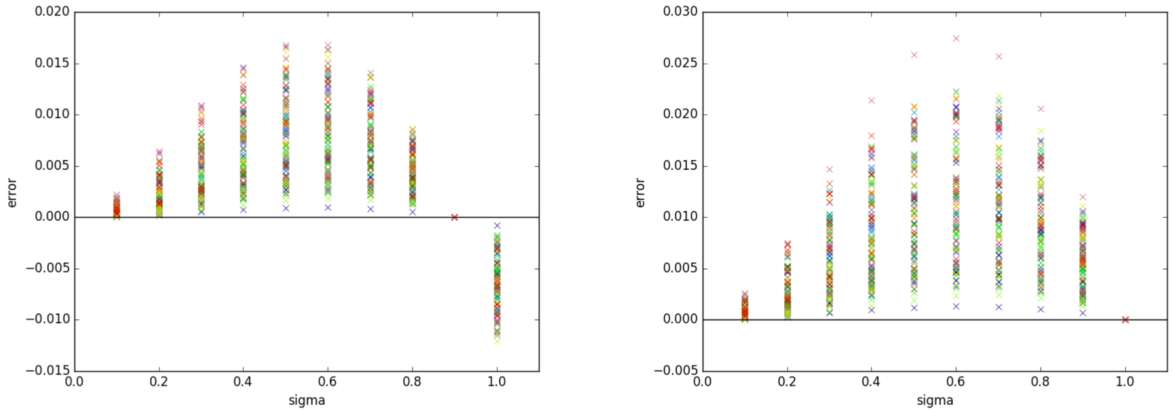
(f) $\sigma_* = 0.6$



(g) $\sigma_* = 0.7$



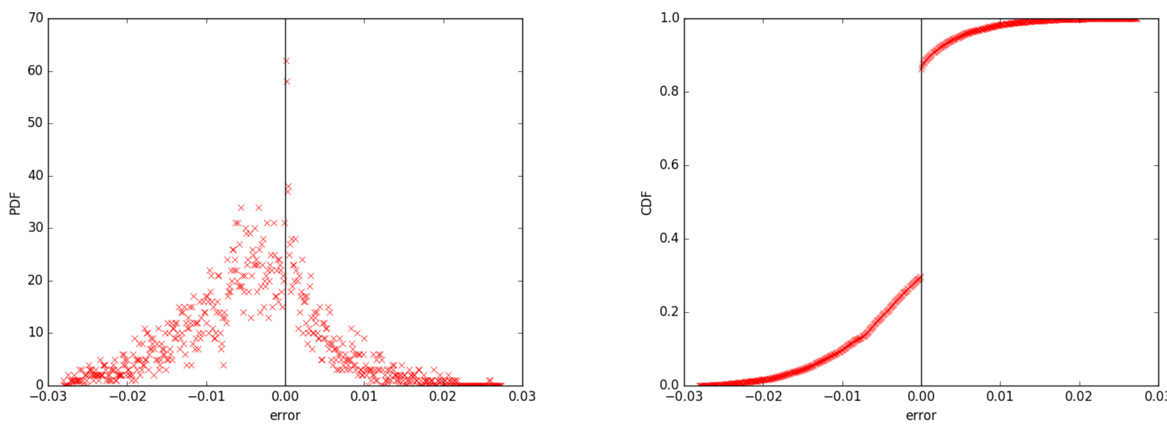
(h) $\sigma_* = 0.8$



(i) $\sigma_* = 0.9$ **(j)** $\sigma_* = 1.0$

Figure 12: Weighted errors per σ for a European call option with different σ_*

The approximated probability density and cumulative distribution functions are plotted in Figure 13. The relative probability, that an error with value within x_i and x_{i+1} with $\Delta x = 0.0001$ occurs, is shown in Figure 13(a). In this graph $f(0) = 5648$ is not plotted. It is much higher than all other values. The huge jump at 0 in Figure 13(b) expresses that behavior. About 50% of all errors are zero.



(a) Approximated probability density function **(b)** Approximated cumulative distribution function

Figure 13: Approximated probability density and cumulative distribution function for weighted errors of a European call option

Still there occur errors greater than zero. As explained previously, positive deviations from zero imply a shortfall. Applying the risk measure from Section 4.1, 4.2, and 4.3 gives an appraisal of how high the risks for shortfalls are.

The Upper Partial Moment (UPM) with border $b = 0$ and order $m = 0$ is one for all options. So, there are positive deviations from zero for all options and the probability that there might be a shortfall is one.

European call option: $UPM_0 = 1.0$ European put option: $UPM_0 = 1.0$

digital call option: $UPM_0 = 1.0$ digital put option: $UPM_0 = 1.0$

How high the risks for different measurements are can be seen below.

European call option: $VaR_{0.95} = 0.00483$ European put option: $VaR_{0.95} = 0.00483$

digital call option: $VaR_{0.95} = 0.00101$ digital put option: $VaR_{0.95} = 0.00027$

European call option: $CVaR_{0.95} = 0.01$ European put option: $CVaR_{0.95} = 0.01$

digital call option: $CVaR_{0.95} = 0.001$ digital put option: $CVaR_{0.95} = 0.001$

European call option: $UPM_1 = 0.001$ European put option: $UPM_1 = 0.001$

digital call option: $UPM_1 = 0.0$ digital put option: $UPM_1 = 0.0$

European call option: $UPM_2 = 0.0$ European put option: $UPM_2 = 0.0$

digital call option: $UPM_2 = 0.0$ digital put option: $UPM_2 = 0.0$

With 95% confidence the maximum loss which can occur (VaR) is 0.00483 for European options, 0.00101 for digital call options and 0.00027 for digital put options. In the most pessimistic prediction (CVaR) the risk for European options is 0.01 and 0.001 for digital options. The expected deviation from zero (UPM_1) is 0.001 for European options and zero for digital options. For all options the expected squared deviation from zero (UPM_2) is zero.

Knowing a distribution for the volatility of the underlying asset can narrow the risk for a shortfall. Now, it might be interesting to analyze how much a higher amount of historical values may affect the risk for shortfalls. Maybe it is possible to eliminate the risk by using enough data points.

Chapter 6

Conclusion

6.1 Summary

In this thesis the influence of volatility in the Black-Scholes model was analyzed. The model is used to calculate the theoretical price of European options. There are a few assumptions. One of those is a constant volatility. The real volatility of a stock or a portfolio is not predictable. For calculation purposes, it is assumed by a broker. This approximation may not match with the development in real life. Simulating different stocks and using an assumed volatility (σ) and the real one (σ_*), led to the results, that there are risks for shortfalls.

For European call and put options the errors in the computation behave alike. In the case, where $\sigma = \sigma_*$ the errors are zero. But if σ is smaller than σ_*, the errors are positive. Those imply a shortfall. The option exceeds the amount of cash that is available. If σ is greater than σ_*, the errors are negative and imply a surplus.

In difference to that, it is not possible to say, that all errors of digital call options are positive for σ smaller than σ_* or negative for σ greater than σ_*. For both cases the errors are positive and negative. The greater the difference between σ and σ_* is, the bigger are the absolute values. Hence, for all σ and σ_* occur positive errors and consequently, for all combinations of σ and σ_* except of $\sigma = \sigma_*$ it is possible to have a shortfall. The errors for digital put options are mirrored at the x-axis compared to those of digital call options.

Additionally, the approximated probability density functions of all options were compared with a normal and a Gumbel distribution with estimated parameters. Both covered the calculated errors well. So, it might be possible to calculate the probability of a portfolio error with one of these distributions.

In Chapter 4 three different risk measures were discussed: the Value at Risk (VaR), the conditional Value at Risk (CVaR) and the Lower or Upper Partial Moments (UPM). The risk of a shortfall was of interest. VaR measures the worst expected loss. For a confidence level of $\alpha = 0.95$ and for all assumed σ this was 6.8 for European options, 0.5 for digital call options and 0.7 for digital put options. CVaR computes the expected loss, given that the loss exceeds VaR. In the most pessimistic prediction CVaR considers how bad things will go. For a confidence level $\alpha = 0.95$ CVaR was 9.776 for European options, 0.768 for a digital call option and 1.393 for a digital put option. UPMs gather the deviations from a boundery b. The expected deviation from $b = 0$ for all simulations is 1.426 for European options, 0.145 for digital call options and 0.137 for digital put options.

Further the risks for individual assumed volatilities σ were computed. For $\sigma = 0.1$ the risks are the highest. They decrease for all other σ. The broker does not have knowledge about the real volatility of the stock. With this computations it is possible to get a notion of how high the risk of a shortfall for the chosen estimated volatility is.

Based on historical data it is possible to estimate the volatility of the underlying asset. As stated in Chapter 5 the estimator for σ_*^2 is χ-squared distributed. Weighting the hedged errors by the probability of using a particular σ results in smaller errors and many errors equal zero. Hence, the risk of getting a shortfall narrowed, but it might exist for all options. For European options $VaR_{0.95}$ is equal to 0.00483. The maximum loss that can occur is 0.00101 for digital call options and 0.00027 for digital put options. The $CVaR_{0.95}$ for European options is 0.01 and 0.001 for digital options. For European options the expected deviation from zero is 0.001 and for digital options it is zero. The squared deviation from zero is zero for all options considered in this thesis. So, knowing the distribution of the volatility of the underlying stock can reduce the risk for a shortfall.

In summary, calculating option prices with a wrong assumed volatility might end in a shortfall. How high a risk is depends, among others, on the used σ and the real volatility. By estimating the used σ from historical data the risk of a shortfall can be scaled-down.

6.2 Further Investigations

The results in this thesis were for very special input values. A stock starting value of $S_0 = 20$ and a striking price of $K = 16$ were assumed. Additionally, the risk-free rate was supposed to be $r = 0$. For further improvements one could calculate the errors for different r and other input values. Another option is to analyze more theoretically.

A further analysis might bring clarity to the assumptions of the probability distribution. Maybe, it is possible to find an appropriate one for those errors.

As seen above, the risk of a shortfall can be minimized by estimating the assumed volatility. The estimator $\hat{\sigma}^2$ is χ-squared distributed. Within further investigations one might find out in what extent a change of the degrees of freedom affect the risk of a shortfall. In this thesis 10 historical stock values ($n = 11$) were assumed. Naturally the question arises, whether it is possible to eliminate the risk of a shortfall by using enough historical values.

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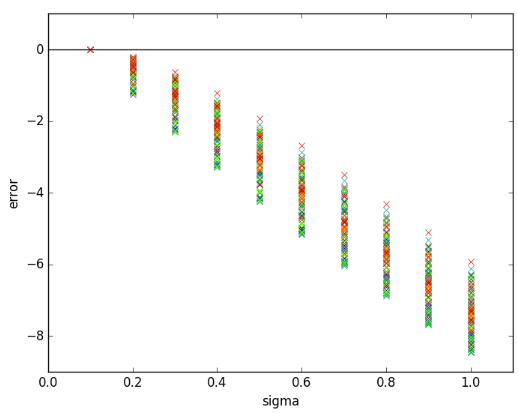
Appendices

