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Strategic Inventories in a Supply Chain with Vertical Control and Downstream Cournot Competition

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STRATEGIC INVENTORIES IN A SUPPLY CHAIN WITH VERTICAL CONTROL AND DOWNSTREAM COURNOT COMPETITION

by

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ABSTRACT

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Strategic Inventory (SI) has been an area of increased interest in theoretical supply chain literature recently. Most of the work so far however, has only considered a supply chain without downstream competition between retailers. Competition is ubiquitous in most market situations, hence, interactions between SI and retailer competition merits study as a first step in bringing the conversations and insights from this stream of literature to the real world.

We present here a two-period and a three-period model of one manufacturer supplying an identical product to two retailers who form a Cournot duopoly. We also study a Commitment contract, where the manufacturer commits to all the selling seasons' wholesale prices at the beginning of the 1st period. Commitment contracts have been shown to eliminate SI carriage over two selling seasons in the absence of retailer competition (Anand et al. (2008)). We aim to deduce if this type of contract has the same effect in the presence of downstream competition. We determine closed-form Nash Equilibrium decision variable values for each of these models using game-theoretic modeling, a price-dependent linear demand function, and backward induction.

We find that, the introduction of downstream Cournot duopoly competition leads to

lower profits for both the manufacturer and retailer. This holds, whether the number of selling season is two or three. Consumer Surplus is also uniformly lower under retailer competition, compared to a downstream monopoly supply chain.

When we try to deduce the effect of SI carriage under Cournot duopoly competition, by comparing an SC with Cournot duopoly competition and SI allowed between periods, to a similar SC with a Cournot duopoly downstream and a static, repeating, one-shot game in each period, with no SI carried – we find again that manufacturer and retailer profits are both lower when SI carriage is allowed. This holds whether the number of selling seasons is two or three. Consumer Surplus is also lower uniformly over both two and three selling seasons.

Under a Commitment contract, over two selling seasons, the manufacturer ends up with an advantage, making a higher profit with downstream retailer competition, than compared to supplying to a monopoly downstream under the same contract. The retailers, while competing as a Cournot duopoly, are not able to use the relative advantage that comes from a Commitment contract to make a higher profit, as they are, when the downstream is a single retailer monopoly. The consumer also is disadvantaged by the introduction of downstream Cournot competition under a Commitment contract.

When we compare a manufacturer supplying to a Cournot duopoly downstream of retailers, with, and without a Commitment contract (dynamic ordering), we see that the manufacturer and consumer benefit under a Commitment contract, making higher profits, but the retailer is at a disadvantage.

It would be an interesting extension of this work to generalize the results from two and three selling seasons, presented here, to the "n" period case. It would also be beneficial to run empirical studies in real-world supply chains to validate if and to what extent the insights developed by this kind of game-theoretic modeling hold in a real-world supply chain setting. Development of contracts that are more effective than a Commitment contract in coordinating this supply chain would be another possible area for further research.

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1. INTRODUCTION AND MOTIVATION

Researchers have proposed various definitions of supply chains over the years. Two examples are:

- *A supply chain is a set of firms that passes materials forward* (Lalonde and Masters, 1994)

- *Several independent firms are involved in manufacturing a product and placing it in the hands of the end-user in a supply chain— raw material and component producers, product assemblers, wholesalers, retailer merchants and transportation companies, are all members of a supply chain* (Lambert et al., 1998).

Inventory holding in supply chains is inevitable. Inventory helps smooth production and demand uncertainty, transportation delays and price fluctuations. On the contrary, inventory incurs holding costs (warehouse rental, administrative, refrigeration etc.), usually expressed either as a per-unit cost or a fixed cost plus a per-unit cost, and hence, larger the inventory, larger the holding cost. Due to these two competing effects of inventory, determining the optimal amount of inventory to carry in a supply chain has been a topic of active interest in supply chain research. A review of the relevant research in this area, both from a single-decision maker optimization as well as game-theoretic (multiple independent decision makers) approaches are summarized in sections 2.1 and 2.2 of this document respectively.

Economic competition is another important research area in business and engineering literature since the 1800s. Seldom does one firm dominate a market and hence, competition is an inevitable artifact of any free-market economic system. One of the most basic and widely used economic competition models – the Cournot model dates back to the 18th century French philosopher and mathematician, Antoine Augustin Cournot (1801-1887) and is still used widely to model economic competition in various kinds of models, due to its versatility and simplicity. Research on economic competition has typically focused on expressing competitive scenarios found in the real world in terms of an adequate model, either one of the standard competition models like Bertrand, Cournot and Stackelberg, or a hybrid model that represents the real-world situation best. Research in this area has focused on the interplay of control and incentives – situations where one of the competing entities is in a position of strength compared to the others and then the design of appropriate incentives to level the playing field – coordination, or the analysis of different competitive scenarios – entry of a new firm into an established market, mergers and acquisitions et al. A review of the standard economic competition models is presented in section 2.3 of this document, and a review of the literature on inventory as a strategic weapon is presented in section 2.4.

The focus of this work is the effect of competition on the retailers' tendencies to carry strategic inventory into the subsequent period. Strategic inventories are inventories carried by supply chain entities for purely strategic reasons, even in the absence of “traditional” reasons to hold inventory. Traditional reasons for holding inventory at a supply chain entity (e.g. Manufacturer, retailer, and distributor) have been economies of scale in production, resulting in cycle inventories; to hedge against production or distribution delays and ensure timely availability of goods, resulting in pipeline inventories; inventories held as safety stock, to hedge against demand and supply uncertainty; inventories held to hedge against price fluctuations, termed speculative inventory (Anand et al., 2008); and inventories held to smooth production and thus lower production costs (Holt et al., 1960) . Vertical control refers to an upstream supply chain entity controlling one of the operating parameters of a downstream entity like price or inventory. In this thesis, vertical control always implies price control, where an upstream manufacturer can set a different wholesale price in each period, while supplying to two competing downstream retailers, with multi-period ordering. Recent literature in the area of strategic inventories in a supply chain with vertical control is reviewed in section 2.5 of this document.

In Chapter three, we completely characterize and derive closed-form Nash equilibrium decision variable values for a two and three-period model of one manufacturer supplying identical product to a Cournot duopoly of retailers with SI carrying allowed between periods. We formulate this problem as a dynamic game and derive closed-form equilibrium decision variable values for both the manufacturer and retailers in every period. We present comparative analyses of these two and three period models to two and three period models with a monopoly downstream and a Static Cournot duopoly downstream to deduce the effect of competition and Strategic Inventory respectively.

In Chapter four, we completely characterize and derive Nash equilibrium decision variable values for a two period Cournot duopoly downstream model with SI, under a "Commitment contract" where the manufacturer commits to the wholesale price for every selling season, at the beginning of the first period itself. We then present comparative analysis of this model to an identical model under Commitment contract, but with a monopoly downstream and a dynamic 2-period Cournot duopoly model, to deduce the effect of downstream competition under a Commitment contract, and the effectiveness of a Commitment contract with a downstream Cournot duopoly respectively.

2. LITERATURE REVIEW

In the management of supply chain processes, inventory management is challenging because it directly impacts costs and service (Felea, 2008). Inventory is commonly seen as a tradeoff between holding costs and service levels, and the efficient balancing of these two competing parameters constitutes effective inventory management – keeping holding costs low, while keeping service levels high enough to attract customers consistently.

Inventory decision making differs significantly with whether the supply chain is centralized or decentralized. In a centralized supply chain, there is one decision maker and the decisions of this entity are binding on all the other entities in the supply chain. However, a decentralized supply chain is one where each supply chain entity – manufacturer, retailer, intermediary are free to make their own operational decisions.

2.1 Supply chain inventory optimization under a single decision maker

The news-vendor problem is a classic example of inventory theory focusing on optimization under a single decision maker. It is typically characterized by fixed prices and uncertain demand. It is named so, since it describes the dilemma faced by a newspaper vendor, who needs to decide every morning how many newspapers to buy for the day without knowing the characteristics of the demand. Also newspapers bought yesterday are useless today, and will not be sold. By analogy, in a news-vendor problem, unsold product at the end of each day is assumed to be disposed off at a salvage price s ($s \geq 0$).

The classical single-period news-vendor problem (SPP) is to find a product's order quantity that maximizes the expected profit under probabilistic demand. The SPP model assumes that if any inventory remains at the end of the period, a discount is used to sell it or it is disposed (Khouja, 1999). Here, there is only one decision maker and a single selling season. There are also various extensions to the classic problem that have been addressed by various researchers over the years. One early example is Kabak and Schiff (1978) who solve the news-vendor problem with the objective of profit-satisficing - maximizing the probability of achieving a certain target profit. Petruzzi and Dada (1999) summarize and critique of several works related to the news-vendor model, with parameters like demand and selling price, being endogenously supplied, in contrast to traditional news-vendor models, where these parameters are exogenous to the model.

In more recent research, Boute et al. (2006) consider a two-echelon supply chain with a single retailer that holds finished goods inventory to meet independent, identically distributed (i.i.d) customer demand and a manufacturer that produces the goods on a made-to-order basis. They show that by including the impact of the retailer's order decision on lead times, the order pattern can be smoothed to a considerable extent without increasing stock levels. This reduces inventory and hence inventory costs. In more recent research, Lam and Ip (2011) integrate the concept of customer satisfaction into inventory management and a “customer satisfaction inventory” model which integrates probabilistic concepts of Markov chains to abstract the value of retention vs. migration of customers as a decision variable into an inventory model such that the replenishment inventory level can be tailored to future expected customer demand without keeping excessive inventory. This represents one of the first attempts to couple inventory levels with customer satisfaction directly.

The overarching theme in these models is that they focus on optimization under a single decision maker and hence incentives to either hold or not hold inventory. Specific incentive schemes to discourage/encourage holding of inventory are not discussed.

2.2 Game-theoretic inventory modeling

Cachon and Zipkin (1999) compare competitive and collaborative inventory policies in a two-stage supply chain with stationary stochastic demand, where inventory holding costs are charged at each stage and back-orders incur a penalty. They propose two game-theoretic models – one in which the firms track their echelon inventories and another in which they track only local inventories and compare the optimal policies chosen in either scenario. The optimal policy in the competitive case (each firm only watches local inventory) minimizes every agent's inventory costs whereas an optimal policy in the cooperative scenario (each firm watches echelon inventory) minimizes the cost of the entire supply chain. They show that competition reduces supply chain efficiency and that a system-optimal solution can be achieved as a Nash Equilibrium by using a linear transfer payment scheme. They develop a set of optimal transfer payment schemes based on easily verifiable metrics like inventory and back-orders that eliminate incentives to deviate and achieve a supply chain efficiency level comparable to the cooperative equilibrium. Chen et al. (2009) analyze the cost/benefit allocation among several retailers in various *inventory centralization games* (Distribution System with multiple retailers who can place joint orders and place inventory in a central warehouse location). Unlike previous related work like Chen and Zhang (2006) and Slikker et al. (2005), this work assumes that the pricing decision is endogenous. They employ convex programming techniques to prove that there exists at least one feasible allocation that cannot be improved upon, (non-empty core) in an inventory centralization game with price-dependent linear demand. Bichescu and Fry (2009) analyze a decentralized supply chain following a periodic review (Q, R) inventory policy with a VMI arrangement. They explore the division of channel power between the different agents by analyzing 3 different models – one model of SC with a powerful supplier that can influence the decision-making process, a second with a powerful retailer and a third where both have approximately equal power, by modeling the powerful agent scenarios as a Stackelberg game where either agent moves first, and the third scenario as a simultaneous move game. They show that merely opting for a VMI arrangement leads to savings, irrespective of the power scenario.

In all this literature, inventory has been viewed as a parameter that needs to be optimized in different ways to positively impact supply chain efficiency and profits– centralization, VMI, optimization under a single decision-maker to maximize or minimize an operational parameter like total profit or cost respectively, game-theoretic perspectives to address decentralized decision-making, incentives to promote equilibria that benefit the entire supply chain vs. any single agent etc. None of the papers reviewed in the previous two sections view inventory as conferring a strategic advantage to any agent. The traditional view has always been that inventory incurs a holding cost and hence is a liability that needs to be minimized, or at least managed efficiently with a holding cost trade-off to act as a hedge against uncertain supply and/or demand, channel delays etc. However, there is an entire line of literature that views inventory as a strategic weapon and then a further line of developing literature that extends this notion from economics to supply chains, exploring the strategic value of inventory in supply chains. We review these two strands of literature in the following two sections and posit that this thesis furthers the emerging line of research in the latter strand –strategic role of inventories in supply chains.

2.3 Classic economic competition models

The Miriam-Webster dictionary defines competition as, "the effort of two or more parties acting independently to secure the business of a third-party by offering the most favorable terms." There are various basic economic competition models that are usually used in the literature (for simplicity of analysis), to model various competition scenarios in the real-world.

Some of the most important ones are:

<u>Cournot competition:</u> Cournot competition is an economic model where companies compete on the amount of output they will produce, which they decide independently of each other and simultaneously. It is named after Antoine Augustin Cournot (1801-1877). A key assumption of this model is the Cournot Conjecture which states that each firm aims to maximize profits, based on the expectation that its own output decision will not have an effect on the decisions of its rivals.

<u>Stackelberg competition:</u> Stackelberg competition model is used to describe a situation where one of the firms in a market, has a natural advantage over the other competing firms and makes the first move, ahead of the other players. This firm is called the *Stackelberg Leader*. The advantage the Stackelberg leader firm holds over the other firms in the market

(followers) could be due to the fact that the leader held a monopoly in the market and the followers are new entrants. It could also be based on other forms of leadership of one firm over others in an Oligopoly, like, the leader firm holding excess inventory, which helps it to make a production decision before the other firms in the market. This form of competition is named after the German economist Heinrich Freiherr von Stackelberg who published Market Structure and Equilibrium (Marktform und Gleichgewicht) in 1934 which described the model.

Bertrand competition: Bertrand competition is a model of competition used in economics, named after the French economist Joseph Louis François Bertrand (1822-1900). It is fundamentally different from the Cournot and Stackelberg models of economic competition described previously in that, firms do not compete over output quantity, but act as price-setters i.e., they set prices at which each firm is going to sell goods in the market, simultaneously, and each firm is assumed to produce/be able to procure enough product to meet end-customer demand at that price point. Also, this is a simultaneous-ordering situation like the Cournot model and unlike the Stackelberg model.

1.4 Inventory as a strategic weapon

Inventories as a strategic weapon has been frequently studied in economic literature for over 30 years now and more recently from a marketing perspective (operations-marketing interface). Murphy, Toman and Weiss (1989) represents one of the first attempts at a game-theoretic model of oil market disruption and the role of «inventory stock-piling» in this environment. Rotemberg and Saloner (1989), analyzes the role of inventories in supporting collusion between supply chain entities. The model considered in this paper is that of duopoly retailers, who themselves are the producers, selling product over two consecutive seasons with inventory carrying being allowed between seasons. The demand function considered is price-dependent linear. One of the key intuitions they derive from the model is that, when demand is high, there is increased incentive to deviate from implicitly collusive arrangement, so there is increased strategic inventory carrying by the two retailers. Balachander and Farquhar (1994) represent an early attempt in the operations-marketing interface literature that considers the *strategic value* of a particular inventory strategy. In this case, they reason that though a firm might lose from foregone sales in case of a stockout, it might indirectly also benefit from the higher price a competititor is able to charge. They thus argue that customers are more prone to search elsewhere for a product upon encountering a stockout, from this reduced price-competition between the firms. This would then provide an incentive for competing firms to actually induce stockouts (deliberately stocking less) thereby resulting in inventory now acquiring a strategic dimension.

Matsumura(2002) similarly studies inventories as a strategic weapon, but from the standpoint of it used as a co-ordinating device in a duopoly, competing-retailer environment. Matsumura's model is a finitely repeated competition model with a finite number of selling seasons through which inventory carrying is allowed, as opposed to the more restrictive one-shot, two-selling season model of Rotemberg and Saloner (1989). One of the important insights derived in this paper is that, if a firm deviates from collusive behavior, the rival increases it's inventories to punish the deviator. Large inventory holding by the punishing firm effectively makes it the Stackelberg leader, forcing the defecting firm to follow. In this situation, the second mover's equilibrium optimal strategies, *always* place it at a disadvantage compared to the punishing firm and hence inventory holding acts as a strategic deterrent to firms taking non cooperative actions. The two competing firms are allowed to carry inventory at any given point in time,only one period forward.

Mollgaard, Poddar and Sasaki (2000) analyze the strategic role of inventories in a competing retailer environment. Their analysis again looks at the "strategic value" of inventory, from yet another angle - strategic inventory carrying allowing a firm to raise it's latter period output. Using a two-period model, they establish that the *strategic* value of inventories depend on the convexity of the cost function, on the cost of storage, and on the slopes of each firm's individual supply schedules. This makes their analysis more granular than earlier attempts. In addition to exploring what kind of strategic role inventories can play in a competing-retailer environment, they establish explicitly, factors that affect the strategic value of inventory in such situations.

In summary, all these attempts at analyzing the strategic value of inventories do so, in the sense of inventories giving one of the firms a strategic edge over the other in a competitive environment, or inventory carrying either aiding or abetting collusive behavior in a cooperative environment. All of the above models assume that the producer himself is the seller of the goods and the *supply chain* angle is explicitly absent from this strand of literature.

Table 2.1 Summary of literature on inventory as a strategic weapon

Author (Year)	**Types of strategic roles analyzed for inventory**	**Types of competition Models Considered**	**Vertical control present?**	**Single period/Multi-period?**	**Number of players**
Murphy et al. (1989)	Stockpiling by entities	Simultaneous move	No	Single Period	N-player
Rotermberg & Saloner (1989)	Punish firms that deviate from a collusive arrangement.	Simultaneous move	No	Two Period	N-player oligopoly
Balachander & Farquhar (1994)	Strategically inducing stock-out	Monopoly, Simultaneous move	No	Single Period	n-player oligopoly
Mollgaard et al. (2000)	Allow firm to raise its latter period output	Simultaneous move	No	Two-period	two-player
Matsumura (2002)	Firm can commit to larger sales in a later period – encourages firms to take collusive action	Cournot competition	No	Two-period	two-player

2.5 Strategic inventories in supply chains

One of the most significant works in this area is Anand et al. (2008), who conjecture among other things that strategic inventories are optimal for a wide range of contractual structures in an n-period ordering environment, first for markets with price-dependent linear end demand as well as later for, arbitrary demand functions. This is the case, even when all the traditional reasons for holding inventory at a supply chain entity do not exist. Traditional reasons for holding inventory are demand uncertainty, pipeline delays in getting the product from producer to end consumer on time, economies of scale in production and distribution and such other factors. Throughout their models they assume that the nature of end-demand is known and replenishment is instantaneous, no back-ordering exists and production costs are the same across periods, but still prove that inventory carriage between periods is optimal in certain cases and make the important argument that design of coordinating supply chain contracts has to take the possibility of strategic inventory carriage into account.