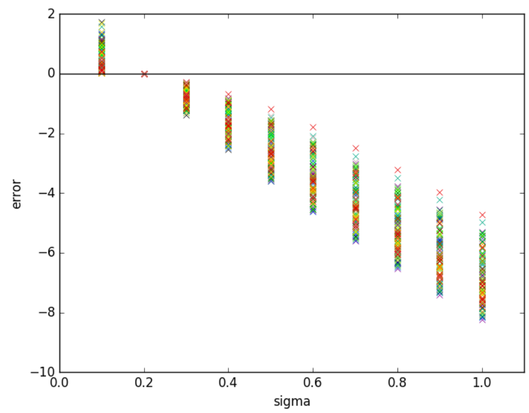
Appendix A

Plots of Errors for $\boldsymbol{\sigma}$ of Different Simulations

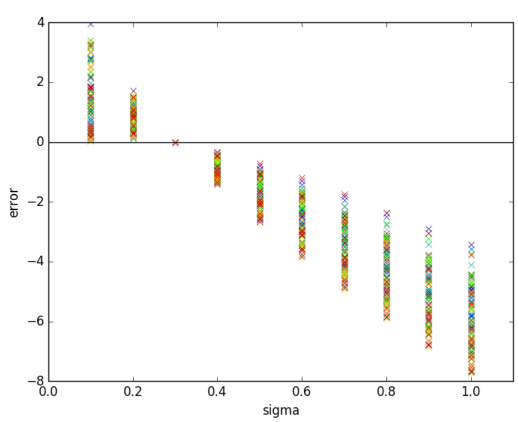
A.1 European Put Option



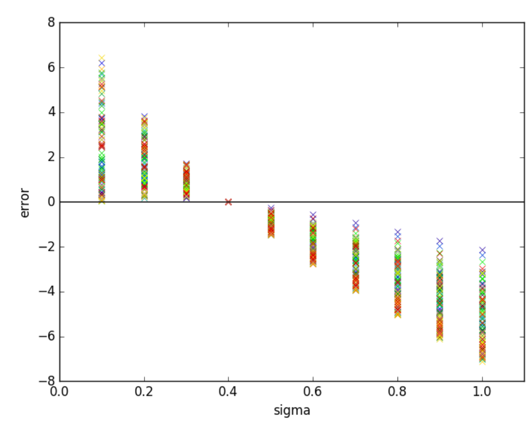
(a) $\sigma_* = 0.1$



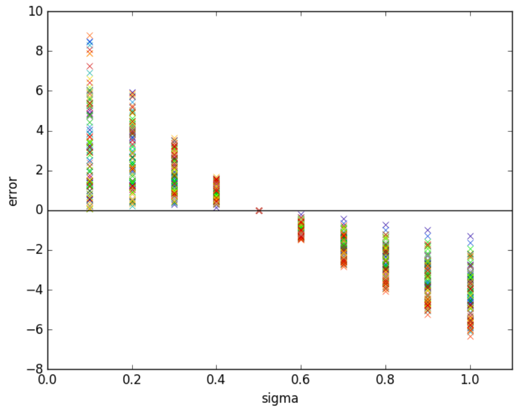
(b) $\sigma_* = 0.2$



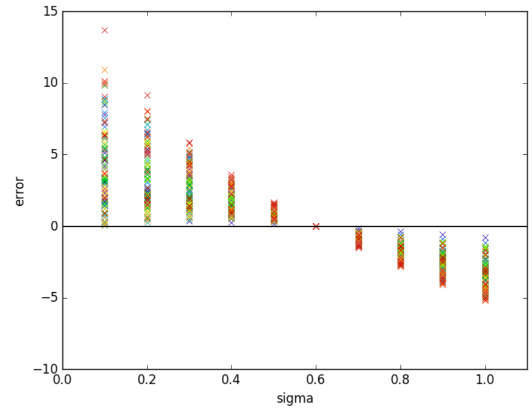
(c) $\sigma_* = 0.3$



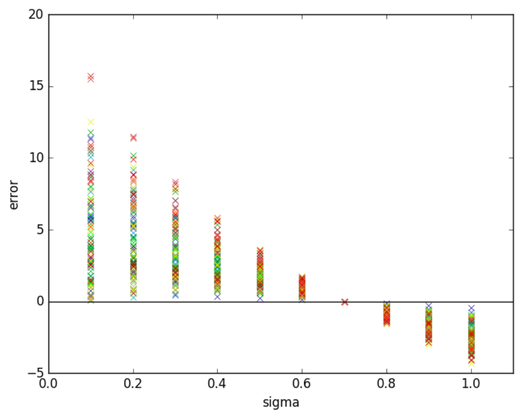
(d) $\sigma_* = 0.4$



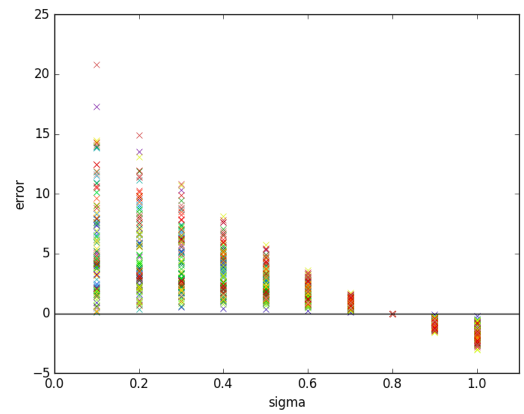
(e) $\sigma_* = 0.5$



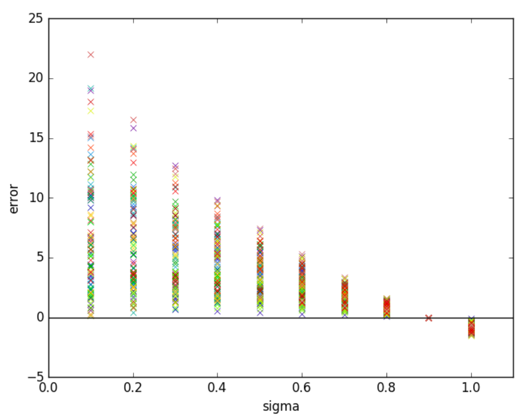
(f) $\sigma_* = 0.6$



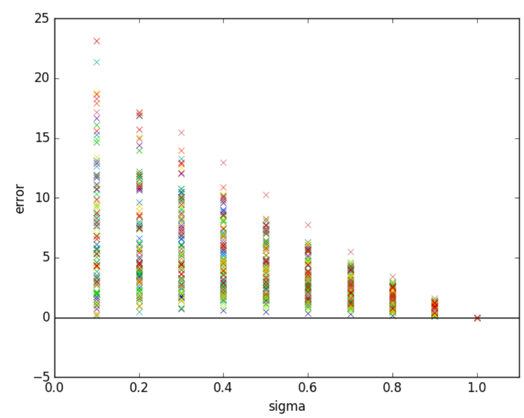
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$



(i) $\sigma_* = 0.9$



(j) $\sigma_* = 1.0$

Figure 14: Hedged portfolio errors per σ for a European put option with different σ_*

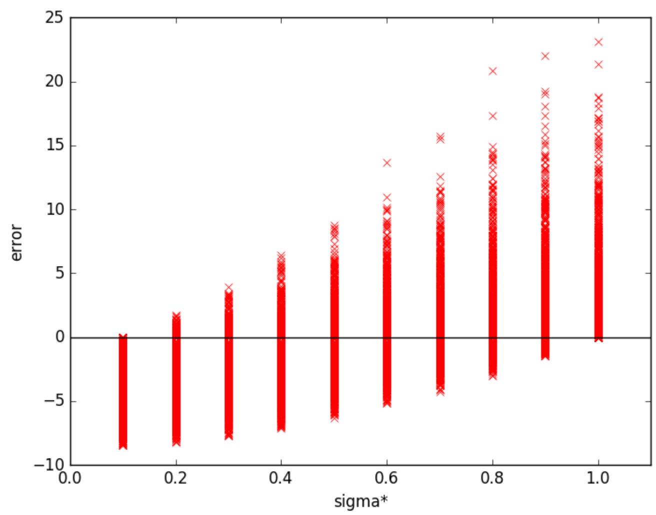
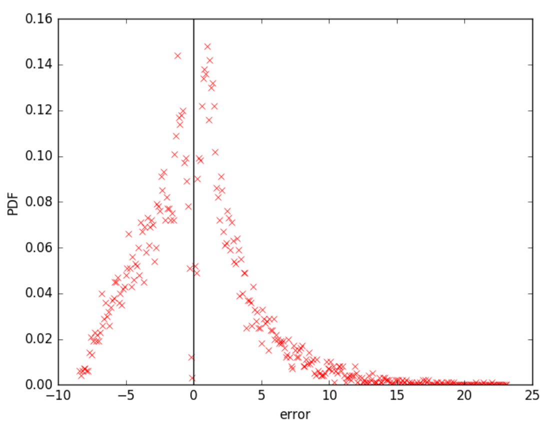
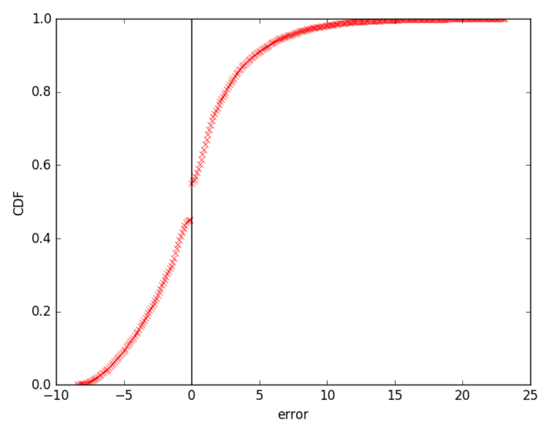


Figure 15: Range of hedged errors per σ_* for a European put option

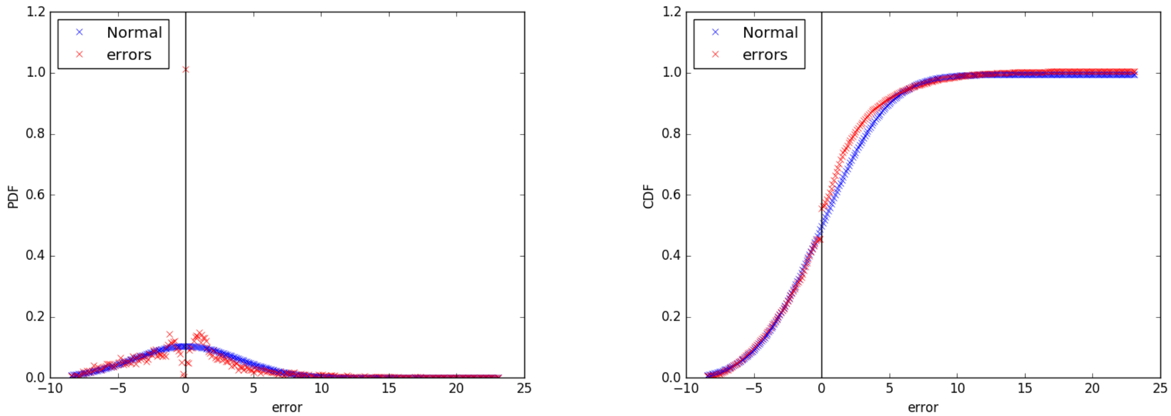


(a) Approximated probability density function



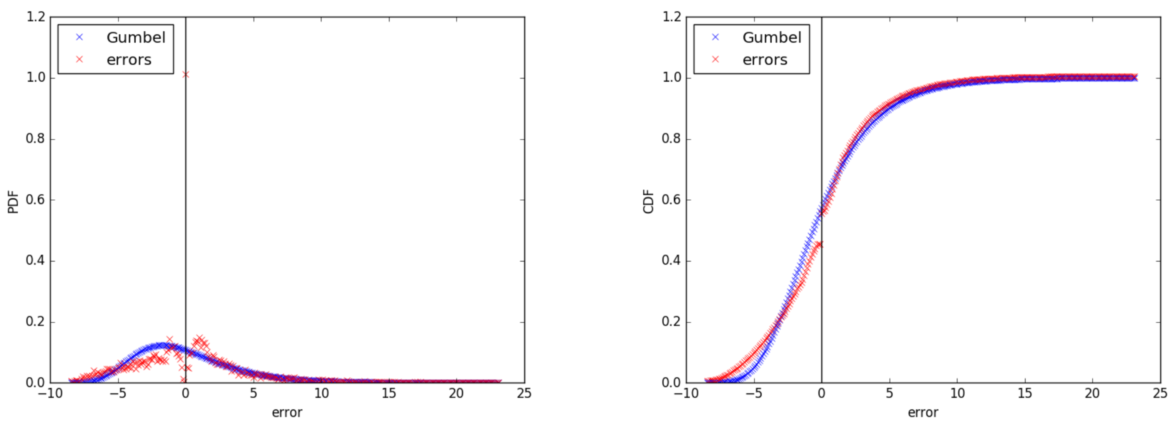
(b) Approximated cumulative distribution function

Figure 16: Approximated probability density and cumulative distribution function for a European put option



(a) PDF **(b)** CDF

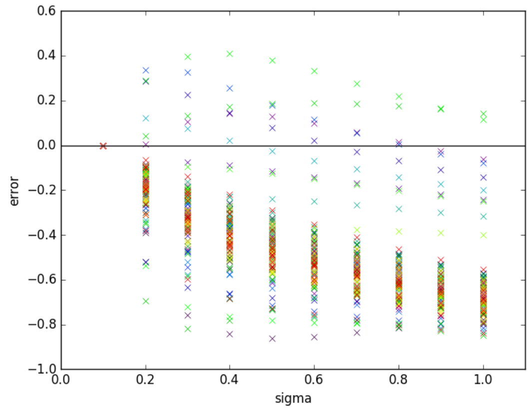
Figure 17: Comparison of errors of a European put option to Normal distribution



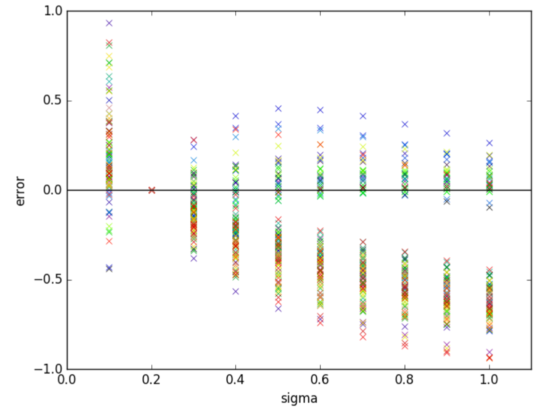
(a) PDF **(b)** CDF

Figure 18: Comparison of errors of a European put option to Gumbel distribution

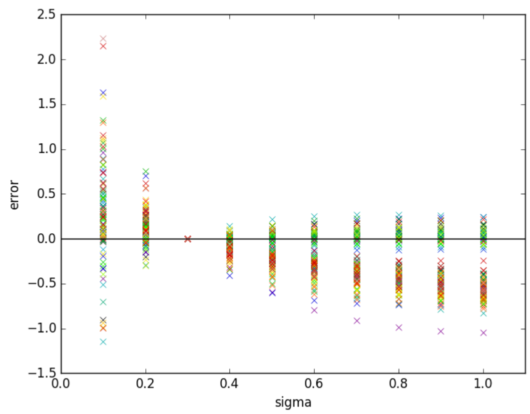
A.2 Digital Put Option



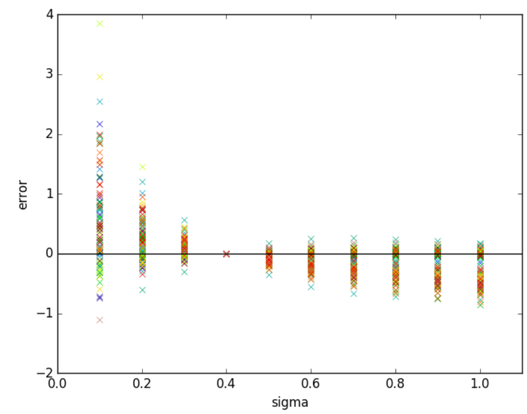
(a) $\sigma_* = 0.1$



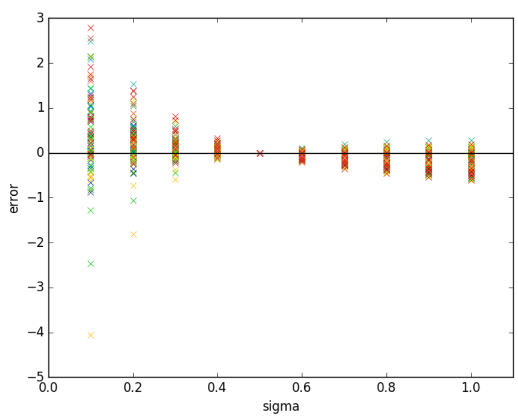
(b) $\sigma_* = 0.2$



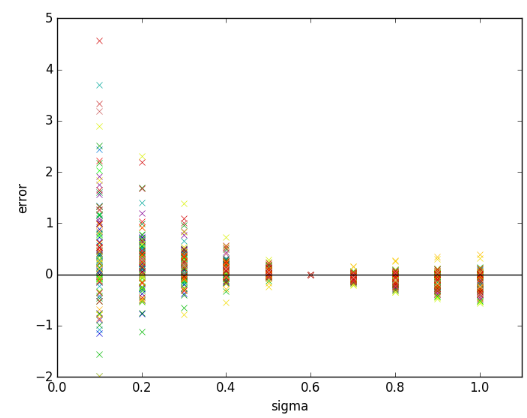
(c) $\sigma_* = 0.3$



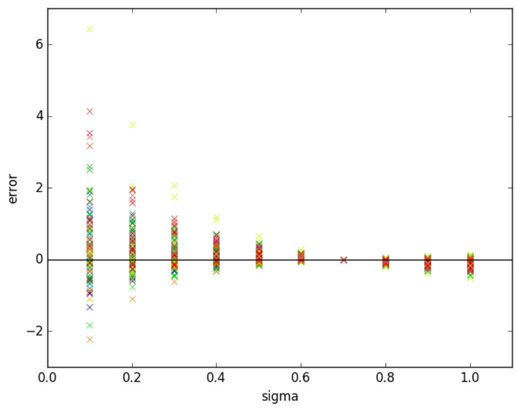
(d) $\sigma_* = 0.4$



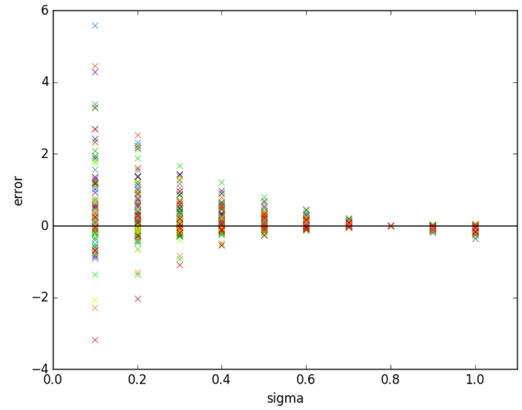
(e) $\sigma_* = 0.5$



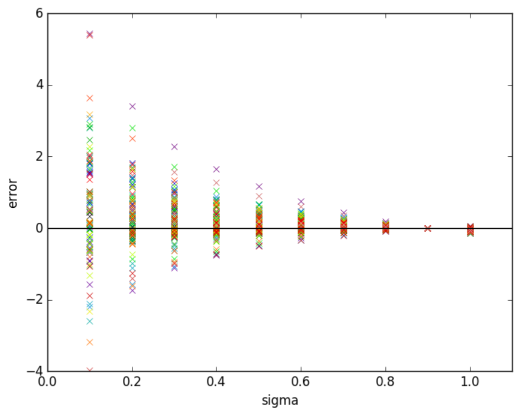
(f) $\sigma_* = 0.6$



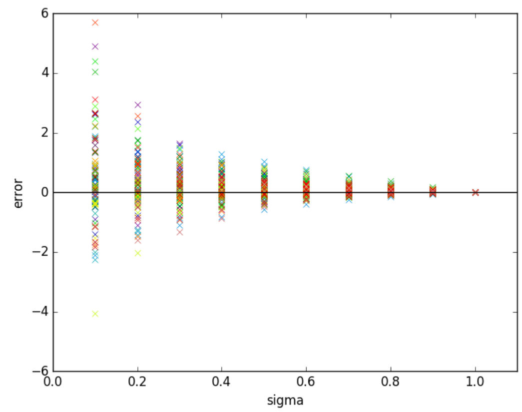
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$



(i) $\sigma_* = 0.9$



(j) $\sigma_* = 1.0$

Figure 19: Hedged portfolio errors per σ for a digital put option with different σ_*

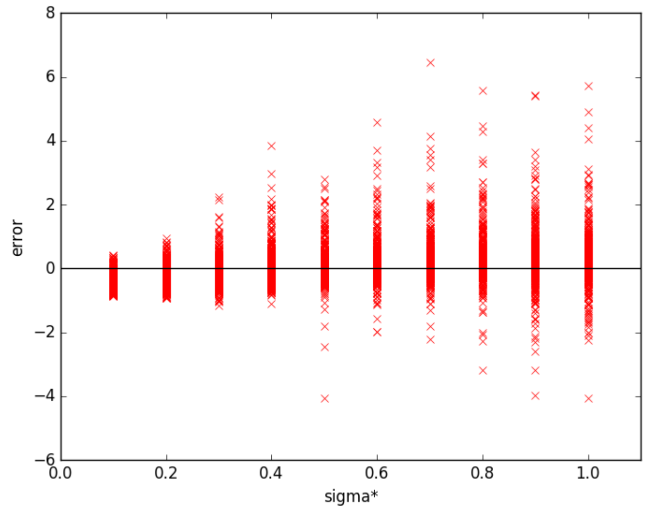
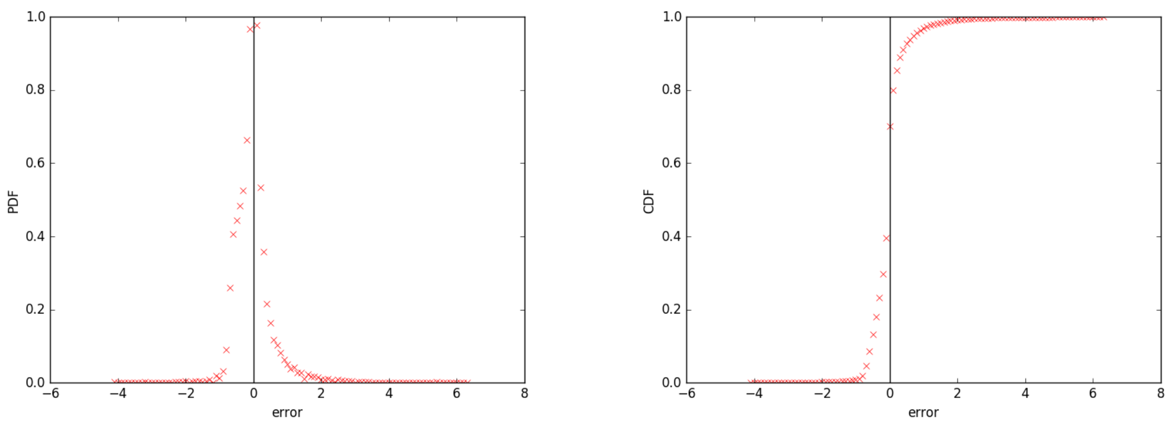


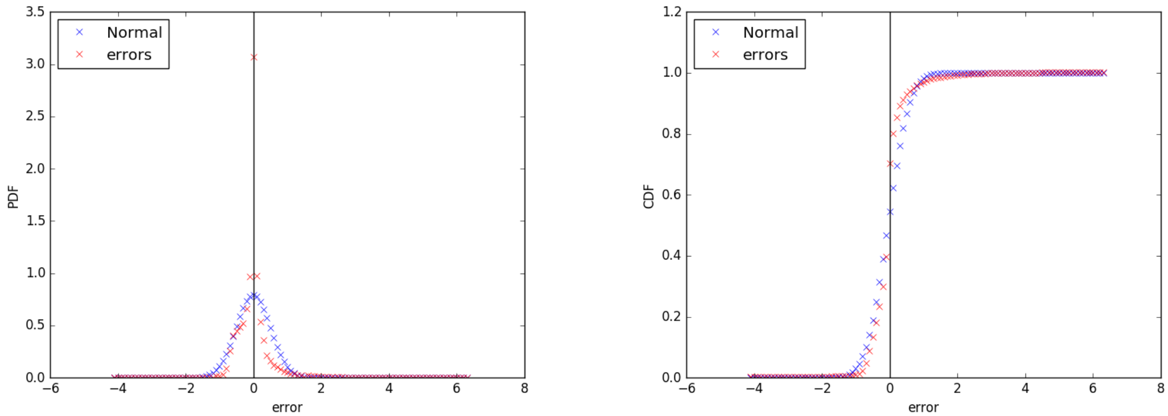
Figure 20: Range of hedged errors per σ_* for a digital put option



(a) Approximated probability density function

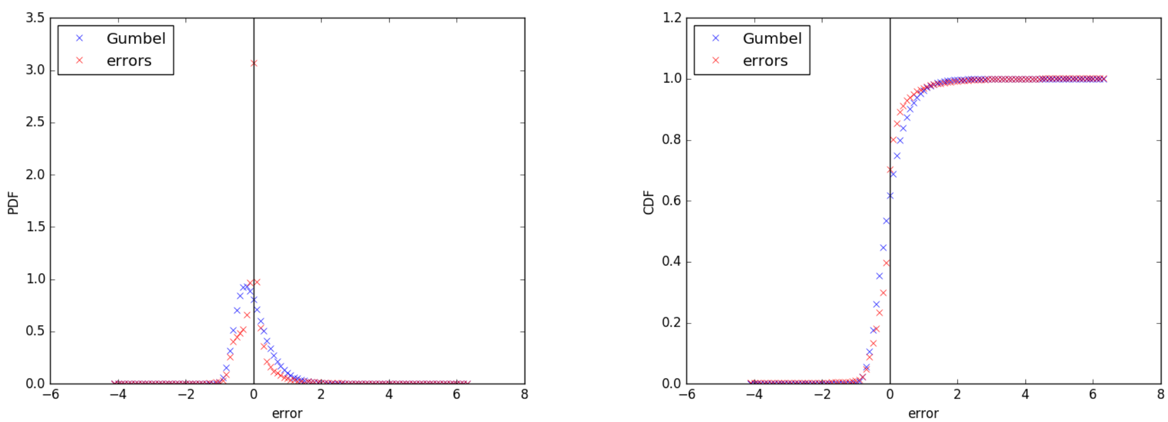
(b) Approximated cumulative distribution function

Figure 21: Approximated probability density and cumulative distribution function for a digital put option



(a) PDF **(b)** CDF

Figure 22: Comparison of errors of a digital put option to Normal distribution



(a) PDF **(b)** CDF

Figure 23: Comparison of errors of a digital put option to Gumbel distribution

Appendix B

Tables of Minimum and Maximum Errors for σ of Different Simulations

B.1 European Put Option

σ	0.1		0.2		0.3		0.4		0.5	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−0.0	0.0	−1.26317	−0.20272	−2.29893	−0.62572	−3.29365	−1.21195	−4.24101	−1.91469
0.2	0.01725	1.74193	−0.0	0.0	−1.37948	−0.2881	−2.54286	−0.6728	−3.5895	−1.17719
0.3	0.03103	3.95978	0.10827	1.7183	−0.0	0.0	−1.41563	−0.32582	−2.68168	−0.70953
0.4	0.04576	6.43765	0.16839	3.82998	0.18028	1.71909	−0.0	0.0	−1.46822	−0.2521
0.5	0.06153	8.79896	0.21586	5.95566	0.28417	3.63902	0.15769	1.68743	−0.0	0.0
0.6	0.0786	13.68867	0.2632	9.16158	0.37344	5.87134	0.261	3.62161	0.13984	1.69377
0.7	0.09748	15.72432	0.3141	11.48889	0.46387	8.36547	0.34761	5.81608	0.2392	3.66835
0.8	0.11874	20.84193	0.36935	14.90892	0.55814	10.85935	0.43068	8.14752	0.32273	5.76965
0.9	0.14294	22.02865	0.42886	16.52301	0.65563	12.68412	0.51477	9.81151	0.4007	7.41496
1.0	0.17058	23.15056	0.49293	17.17745	0.75632	15.44842	0.60077	12.94376	0.47764	10.27263

σ	0.6		0.7		0.8		0.9		1.0	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−5.16345	−2.6832	−6.04396	−3.48492	−6.88422	−4.29916	−7.68423	−5.11172	−8.44617	−5.91205
0.2	−4.61004	−1.78796	−5.5942	−2.47496	−6.52925	−3.20922	−7.40969	−3.96793	−8.2379	−4.73422
0.3	−3.84038	−1.17984	−4.92047	−1.73908	−5.92255	−2.36188	−6.85537	−2.89681	−7.72461	−3.42139
0.4	−2.77036	−0.56958	−3.94844	−0.93235	−5.05927	−1.32116	−6.12589	−1.72149	−7.11762	−2.12297
0.5	−1.47025	−0.19815	−2.82984	−0.43597	−4.08846	−0.70478	−5.2506	−0.9936	−6.32478	−1.29238
0.6	−0.0	0.0	−1.4998	−0.16272	−2.84257	−0.34831	−4.05612	−0.55281	−5.16187	−0.77021
0.7	0.12488	1.74405	−0.0	0.0	−1.57478	−0.13715	−3.00006	−0.28642	−4.29473	−0.44578
0.8	0.21878	3.63057	0.11191	1.71712	−0.0	0.0	−1.56533	−0.11748	−3.04521	−0.24018
0.9	0.29856	5.27866	0.19966	3.3392	0.10044	1.59774	−0.0	0.0	−1.51064	−0.10165
1.0	0.37215	7.75293	0.27515	5.47548	0.18177	3.44154	0.09016	1.62599	−0.0	0.0

Table 3: Minimum and maximum hedged errors for a European put option

B.2 Digital Put Option

σ	0.1		0.2		0.3		0.4		0.5	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−0.0	0.0	−0.69596	0.33553	−0.81909	0.39598	−0.84093	0.4089	−0.86144	0.38057
0.2	−0.43874	0.93597	−0.0	0.0	−0.38146	0.28152	−0.56269	0.41608	−0.66098	0.4583
0.3	−1.14688	2.23362	−0.29879	0.75207	−0.0	0.0	−0.40826	0.14206	−0.60346	0.21797
0.4	−1.1118	3.85218	−0.60089	1.45442	−0.30849	0.56028	−0.0	0.0	−0.3554	0.17327
0.5	−4.05266	2.78322	−1.81307	1.53309	−0.58599	0.81913	−0.14907	0.32932	−0.0	0.0
0.6	−1.9838	4.57163	−1.11357	2.31187	−0.77661	1.38601	−0.55312	0.73243	−0.24524	0.29455
0.7	−2.22838	6.44623	−1.10072	3.7527	−0.60759	2.06799	−0.31489	1.18136	−0.17748	0.67626
0.8	−3.16849	5.57665	−2.02159	2.53266	−1.08427	1.68064	−0.5402	1.21509	−0.27258	0.79377
0.9	−3.97245	5.44333	−1.73042	3.4115	−1.10118	2.27686	−0.75955	1.64622	−0.5092	1.16224
1.0	−4.06376	5.7122	−2.02218	2.94918	−1.33377	1.63871	−0.86331	1.2812	−0.58233	1.03195