Keskinocak et al. (2008) extend the two-period model proposed by Anand et al. (2008) to case where the manufacturer's first period capacity is limited, thereby analyzing the effect of strategic inventory carriage in a capacitated production environment. Zhang, Natarajan and Sosic (2008) extend the two period model from Anand et al. (2008) to the case with asymmetric information in an n-period model. Their key assumption is that the inventory level of the retailer from the previous period is invisible to the supplier, when the supplier is setting his wholesale price for the period. This is different from the other related efforts in analyzing the impact of strategic inventories in a vertically-controlled supply chain. All the previous studies (Anand et al. 2008, Keskinocak et al. 2008) assume full information at all stages and time-periods of the game. They analyze the kind of contracts that can minimize the informational advantage the retailer has in this scenario (by not sharing the inventory information with the manufacturer)

These papers are the first to analyze the effect of strategic inventories in a supply chain. Strategic inventory carriage as a part of single-echelon inventory games among market competitors have been analyzed before in the literature by papers like Rotemberg and Saloner (1989), but a key difference to Anand et al. (2008) is the absence of vertical control, or a multi-echelon supply chain with procurement from an upstream manufacturer. In previous economics literature, typically, the producer himself is the seller in the market.

Krishnan and Winter (2007) propose a scheme of joint price and inventory control in a one-manufacturer, two-retailer supply chain where the retailers compete as differentiated duopolists under uncertain demand. They find that a combination of a buy-back contract with a resale price ceiling leads to maximization of joint profits. This work is close, but different to the work presented in the thesis. A very crucial differentiator is the fact that we do not focus on joint price and inventory control. In our model, inventory decisions are taken independently by the retailers and the role of strategy inventory carrying by retailers acting independently is explored, for its interaction with price-control and retailer competition.

Another recent work in the area, Desai, Koenigsberg and Purohit (2010) focus on the effects of forward buying by retailers and proves that the motivations for forward buying are more complex than just manufacturer trade promotions or that retailer stock piling only helps the retailer but hurts the manufacturer (a view shared by Anand (2008) who subsequently develop co-ordinating contracts to remedy this inequality). This work adds to the model considered by Anand (2008), the possibility of manufacturer trade promotions as well as uncertain demand and find that, regardless of whether the manufacturer offers a trade promotion, forward buying can be beneficial, both to the retailer and the manufacturer. They also consider a competitive model with two retailers facing identical but different demand functions and that the retailers are differentiated (they aren't selling the same end-product). Also, they find that, in the case of uncertain demand, strategic forward buying is encouraged and find that the retailer orders a quantity higher than they expect to sell, even in the most optimistic demand scenario. This work further bolsters evidence that inventory can play a strategic role in the supply chain in some very interesting ways that merit more careful research and analysis.

Hartwig, et al. (2015) present an experimental study on the effect of SI on supply chain performance. They show that the positive effects of SI are more pronounced than theoretically predicted – reducing average wholesale prices and the double marginalization effect which leads to benefits for both manufacturer and retailer alike in a one-manufacturer, one-retailer SC two selling seasons. Downstream competition is not considered.

Table 2.2 Summary of literature on Strategic inventories in a supply chain

Author (year)	**Vertical control Present?**	**Competititve downstream present?**	**Type of demand functions Considered**	**Number of periods**	**Co-ordinating contracts developed?**
Krishnan and Winter (2007)	Yes	Yes	Uncertain demand	Single period	Yes
Anand (2008)	Yes	No	Price-dependent linear, general	2 period, n-period	Yes
Keskinocak (2008)	Yes	No	Price-dependent linear	2-period	Yes
Desai et al. (2010)	Yes	Yes. 2 manufacturer one retailer and one manufacturer 2-retailer configurations	Uncertain, price-dependent linear, depending on only the particular retailer's prior retail price	2-period	No
Viswanathan and Jang (2009)	Yes	Yes, Cournot Competing downstream duopoly	Price-dependent linear, depending on the total quantity on sale in the market in the given period.	2-period	No
Viswanathan and Jang (2010)	Yes	Yes, Stackelberg Competing downstream duopoly	Price-dependent linear,depending on the total quantity on sale in the market in the given period.	2-period	No

2.6. Game Theory in supply chains

Game theory is a powerful tool used to analyze situations where there are multiple players ("agents" or "stake-holders") and each stake-holder's payoff is affected by the decisions of the other players. It is easy to see that a supply chain can easily be cast into a game theoretic model, since it typically contains retailers, distributors, manufacturers all of whom make different kinds of strategic, tactical and operational decisions that can have a direct effect on the strategic, tactical and operational decisions of every other entity in the supply chain. This fact has led to a lot of game theoretic concepts being used to analyze supply chains and evolve coordination mechanisms, determine optimal decisions for each party and such other things.

One of the most important reviews that covers a lot of significant work in the use of game theoretic techniques in supply chain analysis was Cachon and Netessine (2003). This work introduces the basics of game theory that apply to supply chain research and reviews in fair detail and an accessible format, a set of game theoretic tools that have been used/can be used in future to analyze supply chain problems. Most of the review focuses on static, non-cooperative, non-zero sum games.

In the following, we provide an introduction to some key game theoretic concepts that are essential to understanding the analytical techniques used in this dissertation.

Basic definitions, results and concepts: (Fudenberg & Tirole, 1991)

1. **Strategic form game:** Strategic form games have three elements

- A set of players $i \in I$
- The Pure Strategy Profile (S_i) for each player i
- Payoff Functions u_i that give the player i's von Neumann-Morgenstern utility $u_i(s)$ for each profile $s = (s_1, s_2, s_3,...,s_I)$
- Each player's objective is to maximize his own pay-off function.

2. **Pure strategy:** A pure strategy is a predetermined plan of action in a game of which strategy to choose, from a strategy profile, after considering all the strategies from the profile.

3. **Mixed strategy:** A mixed strategy (σ_i) can be considered a probability distribution over pure strategies. Each player's randomization is statistically independent of those of his opponents and the payoffs to a profile of mixed strategies are the expected values of the corresponding pure-strategy payoffs.

4. A **Nash equilibrium (NE)** is a profile of strategy such that each player's strategy is an optimal response to other players' strategies. A mixed strategy profile σ* is Nash Equilibrium if, for all players *i*, $u_i(\sigma^*_i, \sigma^*_{-i}) >= u_i(s_i, \sigma^*_{-i})$ for all $s_i \in S_i$

5. A **Pure Strategy Nash Equilibrium** is a pure strategy satisfying the above condition.

6. **Strict Nash equilibrium:** A Nash Equilibrium is *strict* (Harsanyi, 1973), if each player has a best response to his rivals' strategies. i.e., s* is a strict NE if $u(s_i^*, s_{-i}^*) >$

$u(s_i, s^*_{-i})$ for all i and $s_i \neq s_i^*$. By definition, a strict Nash Equlibrium is a pure strategy Nash Equlibrium.

7. **Domination of strategies:** Pure strategy s_i is strictly dominated for player *i* if there exists $s_i' \in i$ such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$

8. **Iterated elimination of pure strategies:** When one round of elimination of strictly dominated strategies yields a unique strategy profile $s^* = (s_1^*, s_2^*, \ldots s_i^*)$, this strategy profile is necessarily a Nash equilibrium.

9. **Pareto optimality:** Given a set of alternative allocations of, say, goods or income for a set of individuals, a change from one allocation to another that can make at least one individual better off without making any other individual worse off is called a Pareto improvement. An allocation is Pareto efficient or Pareto optimal when no further Pareto improvements can be made.

2.7 Multiple Nash Equilibria in Games

The problem of selecting one unique Nash Equilibrium from many possible ones has been a pertinent problem for researchers for years now. There are many situations in a game theoretic setting, where there are multiple possible Nash Equilibria. This poses a problem, especially in real-world applications. E.g., Consider a supply chain Game where the Inventory level at a manufacturer and retailer needs to be decided. If we solve this game using backward induction and find that there are multiple possible equilibrium inventory values at each entity, then we face a dilemma as to which of these values to use, while physically setting inventory levels.

There are many approaches to deal with the problem of multiple Nash equilibria. One of the more common ones is to seek a mixed strategy equilibrium (a randomization over the multiple pure strategies) in cases where there are multiple pure strategy equilibria possible. Though this approach is simple and elegant, and yields one mixed strategy equilibrium, when presented with multiple pure strategy equilibria, it is not useful while modeling for real-world applications, since it would be meaningless to say that the optimal inventory value for a supply chain entity would be 70 units 10% of the time and 40 units 30% of the time, randomly.

Researchers have always been interested in various creative approaches to this problem ever since John Nash's seminal work introducing the notion of the Nash Equilibrium, in 1950.

One of the approaches is to choose a Nash Equilibrium randomly from the multiple available equilibria. Bjorn and Vuong (1984) present one such model where they choose a Nash Equilibrium among many, randomly. They analyze the econometric decision of a husband and wife to participate or not in the labor force, together and model the behavior of this couple, using a game-theoretic framework. They distinguish their work from the previous efforts on this problem in that, the previous efforts all considered the husband's decision to work or not work, as exogenous to the model, but the authors here assume this decision happens within the framework of the problem itself. Another contribution of this work is that they assume a utility function that specifies the labor supply of a husband and wife using individual utility functions in contrast to previous work that specifies the labor supply of a husband and wife from the outcome of a joint utility function. In other words, the decisions of the husband and wife are not independent and the decision of the husband has an impact on the utility function of the wife and vice-versa. They find that there exist multiple Nash equilibria to this game. i.e., both people not participating in the workforce as well as both people simultaneously participating are both Nash equilibria. They resolve this situation by proposing that the probability of occurrence of each of these pairs of outcome (equilibria) is distributed according to certain weights and provide a log-likelihood function formulation for estimating these weights and probabilities.

An approach that is a little more sophisticated than random choosing is one of choosing extremal equilibria. Jia (2008) develops game-theoretic models quantify the impact of national discount chains on the profitability and entry and exit decisions of small retailers from the late 1980 to the late 1990s. He also examines the entry decisions of a small chain of stores vs just a single store, thereby relaxing the assumption that the entry/exit decision of a particular store is independent of the entry/exit decision of every other simultaneous entrant. They find that Walmart's expansion from the late 80s to the 90s explains about 40-50% of the net change in the number of small discount retailers, during that period. They remark that modeling the entry decisions of a chain of retail stores, as opposed to one store causes the profits of the stores to be spatially related. This leads to huge problem sizes – they cite an example with the entry decision that has 2^{2000} choices for a market size of 2000 stores. They encounter multiple equilibria, while solving for a Nash equilibrium to this entry game and solve this by transforming the profit-maximization problem into a search for fixed points of the necessary conditions. This transformed problem then has much smaller dimensions than 2^{2000} and is now much easier to solve for a Nash equilibrium. They solve for a Nash equilibrium in this situation using a search technique that relies on the super-modularity property of games. A super-modular game is one in which the marginal utility of increasing a player's strategy increases with the increase of other player's strategies. In other words, the best response of each player is a non-decreasing function of other players' strategies.

Tamer (2003) studies a bi-variate (only two decision choices per player), simultaneous response model, which is a stochastic representation of the equilibria in a two-person game. He finds that, in this situation there are multiple equilibria and categorize the approaches taken by previous works on similar models (with multiple equilibria) into two classes – incomplete and incoherent models. Incomplete models are ones where the model predicts a non-unique outcome that maps to a certain region of the exogenous variables. They observe that researchers dealing with these models have either made simplifying assumptions such that the outcome space changes, or impose ad-ho selection mechanisms in the regions of multiplicity. The author studies these kind of incomplete models in the regions that exhibit multiplicity of equilibria. Using restrictions on the probabilities of the non-unique outcomes and by identifying the parameters in the model that can be consistently estimated, the authors develop a maximum likelihood estimator for the parameters and hence resolve the multiplicity problem.

Bresnahan and Reiss (1990) develop empirical models of market structure from qualitative choice models of a firm's entry decision into a market. These models assume that the author does not observe market entrant's revenues or costs but draw inferences from a firm's unobservable profits that describe potential entrants' strategies. They then use these models to study the market concentration in retail markets for new automobiles, thus determining the entry strategy for a new entrant into the market. They find that there exist multiple Nash equilibria that make it impossible in this situation to use qualitative choice models. They overcome this situation by reinterpreting their game theoretic model from one that predicts the individual choices of each of the entrants to one that predicts the total number of entrants in the market, in which case, they find, there exist unique Nash Equilibria. This is an example of another approach used in a situation with multiple Nash equilibria is to recast the game such that there exists a unique equilibrium. They also find that recasting the game as a sequential decision game, rather than simultaneous move, also eliminates the problem of multiple Nash equilibria. They argue that this decision also makes physical sense because one can always think of entry decisions of firms in a market happening one after another, and not all at the same time. Another similar example is Berry (1992) who present a market entry model for the airline industry. Here also, they encounter the problem of Multiple Nash Equilibria for firms' entry/exit decisions. Here too, they try to recast the game to model the number of new entrants (and not the identities of the individual entrants) versus a model that tries to predict each firm's entry/exit decisions based on the decisions of the other potential entrants. They find that this recast game has a unique equilibrium, and thus they are able to deal with the situation where multiple Nash equilibria exist. Lung (2010) presents an alternative to all these methods discussed above, using the *Nash Ascendancy Relation.* They postulate that if a strategy profile *x* dominates a strategy profile *y*, it can be said that strategy *x* is more stable (closer to equilibrium) than strategy *y.* In this case, it can be said that *x Nash Ascends y* and is more likely to be a Nash Equilibrium than *y*. By formulating an iterative method that does an approximate computation of how many strategies, each strategy in a strategy space ascends, they propose a method to compute a unique Nash equilibrium for a game. In summary, all of these works, represent various methods and approaches to choose a single Nash equilibrium in a situation where there exist multiple possible equilibria. It can be seen that this is still an open research area and none of the methods proposed above provide a definite or fool-proof way of choosing one Nash equilibrium from many available choices and the final decision of method followed depends on the problem on hand, the objectives of the research (whether recasting the problem in a different way still fulfills the research objectives), time constraints (that prohibit computationally costly algorithms), amount of accuracy required etc., and hence the technique used has to be tailored specifically to the problem on hand, based on the unique characteristics of the problem, most of the time.

2.8 Supply chain coordination with contracts

In a supply chain, operational decisions of the individual entities are usually made, keeping in mind, only their own best interest and these interests need not always be the same as the best interests of the supply chain as a whole (Cachon et al., 2003). Optimal supply chain performance can be achieved if firms coordinate by contracting on a set of transfer payments that align each firm's objective to the supply chain objective. This mechanism is termed decentralized supply chain coordination using contracts.

In the following paragraphs, we review some of the important contracts found in recent supply chain literature (Cachon et al., 2003).

- Wholesale price contract: With a wholesale price contract, the supplier charges the retailer a fixed price w per-unit of product purchased. Lavaliere and Porteus (2001) analyze a wholesale price contract in the context of the news-vendor model. They consider a manufacturer producing a single good at a marginal cost *c* which is sold at the fixed retail price $r>c$. Salvage value is assumed to be zero. A single selling season is assumed with demand drawn from a continuous distribution Φ, with density φ. Unmet demand (stock-out) is assumed to be lost, resulting in lost margin to the retailer but without any additional stock-out penalty. The manufacturer is the Stackelberg leader, presenting the wholesale price *w* that the retailer can either accept or reject. An optimal contract in this situation is one that maximizes the manufacturer's profit subject to the retailer's acceptance. The retailer accepts any terms that allow an expected profit greater than zero (the model does not assume any opportunity cost). This model does not consider supplier competition, retailer power and retailer pricing policies that affect wholesale prices in the real world. The authors show that manufacturer's profit and sales quantity increase (mostly) with increase in market size but the resulting wholesale price depends on how the market grows. They also show that, if the market becomes more variable, the retailer becomes more price-sensitive and hence the wholesale price decreases. Gerchak and Wang (2004) show that in assembly systems (where multiple components need to be mated together to form the finished retail product) with random demand. They find, among other things that the wholesale price contract is inferior to the revenue sharing contract, as the number of suppliers increase, in the case of assembly systems. Using an Exponential demand distribution, they provide an example for which the VMI with revenue sharing dominates the wholesale price contract, with a single supplier. Unlike in Lariviere and Porteus (2001), here, unsold items are returned to the manufacturer for a pre-arranged price. In this model, the component lot sizes (Q_i) are selected by the manufacturers (suppliers) and the assembly quantity (Q_o) is chosen by the assembler. A basic revenue-sharing contract as described in this paper specifies that, for each unit of final product sold, the assembler pays supplier i, α_i i=1, 2, 3 ...n and 0< α_i<1, out of the total $1 of revenue. That revenue sharing scheme is clearly known to suppliers. So, the model considered in this paper is a full information game. The wholesale price contract considered here is a multiple-supplier generalization of the model described in Lariviere and Porteus (2001), reviewed above. Recent research on the use of wholesale price contracts in supply chains continues to innovate on the type of markets considered, types of incentives researched and types of insights gained – all with a simple wholesale price type arrangement. Case in point is Hu and Gan (2010) who research the impact of credit on stimulating demand and hence study a wholesale price contract model of a supply chain with "credit sale". They find that if suppliers provide retailer with credit (incentive to increase sales) and the retailers only provide account sales (no credit), this scenario improves the overall supply chain performance.

- <u>Buyback contract:</u> A buyback contract is one where the retailer buys the product from the supplier at a unit price w but the supplier pays the retailer b per unit remaining at the end of season. The model assumes that the retailer can never profit from the leftover inventory (and hence the *"buy-back"* is a legitimate incentive) i.e., $w \geq b$always. Another important assumption in these contracts is that the supplier is able to verify the number of units remaining at the end of each selling season.

Padmanabhan and Png (1995) study the strategic effects of retailer returns policies on competing retailers. In this work, show that when there is no uncertainty in demand, a returns policy induces retailers to compete more intensely, since it reduces retail prices without affecting wholesale prices, which serves to reduce retailer profits leaving manufacturer margins intact. When demand is uncertain though, they find that a returns policy induces the retailer to overstock, reducing the upstream manufacturer profits. Emmons and Gilbert (1998) also focus on the strategic effects of returns policies that provide for conditions under which a retailer can return unsold merchandise for a full or partial refund from the supplier, and the role such a policy can play in aligning the self-interested behavior of the retailer with the best interests of a manufacturer. They opine that these kinds of policies are often used to encourage larger orders from retailers, of style goods, which are characterized by uncertain demand, long production times and short selling seasons. A key assumption of this model is that the retailer sticks to one profit-maximizing retail price through the selling season (as opposed to related work like Gallego and vanRyzin (1994) who consider the prospect of the retailer being able to change the retail price in the middle of the selling season). Like the recent research on wholesale price contracts, the research on buy-back contracts has also tended towards more innovative and interesting variations of these inherently simple contracts. E.g.: Hou et al. (2010) who study a buyback contract between a buyer and a backup supplier, when the buyer's main supplier experiences supply disruptions. The primary source in this model is assumed to be cheaper than the backup source. They solve for the optimal backup supplier return price and the buyer's optimal order quantity and deduce among other things that if the disruption probability increases (both the probability of demand and supply uncertainty), it is better for the buyer to order more from the backup supplier till the supply meets demand and in this scenario, the buyer's expected profit decreases whereas that of the backup supplier increases. Also, they deduce that the optimal order quantity for the buyer under supply uncertainty is larger than under demand uncertainty, but the total expected profit of the buyer and supplier combined is larger under supply uncertainty than demand uncertainty, especially when the supply disruption probability is large.

- Revenue sharing contract: In a revenue sharing contract the supplier charges w_r per unit purchased plus the retailer shares a percentage of his revenue, back with the supplier. Cachon and Lariviere (2000) analyze revenue sharing contracts for the video rental industry. They show that a revenue sharing contract co-ordinates the supply chain consisting of one manufacturer supplying to a retailer, in both deterministic and stochastic demand situations. They also show that a revenue sharing contract can coordinate a news-vendor style supply chain with price-dependent demand, while a buy-back contract cannot. Wang et al. (2004) study a *consignment contract with revenue sharing,* in which, akin to VMI, a supplier decides on a retail price and delivery quantity and retains ownership of the goods. For each item of goods sold, the retailer then deducts a certain percentage amount from the selling price and returns the balance to the supplier. This kind of arrangement, the authors opine, serves to shift the risk of inventory ownership from the retailer to the supplier, in contrast to a wholesale price-contract, discussed in the earlier section, where the inventory risk is concentrated wholly at the hands of the retailer. Hence, this kind of contract is more advantageous to the retailer when demand uncertainty is high, since he can order exactly how much he needs, depending on how much information he has about demand in the forthcoming selling season. Dana and Spier (2001) consider a related model of an upstream firm that sells a good to a downstream firm who then resell it or rent It for one period (video cassette rental industry). Perfect competition is assumed and the supplier offers a contract {t,r} to the retailers where *t* is the transfer price per unit of good sold to the retailer and *r* is the royalty percent of total revenue that the retailers each share with the manufacturer. They consider two models of this kind in which the upstream firm (supplier) has an interest in softening price-competition between the downstream competing retailers and show that revenue-sharing contracts used with a linear input price leads to increased *vertical integration* (increased control of the downstream retailers' operating parameters by the upstream firm).

- <u>The quantity flexibility contract:</u> A quantity flexibility contract is one where the supplier charges the retailer *w* per unit of product purchased and compensates the retailer for unsold units left over after the selling season. Tsay et al. (1999) provide a comprehensive framework for a quantity flexibility contract in a supply chain consisting of one manufacturer supplying to a downstream retailer. Tsay opines here that the quantity flexibility contract fulfils a specific niche in a typical supply chain – the scenario of a customer that provides a planning forecast of intended purchase for the next selling season, but does not commit to a specific quantity. In this kind of situation, the customer has incentive to over-forecast to ensure adequate supply, but the supplier now bears the risk of over-production. With a quantity flexibility contract, the supplier agrees to supply up to a certain percentage above the forecast quantity, in return for the retailer's promise to not buy anything less than a certain percentage below the forecast quantity. Under this contract, the retailer relays an order quantity forecast to the manufacturer much before the start of the selling season. He then places his final order, just before the selling season starts, based on an updated demand signal. Market demand occurs and is fulfilled by the retailer to the extent possible, based on this order quantity and retailer surplus, if any, is salvaged. Li and Kouvelis (1999) study a similar quantity flexibility contracts but of two primary types – a *time inflexible* contract that requires the retailer to not only specify the order quantity, but also the exact time of the ordering decision and a *time flexible* contract which eases the temporal limitation of the earlier case and permits ordering of a specific order quantity anytime within a mutually agreed time window. They also consider the situation of pure quantity flexibility – the ability to order a different quantity at any point in time. They claim to incorporate *risk-sharing* in the model by providing for the supplier being able to charge a different price each time, based on the time of the order and the quantity ordered. They derive the optimal time and order quantity decisions for the retailer and conclude that *contractual flexibility* in sourcing decisions can effectively reduce sourcing costs in an environment of price uncertainty. Lian and Deshmukh (2009) study quantity flexibility contracts that are a little more sophisticated than the ones studied by Tsay (2001) and Li and Kouvelis (1999). In this work, the time horizon between the initial demand forecast and the start of the selling season, is treated as a continuous opportunity for the retailer to place an order, whenever the retailer is comfortable placing a firm one. The further in advance the order is placed, the larger the discount the retailer gets. The retailer is also given the opportunity here to update quantities for future rolling time horizons at any time, but he pays a premium for the incremental units. The paper develops heuristics for optimal order quantity and timing decisions for the retailer in this situation and evaluates the efficacy of different strategies possible under this contract, for optimal profit. One important limiting assumption in this work is that the retailer can only increase his order quantity for future periods at the present time, he can never decrease it. Relaxing this assumption might affect the insights drawn on optimal strategies using this contract, considerably. Shi and Chen (2008) study quantity flexibility contracts with satisficing objectives, by assuming that supply chain agents are risk-averse, rather than neutral, as assumed by most of the previous analyses. With satisficing objectives, the goal is to maximize the probability of achieving a certain target. They obtain contracts based on the Pareto optimality criterion, which is again different from the Nash Equilibrium criterion usually used to derive optimal contract parameters. The authors setup a definition of a Pareto contract as: *"Within a contractual form, a contract is said to be Pareto if its parameter set is Pareto, that is, there does not exist an alternative parameter set such that no agent is worse off and at least one agent is strictly better off."* They also note that Pareto contracts do not co-ordinate a supply chain, because there is one agent that is always strictly better off than other agents and for a contract to coordinate a supply chain, we need that the optimal actions of the agents under the contract lead to pareto optimality for the supply chain, as a whole, and not any individual agent.

- Sales rebate contract: A sales rebate contract is one where the supplier charges the retailer *w* per unit of product purchased, but gives the retailer a rebate *r* per unit of product sold above a threshold *t*. Taylor (2002) considers two kinds of channel sales rebates – linear and target rebates. A linear rebate is one where a rebate is paid for every unit sold and a channel rebate where a rebate is paid only for every unit sold beyond a target level. In this work, they deduce that when demand is not influenced by sales effort, a properly designed channel rebate achieves channel coordination and leads to a win-win outcome. They also note that an implementable coordination mechanism cannot be developed using just linear rebates. Another significant observation the authors make in the paper is that a rebate is distinct from an upfront reduction in the wholesale price (discount), since the rebate is realized only after the item is sold by the retailer. They observe that rebates are used primarily in industries like computer hardware characterized by high demand variability and short cycle times. Wong, Qi and Leung (2009) look at how sales rebate contracts co-ordinate supply chains. They consider a two-echelon supply chain with one vendor supplying to multiple retailers with a VMI arrangement. They argue that VMI actually facilitates a sales rebate contract, since with a VMI arrangement, the retailers' real-time inventory information is available to the vendor. They show that, with this kind of arrangement – sales rebate contract under a VMI mechanism, the retailers achieve perfect coordination, i.e., retailers acting strategically to maximize their best interest can also make price decisions to maximize the aggregate supply chain profit. They claim that the proper rebate contract makes retailers lower their prices to system-optimal prices, which increases demand and hence aggregate supply chain profit also. He et al. (2009) look at a supply chain facing stochastic demand which is sensitive to both sales effort and retail price. They show that when demand is sensitive to both these quantities neither a returns policy nor a wholesale price contract coordinates the supply chain. They show that a returns policy coupled with sales rebate and penalty leads to a coordinating outcome for the supply chain. The penalty comes into effect, anytime the retailer sells below the target level *T*, which the supplier sets for the retailer. For every unit the retailer sells above the target level, the retailer gets a rebate *r.*

- Quantity discount contract: A general quantity discount contract can be defined as one where there is a transfer payment T = w (q)*q, where w (q) is the wholesale price per-unit, a decreasing function of *q*. Cachon (2003), in their review of supply chain coordination using contracts opine interestingly that *roughly speaking, quantity discount contracts achieve coordination by manipulating the retailer's marginal cost curve, while leaving the retailer's marginal revenue curve untouched.* This would be a simplistic but elegant way to understand the basic working of a quantity discount contract. Ghandfourish and Loo (1992) propose a non-linear quantity discount based procurement model for a multi-national oil company with affiliated plants all over the world. They use a non-linear programming model to reduce overall procurement cost, using this model. It would be apt to note here though, that in this model, the oil company exercises significant control over the affiliated plants' operational parameters and hence the situation considered in this problem is one that concerns a single decision maker rather than a decentralized supply chain that forms the backdrop of most of the efforts in supply chain coordination using contracts. Chen, Federgruen and Cheng (2001) study mechanisms for coordinating a two-echelon distribution system consisting of one supplier supplying to multiple downstream retailers, where the sales volumes are derived endogenously using known demand functions. They show that this decentralized system achieves equivalent performance (in terms of maximized system-wide profits), as a centrally controlled system, when coordination is achieved via periodically charged fixed fees and a discount pricing scheme where the discount given is the sum of 3 components based on annual sales volume, order quantity and order frequency respectively. They also show that a discounting scheme based solely on order quantity is not sufficient to optimize channel-wide profits in the presence of multiple identical downstream retailers. Su and Shi (2002) present a game-theoretic framework for the quantity discount problem with return contracts. They postulate a two-stage game model, where, in the first stage, the manufacturer and retailer determine their inventory levels cooperatively (manufacturing is make-to-order, end-user demand is stochastic) and in the second stage, in an attempt to increase channel efficiency, the manufacturer designs an optimal incentive scheme (quantity discount) to entice the retailer(buyer) to change the ordering decision. The retailer can return unsold inventory to the manufacturer at the end of the selling season for a pre-determined buyback price. The authors develop a menu of optimal discount-return configurations that balance an optimal returns policy with a quantity discount and find among other things that a higher wholesale price can result in a more liberal returns policy.

In summary, in this chapter, we have focused on reviewing the extant literature related the problems presented and solved in this dissertation. We started with a review of inventory decision-making models under a single decision maker, moving on to game-theoretic inventory models and the notion of inventories as a strategic weapon. Subsequently we focused on the work that focuses on inventories as a strategic weapon in a supply chain setting (with vertical control). Then we moved on to provide an overview of basic game theory concepts used in this dissertation as well as the techniques used to deal with multiple Nash equilibria. Finally, we conclude this review with a compilation of recent research on supply chain coordination with contracts.

3. DYNAMIC COURNOT DUOPOLY MODEL WITH VERTICAL CONTROL AND SI CARRIAGE BETWEEN PERIODS

This chapter presents two analytical models to determine the equilibrium SI levels and associated profits for the manufacturer and retailers in a supply chain over two and three periods, respectively. In both cases, we have one manufacturer and two retailers along with vertical control, whereby the manufacturer exerts control over any of the decision variables of the retailer, selling quantities and SI levels in this case, by setting the wholesale prices at certain points of time. The manufacturer and retailers all compete for profit. The two retailers are in a Cournot competition with each another, i.e., they compete on quantity. The demand is a known function of the product price.

The demand function in these models is a price-dependent linear function, a common function form used in the supply chain, inventory, and economic modeling literature. That is, the unit price of a product sold in the market is $p(Q) = a-bQ$. Here "a" is the reservation price, the maximum a customer is willing to pay for this product. It is an exogenous strictly positive model parameter. The parameter "b" is another strictly positive model parameter with $b \ll a$. Q is the total quantity on the market for sale in a particular selling season.

We approach the problem using a game-theoretic modeling and a backward induction reasoning - observing the last period's decisions first and then working backward to the previous periods' decisions. The equilibria obtained at every decision stage are sub-game perfect Nash equilibria, i.e., the Nash equilibrium obtained at each decision stage is not only the Nash equilibrium for that particular stage, but also a Nash equilibrium for the entire game (Selten, 1975).

3.1 Two-period Cournot duopoly model with vertical control and SI carriage between periods

The first model we will present in this dissertation consists of one manufacturer supplying a product to two Cournot duopoly retailers at its downstream in two consecutive selling seasons. The retailers are allowed to carry inventory between periods. The two retailers are identical in all respects. All products are sold at the end of the 2nd period. Figure 3.1 illustrates the model.

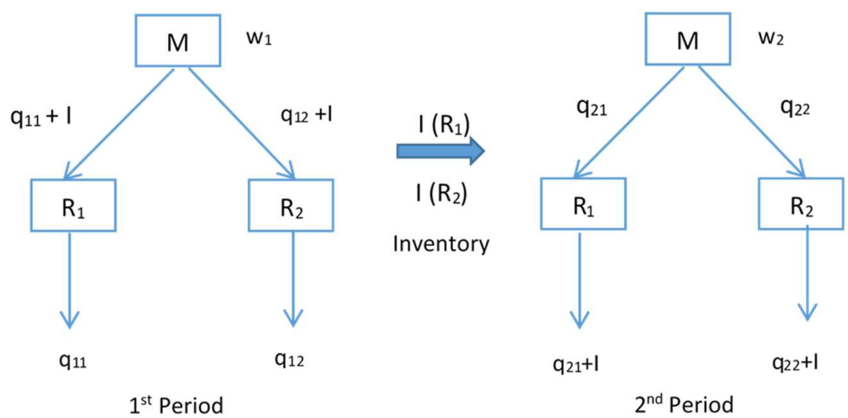

Figure 0.1 Illustration of a two-period Cournot duopoly model with Strategic Inventory

The business flow in the two periods is as follows:

Period 1:

Step 1: The manufacturer announces its 1st period wholesale price (w_1).

Step 2: Retailer 1 and Retailer 2 simultaneously announce their 1st period order quantities, which is the sum of the quantity they want to sell in Period 1 (q_{11} and q_{12}, respectively), and the quantity they want to carry to Period 2, I, for each of the retailer. Inventories carried to the 2nd period are held at a holding cost of h/unit.

Step 3: The market decides the retail price, p, of each unit of product sold: p $(q_{11}+q_{12})$ = a-b $(q_{11}+q_{12})$.

Step 4: Quantities q_{11} and q_{12} are sold simultaneously by Retailer 1 and Retailer 2 at the price p $(q_{11}+q_{12})$ per unit and revenues are realized. Inventory I is carried by each retailer over to the second period.

Period 2:

Step 1: The manufacturer announces the 2nd period wholesale price (w_2).

Step 2: Retailer 1 and Retailer 2 simultaneously announce their 2nd period order quantities (q_{21} and q_{22}).

Step 3: The market decides the retail price of each unit of product sold: p $(q_{21}+q_{22}+2I)$ = a-b$(q_{21}+q_{22}+2I)$.

Step 4: Quantities $(q_{21} + I)$ and $(q_{22} + I)$ are sold simultaneously by Retailer 1 and Retailer 2 at the retail price and revenues are realized.

The complete work for all the calculations of the equilibrium values is shown in the Appendix. As an illustration, we briefly present below the procedure of determining the retailers' 2nd equilibrium period order quantity.

<u>Nomenclature (all variables are non-negative):</u>

i**:** period ; ***j*****:** retailer; ***a*****:** model parameter reservation price (a>0) ; ***b***: model parameter (b>0); $\boldsymbol{q_{ij}}$ = order quantity of retailer *j* in period *i* (i = 1,2; j = 1,2); $\boldsymbol{I_j}$: inventory carried by retailer *j* from period 1 *to* period *2*; $\boldsymbol{w_i}$: wholesale price set by the manufacturer in period i (i=1,2); ***h****:* holding cost of each retailer to carry one unit of inventory from period 1 to period 2 (it is assumed that both retailers have the same holding cost rate); ***ΠR_{ij}*** : profit of retailer j in period I; ***ΠM_i*** : profit of manufacturer in period *i.*

Since the two retailers are identical in all respects and have equal holding costs, we can say that, in equilibrium, they carry equal amounts of inventory from the 1st period to the second. i.e., $I_1=I_2 = I$

<u>Retailer 1's second period order quantity decisions</u>

The conditions below are met by the decisions of the earlier periods' business flow i.e.

Given:

$0< w_2< a$ (1); $I_1, I_2\geq 0, h>0$ (2); $a-b\ (I_1+I_2) \geq 0$ (3)

The only decision variable here is q_{21}, and the decision variable needs to meet the following conditions:

$a - b\ (q_{21}+q_{22}+2I) \geq 0$ (2nd period retail price is non-negative) (4)

$q_{21} \geq 0$ (2nd period retailer 1 sales quantity is non-negative) (5)

Profit for the retailer is revenue minus cost, and we can write the 2nd period profit function for retailer 2 as:

$$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2I))(q_{21} + I) - (w_2 q_{21}) \quad (6)$$

Taking the first derivative of (6) with respect to q_{21}, we get:

$$\frac{\partial \Pi R_{21}}{\partial q_{21}} = a - b(2q_{21} + q_{22} + 3I) - w_2 \quad (7)$$

Differentiating (7) again with respect to q_{21}, we have:

$$\frac{\partial^2 \Pi R_{21}}{\partial q_{21}^2} = -2b < 0 \quad (8)$$

Equation (8) shows that (6) is concave with respect to q_{21}. Setting (7) at zero, we get the profit-maximizing 2nd period order quantity for retailer 1: $q_{21}^o = \frac{a-w_2}{2b} - \frac{q_{22}+3I}{2}$ (9)

We need to check if (9) meets (4) and (5).