σ	0.6		0.7		0.8		0.9		1.0	
σ_*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*	*min*	*max*
0.1	−0.85372	0.33226	−0.83539	0.27671	−0.8148	0.22049	−0.82985	0.1666	−0.84713	0.14307
0.2	−0.74035	0.44978	−0.81845	0.41522	−0.87075	0.36862	−0.90729	0.31761	−0.93459	0.26625
0.3	−0.79246	0.25453	−0.91565	0.26901	−0.98766	0.27012	−1.02661	0.26181	−1.04503	0.24641
0.4	−0.554	0.24387	−0.6606	0.25682	−0.71692	0.24026	−0.75958	0.20948	−0.84994	0.17255
0.5	−0.21036	0.11137	−0.35326	0.19352	−0.45884	0.24473	−0.54089	0.27144	−0.60576	0.28106
0.6	−0.0	0.0	−0.20292	0.1606	−0.34652	0.26673	−0.46836	0.34257	−0.568	0.38583
0.7	−0.0741	0.28913	−0.0	0.0	−0.21727	0.05413	−0.38315	0.10043	−0.51204	0.13708
0.8	−0.12523	0.45694	−0.04726	0.21292	−0.0	0.0	−0.18678	0.03256	−0.34621	0.05582
0.9	−0.33829	0.75885	−0.19917	0.43591	−0.08753	0.18751	−0.0	0.0	−0.14186	0.06713
1.0	−0.39577	0.76856	−0.25057	0.57006	−0.13962	0.38601	−0.05787	0.18987	−0.0	0.0

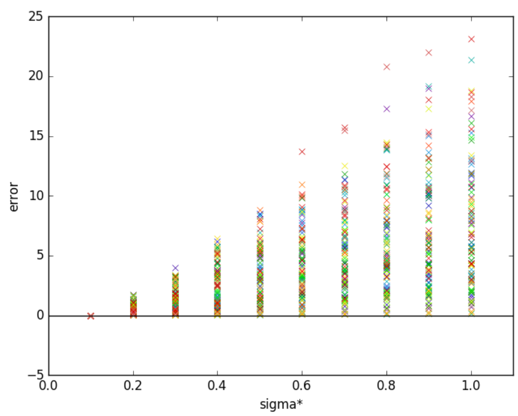
Table 4: Minimum and maximum hedged errors for a Digital put option

Appendix C

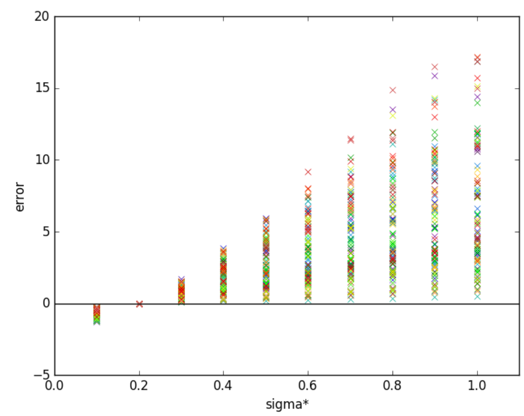
Plots and Risk Measures for σ

C.1 Plots for σ

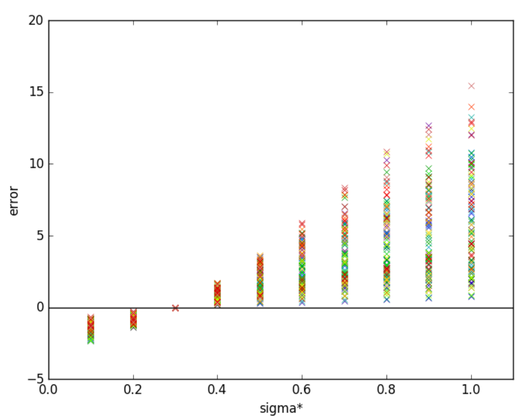
C.1.1 Plots for European call option



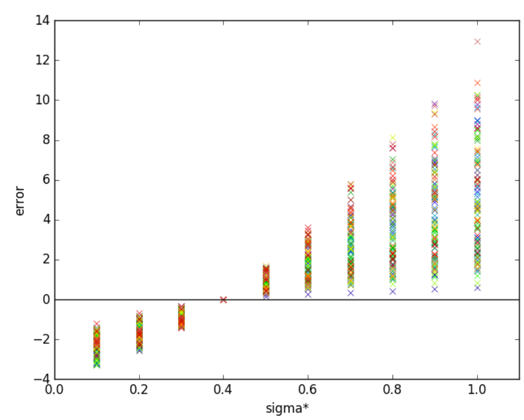
(a) $\sigma = 0.1$



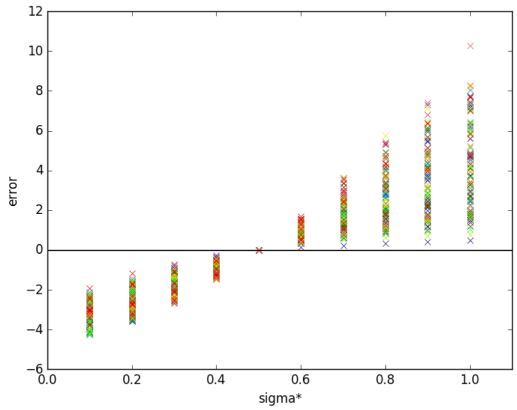
(b) $\sigma = 0.2$



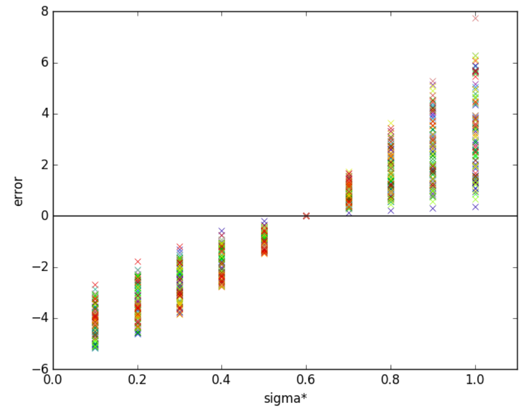
(c) $\sigma = 0.3$



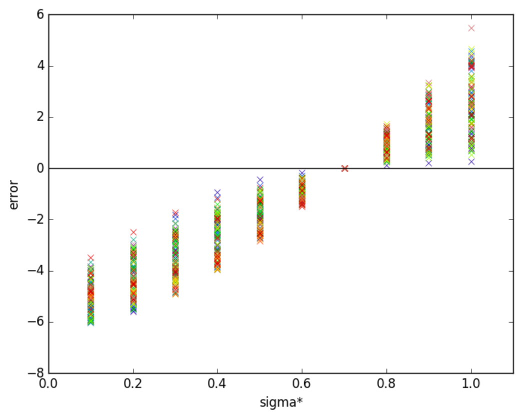
(d) $\sigma = 0.4$



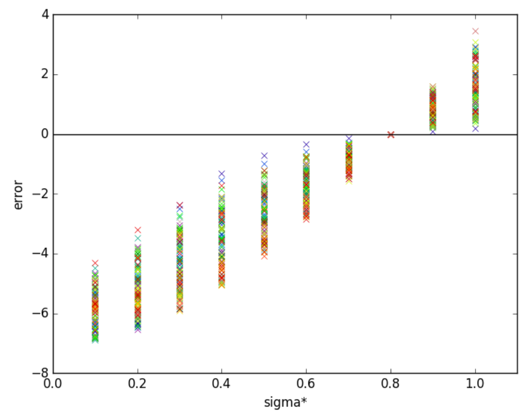
(e) $\sigma = 0.5$



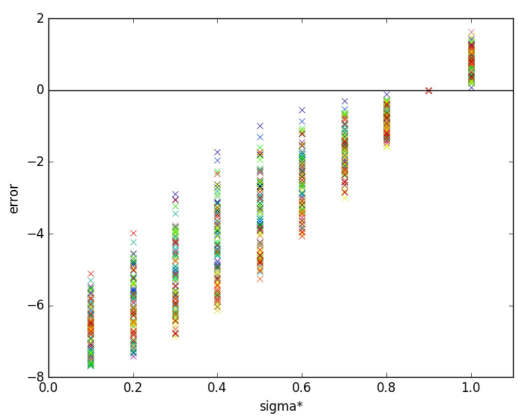
(f) $\sigma = 0.6$



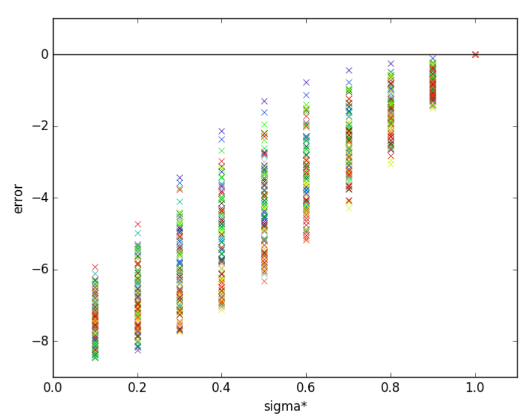
(g) $\sigma = 0.7$



(h) $\sigma = 0.8$



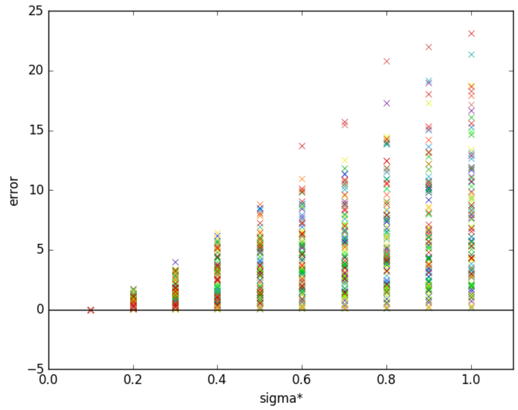
(i) $\sigma = 0.9$



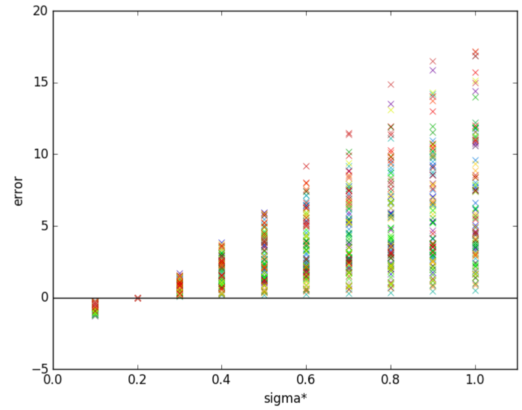
(j) $\sigma = 1.0$

Figure 24: Hedged portfolio errors per σ_* for a European call option with different σ