Re-arranging (4), we have: $q_{21} \leq \frac{a}{b} - (q_{22} + 2I)$ (10)

Equation *(*9) can be re-written as:

$$q_{21}^o = (\frac{a}{2b} - \frac{q_{22}}{2} - I) - (\frac{I}{2} + \frac{w_2}{2b}) \leq \frac{a}{2b} - \frac{1}{2}(q_{22} + 2I) \leq \frac{a}{b} - (q_{22} + 2I) \quad (11)$$

Equation (11) shows that (9) always meets (10).

Next, we check if (9) fulfills (5), which leads to the following sub-cases for the optimal q^*_{21} decision:

Case 1.1(a): q^o_{21} *(9)* ≤ 0, In this case we set $q^*_{21} = 0$ (12)

Case 1.1(b): q^o_{21} (9) > 0, In this case $q^*_{21} = q^o_{21}$ (13)

Retailer 2's 2nd period order quantity decisions

Retailer 2’s 2nd period decisions are made using an identical procedure to Retailer 1, and we can obtain the profit-maximizing 2nd period retailer order quantity decision (q^o_{22}) as:

$$q_{21}^o = \frac{a - w_2}{2b} - \frac{q_{22} + 3I}{2} \quad (14)$$

and the following two cases (following an identical procedure to the previous section).

Case 1.2 (a): If $q^o_{22}(14) \leq 0$, $q^*_{22} = 0$ (15)
Case 1.2 (b): If $q^o_{22}(14) > 0$, $q^*_{22} = q^o_{22}(14) = \frac{a - w_2}{2b} - \frac{q_{21} + 3I}{2}$ (16)

Combined equilibrium analysis – q_{21} and q_{22} decisions

Since the two retailers are identical in all respects, symmetrical, in Cournot competition with each other and, take their decisions simultaneously, we can postulate that their equilibrium 2nd period order quantities are equal. There are only two possible equilibria:

Case (a): $q^o_{21} \leq 0$ and $q^o_{22} \leq 0$
In this case, $q^*_{21} = q^*_{22} = 0$ (from (12) and (14)) (17)
Case (b): $q^o_{21} > 0$ and $q^o_{22} > 0$ (Case 1.1(b) and Case 1.2(b))

In this case, $q^*_{21} = \frac{a-w_2}{2b} - \frac{q_{22}+3I}{2}$ from (13)) (18)

$q^*_{22} = \frac{a-w_2}{2b} - \frac{q_{21}+3I}{2}$ (from (16)) (19)

Solving (18) and (19) together, we obtain,

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I$ (20)

Equation (20) is the equilibrium order quantity decision for Retailer 1 and Retailer 2, in the two period Cournot duopoly model with SI allowed between the two selling seasons.

3.1.1 Effect of downstream retailer competition over two periods

In this section we analyze the effect of retailer competition on equilibrium values. Figure 3.2 below presents an illustration of this comparison:

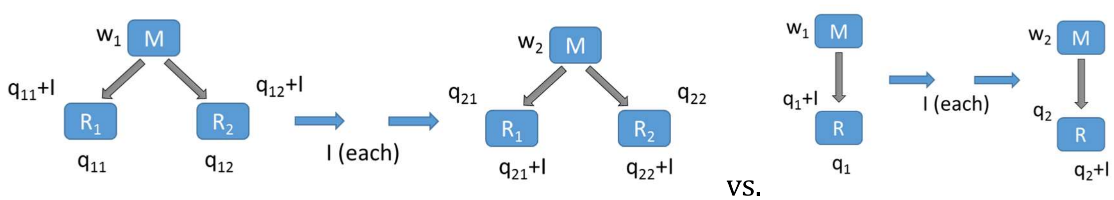

Figure 0.2 Illustration of the comparison between a 2-period Cournot duopoly downstream and a monopoly downstream model.

Table 3.1 presents a comparison of the equilibrium values of our model (two competing retailers) with those of Anand et al. (2008) (one retailer). In the table, all values for the downstream monopoly case are referenced from Anand et al. (2008). For ease of comparison we express our results in Table 1 using the same nomenclature as theirs Q_1 = (total purchase quantity in the 1st period) = $q_{11} + q_{12} + 2I$, p_1 = (retail price-per-unit in the 1st period) = $a-bq_1$, q_1 = (total sale quantity) = $q_{11}+q_{12}$, Q_2 = (total purchase quantity in the 2nd period) = $q_{21} + q_{22}$, q_2 = (total sale quantity in the 2nd period) = $q_{21}+q_{22}+2I$, $p_2=a - bq_2$

(customer retail price in the 2nd period) and (the SI quantity) $I^* = 2I$, since both retailers carry identical SI of I into the 2nd period. We use this convention for all comparison tables in this dissertation.

Table 0.1 *Comparison of equilibrium values between a two period Cournot duopoly and a monopoly downstream models depending on the holding cost range (the results in column 2 are the opposite for the other side of the holding cost ranges in column 3)*

Quantities compared	Result	Holding cost range	Cournot duopoly down-stream value	Monopoly downstream value
w^*_1	lower	Always	$\frac{12a-3h}{23}$	$\frac{9a-2h}{17}$
w^*_2	higher	h< 0.906a	$\frac{19a+24h}{46}$	$\frac{6a+10h}{17}$
$\Pi M_1 + \Pi M_2$	lower	Always	$\frac{120a^2-198ah+42h^2}{529b}$	$\frac{9a^2-4ah+8h^2}{34b}$
Q^*_1	lower	h<=0.66a	$\frac{10a-14h}{23b}$	$\frac{13a-18h}{34b}$
q^*_1	higher	Always	$\frac{22a+6h}{69b}$	$\frac{4a+h}{17b}$
Q^*_2	lower	Always	0	$\frac{3a+5h}{17b}$
q^*_2	lower	Always	$\frac{8(a-6h)}{69b}$	$\frac{11a-10h}{34b}$
I^*	lower	Always	$\frac{8(a-6h)}{69b}$	$\frac{5(a-4h)}{34b}$
$Q^*_1+Q^*_2$	lower	Always	$\frac{10a-14h}{23b}$	$\frac{19a-8h}{34b}$
p^*_1	lower	Always	$\frac{(47a-6h)}{23}$	$\frac{13a-h}{17}$
p^*_2	higher	Always	$\frac{61a+48h}{69}$	$\frac{23a+10h}{34}$
$\Pi R_1 + \Pi R_2$	lower	Always	$\frac{594h^2-1164ah+442a^2}{4761b}$	$\frac{155a^2-118ah+304h^2}{1156b}$

From Table 3.1, we see that the manufacturer's and retailer's profits are always lower across the two selling seasons combined in the Cournot duopoly downstream than in the monopoly downstream. We can attribute this to increased double marginalization effect due to the retailers needing to compete with one another and maximize their individual profits. We observe that the introduction of the Cournot competition leads the manufacturer setting a lower wholesale price in the first period and mostly higher (when h<0.91a; which mostly holds) wholesale price in the 2nd period, in equilibrium, compares to when he supplies a similar monopoly downstream. The equilibrium SI quantity set by the retailers, in response, is lower.

We compare the wholesale and retail prices over the two periods combined. In each case, the average is a weighted average, weighted by the quantity bought in each period for the wholesale price and the quantity sold in the period for the retail price.

$w^{avg.}{}_{Cournot} = \frac{12a-3h}{23}$ (21); $w^{avg.}{}_{monopoly} = \frac{8h^2-4ah+9a^2}{19a-8}$ (22).

Subtracting these two quantities; (21)-(22) yields: $\frac{160h^2+61ah-21a^2}{184h-437a}$(23).

Since a>>h usually, we ignore the *h* terms of (23) to obtain: $\frac{-21a^2}{-437a}$ > 0 always. Hence we can conclude that the Cournot duopoly downstream leads to mostly higher average wholesale prices. We compute weighted average retail prices similarly, to obtain:

$p^{avg}{}_{Cournot} = \frac{-(241\quad{}^{2}+2094ah-3590a^2)}{69(38a-9\quad)}$ (24); $p^{avg}{}_{monopoly} = \frac{152h^2+247ah-288\quad{}^{2}}{476h-408}$ (25).

Subtracting these two quantities, we get (24)-(25) = $\frac{102096h^3-572049ah^2-5435\quad{}^{2}h+354792a^3}{1477980h^2-1890876ah+534888a^2}$

(26). Ignoring *h* terms from equation (26), we get $\frac{354792a^3}{534888a^2}$>0 always. Thus, we conclude that a Cournot duopoly downstream leads to mostly higher equilibrium wholesale and retail prices. However, neither of these, higher wholeasale prices set by the manufacturer on average across the two periods, or the higher retail prices commanded by either retailer over the two periods, leads to either the manufacturers or retailers making a higher profit in the presence of downstream retailer competition, compared to a monopoly downstream. This is because, the order quantity for both periods combined (Q_1+Q_2 from Table 1) is lower and this offsets the higher wholesale and retail prices commanded by the manufacturer and the retailers respectively, to drag down both their profits compared to the monopoly case, in aggregate.

Next, we check the impact of Cournot competition on Consumer Surplus over two selling seasons. Consumer Surplus is the difference between the total amount the consumers are willing to pay for the good and the price they actually pay. Since the maximum possible price of the good is the reserve price “a”, the total Consumer Surplus over the two periods is given by CS = $\frac{1}{2}(a - p_1)q_1 + \left(\frac{1}{2}\right)(a - p_2)q_2$. Computing Consumer Surplus for the two cases compared – Cournot duopoly and Monopoly downstream we have:

$CS_{\text{CournotDuopoly}} = \frac{1206h^2 - 402ah - 760a^2}{4761}$ (28) and $CS_{\text{Monopoly}} = \frac{185a^2 - 188ah + 104h^2}{2312b}$ (29).

Subtracting these two quantities: (28) – (29) yields: $\frac{2293128h^2 - 34356ah - 2637\quad{}^2}{11007432b} < 0$ when a>>h, which mostly holds. Thus, we can conclude that the introduction of downstream Cournot competition mostly reduces Consumer Surplus.

In summary, over two selling seasons, both the Manufacturer and downstream Cournot competing retailers make lower profit compared to when the downstream is a single retailer monopoly. This is due to the presence of downstream retailer competition – double marginalization effect. Consumers do not benefit either, as evidenced by the lower Consumer Surplus in the Cournot duopoly case. Strategic Inventory is carried at a finite but lower level than the monopoly downstream, by the Cournot duopoly retailers and this doesn’t help the retailers make a higher profit in aggregate, than the monopoly downstream.

3.1.2: Effect of Strategic Inventory over two periods

In Table 3.2 we present a comparison of the equilibrium values of our model with a static Cournot duopoly model, i.e., a model that does not allow a Strategic Inventory. This comparison shows the effect of SI in the presence of a Cournot competing retailer downstream. Fig 3.3 is an illustration of this comparison.

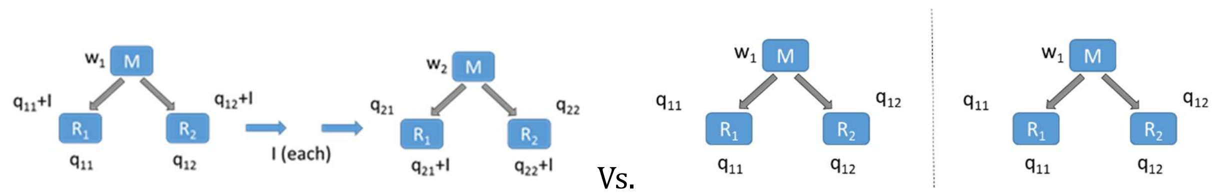

Figure 0.3 Illustration of the comparison between a two-period Cournot duopoly model with SI and a Static Cournot duopoly model

Table 0.2 Comparison of equilibrium values between a two period Cournot duopoly with SI and a static Cournot duopoly (no SI) depending on the holding cost range (for the other side of the holding cost range in column 3, the result in column 2 is the opposite)

Quantities compared	Result	Holding cost range	Cournot duopoly downstream value	Static Cournot duopoly value
w^*_1	Lower	always	$\frac{12a-3h}{23}$	$\frac{a}{2}$
w^*_2	Lower	h≤a/6	$\frac{19a+24h}{46}$	$\frac{a}{2}$
$\Pi M_1 + \Pi M_2$	Lower	always	$\frac{120a^2-198ah+42h^2}{529b}$	$\frac{a^2}{3b}$
Q^*_1	Higher	h≤a/6	$\frac{10a-14h}{23b}$	$\frac{a}{3b}$
q^*_1	Higher	h≤a/6	$\frac{22a+6h}{69b}$	$\frac{a}{3b}$
Q^*_2	Lower	always	0	$\frac{a}{3b}$
q^*_2	Lower	always	$\frac{8(a-6h)}{69b}$	$\frac{a}{3b}$
I*	Higher	always	$\frac{8(a-6h)}{69b}$	N/A
$Q^*_1+Q^*_2$	Lower	always	$\frac{10a-14h}{23b}$	$\frac{19a-8h}{34b}$

p*₁	Lower	always	$\frac{(47a-6h)}{23}$	$\frac{2a}{3}$
p*₂	Higher	always	$\frac{61a+48h}{69}$	$\frac{2a}{3}$
$\Pi R_1 + \Pi R_2$	Lower	always	$\frac{594h^2-1164ah+442a^2}{4761b}$	$\frac{a^2}{18b}$

From Table 3.2 we see that the manufacturer and retailer profits are always lower across the two selling seasons combined when SI is allowed. We observe that the manufacturer mostly sets a lower wholesale price in the first period, and a mostly lower wholesale price in the second period, as well, (when h<=a/6; which mostly holds) when SI is allowed. SI is carried in equilibrium when it is allowed, compared to a static Cournot duopoly downstream where each period is modeled as a one-shot game with no inventory allowed to be carried between periods. We compare the wholesale and retail prices over the two periods combined. In each case, the average is a weighted average, weighted by the quantity bought in each period for the wholesale price and the quantity sold in the period for the retail price.

$w^{avg}{}_{CournotSI} = \frac{12a-3h}{23}$ (30); $w^{avg}{}_{staticCournot} = \frac{a}{2}$ (31).

(30)- (31) yields: $\frac{a-6h}{46}$(32). Ignoring the h terms from (32), since a>>h, we get: $\frac{a}{46}$ >0 always. We can hence conclude that the manufacturer mostly sets a higher wholesale price on average, over the two selling season, when SI is allowed. We similarly compute weighted average retail prices, to obtain: $p^{avg}{}_{CournotSI} = \frac{-(2412\ \ {}^{2}+2094ah-3590a^2)}{69(38a-9\ \ )}$ (33); $p^{avg}{}_{staticCournot} = \frac{2a}{3}$ (34). Subtracting the two terms we computed above, (33)-(34) yields: $\frac{675h^2-639ah+220a^2}{3105h-1311a}$.

Ignoring *h* terms again. we get $\frac{220a^2}{-1311}$<0 always since a > 0. We can hence conclude that allowing SI leads to lower retail prices on average. Next, we ascertain the impact of introduction of SI in a Cournot duopoly downstream on Consumer Surplus. Computing Consumer Surplus for the two cases compared we have:

$CS_{CournotSI} = \frac{1206h^2 - 402ah - 760\ ^2}{4761}$ (35) and $CS_{StaticCournot} = \frac{a^2}{9b}$ (36). (35)- (36) yields: $\frac{1206h^2 - 402ah - 1289a^2}{4761b}$ (37). Ignoring h terms in (37) we get: $\frac{-1289a^2}{4761}$ which is <0 always. This implies that Consumer Surplus is mostly lower when SI is allowed over two selling seasons.

In summary, we see that allowing SI under Cournot duopoly downstream over two selling seasons does not help any of the SC entities in any way with reduced manufacturer profit, reduced retailer profit, and reduced Consumer Surplus. Though the retailers carry SI in equilibrium, and enough of it to not order at all in the 2nd period and sell only the carried SI, this strategy is insufficient for them to make a better profit than the static Cournot case.

3.2 Three-period Cournot duopoly model with vertical control and SI carriage between periods

In this section, we present a comparison of a three period Cournot duopoly model and a three period model with one manufacturer selling to one retailer with SI carriage allowed between periods, using the framework established by Anand et al. (2008). Anand et al. (2008) do not provide closed form expressions for the equilibrium decision variable values for a three-period model; we derive them here for ease of comparison, using the same framework and assumptions established in their study.

3.2.1 Effect of Cournot competition over three selling seasons

In Table 3.3 we present a comparison of equilibrium manufacturer wholesale prices between the 3-period Cournot duopoly model and a 3-period model with one manufacturer supplying to a single retailer monopoly downstream. This yields the effect of Cournot duopoly competition over three periods. Fig 3.4 below is an illustration of this comparison.