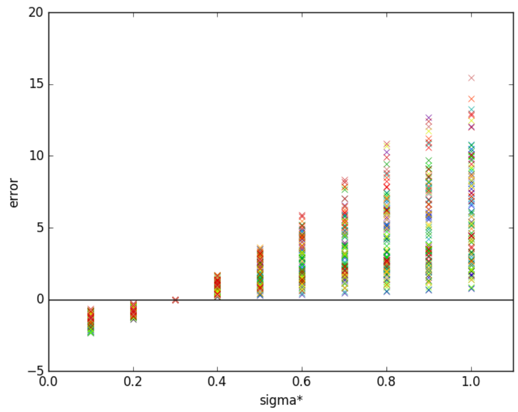
C.1.2 Plots for European put option



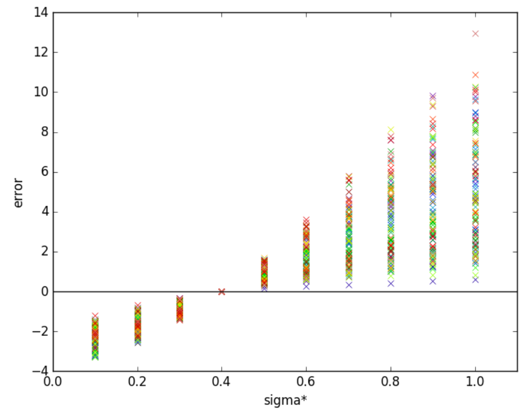
(a) $\sigma = 0.1$



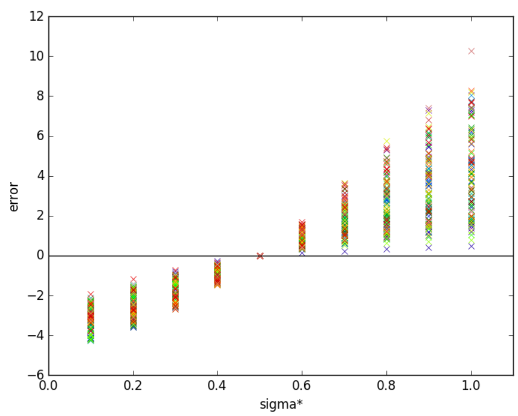
(b) $\sigma = 0.2$



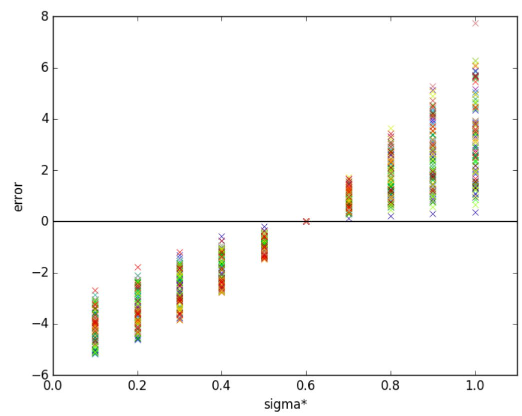
(c) $\sigma = 0.3$



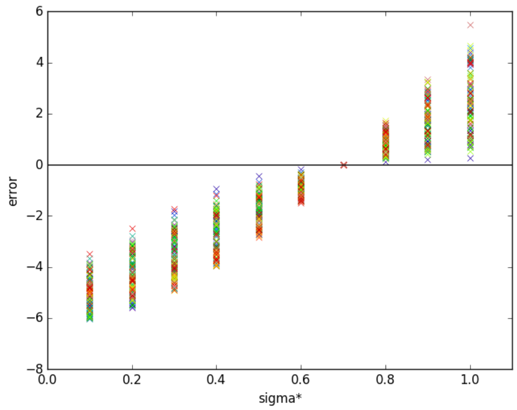
(d) $\sigma = 0.4$



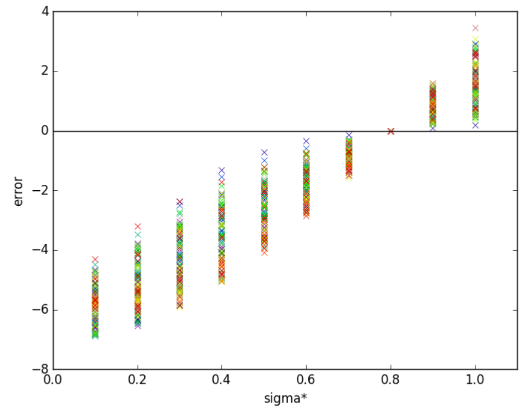
(e) $\sigma = 0.5$



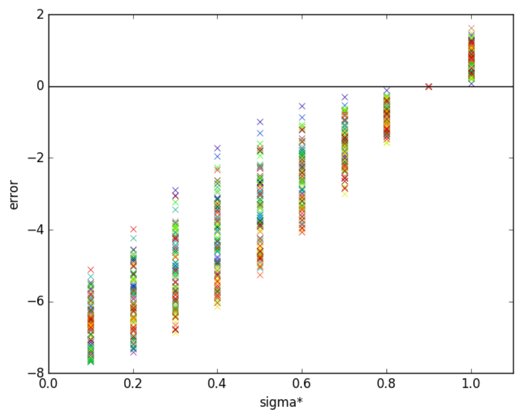
(f) $\sigma = 0.6$



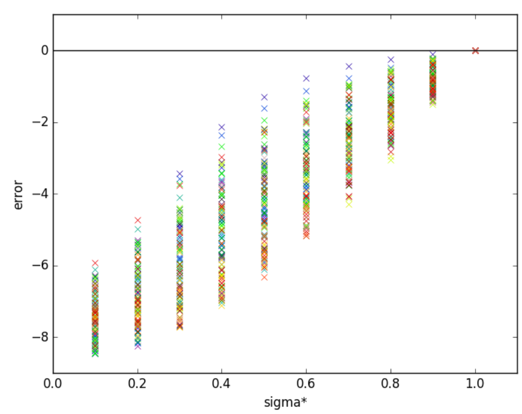
(g) $\sigma = 0.7$



(h) $\sigma = 0.8$



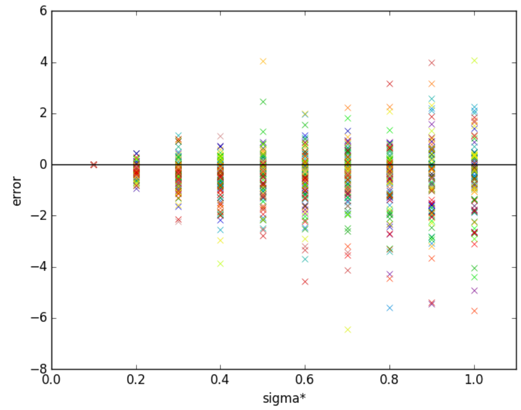
(i) $\sigma = 0.9$



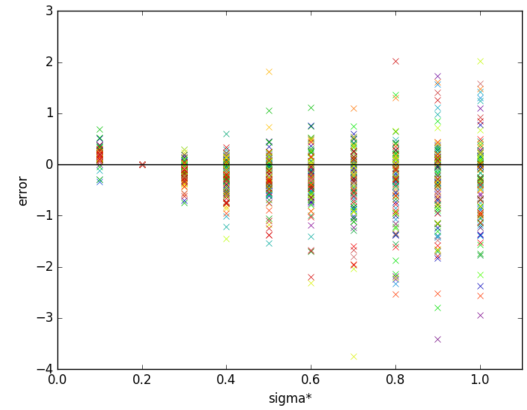
(j) $\sigma = 1.0$

Figure 25: Hedged portfolio errors per σ_* for a European put option with different σ

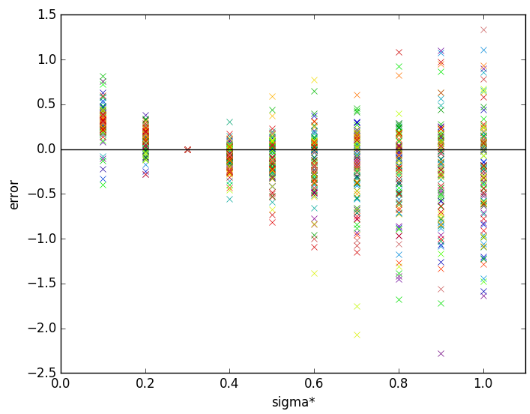
C.1.3 Plots for digital call option



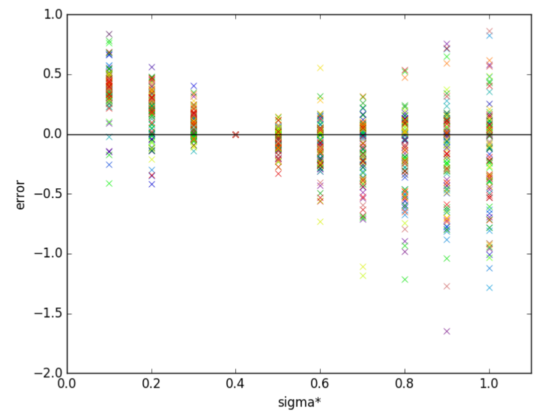
(a) $\sigma = 0.1$



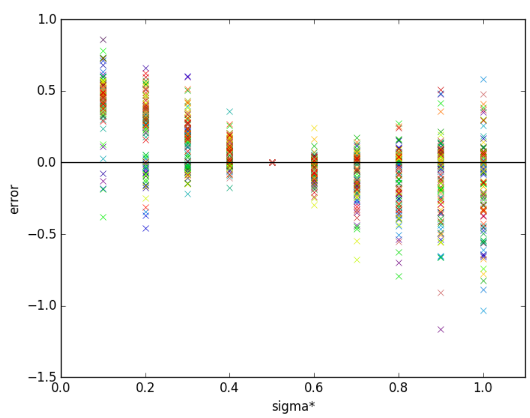
(b) $\sigma = 0.2$



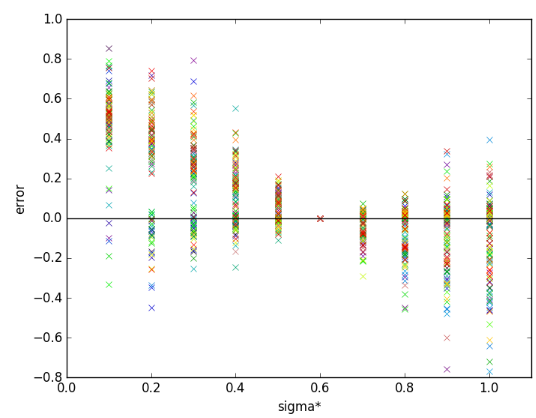
(c) $\sigma = 0.3$



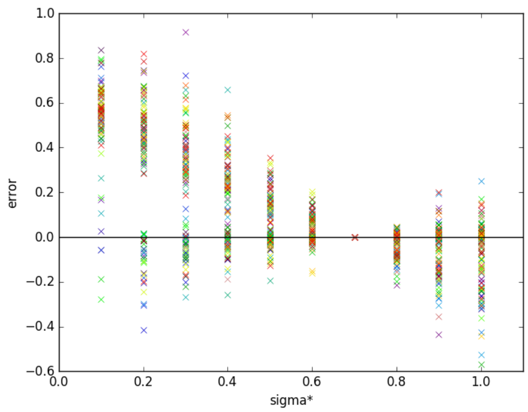
(d) $\sigma = 0.4$



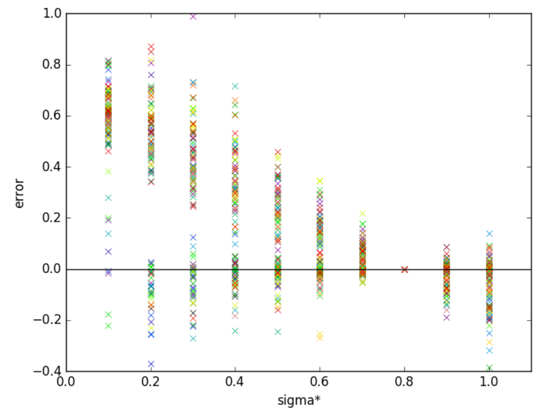
(e) $\sigma = 0.5$



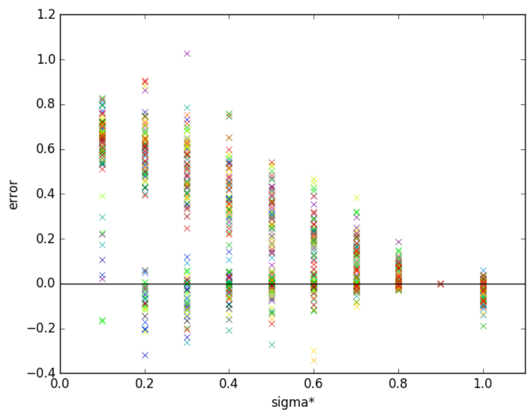
(f) $\sigma = 0.6$



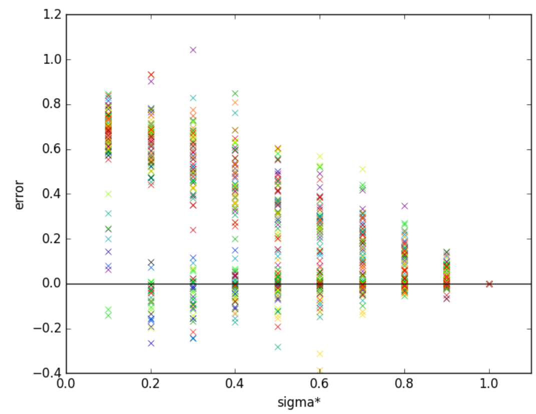
(g) $\sigma = 0.7$



(h) $\sigma = 0.8$



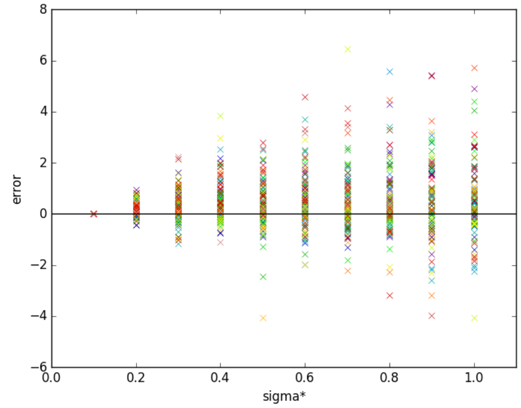
(i) $\sigma = 0.9$



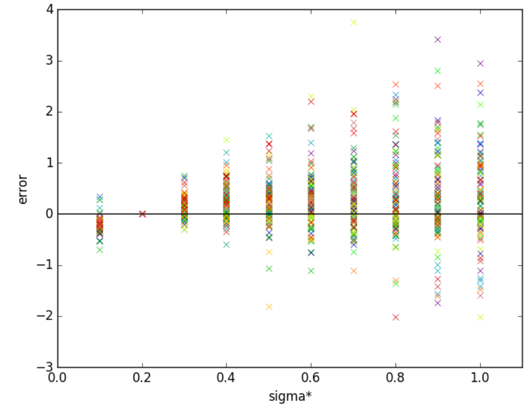
(j) $\sigma = 1.0$

Figure 26: Hedged portfolio errors per σ_* for a digital call option with different σ