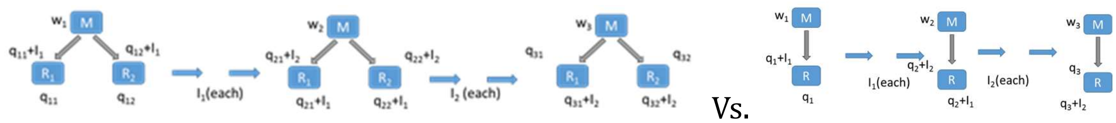

Figure 0.4 Illustration of the comparison between a three-period Cournot duopoly model with SI and a 3-period monopoly downstream model with SI

Table 0.3 Comparison of equilibrium values between three period Cournot duopoly with SI and monopoly downstream model with SI, depending on the holding cost range (the result in column 2 is the opposite for the other side of the holding cost range in column 3)

Quantities compared	Result	Holding cost range	Three-period Cournot duopoly value	Monopoly downstream value
w^*_1	higher	always	a	$\frac{a}{2}-\frac{h}{4}$
w^*_2	lower	always	$\frac{12a-3h}{23}$	a
w^*_3	lower	always	$\frac{24h+19a}{46}$	a
$\Pi M_1 + \Pi M_2 + \Pi M_3$	lower	always	$\frac{792h^2-126ah+1586a^2}{4761b}$	$\frac{20a^2-4ah+h^2}{16b}$
Q^*_1	higher	always	$\frac{2a}{9b}+\frac{2h}{3b}$	$\frac{2a-h}{4b}$
q^*_1	lower	always	0	$\frac{a}{4b}+\frac{h}{8b}$
Q^*_2	lower	always	$\frac{44a-264h}{207b}$	$\frac{a}{b}$
q^*_2	lower	always	$\frac{6h+22a}{69b}$	$\frac{2a-3h}{8b}$
Q^*_3	same	always	0	0
q^*_3	lower	always	$\frac{8a-48h}{69b}$	$\frac{a}{b}$
I^*_1	higher	always	$\frac{2a}{9b}+\frac{2h}{3b}$	$\frac{2a-3h}{8b}$
I^*_2	lower	always	$\frac{8a-48h}{69b}$	$\frac{a}{b}$
p^*_1	higher	always	a	$\frac{6a-h}{8}$
p^*_2	lower	always	$\frac{47a-6h}{69}$	$\frac{6a+3h}{8}$
p^*_3	higher	always	$\frac{61a+48h}{69}$	0
$\Pi R_1 + \Pi R_2 + \Pi R_3$	lower	always	$-\ \frac{8154h^2+16476ah+1898a^2}{14283b}$	$\frac{19h^2-36ah-20a^2}{64b}$

From Table 3.3 we see that the manufacturer profits and retailer profits are lower in a Cournot duopoly downstream with SI allowed, compared to a monopoly downstream, over three selling seasons. This is similar to the result obtained in Section 3.1 over two selling seasons, further bolstering the evidence that double marginalization erodes profits in the presence of downstream Cournot competition, irrespective of the number of selling seasons.

We observe that the manufacturer sets a higher wholesale price in the first period and lower in the two subsequent ones in a Cournot duopoly downstream compared to a monopoly downstream. SI is carried at a higher level, from the 1st to the 2nd period and is carried at a lower level from the 2nd to the 3rd period.

We compare the average wholesale and retail prices over the three periods combined. $w^{avg.}{}_{3PDCournotSI} = -\frac{396h^2-63ah+793a^2}{1449h-1035a}$(39); $w^{avg}{}_{3PDMonopolySI} = \frac{-(h^2-4ah+20a^2)}{4h-24a}$ (40).

(39)- (40) yields: $\frac{-(135\ \ ^3-2925ah^2-28436ha^2+16\ \ ^3)}{5796\ \ ^2-38916ah+2484a^2}$ (41). Ignoring the h terms from (41) since a>>h, we see that (41) <0 always.

Hence we conclude that the manufacturer mostly sets a lower wholesale price on average, over the three selling seasons, when supplying to a Cournot duopoly downstream versus a monopoly over three selling seasons.

We compute average retail prices similarly to obtain:

$p^{avg}{}_{3PdCournotSI} = \frac{1170h^2+1197ah-761a^2}{1449h-1035}$(43); $p^{avg}{}_{staticCournot} = \frac{5h^2+4ah-12a^2}{8h-16}$ (44).

Subtracting the two equations above: (43)-(44) yields:

$$\frac{-(2115h^3-9765ah^2-3712a^2h-244a^3)}{11592h^2-31464ah+16560a^2}$$

(45). we observe that (45) > 0, when a>>h, which mostly holds. Thus we can say that retail prices are mostly higher in a Cournot duopoly downstream with SI allowed compared to a monopoly downstream over three selling seasons.

Next, we ascertain the impact of Cournot competition on Consumer Surplus over three selling seasons. Computing Consumer Surplus for the two cases compared we have:

$CS_{3PDCournotSI} = \frac{2340h^2+(3312ab+2808\quad)h+4209a^2b-4a^2}{9522b}$ (46) and $CS_{Monopoly} = \frac{5h^2-4ah+4a^2}{64b}$ (47).

Subtracting the two equations above, (46)-(47) yields:

$\frac{(51075h^2+(105984ab+108900ah+13468\quad{}^2b-19172a^2}{304704b}$(48)

We see readily that (48) is < 0 when a>>h, which mostly holds. This implies that, Consumer Surplus is mostly lower in a Cournot duopoly downstream with SI allowed over three selling seasons.

In summary, we see that the effect of competition over three selling seasons with SI carriage allowed between periods, mirrors the results for the two-period case. Like we saw, over two selling seasons in Section 3.1, manufacturer and retailer profits are both uniformly lower in the presence of a Cournot competing retailer downstream versus a monopoly downstream, over three selling seasons. Consumer Surplus is also lower, as in the two-selling season model. The retail prices are higher compared to the monopoly downstream supply chain, but like in the two-period case, the higher retail prices, fail to translate into higher profits for the retailer in aggregate. Average wholesale prices are lower in the 3-selling season model with a Cournot duopoly downstream, than the two-selling season model, where they are higher than the corresponding monopoly downstream model with two selling seasons as well. This seems to indicate that, as the number of selling seasons increases, this has a depressive effect on the average wholesale price, with all other things being equal.

3.2.2 Effect of Strategic Inventory in a Cournot Competing duopoly downstream supply chain over three selling seasons.

In Table 3.4, we present a comparison of a three-period model with SI allowed to a three-period static model (one-selling season model with no SI allowed, replicated thrice). This comparison allows us to get an insight into which of the effects of SI allowance in a Cournot duopoly downstream are sustained when the number of selling seasons goes from two to three. Figure 3.5 below illustrates this comparison.

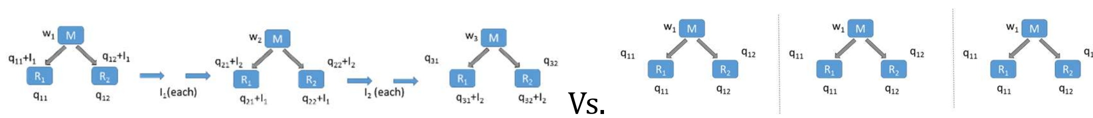

Figure 0.5 Illustration of the comparison between a three period Cournot duopoly model with SI and a Static Cournot duopoly model over three periods

Table 0.4 Comparison of equilibrium values between three period Cournot duopoly with SI and static Cournot duopoly (no SI) depending on the holding cost range (the result in column 2 is the opposite for the other side of the holding cost range in column 3)

Quantities compared	Result	Holding cost range	3-period Cournot duopoly value	Static Cournot value
w^*_1	higher	always	a	0.5a
w^*_2	lower	always	$\frac{12a - 3h}{23}$	0.5a
w^*_3	Lower	always	$\frac{24h + 19a}{46}$	0.5a
$\Pi M_1 + \Pi M_2 + \Pi M_3$	lower	always	$\frac{792h^2 - 126ah + 1586a^2}{4761b}$	$\frac{a^2}{3b}$
Q^*_1	lower	always	$\frac{2a}{9b} + \frac{2h}{3b}$	$\frac{0.334a}{b}$

q^*_1	lower	always	0	$\frac{0.334a}{b}$
Q^*_2	lower	always	$\frac{44a-264h}{207b}$	$\frac{a}{3b}$
q^*_2	Lower	h≤a/6	$\frac{6h+22a}{69b}$	$\frac{a}{3b}$
Q^*_3	lower	always	0	$\frac{a}{3b}$
q^*_3	lower	Always	$\frac{8a-48h}{69b}$	$\frac{a}{3b}$
I^*_1	higher	always	$\frac{2a}{9b}+\frac{2h}{3b}$	0
I^*_2	higher	always	$\frac{8a-48h}{69b}$	0
p^*_1	higher	Always	a	0.67a
p^*_2	lower	h≤a/6	$\frac{47a-6h}{69}$	0.67a
p^*_3	higher	Always	$\frac{61a+48h}{69}$	0.67a
$\Pi R_1 + \Pi R_2 + \Pi R_3$	lower	Always	$-\frac{8154h^2+16476ah+1898a^2}{14283b}$	$\frac{a^2}{12b}$

From Table 3.4 we see that both manufacturer and retailer profits are always lower in equilibrium when the manufacturer supplies to a Cournot duopoly downstream with SI allowed, over three selling seasons compared to a similar downstream where SI is allowed Static Cournot duopoly model).

We observe that the manufacturer sets a higher wholesale price in the first period than the static solution and lower in the two subsequent ones with a Cournot duopoly downstream. SI is the higher than the static solution (zero) from 1st to 2nd period and higher again than the static solution from the 2nd to 3rd period. We compare the wholesale and retail prices over the three periods combined using the average wholesale and retail price in both cases.

$w^{avg.}{}_{3PDCournotSI}$ = - $\frac{396h^2-63ah+793a^2}{1449h-1035a}$ (49); $w^{avg}{}_{3PDStatticCournot}$ = $\frac{a}{2}$ (50).

Subtracting these two equations above, (49)-(50) yields: $\frac{-(792h^2+1323ah+551a^2)}{2898h-2070a}$(51). We see that (51) is > 0 when a>>h, which holds in most real-world supply chains. Thus, we conclude that the manufacturer mostly sets a higher wholesale prices on average when SI carriage is allowed in a supply chain consisting of a manufacturer supplying product to a Cournot duopoly downstream of retailers over three seasons, compared to when it is not allowed.

We compute weighted average retail prices similarly, to obtain:

$p^{avg}{}_{3PdCournotSI}$ = $\frac{1170\ \ ^2+1197ah-761a^2}{1449h-1035}$(52); $p^{avg}{}_{staticCournot}$ =$\frac{2a}{3}$(53).

Subtracting the two equations above, (52)-(53) yields: $\frac{1170\ \ ^2+231ah-71a^2}{1449h-1035a}$(54). We observe that (55) > 0 (when a>>h). We can hence conclude that retail prices are mostly higher in a Cournot duopoly downstream with SI allowed compared to a static Cournot duopoly downstream (no SI) over three selling seasons. Next, we ascertain the impact on SI allowance in a Cournot duopoly downstream on Consumer Surplus over three selling seasons. Computing Consumer Surplus for the two cases compared we have: $CS_{3PDCournotSI}$ = $\frac{2340h^2+(3312ab+280\ \ \ \ )h+4209a^2b-4a^2}{9522b}$ (56) and $CS_{Monopoly}$ =$\frac{0.167\ a^2}{b}$ (57).

Subtracting the two equations above, (56)-(57) yields:

$\frac{(1170000h^2+(1656000ab+1404000a)h+2104500a^2b-797087a^2}{476100b}$ (58).

We see readily that (59) is <0 always when a>>h, which mostly holds. This implies that, Consumer Surplus is mostly lower in a Cournot duopoly downstream with SI allowed, versus a similar Cournot duopoly with no SI, over three selling seasons.

In summary, over three selling seasons, the effect of allowing SI carriage in a supply chain consisting of a manufacturer supplying to a Cournot duopoly downstream of retailers, mirrors the effects observed when the number of selling seasons is two, in Section 3.2. Both manufacturer and retailer profits are lower, over three selling seasons when SI is allowed than the static one-shot game in each period where no SI carried, as in the two-period case. SI is carried in equilibrium, when it is allowed, in the 3 selling season model, both from period 1-2 and from period 2-3. However, this does not seem to be enough to confer a strategic advantage to the competing downstream retailers, as their profits are consistently lower than the static solution. Consumers also do not seem to be benefited (lower Consumer Surplus) by allowing SI carriage by downstream retailers. Again, this mirrors what is observed in the two-period case, where the consumers are worse-off with SI allowed. Average retail prices in the 3-period model are higher than the static solution, whereas over two-selling seasons, they are lower, implying that the retailers are able to command a higher retail price in the market, on an average, as the number of selling seasons increases from two to three. It is remarkable however, that, this does not translate into higher retailer profits.

3.3 Conclusions

In this chapter, we presented a two-period and three-period model of a supply chain, a manufacturer supplying to a downstream Cournot duopoly of retailers with SI carrying allowed between periods. This problem is formulated as a dynamic game and closed-form equilibrium decisions variable values for both the manufacturer and retailers in every period, are derived.

We find that, the introduction of downstream Cournot duopoly competition of a supply chain consisting of one manufacturer supplying identical product to the retailers leads to lower profits for both the manufacturer and retailer. This holds, whether the number of selling season is two or three. Consumer Surplus is also uniformly lower under downstream retailer competition, compared to a downstream monopoly supply chain. This implies that the introduction of downstream retailer competition leaves all the SC entities – manufacturer, retailer and consumer, worse off, due to the double marginalization effect.

When we try to deduce the effect of SI carriage under Cournot duopoly competition by comparing an SC with Cournot duopoly competition and SI allowed between periods to a similar SC with a Cournot duopoly downstream and a static, repeating one-shot game in each period with no SI carried, we find again that manufacturer and retailer profits are both lower when SI carriage is allowed. This holds whether the number of selling seasons is two or three. Consumer Surplus is also lower uniformly over both two and three selling seasons. This indicates that allowing SI carriage, in the presence of downstream retailer competition does not benefit either the manufacturer or retailer in contrast to what is observed in the monopoly downstream case, where the retailer clearly benefits.

Further investigation is necessary to determine if and how much the effects observed in this research with the interaction of Strategic Inventories and competition are an artifact of the type of downstream retailer competition chosen to be modeled – Cournot. It would be meaningful to examine this system’s behavior under other models of competition like Bertrand and Stackelberg (one of the retailers is the leader) to deduce the effect of the mode of competition on SI and other key performance indicators for the manufacturer and downstream retailers. Another logical extension of the study would be to try to generalize the results for two and three selling seasons to arbitrary selling season lengths and see if and how well the insights derived for two and three selling seasons hold for the “n” period case. It would also be valuable to validate empirically some of the observations from this paper, and others in this line of research, to see if the insights predicted by game-theoretic modeling hold up in the real-world; if yes, how strongly and if not, where the key points of divergence are and how that can inform further theoretical model-building in this area.

4. COURNOT DUOPOLY MODEL UNDER A COMMITMENT CONTRACT

In this chapter, we aim to analyze the impact of a "Commitment contract" on Strategic Inventory in a one-manufacturer two-retailer supply chain, with the two retailers competing as a Cournot duopoly. When the manufacturer agrees to use a "Commitment contract" he/she is agreeing to set the wholesale prices for both the 1st and 2nd period at the beginning of the 1st period itself. Anand et. al. (2008) use this type of contract in a supply chain without downstream retailer competition (one manufacturer supplying to a monopoly downstream retailer) and find that a Commitment contract eliminates Strategic Inventory carriage. In other words, the Commitment contract ends up being a "strategic tool" for the manufacturer to successfully dissuade the downstream retailer from carrying Strategic Inventory. One of our objectives here is to see if the same holds when the downstream is a Cournot-competing retailer duopoly.

4.1 Two-period Cournot duopoly Model with Commitment contract:

The model considered here consists of one manufacturer supplying a product in two consecutive selling seasons to two Cournot duopoly retailers at its downstream. The retailers are allowed to carry inventory between periods. The two retailers are identical in all respects. All product is sold at the end of the 2nd period. Figure 4.1 illustrates this model.

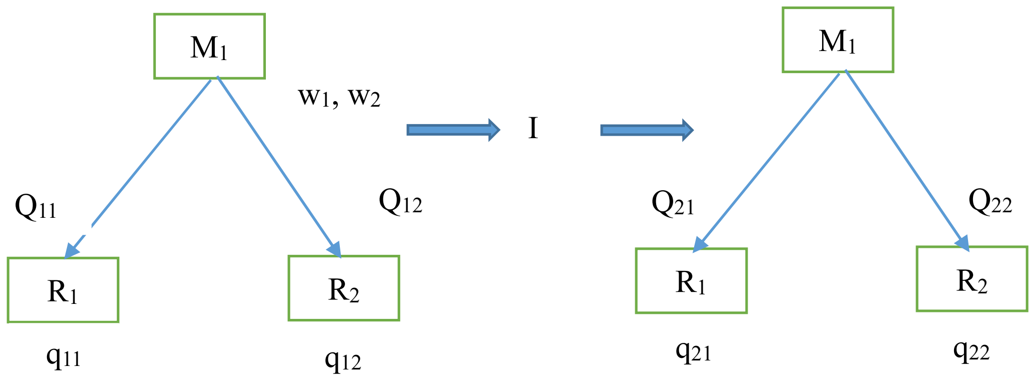

Figure 0.1 Illustration of two-period Cournot duopoly model under Commitment contract

The business flow in the two periods is as follows:

Period 1:

Step 1: The manufacturer announces its 1st period wholesale price (w_1) and 2nd period wholesale price at the beginning of period 1

Step 2: Retailer 1 and Retailer 2 simultaneously announce their 1st period order quantities, which is the sum of the quantity they want to sell in Period 1 (q_{11} and q_{12}, respectively), and the quantity they want to carry to Period 2, I for each of the retailer. Inventories carried to the 2nd period are held at a holding cost of h/unit.

Step 3: The market decides the retail price, p, of each unit of product sold: p $(q_{11}+q_{12})$ = a-b $(q_{11}+q_{12})$.

Step 4: Quantities q_{11} and q_{12} are sold simultaneously by Retailer 1 and Retailer 2 at the price p $(q_{11}+q_{12})$ per unit and revenues are realized. Inventory I is carried by each retailer over to the second period.

Period 2:

Step 1: Retailer 1 and Retailer 2 simultaneously announce their 2nd period order quantities (q_{21} and q_{22}).

Step 2: The market decides the retail price of each unit of product sold: p ($q_{21}+q_{22}+2I$) = a-b ($q_{21}+q_{22}+2I$).

Step3: Quantities (q_{21} + I) and (q_{22} + I) are sold simultaneously by Retailer 1 and Retailer 2 at the price p ($q_{21}+q_{22}$ + 2I) per unit and revenues are realized.

We now determine the profit-maximizing selling quantity in both periods and the wholesale prices of the manufacturer in both periods. We start with the 2nd period retailer decisions.

Nomenclature (all variables are non-negative): **i**: period (i=1,2,3) , **j**: retailer (i=1,2), **a**: model parameter reservation price (a>0) , **b**: model parameter (b>0, a>>b), $\mathbf{q_{ij}}$: buying quantity of retailer j in period i, to be sold in the same period (i.e., excluding SI) **I**: inventory carried by each retailer from Period 1 to Period 2, $\mathbf{w_i}$: wholesale price set by the manufacturer in period i, **h**: holding cost of each retailer to carry one unit of inventory from Period 1 to Period 2, $\mathbf{\Pi R_{ij}}$: profit of retailer j in period I, $\mathbf{\Pi M_i}$: profit of the manufacturer in period i.. Determination of the equilibrium quantities start from the retailer's decision in the second period. The complete work for the determination is shown in the Appendix.