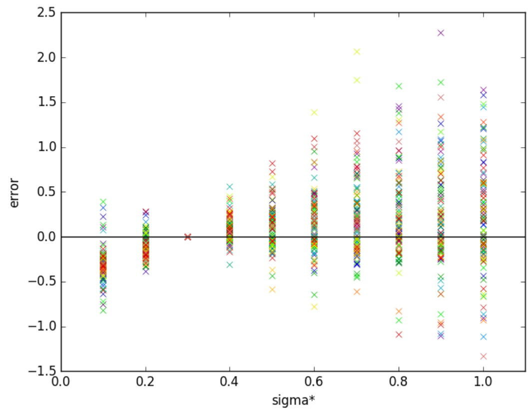
C.1.4 Plots for digital put option



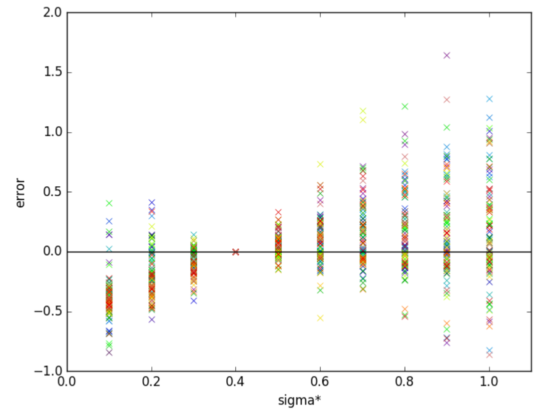
(a) $\sigma = 0.1$



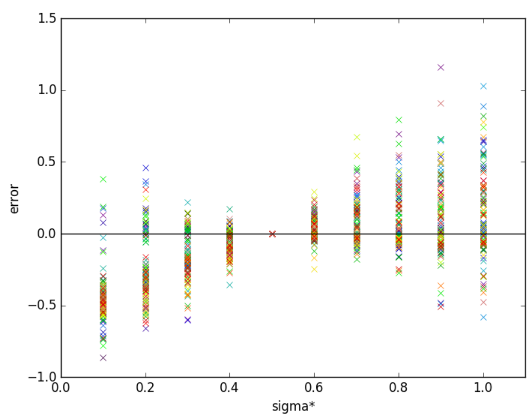
(b) $\sigma = 0.2$



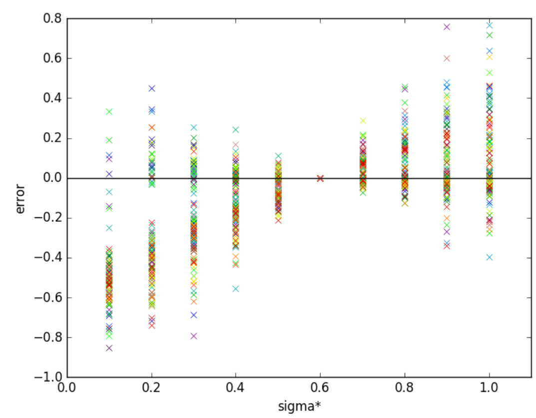
(c) $\sigma = 0.3$



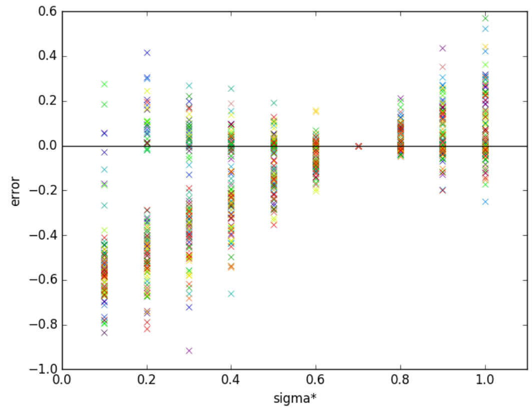
(d) $\sigma = 0.4$



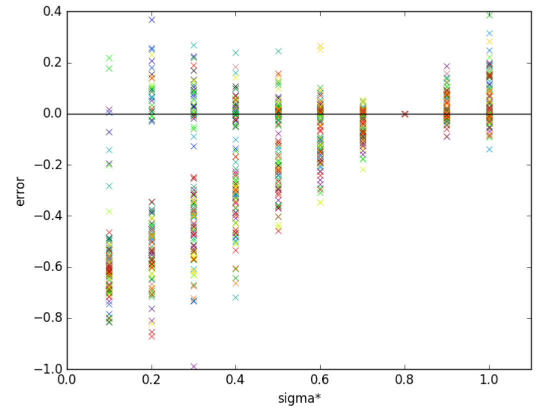
(e) $\sigma = 0.5$



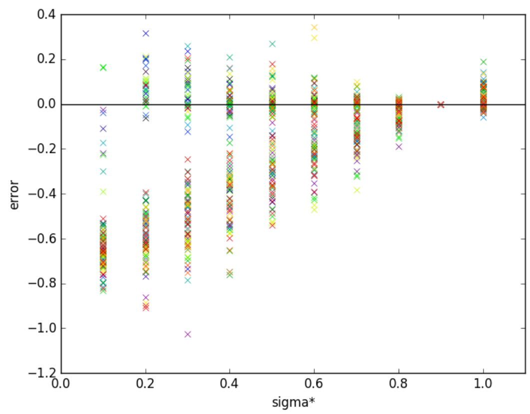
(f) $\sigma = 0.6$



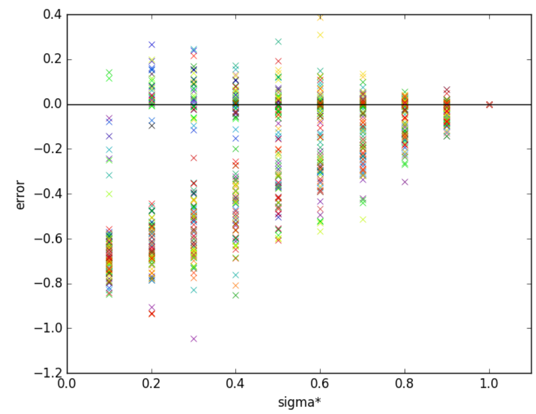
(g) $\sigma = 0.7$



(h) $\sigma = 0.8$



(i) $\sigma = 0.9$



(j) $\sigma = 1.0$

Figure 27: Hedged portfolio errors per σ_* for a digital put option with different σ

C.2 Risk Measures for all $\boldsymbol{\sigma}$

C.2.1 Value at Risk for all $\boldsymbol{\sigma}$

$\sigma = 0.1$

European call option: $VaR_{0.95} = 11.92$ European put option: $VaR_{0.95} = 11.92$

digital call option: $VaR_{0.95} = 0.92$ digital put option: $VaR_{0.95} = 2.33$

$\sigma = 0.2$

European call option: $VaR_{0.95} = 10.61$ European put option: $VaR_{0.95} = 10.61$

digital call option: $VaR_{0.95} = 0.46$ digital put option: $VaR_{0.95} = 1.36$

$\sigma = 0.3$

European call option: $VaR_{0.95} = 8.54$ European put option: $VaR_{0.95} = 8.54$

digital call option: $VaR_{0.95} = 0.4$ digital put option: $VaR_{0.95} = 0.87$

$\sigma = 0.4$

European call option: $VaR_{0.95} = 6.85$ European put option: $VaR_{0.95} = 6.85$

digital call option: $VaR_{0.95} = 0.44$ digital put option: $VaR_{0.95} = 0.62$

$\sigma = 0.5$

European call option: $VaR_{0.95} = 5.21$ European put option: $VaR_{0.95} = 5.21$

digital call option: $VaR_{0.95} = 0.5$ digital put option: $VaR_{0.95} = 0.4$

$\sigma = 0.6$

European call option: $VaR_{0.95} = 3.82$ European put option: $VaR_{0.95} = 3.82$

digital call option: $VaR_{0.95} = 0.56$ digital put option: $VaR_{0.95} = 0.27$

$\sigma = 0.7$

European call option: $VaR_{0.95} = 2.7$ European put option: $VaR_{0.95} = 2.7$

digital call option: $VaR_{0.95} = 0.61$ digital put option: $VaR_{0.95} = 0.18$

$\sigma = 0.8$

European call option: $VaR_{0.95} = 1.55$ European put option: $VaR_{0.95} = 1.55$

digital call option: $VaR_{0.95} = 0.65$ digital put option: $VaR_{0.95} = 0.13$

$\sigma = 0.9$

European call option: $VaR_{0.95} = 0.74$ European put option: $VaR_{0.95} = 0.74$

digital call option: $VaR_{0.95} = 0.68$ digital put option: $VaR_{0.95} = 0.09$

$\sigma = 1.0$

European call option: $VaR_{0.95} = -0.01$ European put option: $VaR_{0.95} = -0.01$

digital call option: $VaR_{0.95} = 0.71$ digital put option: $VaR_{0.95} = 0.07$

C.2.2 Conditional Value at Risk for all σ

$\sigma = 0.1$

European call option: $CVaR_{0.95} = 15.46$ European put option: $CVaR_{0.95} = 15.46$
digital call option: $CVaR_{0.95} = 1.753$ digital put option: $CVaR_{0.95} = 3.401$

$\sigma = 0.2$

European call option: $CVaR_{0.95} = 12.83$ European put option: $CVaR_{0.95} = 12.83$
digital call option: $CVaR_{0.95} = 0.947$ digital put option: $CVaR_{0.95} = 1.902$

$\sigma = 0.3$

European call option: $CVaR_{0.95} = 10.474$ European put option: $CVaR_{0.95} = 10.474$
digital call option: $CVaR_{0.95} = 0.667$ digital put option: $CVaR_{0.95} = 1.218$

$\sigma = 0.4$

European call option: $CVaR_{0.95} = 8.426$ European put option: $CVaR_{0.95} = 8.426$
digital call option: $CVaR_{0.95} = 0.565$ digital put option: $CVaR_{0.95} = 0.84$

$\sigma = 0.5$

European call option: $CVaR_{0.95} = 6.626$ European put option: $CVaR_{0.95} = 6.626$
digital call option: $CVaR_{0.95} = 0.578$ digital put option: $CVaR_{0.95} = 0.587$

$\sigma = 0.6$

European call option: $CVaR_{0.95} = 4.987$ European put option: $CVaR_{0.95} = 4.987$
digital call option: $CVaR_{0.95} = 0.634$ digital put option: $CVaR_{0.95} = 0.403$

$\sigma = 0.7$

European call option: $CVaR_{0.95} = 3.513$ European put option: $CVaR_{0.95} = 3.513$
digital call option: $CVaR_{0.95} = 0.68$ digital put option: $CVaR_{0.95} = 0.272$

$\sigma = 0.8$

European call option: $CVaR_{0.95} = 2.196$ European put option: $CVaR_{0.95} = 2.196$
digital call option: $CVaR_{0.95} = 0.717$ digital put option: $CVaR_{0.95} = 0.191$

$\sigma = 0.9$

European call option: $CVaR_{0.95} = 1.063$ European put option: $CVaR_{0.95} = 1.063$
digital call option: $CVaR_{0.95} = 0.75$ digital put option: $CVaR_{0.95} = 0.15$

$\sigma = 1.0$

European call option: $CVaR_{0.95} = -0.01$ European put option: $CVaR_{0.95} = -0.01$
digital call option: $CVaR_{0.95} = 0.777$ digital put option: $CVaR_{0.95} = 0.141$

C.2.3 Upper Partial Moments with $m = 0$ for all σ

$\sigma = 0.1$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.2$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.998$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.3$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.4$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.5$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.6$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.7$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.8$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 0.9$

European call option: $UPM_0 = 0.999$ European put option: $UPM_0 = 0.999$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

$\sigma = 1.0$

European call option: $UPM_0 = 0.9$ European put option: $UPM_0 = 0.9$
digital call option: $UPM_0 = 0.999$ digital put option: $UPM_0 = 0.999$

C.2.4 Upper Partial Moments with $m = 1$ for all σ

$\sigma = 0.1$

European call option: $UPM_1 = 3.73277$ European put option: $UPM_1 = 3.73277$
digital call option: $UPM_1 = 0.15507$ digital put option: $UPM_1 = 0.56593$

$\sigma = 0.2$

European call option: $UPM_1 = 3.18296$ European put option: $UPM_1 = 3.18296$
digital call option: $UPM_1 = 0.0972$ digital put option: $UPM_1 = 0.30466$

$\sigma = 0.3$

European call option: $UPM_1 = 2.48589$ European put option: $UPM_1 = 2.48589$
digital call option: $UPM_1 = 0.09226$ digital put option: $UPM_1 = 0.18028$

$\sigma = 0.4$

European call option: $UPM_1 = 1.84293$ European put option: $UPM_1 = 1.84293$
digital call option: $UPM_1 = 0.09791$ digital put option: $UPM_1 = 0.11359$

$\sigma = 0.5$

European call option: $UPM_1 = 1.29313$ European put option: $UPM_1 = 1.29313$
digital call option: $UPM_1 = 0.11224$ digital put option: $UPM_1 = 0.0745$

$\sigma = 0.6$

European call option: $UPM_1 = 0.84354$ European put option: $UPM_1 = 0.84354$
digital call option: $UPM_1 = 0.13104$ digital put option: $UPM_1 = 0.04872$

$\sigma = 0.7$

European call option: $UPM_1 = 0.493$ European put option: $UPM_1 = 0.493$
digital call option: $UPM_1 = 0.15322$ digital put option: $UPM_1 = 0.03156$

$\sigma = 0.8$

European call option: $UPM_1 = 0.23898$ European put option: $UPM_1 = 0.23898$
digital call option: $UPM_1 = 0.17798$ digital put option: $UPM_1 = 0.02068$

$\sigma = 0.9$

European call option: $UPM_1 = 0.07666$ European put option: $UPM_1 = 0.07666$
digital call option: $UPM_1 = 0.20414$ digital put option: $UPM_1 = 0.0144$

$\sigma = 1.0$

European call option: $UPM_1 = 0.0$ European put option: $UPM_1 = 0.0$
digital call option: $UPM_1 = 0.23135$ digital put option: $UPM_1 = 0.01146$

C.2.5 Upper Partial Moments with $m = 2$ for all σ

$\sigma = 0.1$

European call option: $UPM_2 = 30.235$ European put option: $UPM_2 = 30.235$
digital call option: $UPM_2 = 0.21$ digital put option: $UPM_2 = 1.062$

$\sigma = 0.2$

European call option: $UPM_2 = 21.887$ European put option: $UPM_2 = 21.887$
digital call option: $UPM_2 = 0.061$ digital put option: $UPM_2 = 0.326$

$\sigma = 0.3$

European call option: $UPM_2 = 14.268$ European put option: $UPM_2 = 14.268$
digital call option: $UPM_2 = 0.037$ digital put option: $UPM_2 = 0.13$

$\sigma = 0.4$

European call option: $UPM_2 = 8.752$ European put option: $UPM_2 = 8.752$
digital call option: $UPM_2 = 0.035$ digital put option: $UPM_2 = 0.059$

$\sigma = 0.5$

European call option: $UPM_2 = 4.997$ European put option: $UPM_2 = 4.997$
digital call option: $UPM_2 = 0.043$ digital put option: $UPM_2 = 0.028$

$\sigma = 0.6$

European call option: $UPM_2 = 2.575$ European put option: $UPM_2 = 2.575$
digital call option: $UPM_2 = 0.056$ digital put option: $UPM_2 = 0.013$

$\sigma = 0.7$

European call option: $UPM_2 = 1.133$ European put option: $UPM_2 = 1.133$
digital call option: $UPM_2 = 0.072$ digital put option: $UPM_2 = 0.006$

$\sigma = 0.8$

European call option: $UPM_2 = 0.38$ European put option: $UPM_2 = 0.38$
digital call option: $UPM_2 = 0.089$ digital put option: $UPM_2 = 0.003$

$\sigma = 0.9$

European call option: $UPM_2 = 0.071$ European put option: $UPM_2 = 0.071$
digital call option: $UPM_2 = 0.107$ digital put option: $UPM_2 = 0.002$

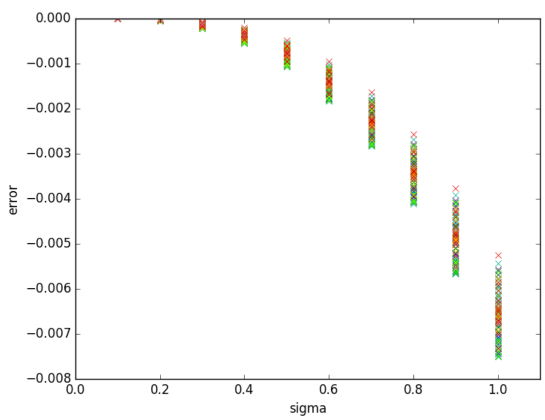
$\sigma = 1.0$

European call option: $UPM_2 = 0.0$ European put option: $UPM_2 = 0.0$
digital call option: $UPM_2 = 0.127$ digital put option: $UPM_2 = 0.001$

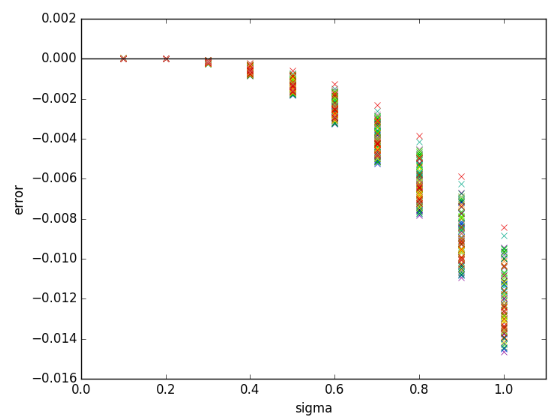
Appendix D

Plots of Weighted Errors

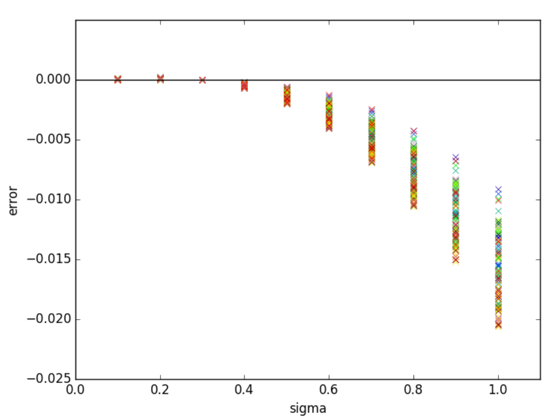
D.1 European Put Option



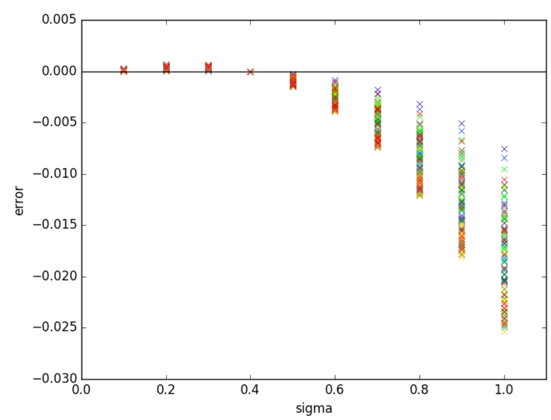
(a) $\sigma_* = 0.1$



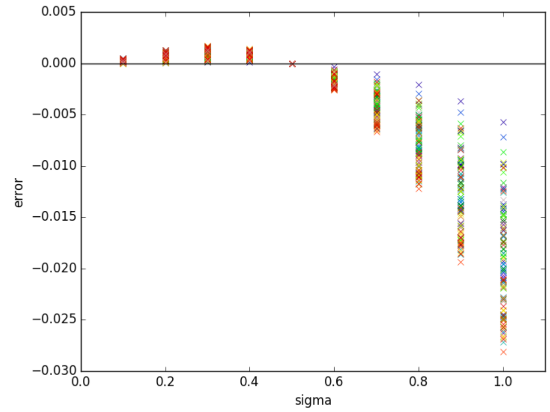
(b) $\sigma_* = 0.2$



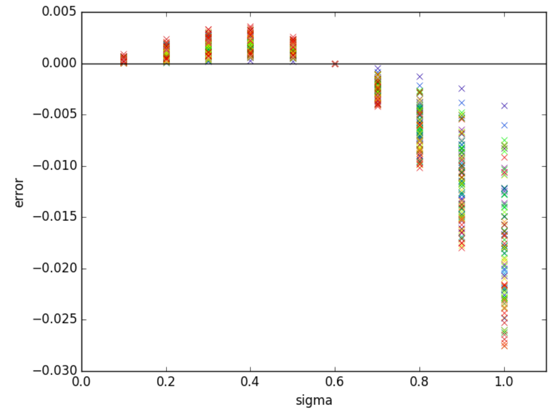
(c) $\sigma_* = 0.3$



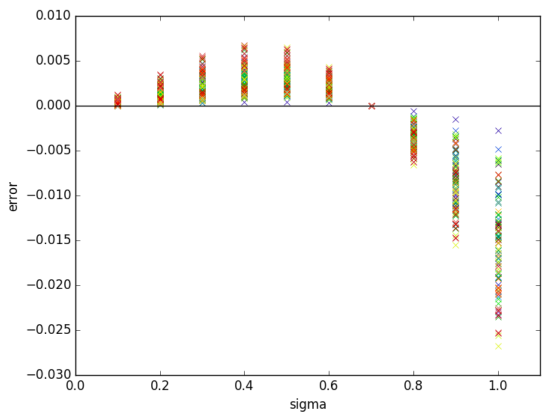
(d) $\sigma_* = 0.4$



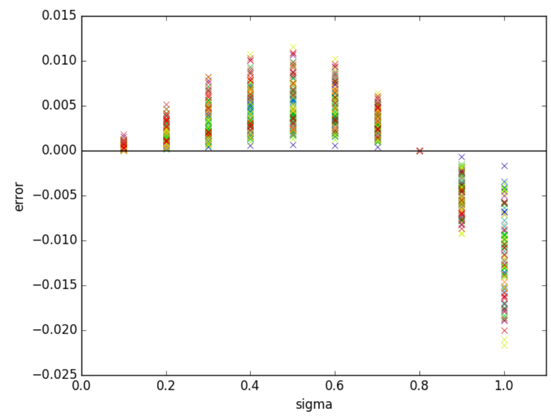
(e) $\sigma_* = 0.5$



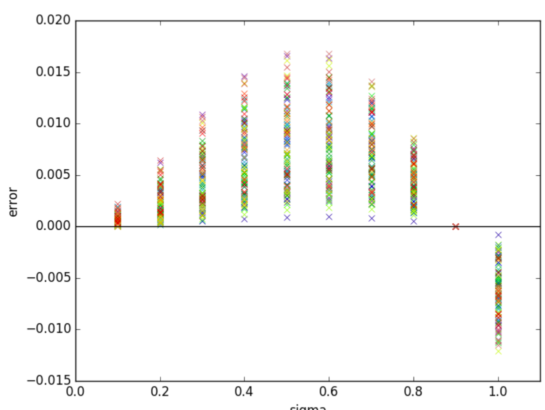
(f) $\sigma_* = 0.6$



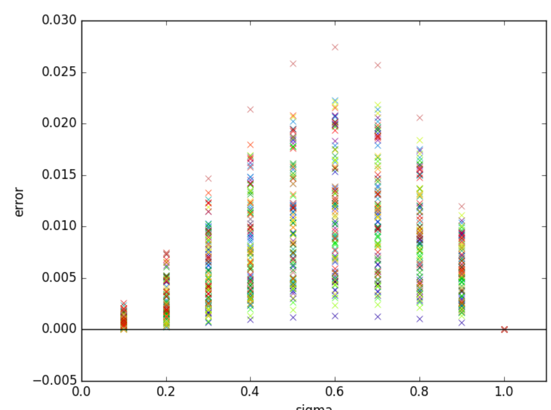
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$

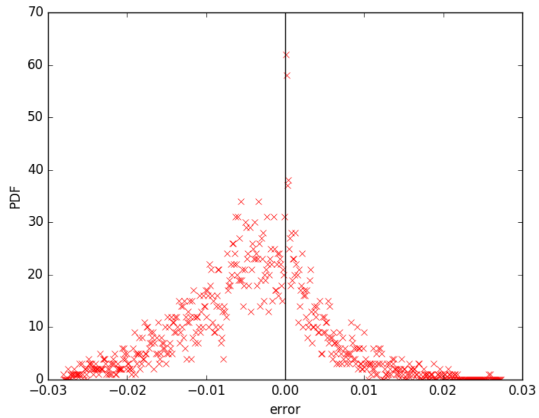


(i) $\sigma_* = 0.9$

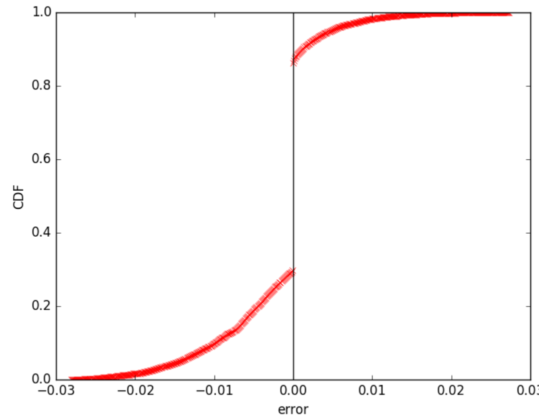


(j) $\sigma_* = 1.0$

Figure 28: Weighted errors per σ for a European put option with different σ_*



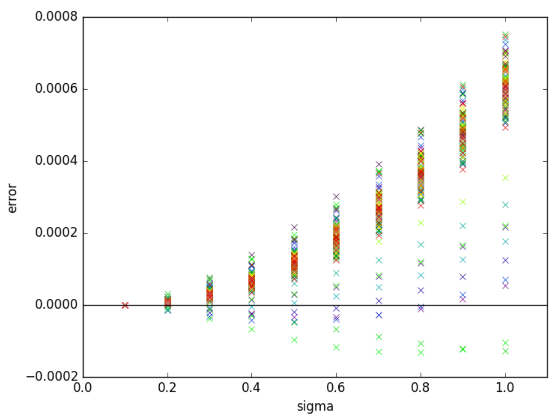
(a) Approximated probability density function



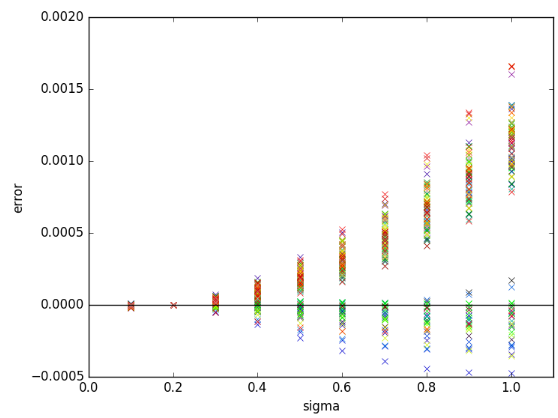
(b) Approximated cumulative distribution function

Figure 29: Approximated probability density and cumulative distribution function for weighted errors of a European put option