4.1.1 Effect of a Cournot duopoly competition on a Commitment contract over two periods

Anand et al. (2008) find that a Commitment contract completely eliminates SI in a two-period model with one manufacturer supplying to a single downstream retailer. Under a Commitment contract, the manufacturer announces the wholesale price for the 1st and 2nd period at the beginning of the 1st period. In this section, we study the effect of this contract on SI in the case with downstream retailer competition. The equilibrium decision variable values for the manufacturer and retailers under a Commitment contract are as follows:

$w^*_1 = w^*_2 = \frac{a}{2}$, $q^*_{11} = q^*_{12} = \frac{a-w_1}{3b} = \frac{a}{6b}$, $q^*_{21} = q^*_{22} = 0$, $I^* = \frac{a-w_2}{3b} = \frac{a}{6b}$

In Table 4.1, we present a comparison of the equilibrium decision variable values of the Cournot duopoly downstream model under a Commitment contract with those of the one-manufacturer, one-retailer model under a Commitment contract. Figure 4.2 below illustrates this comparison.

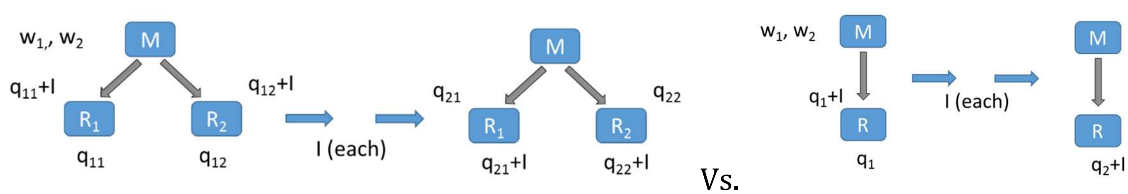

Figure 0.2 *Illustration of the comparison between a two-period Cournot duopoly with SI under Commitment contract and a two-period monopoly downstream with SI under Commitment contract*

Table 0.1 Comparison of equilibrium values between a two period Cournot duopoly with Commitment contract and monopoly downstream model with Commitment contract depending on the holding cost range (the results in column 2 are the opposite for the other side of the holding cost ranges in column 3)

Quantities compared	**Result**	**Holding cost range**	**Cournot duopoly downstream under a Commitment contract**	**Monopoly downstream under a Commitment contract**
w^*_1	same	always	$\frac{a}{2}$	$\frac{a}{2}$
w^*_2	same	always	$\frac{a}{2}$	$\frac{a}{2}$
$\Pi M_1 + \Pi M_2$	higher	always	$\frac{a^2}{3b}$	$\frac{a^2}{4b}$
Q^*_1	higher	always	$\frac{2a}{3b}$	$\frac{a}{4b}$
q^*_1	higher	always	$\frac{a}{3b}$	$\frac{a}{4b}$
Q^*_2	lower	always	0	$\frac{a}{4b}$
q^*_2	higher	always	$\frac{a}{3b}$	$\frac{a}{4b}$
I^*	higher	always	$\frac{a}{3b}$	0
$Q^*_1+Q^*_2$	higher	always	$\frac{2a}{3b}$	$\frac{a}{2b}$
p^*_1	lower	always	$\frac{2a}{3}$	$\frac{3a}{4}$
p^*_2	lower	Always	$\frac{2a}{3}$	$\frac{3a}{4}$
$\Pi R_1 + \Pi R_2$	lower	always	$\frac{a^2}{18b} - \frac{ah}{6b}$	$\frac{a^2}{8b}$

From Table 4.1, we see that the manufacturer's profit is always higher and retailer' profits always lower across the two selling seasons when supplying to a Cournot duopoly downstream than in a monopoly downstream, under a Commitment contract. The introduction of the Cournot competition does not change the manufacturer wholesale price decision under a Commitment contract. The average retail price paid by the consumer is lower in the Cournot duopoly case in both periods. Next, we check the impact of Cournot competition on Consumer Surplus over two selling seasons under a Commitment contract.

Consumer Surplus is the difference between the total amount the consumers are willing to pay for the good and the price they actually pay. Since the maximum possible price of the good is the reserve price "a", the total Consumer Surplus over the two periods is given by CS $= \frac{1}{2}(a - p_1)q_1 + \left(\frac{1}{2}\right)(a - p_2)q_2$. Computing Consumer Surplus for the two cases compared – Cournot duopoly and Monopoly downstream supply chains under a Commitment contract, we have: $CS_{CournotDuopoly} = \frac{a^2}{9b}$ (1) and $CS_{Monopoly} = \frac{3a^2}{16b}$ (2). From (1) and (2) above, we see that (1) < (2) always, i.e., the consumer surplus in a Cournot duopoly downstream is less under a Commitment contract, compared to a monopoly downstream under the same contractual structure.

In summary, over the two selling seasons, the manufacturer ends up with an advantage, making a higher profit under a Commitment contract with downstream retailer competition, than compared to supplying to a monopoly downstream under the same contract. The retailers, while competing as a Cournot duopoly, are not able to use the relative advantage that comes from a Commitment contract –i.e., knowing the 2nd period wholesale price before-hand to buy product and carry inventory strategically from the 1st period to the 2nd to make a higher profit, as they are, when the downstream. The consumer also seems to be disadvantaged more with the introduction of downstream Cournot competition under a Commitment contract. This can be attributed to increased double-marginalization effect. Strategic Inventory is carried at a finite (and hence) higher level under a Commitment contract when the downstream is a Cournot duopoly rather than a monopoly.

4.1.2 Commitment contract vs. dynamic contract in the presence of Cournot duopoly Competition

In Table 4.2, we present a comparison of the equilibrium values of our model with dynamic two-period model with a Cournot duopoly downstream (without a commitment contract). This comparison shows the effect of a Commitment contract on SI with a Cournot competing retailer downstream. Fig. 4.3 illustrates this comparison.

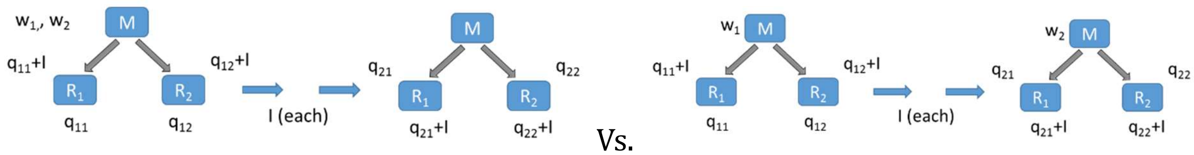

Figure 0.3 Illustration of the comparison between a two-period Cournot duopoly model with SI under a Commitment contract and a two-period Dynamic Cournot duopoly model with SI.

Table 0.2 Comparison of equilibrium values between a two period Cournot duopoly with Commitment contract and a Dynamic two-period Cournot duopoly model depending on the holding cost range (the results in column 2 are the opposite for the other side of the holding cost ranges in column 3)

Quantities compared	**Result**	**Holding cost range**	**2-period Cournot duo-poly model under a Commitment contract**	**Dynamic 2-period Cournot duopoly model**
w^*_1	lower	$h \le \frac{a}{6}$	$\frac{a}{2}$	$\frac{12a-3h}{23}$
w^*_2	lower	always	$\frac{a}{2}$	$\frac{19a+24h}{46}$
$\Pi M_1 + \Pi M_2$	higher	always	$\frac{a^2}{3b}$	$\frac{120a^2-198ah+42h^2}{529b}$
Q^*_1	higher	always	$\frac{2a}{3b}$	$\frac{10a-14h}{23b}$
q^*_1	higher	always	$\frac{2a}{3b}$	$\frac{22a+6h}{69b}$
Q^*_2	same	always	0	0
q^*_2	higher	always	$\frac{a}{3b}$	$\frac{8(a-6h)}{69b}$
I^*	higher	always	$\frac{a}{3b}$	$\frac{8(a-6h)}{69b}$
$Q^*_1+Q^*_2$	higher	always	$\frac{2a}{3b}$	$\frac{10a-14h}{23b}$

p^*_1	lower	h≤95a/18	$\frac{2a}{3}$	$\frac{(47a-6h)}{23}$
p^*_2	lower	always	$\frac{2a}{3}$	$\frac{61a+48h}{69}$
$\Pi R_1 + \Pi R_2$	lower	always	$\frac{a^2}{18b}-\frac{ah}{6b}$	$\frac{594h^2-1164ah+442a^2}{4761b}$

From Table 4.2 we see that manufacturer makes a higher profit, and retailers, lower, under a Commitment contract. We observe that the manufacturer mostly sets a lower wholesale price in the first period (when h≤ $\frac{a}{6}$; which mostly holds) and lower wholesale price in the second period, as well, when supplying under a Commitment contract versus without to a Cournot duopoly of retailers with SI allowed. SI is carried at a higher level under a Commitment contract, implying that the Commitment contract is not effective in dissuading (or completely eliminating as in the monopoly downstream case (Anand et al. 2008)) strategic inventory carriage in the presence of downstream retailer competition.

We compare the wholesale price over the two periods combined. In each case, the average is a weighted average, weighted by the quantity bought in each period i.e.:

$w^{avg}{}_{\cdot CommitmentCournot} = \frac{a}{2}$ (3); $w^{avg}{}_{\cdot DynamicCournot} = \frac{12a-3}{23}$ (4)

Comparing (3) and (4), we can write (3) < (4) always and hence conclude that the average equilibrium manufacturer wholesale prices are lower under a Commitment contract than a Dynamic contract, when the downstream is a competing Cournot duopoly of retailers.

We similarly compute weighted average retail prices, to obtain:

$p^{avg}{}_{CommitmentCournot} = \frac{2a}{3}$ (5); $p^{avg}{}_{DynamicCournot} = p^{avg}{}_{Cournot} = \frac{-(241\ \ {}^2+2094ah-3590a^2)}{69(38a-90h)}$ (6)

Subtracting the two terms we computed above, (5)-(6) yields: $\frac{-804h^2+697ah+614a^2}{2070h-874}$.

Ignoring h terms, we get $\frac{-614\ \ ^2}{-874a}$>0 always since a > 0. We can hence conclude that the consumers pay a higher price on average, over the two selling seasons under a Commitment contract in the presence of downstream Cournot competition.

Next, we ascertain the impact of introduction of SI in a Cournot duopoly downstream on Consumer Surplus. Computing Consumer Surplus for the two cases compared we have: $CS_{CommitmentCourot} = \frac{2a^2}{9b}$ (35) and $CS_{DynamicCournot} = \frac{1206\ \ ^2-402ah-760a^2}{4761b}$ (36). (35) - (36) yields: $\frac{-402h^2+134ah+606\ \ ^2}{1587b}$ (37). Ignoring h terms in (38) we get: $\frac{606a^2}{1587b}$ which is >0 always. We hence conclude that Consumer Surplus is higher under a Commitment contract compared to a Dynamic two-period ordering model with SI allowed between periods, in a Cournot duopoly downstream.

The manufacturer makes a higher profit and retailers lower under a Commitment contract compared a Dynamic contract in the presence of downstream Cournot duopoly competition. Consumers benefit under a Commitment contract vs. Dynamic contract since consumer surplus is higher under a Commitment contract.

4.2 Conclusions

In this chapter, we have presented a two-period model of a manufacturer supplying identical product to a Cournot duopoly downstream of retailers under a "Commitment contract" regime, where the manufacturer quotes the wholesale price for both periods, at the beginning of the 1st period itself. We derive closed-form equilibrium values for all decision variables using game-theoretic modeling.

We find, over the two selling seasons, the manufacturer ends up with an advantage, making a higher profit under a Commitment contract with downstream retailer competition, than compared to supplying to a monopoly downstream under the same contract. The retailers, while competing as a Cournot duopoly, are not able to use the relative advantage that comes from a Commitment contract – i.e.,, knowing the 2nd period wholesale price before-hand, to buy product and carry inventory strategically from the 1st period to the 2nd to make a higher profit, as they are, when the downstream. The consumer also seems to be disadvantaged more with the introduction of downstream Cournot competition under a Commitment contract. This can be attributed to increased double-marginalization effect. Strategic Inventory is carried at a finite (and hence) higher level under a Commitment contract when the downstream is a Cournot duopoly rather than a monopoly. When we compare a manufacturer supplying to a Cournot duopoly downstream of retailers, with, and without a Commitment contract (dynamic ordering), we see that the consumer benefits under a Commitment contract, and that, consumer surplus is higher. The manufacturer makes a higher profit and retailers lower, under a Commitment contract over two selling seasons in the presence of downstream retailer competition. In summary, over two selling seasons, the manufacturer benefits under a commitment contract and the retailers are at a disadvantage.

Development of contracts that are more effective than a Commitment contract in coordinating this supply chain – increasing consumer surplus and further increasing manufacturer and retailer profits can be explored using the results presented in this work. It would also be interesting to run empirical studies in real-world supply chains to validate if and how much the insights developed by this kind of game-theoretic modeling hold in a real-world supply chain setting.

5. CONCLUSIONS AND FUTURE WORK

In Chapter three, we presented a two-period and three-period model of one manufacturer supplying identical product to a Cournot duopoly of retailers with SI carrying allowed between periods. We formulated this problem as a dynamic game and derived closed-form equilibrium decisions variable values for both the manufacturer and retailers in every period. In Chapter four, we presented two and three period Cournot duopoly downstream models with SI, under a "Commitment contract" where the manufacturer commits to the wholesale price for every selling season, at the beginning of the first.

We find that, the introduction of Cournot duopoly competition in the downstream of a supply chain consisting of one manufacturer supplying identical product to the retailers leads to lower profits for both the manufacturer and retailer. This holds, whether the number of selling season is two or three. Consumer Surplus is also uniformly lower under downstream retailer competition, compared to a downstream monopoly supply chain. This implies that the introduction of competition leaves all the SC entities – manufacturer, retailer and consumer, worse off, due to the double marginalization effect.

When we try to deduce the effect of SI carriage under Cournot duopoly competition, by comparing an SC with Cournot duopoly competition and SI allowed between periods, to a similar SC with a Cournot duopoly downstream and a static, repeating, one-shot game in each period, with no SI carried – we find again that manufacturer and retailer profits are both lower when SI carriage is allowed. This holds whether the number of selling seasons is two or three. Consumer Surplus is also lower uniformly over both two and three selling seasons. This indicates that allowing SI carriage, in the presence of downstream retailer competition does not benefit either the manufacturer or retailer in stark contrast to what is observed in the monopoly downstream case, where the retailer clearly benefits.

Under a Commitment contract, over two selling seasons, as we see in Chapter 4, the manufacturer ends up with an advantage, making a higher profit with downstream retailer competition, than compared to supplying to a monopoly downstream under the same contract. The retailers, while competing as a Cournot duopoly, are not able to use the relative advantage that comes from a Commitment contract – i.e., knowing the 2nd period wholesale price before-hand, to buy product and carry inventory strategically from the 1st period to the 2nd to make a higher profit, as they are, when the downstream is a single retailer monopoly. The consumer also seems to be disadvantaged more with the introduction of Cournot competition under a Commitment contract. This can be attributed to increased double-marginalization effect. Strategic Inventory is carried at a finite (and hence) higher level under a Commitment contract when the downstream is a Cournot duopoly rather than a monopoly.

When we compare a manufacturer supplying to a Cournot duopoly downstream of retailers, with, and without a Commitment contract (dynamic ordering), we see that the manufacturer, retailers and consumer all benefit under a Commitment contract in the presence of downstream Cournot competition. The manufacturer and retailer make higher profits under a Commitment contract compared a Dynamic contract in the presence of downstream Cournot duopoly competition. However, this profit for the retailers is less than what they would make, if the downstream was a single retailer monopoly under the same Commitment contract. So, we can say that, though the retailers still benefit under a Commitment contract under downstream Cournot competition, the magnitude of benefit is diminished by the presence of competition. Consumers benefit under a Commitment contract vs. Dynamic contract since consumer surplus is higher under a Commitment contract.

We see from the previous sections that, under a commitment contract (compared to dynamic ordering) in the presence of downstream Cournot competing retailers with SI allowed, the manufacturer is benefitted, making a higher profit than dynamic ordering. This is a counter-intuitive result since one would expect the manufacturer to make a lesser profit in equilibrium under a commitment contract, since he/she locks him/her self up to the 2nd period wholesale price decision at the beginning of the 1st period itself, and does not afford him/her self the opportunity to quote a 2nd period wholesale price, in response to the manufacturer decisions in the 1st period like total quantity ordered, quantity sold in 1st period, quantity carried as Strategic Inventory etc. We however see such counter-intuitive findings in closely related literature. For example, Anand et al. (2008) argue that the whole idea of "Strategic Inventory" being carried in equilibrium, at a finite holding cost h/unit, being a drain on channel profits, is counter-intuitive, and it is inexplicable why SI will be carried, when the traditional reasons for inventory carriage are absent, and the retailer has an opportunity to buy product again in the 2nd period. However, they then explain the counter-intuitive result as a strategy by the retailer to hedge against the uncertainty in 2nd period wholesale price and further demonstrate it works, since once that uncertainty is removed (through a commitment contract) such inventory carrying disappears in equilibrium.

Further investigation is necessary to determine if and how much the effects observed in this research with the interaction of Strategic Inventories, contractual structures, and competition are an artifact of the type of downstream retailer competition chosen to be modeled – Cournot. It would be meaningful to examine this system's behavior under other models of competition like Bertrand and Stackelberg (one of the retailers is the leader) to deduce the effect of the mode of competition on SI and other key performance indicators for the manufacturer and downstream retailers.   Another logical extension of this work would be generalizing the results from two and three selling seasons to see if and how well the insights derived for two and three selling seasons hold for the "n" period case. Running empirical studies in real-world supply chains may validate if and to what extent the insights developed by this kind of game-theoretic modeling hold in a real-world supply chain setting.

Development of contracts that are more effective than a Commitment contract in coordinating this supply chain – increasing consumer surplus and bolstering manufacturer and retailer profits would also be a valuable and interesting extension of the research presented in this dissertation.