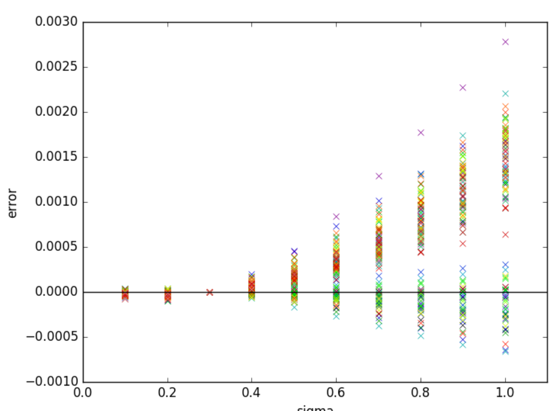
D.2 Digital Call Option



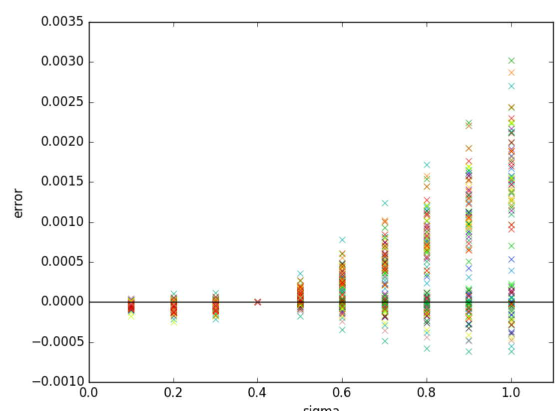
(a) $\sigma_* = 0.1$



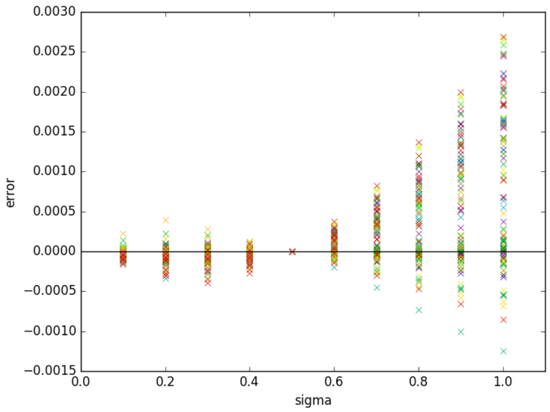
(b) $\sigma_* = 0.2$



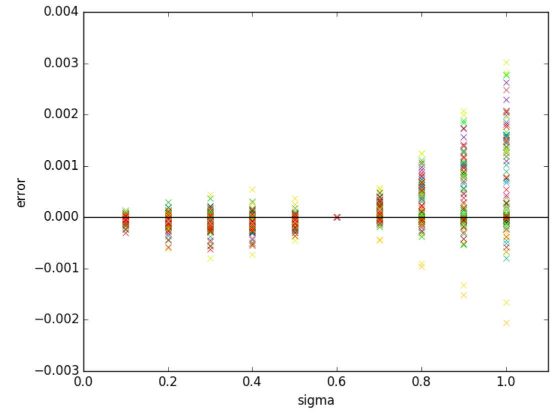
(c) $\sigma_* = 0.3$



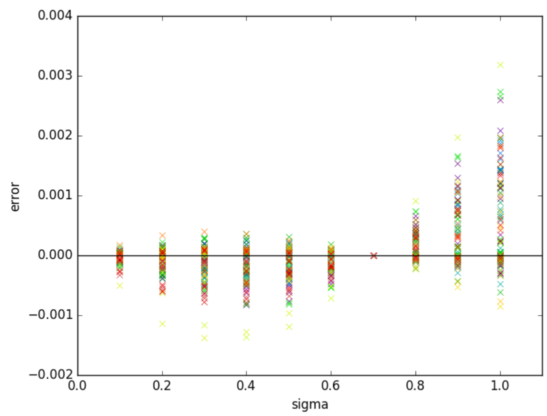
(d) $\sigma_* = 0.4$



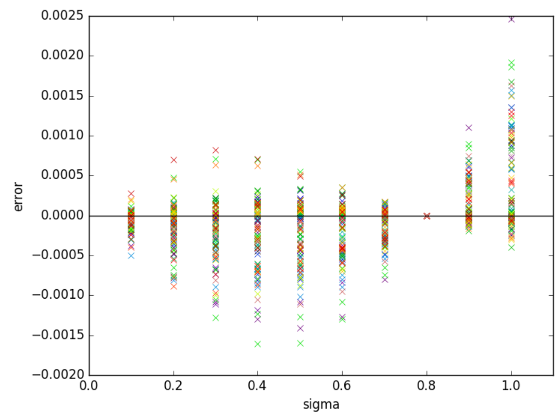
(e) $\sigma_* = 0.5$



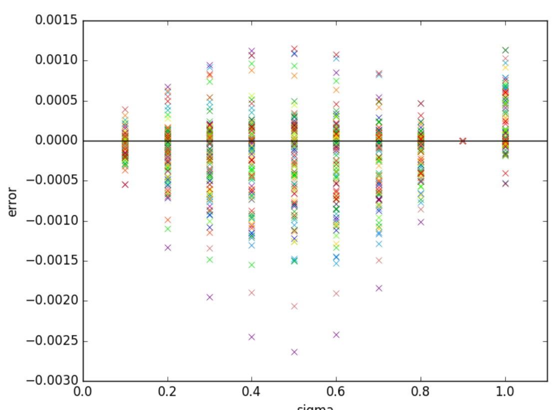
(f) $\sigma_* = 0.6$



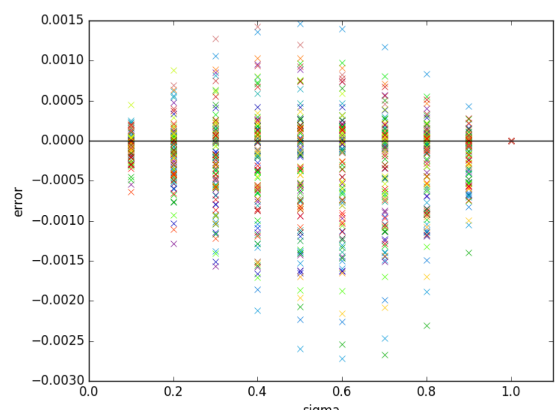
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$

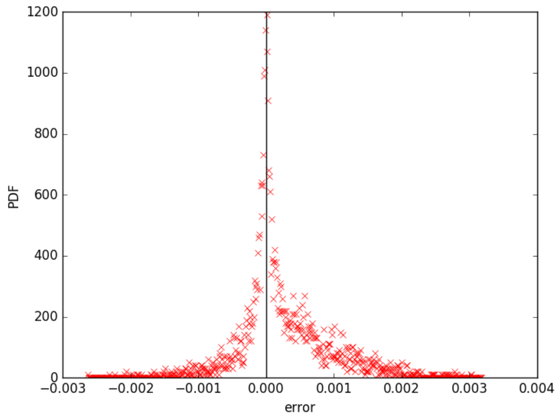


(i) $\sigma_* = 0.9$

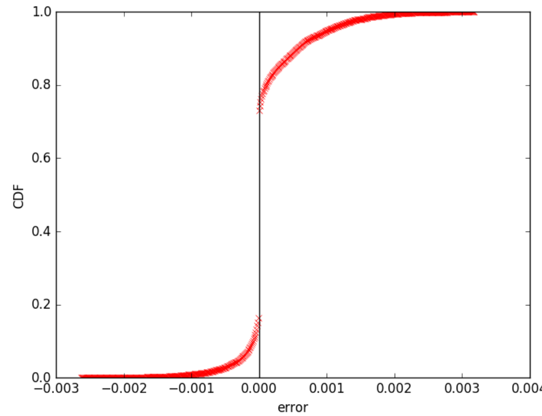


(j) $\sigma_* = 1.0$

Figure 30: Weighted errors per σ for a digital call option with different σ_*



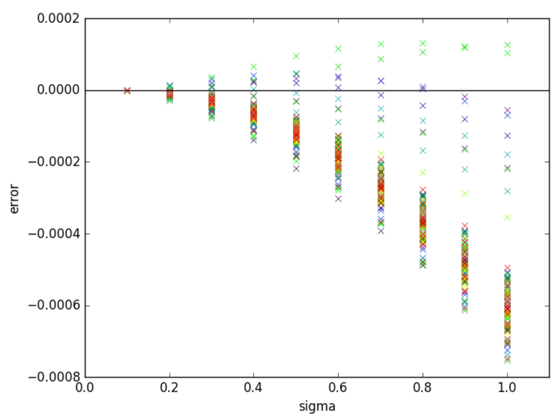
(a) Approximated probability density function



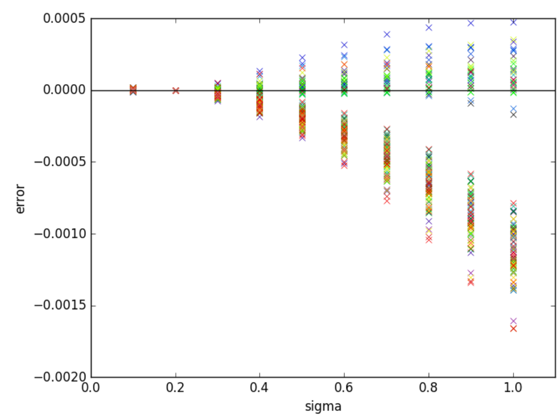
(b) Approximated cumulative distribution function

Figure 31: Approximated probability density and cumulative distribution function for weighted errors of a digital call option

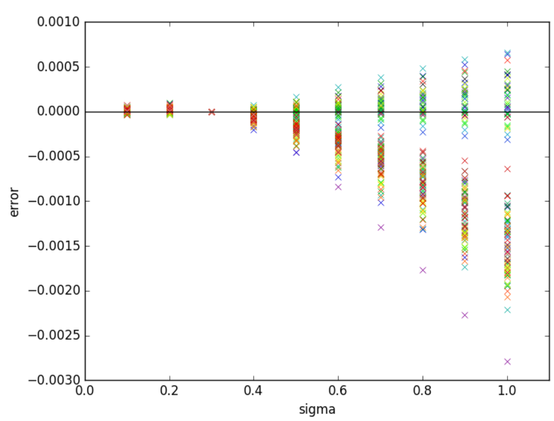
D.3 Digital Put Option



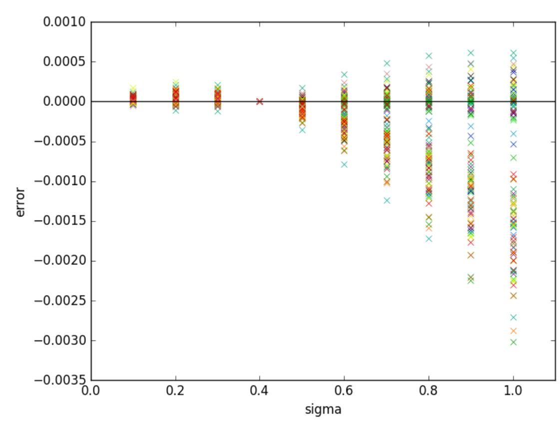
(a) $\sigma_* = 0.1$



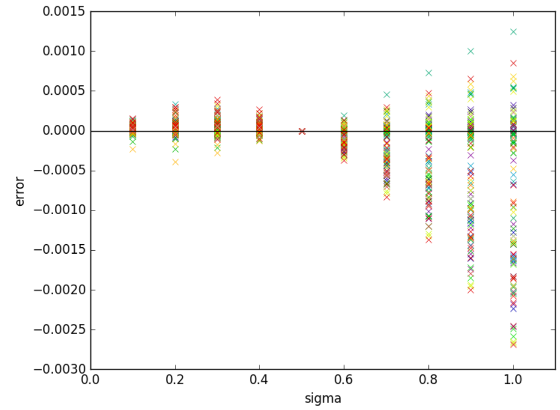
(b) $\sigma_* = 0.2$



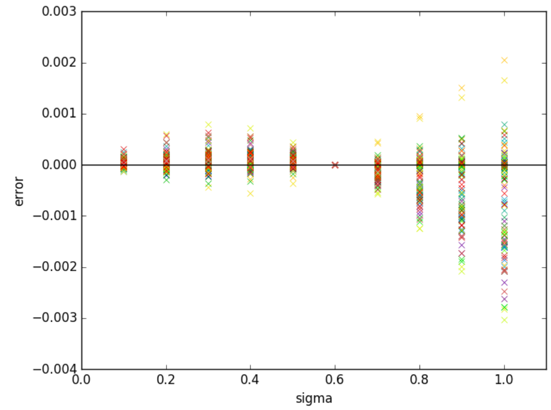
(c) $\sigma_* = 0.3$



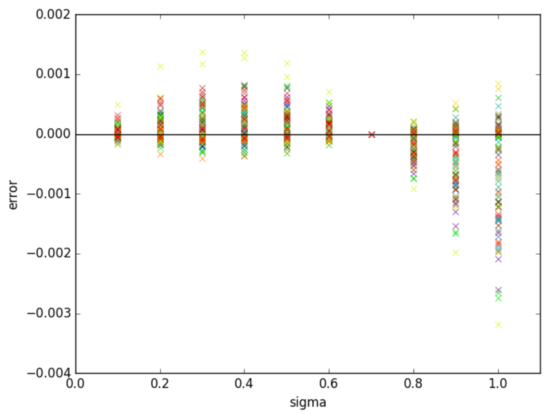
(d) $\sigma_* = 0.4$



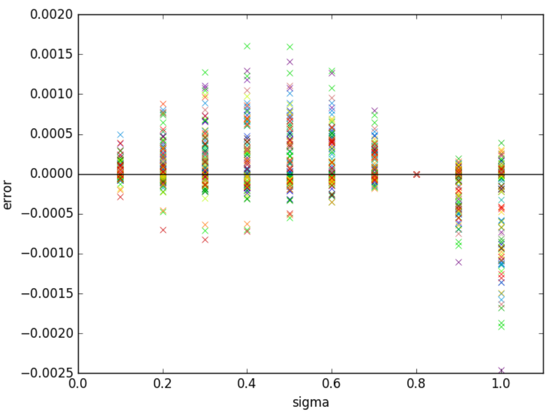
(e) $\sigma_* = 0.5$



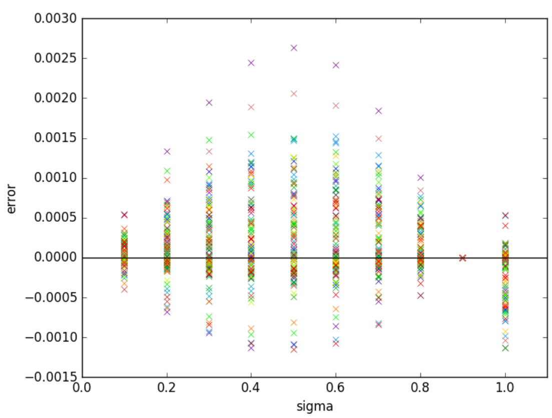
(f) $\sigma_* = 0.6$



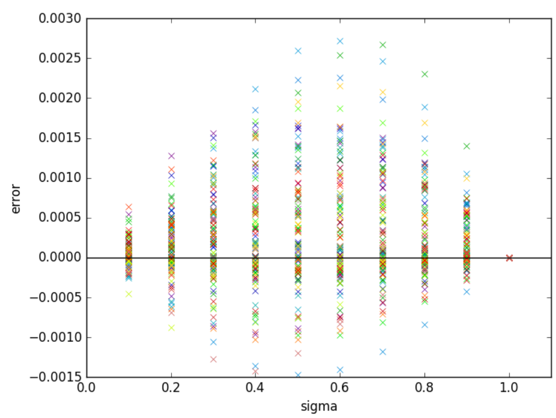
(g) $\sigma_* = 0.7$



(h) $\sigma_* = 0.8$

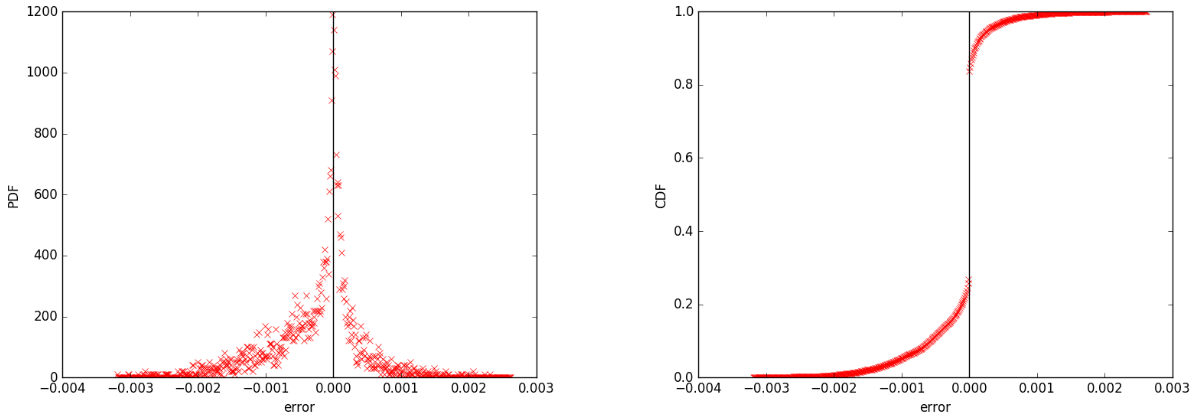


(i) $\sigma_* = 0.9$



(j) $\sigma_* = 1.0$

Figure 32: Weighted errors per σ for a digital put option with different σ_*



(a) Approximated probability density function **(b)** Approximated cumulative distribution function

Figure 33: Approximated probability density and cumulative distribution function for weighted errors of a digital put option