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APPENDIX

1. Static one-period Cournot duopoly with vertical control

Nomenclature:

i*: period, *j*: retailer, *a*:** model parameter reservation price (a>0) ***, b: model parameter (b<a), ***q_{0j}***= order quantity of retailer *j* in period 0 (there is only one period in this model and we denote that as period *0* for convenience), ***w_i*** :wholesale price set by the manufacturer in period i (i=1,2), ***ΠR_{0i}*** : profit of retailer *i* in period 0 ,***ΠM_0*** : profit of manufacturer in period 0.

Following is the analysis of the static single period game, using backward induction. The game starts with the manufacturer announcing a wholesale price *w,* to which the retailers respond with order quantities of q_{o1} and q_{o2} respectively. In keeping with the principle of backward induction, we start with the analysis of the retailers' decisions first.

1.1: Retailers' decisions

Given: (Assumptions)

$0 < w < a, h > 0 , a > 0$ (1-001)

Decision variables: q_{o1}

Requirement (constraints) on the decision variables.

$a - b(q_{o1} + q_{o2}) >= 0$ (1-002)

$q_{01} >= 0$ (1-003)

We can write the profit-function for retailer 1 as:

$$\Pi R_{o1} = (a - b(q_{01} + q_{02}))(q_{01}) - (w q_{01}) \quad (1\text{-}004)$$

$$\frac{\partial \Pi R_{01}}{\partial q_{01}} = a - b(2q_{01} + q_{02}) - w \quad (1\text{-}005)$$

$$\frac{\partial^2 \Pi R_{01}}{\partial q_{01}^2} = -2b < 0 \quad (1\text{-}006)$$

(1-006) shows that (1-004) is concave in q_{o1}.

Setting (1-005) to zero, we get the profit-maximizing retailer 1 order quantity as:

$$q^o_{01} = \frac{a - w}{2b} - \frac{q_2}{2} \quad (1\text{-}007)$$

Rearranging (1-002), we get: $q_{o1} <= \frac{a}{b} - q_{02}$ (1-008)

We need (1-007) to fulfill (1-008) and (1-003) which leads to the following two sub-cases:

Case 3.21(a): If q^o_{01}(1-007)<0, $q^*_{01} = 0$ (1-009)

Case 3.21(b): If q^o_1(1-008) >=0, $q^*_{01} = \frac{a - w}{2b} - \frac{q_2}{2}$ (1-010)

Similarly, for retailer 2 we can write:

$q^{o}_{02} = \frac{a-w}{2b} - \frac{q_{01}}{2}$ (1-011)

Case 3.21(c): If q^{o}_{02}(1-011)<0, $q^*_{02} = 0$ (1-012)

Case 3.21(d): If q^{o}_{02}(1-011) >=0, $q^*_{02} = \frac{a-w}{2b} - \frac{q_{01}}{2}$ (1-013)

3.22: Combined equilibrium analysis:

From (1-009), (1-010), (1-012) and (1-013), we see that there are two possible equilibria for the retailers' decisions:

Case 3.21(s-i): If $\frac{a-w}{2b} < 0$, $q^*_{01} = q^*_{02} = 0$ (1-014)

(1-014) always violates (1-001) and hence it's impossible.

Case 3.21(s-ii): If $\frac{a-w}{2b} \geq 0$,

$q^*_{01} = q^*_{02} = \frac{a-w}{3b}$ (1-015)

We see that Case 3.21(s-ii), always occurs and hence, there is only one equilibrium for the retailer's order quantities i.e., $q^*_{01} = q^*_{02} = \frac{a-w}{3b}$ (1-016)

3.23 Manufacturer's wholesale price decision:

Given:

$a > 0$, $q_{o1}, q_{o2} >= 0$ (1-017)

Decision variable: w

Requirement (constraints) for the decision variables.

$0 <= w <= a$ (1-018)

The objective function is to maximize the manufacturer's period profit given in (1-019)

$$\mathit{\Pi M_2} = w(2^* \frac{a - w}{3b}) \quad (1\text{-}019)$$

$$\frac{\partial \Pi M}{\partial w} = \frac{2a}{3b} - \frac{4w_2}{3b} \quad (1\text{-}020)$$

$$\frac{\partial^2 \mathit{\Pi M_2}}{\partial w_{2^2}} = \frac{-4}{3b} < 0 \quad (1\text{-}021)$$

$$w^* = w^o = \frac{a}{2} > 0 \quad (1\text{-}022)$$

Summary:

Manufacturer's wholesale price decision:

$$w^* = \frac{a}{2} \quad (1\text{-}022)$$

Retailers' order quantity decisions:

$$q^*_{01} = q^*_{02} = \frac{a - w}{3b} = \frac{a}{6b} \quad (1\text{-}023)$$

2. Two period-model – one manufacturer supplying identical product to two identical Cournot duopoly downstream retailers

Nomenclature (all variables are non-negative):

***i*:** period

***j*:** retailer

***a*:** model parameter reservation price (a>0)

b: model parameter (b>0)

q_{ij} = order quantity of retailer *j* in period *i* (i = 1,2; j = 1,2)

I_j : inventory carried by retailer *j* from period 1 *to* period *2*

w_i : wholesale price set by the manufacturer in period i (i=1,2)

***h**:* holding cost of each retailer to carry one unit of inventory from period 1 to period 2 (it is assumed that both retailers have the same holding cost)

ΠR_{ij} : profit of retailer j in period I

ΠM_i : profit of manufacturer in period *i.*

We solve this two-period game by backward induction i.e., solving the second period game first. Since the two retailers are identical in all respects and have equal holding costs, we can say that, in equilibrium, they carry equal amounts of inventory from the 1st period to the second. i.e., $I_1=I_2 = I$

Retailer 1's second period decisions

Given: (Assumptions)

$0< w_2< a$ (2-1)

$I_1, I_2\geq 0,\ , h>0$ (2-2)

$a\text{-}b\ (I_1+I_2) \geq 0$ OR $a\text{-}2bI \geq 0$ (2-3)

Decision variables: q_{21}

Requirement (constraints) on the decision variables.

$a\text{-}b(q_{21}+q_{22}+2I)\geq 0$ (2-4)

$q_{21} \geq 0$ (2-5)

We can write the 2nd period profit function for retailer 2 as:

$$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2\text{I}))(q_{21} + I) - (w_2 q_{21}) \quad (2\text{-}6)$$

$$\frac{\partial \Pi R_{21}}{\partial q_{21}} = a - b(2\text{q}_{21} + q_{22} + 3\text{I}) - w_2 \quad (2\text{-}7)$$

$$\frac{\partial^2 \Pi R_{21}}{\partial q_{21}^2} = -2\text{b} < 0 \quad (2\text{-}8)$$

(2-8) shows that (2-6) is concave with respect to q_{21}

Setting (2-7) to zero, we get the profit-maximizing 2nd period order quantity for retailer 1

as: $q_{21}^{o} = \frac{a-w_2}{2\text{b}} - \frac{q_{22}+3\text{I}}{2}$ (2-9)

We need to check that (2-9) fulfills (2-4) and (2-5)

Re-arranging (2-4), we get:

$$q_{21} \leq \frac{a}{b} - (q_{22} + 2\text{I}) \quad (2\text{-}10)$$

(2-9) can be re-written as:

$$q_{21}^{o} = \left(\frac{a}{2\text{b}} - \frac{q_{22}}{2} - I\right) - \left(\frac{I}{2} + \frac{w_2}{2\text{b}}\right) \leq \frac{a}{2\text{b}} - \frac{1}{2}(q_{22} + 2\text{I}) \leq \frac{a}{b} - (q_{22} + 2\text{I}) \quad (2\text{-}11)$$

From (2-11), we can say that (2-9) always fulfills (2-10) and (2-2).

Next, we only need to check that (2-9) fulfills (2-5), which leads to the following sub-cases for the optimal q^*_{21} decision:

Case 1.1(a): If q^{o}_{21} (9) ≤ 0, $q^*_{21} = 0$ (2-12)

Case 1.1(b): If q^{o}_{21} (9) > 0, $q^{*}_{21} = q^{o}_{21}(9) = \frac{a-w_2}{2b} - \frac{q_{22}+3}{2}$ (2-13)

<u>Retailer 2's 2nd period order quantity decisions</u>

Given: (Assumptions)

$I_1, I_2 \geq 0,\ 0 < w_2 < a, h > 0$ (2-14)

$a\text{-}b\ (I_1+I_2) \geq 0$ OR $a\text{-}2bI \geq 0$ (2-15)

Decision variables: q_{22}

Requirement (constraints) on the decision variables.

$a\text{-}b\ (q_{21}+I_1+ q_{22} +I_2) \geq 0$ OR $a\text{-}b(q_{21}+q_{22}+2I) \geq 0$ (2-16)

$q_{22} \geq 0$ (2-17)

We can write the 2nd period profit function for retailer 1 as:

$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2\text{I}))(q_{22} + I) - (w_2 q_{22})$ (2-18)

Since retailer 2 is symmetrical to retailer 1, we can use a procedure similar to the one employed in Section 1.1 to derive retailer 2's 2nd period profit-maximizing order quantity as

$q_{22}^{o} = \frac{a-w_2}{2b} - \frac{q_{21}+3\text{I}}{2}$ (2-19)

For (2-19) to fulfill (2-16) and (2-17), we have the following two sub-cases, again, using a procedure similar to that used while computing Retailer 1's decisions.

Case (a): If q^{o}_{22}(2-19) ≤0, $q^{*}_{22} = 0$ (2-20)

Case (b): If q^{o}_{22}(2-19) > 0, $q^{*}_{22} = q^{o}_{22}$ (2-19) $= \frac{a-w_2}{2b} - \frac{q_{21}+3\text{I}}{2}$ (2-21)

Combined equilibrium analysis – q_{21} and q_{22} decisions

Since the two retailers are identical in all respects, symmetrical, in Cournot competition with each other and take their decisions simultaneously, we can postulate that their equilibrium 2nd period order quantities are equal. In this case, there are only two possible equilibria:

Case (a): q^o_{21}(2-9) ≤0 AND q^o_{22}(2-19) ≤0 (Case 1.1(a) and Case 1.2(a))

Here, $q^*_{21} = q^*_{22} = 0$ (From (2-12) and (2-16) (2-22)

Case (b): q^o_{21}(9) >0 AND q^o_{22}(19) >0 (Case 1.1(b) and Case 1.2(b))

Here, $q^*_{21} = \frac{a-w_2}{2b} - \frac{q_{22}+3I}{2}$ From (2-13) (2-23)

$q^*_{22} = \frac{a-w_2}{2b} - \frac{q_{21}+3I}{2}$ From (2-21) (2-24)

Solving (2-23) and (2-24) together, we obtain,

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I$ (2-25)

Table 1.1: Summary of 2nd period retailer order quantity decisions

Case No.	Domain Conditions	Equivalent Conditions	Equilibrium Decision (q^*_{21},q^*_{22})
S-(a)	q^o_{21}(2-9) ≤0 AND q^o_{22}(2-19) ≤0	$\frac{a-w_2}{3b} - I \leq 0$	0,0
S-(b)	q^o_{21}(2-9) >0 AND q^o_{22}(2-19) >0	$\frac{a-w_2}{3b} - I \leq 0$	$\frac{a-w_2}{3b} - I, \frac{a-w_2}{3b} - I$

2^{nd} period manufacturer wholesale price decisions

Given:

$I \geq 0$ (2-26)

$a-2bI \geq 0$ (2-27)

Decision variable: w_2

Requirement (constraints) for the decision variables.

$0 \leq w_2 \leq a$ (2-28)

The objective function is to maximize the manufacturer's 2nd period profit given in (29)

$$\Pi M_2 = w_2 (q_{21} + q_{22}) \quad (2\text{-}29)$$

Case (1): $a-3bI > w_2$

In this portion of the w^*_2 domain,

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I$ (From (2-25))

Substituting q^*_{21} and q^*_{22} into (2-29) from (2-25) we get:

$$\Pi M_2 = \left(2\frac{a-w_2}{3b} - 2I\right)(w_2) \quad (2\text{-}30)$$

$$\frac{\partial \Pi M_2}{\partial w_2} = \frac{2a}{3b} - \frac{4w_2}{3b} - 2I \quad (2\text{-}31)$$

$$\frac{\partial^2 \Pi M_2}{\partial w_2^2} = \frac{-4}{3b} < 0 \quad (2\text{-}32)$$

(2-32) shows that (2-30) is concave with respect to w_2.

Equating (2-32) to zero, we get the profit-maximizing 2^{nd} period manufacturer wholesale price as

$$w^o_2 = \frac{a-3bI}{2} \quad (2\text{-}33)$$

(2-33) needs to satisfy the constraint (2-28) as well as the domain conditions of Case 1.4 ($a-3bI>w_2$).

We see that (2-33) always fulfills the RHS of (2-33) and also fulfills the condition, $a-3bI > w^o_2$, as long as $w^o_2>0$, which leads to the following sub-cases:

Case (2-a): If w^o_2(2-33) ≤ 0, $w^*_2 = 0$ (2-34)

Case (2-b): If w^o_2(2-33)>0, $w^*_2 = \frac{a-3bI}{2}$ (2-35)

Case (2): If $a-3bI \leq w_2$, we see from Table 1.1 that neither retailer orders and w^*_2 can be set at any value such that fulfills (2-28) as well as $w_2 \geq a-3bI_1$

This leads to the following two-cases for the optimal w^*_2 decision in this case:

Case 2(a): If w^o_2(2-33)≤ 0, $w^*_2 = 0$ (2-36)

Case 2(b): If w^o_2(2-33) > 0, w^*_2 = any w_2 that fulfills $a-3bI_1 \leq w_2 \leq a$ (2-37)

Table 1.2: Summary of 2nd period manufacturer wholesale price decisions

Case No.	Domain Conditions	w^*_2
1(a)	$I \geq \frac{a}{3b}$	0
1(b)	$I < \frac{a}{3b}$	$\frac{a-3bI}{2}$
2(a)	$a-3bI \leq 0$ OR $I \geq \frac{a}{3b}$	0
2(b)	$a-3bI>0$ OR $I < \frac{a}{3b}$	Any w^*_2 such that $a-3bI_1 \leq w^*_2 \leq a$

From Table 1.2, we see that when $I \geq \frac{a}{3b}$, the equilibrium w^*_2 decision is always $w^*_2 = 0$.

When $I < \frac{a}{3b}$, the manufacturer has a choice of either setting any w^*_2 such that $a-3bI_1 \leq w^*_2 \leq a$ OR $w^*_2 = \frac{a-3bI}{2}$, when $I < \frac{a}{3b}$.

We postulate that the equilibrium decision in this case will be $w^*_2 = \frac{a-3bI}{2}$, as this decision will lead to a higher profit for the manufacturer (both retailers order), versus when $a-3bI_1 \leq w^*_2 \leq a$ (neither retailer orders), in which case his profit is zero. This decision is consistent with the Cournot conjecture that each entity always acts to maximize its own profit. Hence, there are two possible equilibrium values in this case, depending on the I value from the 1st period:

Case (s-i): If $\frac{a}{3b} \leq I \leq \frac{a}{2b}$, $w^*_2 = 0$ (2-38)

Case (s-ii): If $I < \frac{a}{3b}$, $w^*_2 = \frac{a-3bI}{2}$ (2-39)

We further note that I is never greater than $\frac{a}{2b}$ (from (2-27))

<u>1st period order quantity decisions for retailer 1</u>

Given: (Assumptions)

$a, b, h > 0$ (2-40)

$0 \leq w_1 + h < a$ (2-41)

$q_{12} \geq 0$ (2-42)

Decision variables: q_{11}, I

Requirement (constraints) on the decision variables.

$q_{11} \geq 0$ (2-43)

$a - b(q_{11} + q_{12}) \geq 0$ (2-44)

$I_1 \geq 0$ (2-45)

$a - 2bI \geq 0$ (2-46

We can write the 1st period profit function for retailer 1 as:

$\Pi R_{11} = (a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I) - hI$ (2-47)

The 1st period problem for retailer 1 is to set a q_{11} and I to maximize the sum of 1st and 2nd period profits. i.e., retailer 1 needs to maximize:

$\Pi R_{11} + \Pi R_{12}$

There are two possible sub-cases here, based on the 2nd period wholesale price and order quantities (which determine 2nd period profits)

Case (1): $I < \frac{a}{3b}$

In this case, from Table 1.2, $w^*_2 = \frac{a - 3bI}{2}$ and

$q^*_{21} = q^*_{22} = \frac{a - w_2}{3b} - I = \frac{a}{6b} - \frac{I}{2}$ (2-48)

So, the 2nd period profit function becomes:

$$\Pi R_{21} = (\text{a-b}(\frac{a}{3\text{b}} + I))(\frac{a}{6\text{b}} + \frac{I}{2}) - (\frac{a-3\text{bI}}{2})*(\frac{a}{6\text{b}} - \frac{I}{2}) = (\frac{2\text{a}}{3} - bI)(\frac{a}{6\text{b}} + \frac{I}{2}) - (\frac{a-3\text{bI}}{2})(\frac{a}{6\text{b}} - \frac{I}{2}) \qquad (2\text{-}49)$$

Hence, retailer 1's 1st period problem becomes to max:

$$\Pi R_{11} + \Pi R_{21} = (a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I) - hI + (\frac{2\text{a}}{3} - bI)(\frac{a}{6\text{b}} + \frac{I}{2}) - (\frac{a-3\text{bI}}{2})(\frac{a}{6\text{b}} - \frac{I}{2}) \qquad (2\text{-}50)$$

$$\frac{\partial(\Pi R_{11} + \Pi R_{21})}{\partial q_{11}} = a - 2\text{b}\text{q}_{11} - bq_{12} - w_1 \qquad (2\text{-}51)$$

$$\frac{\partial^2(\Pi R_{11} + \Pi R_{21})}{\partial q_{11}^2} = -2\text{b} \qquad (2\text{-}52)$$

$$\frac{\partial(\Pi R_{11} + \Pi R_{21})}{\partial I} = -(w_1 + h) + \frac{2\text{a}}{3} - \frac{5\text{b}}{2}I \qquad (2\text{-}53)$$

$$\frac{\partial^2(\Pi R_{11} + \Pi R_{21})}{\partial I^2} = \frac{-5\text{b}}{2} \qquad (2\text{-}54)$$

(2-52) and (2-54) show that (2-50) is concave in q_{11} and I respectively.

Setting (2-51) and (2-53) to zero respectively, we get the profit-maximizing q_{11} and I_1 values for retailer 1 as:

$$q_{11}^o = \frac{a-w_1}{2\text{b}} - \frac{q_{12}}{2} \qquad (2\text{-}55)$$

$$I^o = \frac{4\text{a}}{15\text{b}} - \frac{2}{5\text{b}}(w_1 + h) \qquad (2\text{-}56)$$

Re-arranging (44), we get:

$$q_{11} \leq \frac{a}{b} - q_{12} \qquad (2\text{-}57)$$

We observe that: $q_{11}^o = \frac{a-w_1}{2\text{b}} - \frac{q_{12}}{2} \leq \frac{a}{2\text{b}} - \frac{q_{12}}{2} \leq q_{11} \leq \frac{a}{b} - q_{12}$ (2-58) i.e., (2-55) always fulfills (2-57) and it's hence enough to check that (55) fulfills (43), which leads to the following sub-cases for the optimal q^*_{11} decision:

Case 1(a): If q^o_{11}(2-55)≤0, $q^*_{11} = 0$ (2-59)

Case 1(b): If q^o_{11}(2-55) > 0, $q^*_{11} = q^o_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (2-60)

Also, I^o_1 (2-56) needs to fulfill (2-45) and (2-46).

we see that I^o_1(2-56) fulfills (2-45) (using (2-41)).

Also, we see that

I^o_1(2-57) $\leq \frac{a}{3b} = \frac{5a}{15b}$. (2-61)

Hence I^o_1(2-57) fulfills all the domain conditions and hence,

$I^*_1 = I^o_1$(2-57)$=\frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$ (2-62)

Case (2): $\frac{a}{3b} \leq I \leq \frac{a}{2b}$

Neither retailer orders in the 2nd period in this case, though $w^*_2 = 0$.

Hence, the 2nd period retailer 1 profit is only gained from selling the inventory carried from the 1st period i.e., ΠR_{21} = (a-2bI)(I)

Hence, $\Pi R_{11} + \Pi R_{21} = (a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I) - hI +$ (a-2bI)(I) (2-63)

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}} = a - 2\mathrm{b}\mathrm{q}_{11} - bq_{12} - w_1 \quad (2\text{-}64)$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}^2} = -2\mathrm{b} \quad (2\text{-}65)$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial I} = -(w_1 + h) + a - 4\mathrm{bI} \quad (2\text{-}66)$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21})}{\partial I^2} = -4\mathrm{b} \quad (2\text{-}67)$$

(2-65) and (2-67) show that (2-63) is concave in q_{11} and I respectively.

Setting (2-64) and (2-66) to zero respectively, we get the profit-maximizing q_{11} and I_1 values for retailer 1 as:

$$q_{11}^{o} = \frac{a-w_1}{2\mathrm{b}} - \frac{q_{12}}{2} \tag{2-68}$$

$$I^{o} = \frac{a-(w_1+h)}{4\mathrm{b}} \tag{2-69}$$

Re-arranging (2-44), we get:

$$q_{11} \leq \frac{a}{b} - q_{12} \tag{2-70}$$

We observe that: $q_{11}^{o} = \frac{a-w_1}{2\mathrm{b}} - \frac{q_{12}}{2} \leq \frac{a}{2\mathrm{b}} - \frac{q_{12}}{2} \leq q_{11} \leq \frac{a}{b} - q_{12}$ (2-71)

i.e., (2-68) always fulfills (44) and it's hence enough to check that (2-68) fulfills (2-43), which leads to the following sub-cases for the optimal q^*_{11} decision:

Case 2(a): If $q^{o}_{11}(68) \leq 0$, $q^*_{11} = 0$ (2-72)

Case 2(b): If $q^{o}_{11}(68) > 0$, $q^*_{11} = q_{11}^{o} = \frac{a-w_1}{2\mathrm{b}} - \frac{q_{12}}{2}$ (2-73)

Also, I^{o}_1 (69) needs to fulfill (2-45) and (2-46).

We see that $I^{o} = \frac{a-(w_1+h)}{4\mathrm{b}} \leq \frac{a}{3\mathrm{b}} \leq \frac{a}{2\mathrm{b}}$ (2-74)

Hence, the domain condition for this case i.e., $\frac{a}{3\mathrm{b}} \leq I \leq \frac{a}{2\mathrm{b}}$, is never satisfied by (69). So, the optimal I* decision is to carry $I^* = \frac{a}{3\mathrm{b}}$, which is the minimum value that fulfills the domain conditions. (2-75)

Retailer 2's 1st period order quantity decision:

Since retailer 2 is symmetrical to retailer 1 (exactly identical with same holding cost and same amount of inventory carried into the 2nd period),

We have the following two sub-cases for the optimal q^*_{12} and I^* decisions in this case:

Case (2-1): $I < \frac{a}{3b}$

Using a procedure exactly similar to Case 1.5, we can obtain the profit-maximizing q^o_{12} and I^o values for this case as:

$$q^{o}_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2} \tag{2-76}$$

$$I^o = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h) \tag{2-77}$$

And the following 2 cases for the optimal q^*_{12} decisions:

Case 2-1(a): If $q^o_{12}(76) \leq 0$, $q^*_{12} = 0$ (2-78)

Case 2-1(b): If $q^o_{12}(76) > 0$, $q^*_{12} = q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (2-79)

and that $I^* = I^o = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$ (2-80)

Case (2-2): $I_1 \geq \frac{a}{3b}$

In this case, again from symmetry to Case 1.61, we can derive the following results:

$$q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2} \tag{2-81}$$

We have the following two sub-cases for the optimal q^*_{12} decision:

Case 2-2(a): If $q^o_{12}(81) \leq 0$, $q^*_{12} = 0$ (2-82)

Case 2-2(b): If $q^o_{12}(81) > 0$, $q^*_{12} = q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (2-83)

and $I^* = \frac{a}{3b}$ (2-84)

Combined Equilibrium Analysis

Solving Case 1(a) and 2-1(a) together, we get:

$q^*_{11} = q^*_{12} = 0$ when $q^o_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2} \leq 0$ and $q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2} \leq 0$

i.e., $q^*_{11} = q^*_{12} = 0$ when $\frac{a-w_1}{2b} \leq 0$OR $w_1 \geq a$ (2-85)

However, this directly contradicts (41), which is a given condition and hence this case is impossible.

We can similarly conclude that the combination of Cases 1.52(a) and 1.62(a) is impossible.

Solving cases 1(b) and 2-1(b) together (1st period retailer 1 and retailer 2 decisions respectively), we get:

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ when $q^o_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2} > 0$ AND $q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2} > 0$

i.e., $q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$when $\frac{a-w_1}{3b} > 0$ or $w_1 < a$,, which we know from (1.52), always holds.

(2-86)

The combination of Case 2(b) and Case 2-2(b) also yields the same result i.e.

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ (2-87)

Hence, the only possible equilibrium for 1st period q_{11} and q_{12} decisions is

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ (2-88)

As for the optimal I* decisions, we have the following:

If $I < \frac{a}{3b}$, $I^* = I^o = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$ (2-89)

If $\frac{a}{3b} \le I \le \frac{a}{2b}$, $I^* = \frac{a}{3b}$ $\qquad \frac{a}{3b} \le \frac{a}{2b}$ (2-90)

We recall from 2-74) that in the case of $\frac{a}{3b} \le I \le \frac{a}{2b}$, the profit-maximizing $I^o{}_1$ value is below the lower bound of the domain. Also, from (2-67), we see that the function is concave and hence, we can say that the profit of retailer 1 is strictly decreasing in I, in the interval $\frac{a}{3b} < I \le \frac{a}{2b}$. (2-91)

Also, we see from (2-65) and (2-67) that, the profit-maximizing $I^o{}_1$ value in the interval $I > \frac{a}{3b}$, always is $< \frac{a}{3b}$, and that the profit function for that interval too is concave, hence, we know that the profit-function for retailer 1 in the interval ($I^o{}_1$(2-62)$\le I \le \frac{a}{3b}$) is also strictly decreasing in I. (2-92)

Further, we see that the profit functions of the retailers in the two intervals, (2-49) and (2-59), respectively, yield the same value at $I = \frac{a}{3b}$, (when $I = \frac{a}{3b}$, $w^*{}_2 = \frac{a-3bI}{2}$=0 and $q^*{}_{21} = q^*{}_{22} = \frac{a-w_2}{3b} - I$=0), so (2-49) reduces to (2-59).

So, we can see that the profit-function for retailer 1 is continuous at $I = \frac{a}{3b}$, and we have shown that the profit-function is strictly decreasing in the interval ($I^o{}_1$(2-80)$\le I \le \frac{a}{3b}$), as well as the interval $\frac{a}{3b} < I < \frac{a}{2b}$. So, we can argue that the profit-maximizing I* decision has to be that I* always lies in the interval $I < \frac{a}{3b}$, since there is no I value that yields a higher profit than the profit when $I^* = I^o{}_1$(2-80).

Hence, the optimal I* decision for retailer 1 is $I^* = I^o{}_1$ (2-80) $= \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$. (2-93)

Similarly, the optimal I* decision for retailer 2 is also $I^* = I^o$ (2-80) $= \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$ (2-94)

1^{st} period manufacturer wholesale price decisions

Given:

a, b, > 0 (2-95)

q_{11}, q_{12}, I ≥0 (2-96)

Decision variable: w_1

Requirement (constraints) for the decision variables.

$0 \le w_1 + h \le a$ (2-97)

The objective of the manufacturer in the 1^{st} period is to maximize the sum of 1^{st} and 2^{nd} period profits i.e., maximize:

$\Pi M_1 + \Pi M_2 = w_1(q_{11}+q_{12}+2I^*)+w_2(q^*_{21}+q^*_{22})$ (2-98)

We see from the previous section, equation that there is only 1 possibility for optimal q^*_{11}, q^*_{12}, I^*, q^*_{21}, q^*_{22}, w^*_2 decisions combined. Hence, we have the following 2 cases for the optimal w^*_1 decision:

In this case, $w^*_2 = \frac{a}{10} + \frac{3}{5}(w_1 + h)$, $q^*_{21} = q^*_{22} = \frac{a}{5b} + \frac{1}{5b}(w_1 + h)$,

$I^* = I^o = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$; $q^*_{11} = q^*_{12} = \frac{a - w_1}{3b}$, $I^* = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h)$

Substituting these into (2-98), we have:

ΠM1 + ΠM2 =(6*h+a+6*w[1])^2/(150*b)+w[1]*(2*((4*a)/(15*b)-(2*(h+w[1]))/(5*b))+(2*(a-w[1]))/(3*b)) (2-99)

$$\frac{\partial(\Pi M_1 + \Pi M_2)}{\partial w_1} = -\frac{(24h - 96a + 184w_1)}{75b} \quad (2\text{-}100)$$

$$\frac{\partial^2 \Pi M_1}{\partial w_1^2} = \frac{-184}{75\text{b}} \le 0 \quad (2\text{-}101)$$

(2-101) shows that (2-99) is concave in w_1. Setting (2-100) to zero, we get the profit-maximizing w^o_1 for this case as:

$w^o{}_1 = \frac{12a-3}{23}$ (2-102)

We see that (2-102) is always ≥0 and is always ≤ a

$\Rightarrow w^*{}_1 = w^o{}_1$ (2-102) $= \frac{12a-3h}{23}$ (2-103)

Summary of decision variable values – 2 period model:

$w^*{}_1 = = \frac{12a-3h}{23}$

$q^*{}_{11} = q^*{}_{12} = \frac{a-w_1}{3b} = \frac{3h+11a}{69b}$

$I^* = \frac{4a}{15b} - \frac{2}{5b}(w_1 + h) = \frac{4a-24}{69b}$

$w^*{}_2 = \frac{19a+2}{46}$; $q^*{}_{21} = q^*{}_{22} = 0$

3. Three-period model with one-manufacturer supplying to one downstream retailer with Strategic Inventory carriage allowed between periods.

<u>3rd period retailer decisions</u>

Given: (Assumptions)

$I_1, I_2 \geq 0$, $0 < w_3 < a$, $h > 0$ (3-1)

$0 < w_2 < a$, $0 < w_1 < a$ (3-2)

$a - bI_2 \geq 0$ (3-3)

$a - bI_1 \geq 0$ (3-4)

$q_1, q_2 \geq 0$ (3-5)

Decision variables: q_3

Requirement (constraints) on the decision variables.

$a - b(q_3 + I_2) \geq 0$ (3-6)

$q_3 \geq 0$ (3-7)

We can write the 3rd period profit function for the retailer as:

$\Pi R_3 = (a - b(q_3 + I_2))(q_{31} + I_2) - (w_3 q_3)$ (3-8)

$\frac{\partial \Pi R_3}{\partial q_3} = -2(q_3 + I_2)b + a - w_3$ (3-9)

$\frac{\partial^2 \Pi R_3}{\partial q_3^2} = -2\text{b}$ (3-10)

(3-9) and (3-10) prove that (3-8) is concave in q_3.

Setting (3-9) to zero, we get the profit-maximizing q_3 decision as:

$q^o_3 = \frac{a - w_3}{2\text{b}} - I_2$ (3-11)

We now need (3-11) to fulfill the constraints (3-6) and (3-7).

Rearranging (3-6), we have: $q_3 < \frac{a}{b} - I_2$. (3-12)

We note that (3-11) readily fulfills the constraint in (3-12). Hence, we have the following sub-cases for the optimal q^*_3 decision:

Case 7.1(a): If q^o_3 (3-11)≤0, $q^o_3 = 0$ (3-13)

Case 7.1(b): If q^o_3(3-11)>0, $q^*_3 = q^o_3(3\text{-}11) = \frac{a-w_3}{2\mathrm{b}} - I_2$ (3-14)

<u>3rd period Manufacturer decisions:</u>

Given:

$I_2 \geq 0$ (3-15)

$a - bI_2 \geq 0$ (3-16)

$q_2 > =0$ (3-17)

Decision variable: w_3

Requirement (constraints) for the decision variables.

$0 \leq w_3 \leq a$ (3-18)

The objective function is to maximize the manufacturer's 3rd period profit given in (3-19)

$\Pi M_2 = w_3(q_3)$ (3-19)

<u>Case (1):</u> $a-2bI_2 > w_3$

In this portion of the w^*_3 domain,

$q^*_3 = \frac{a-w_3}{2\mathrm{b}} - I_2$ (From (3-11)) (3-20)

Substituting q^*_3 from (3-11) into (3-20), we get:

$\Pi M_3 = (\frac{a-w_3}{2\mathrm{b}} - I_2)(w_3)$ (3-21)

$\frac{\partial \Pi M_3}{\partial w_3} = \frac{a}{2\mathrm{b}} - \frac{w_3}{b} - I_2$ (3-22)

$\frac{\partial^2 \Pi M_3}{\partial w_3^2} = \frac{-1}{b} < 0$ (3-23)

(3-23) shows that (3-21) is concave with respect to w_3.

Equating (3-22) to zero, we get the profit-maximizing 3rd period manufacturer wholesale price as $w^o_3 = \frac{a-2bI_2}{2}$ (3-24)

We now need that (3-24) fulfills the constraint (3-18) as well as the particular domain condition for this case i.e., $0 \le w_3 \le a$ AND $a-2bI_2 > w_3$.

We observe that $a-2bI_2 > w_3$ is always satisfied by w^o_3(7-024), hence we only need to check for the condition $0 \le w_3 \le a$. (3-25)

This leads to the following two cases for the optimal w^*_3 decision:

Case 1(a): If w^o_3(3-24) ≤ 0. $w^*_3 = w^o_3 = 0$ (3-26)

Case 1(b): If w^o_3(3-24)>0, $w^*_3 = w^o_3 = \frac{a-2bI_2}{2}$ (3-27)

Case 2: $a-2bI_2 \le w_3$

In this portion of the w^*_3 domain,

$q^*_3 = 0$ (3-28)

Substituting q^*_3 from (3-11) into (3-20), we get:

$\Pi M_3 = 0$ (3-29)

Here it does not matter what the w^*_3 is set at, since $q^*_3 = 0$.

Hence the optimal w^*_3 is the max. allowed by the domain conditions i.e., $0 \le w_3 \le a$ AND $a-2bI_2$.

$\rightarrow w^*_3 = a$ (3-30)

Table 2.1: Summary of w^*_3 decisions

Case No.	Domain Conditions	w^*_3 decision

Case 1(a)	a-2bI_2 > w_3 AND $\frac{a-2bI_2}{2}$≤0	0
Case 1(b)	a-2bI_2 > w_3 AND $\frac{a-2bI_2}{2}$≥0	$w^*_3 = \frac{a-2bI_2}{2}$
Case 2	a-2bI_2 ≤ w_3	w^*_3 = a

From Table 2.1, it is apparent that the manufacturer has a choice in the 3rd period to set the wholesale price either in the range a-2bI_2 > w_3 OR a-2bI_2 ≤ w_3, depending on in which part of the region, the manufacturer makes a better profit.

Computing the profit functions at the three wholesale price levels we have:

$$\Pi M_3(w_3{}^* = 0) = 0 \tag{3-31}$$

$$\Pi M_3(w_3{}^* = \frac{a-2bI_2}{2}) = (\frac{a-w_3}{2b} - I_2)(w_3) = \frac{4I_2^2b^2-4I_2ab+a^2}{8b} \tag{3-32}$$

$$\Pi M_3(w_3{}^* = a) = 0 \tag{3-33}$$

$\frac{4I_2^2b^2-4I_2ab+a^2}{8b} > 0$

$4I_2^2b^2 - 4I_2ab + a^2 \geq 0$

$I_2 \geq \frac{4ab\pm\sqrt{(16a^2b^2-(16a^2b^2))}}{8b^2} = \frac{ab\pm\sqrt{(a^2b^2-(a^2b^2))}}{2b^2}$

$I_2 \geq \frac{a}{2b}$

We see that (3-32) ≥0 if $I_2 \geq \frac{a}{2b}$. We also know that one of the domain conditions for the profit function value in (3-32) to hold is: $\frac{a-2bI_2}{2} \geq 0 \rightarrow I_2 \leq \frac{a}{2b}$

and a-2bI_2>w_3 i.e., a-2bI_2 > $\frac{a-2bI_2}{2}$which always holds.

We see that, at $I_2=\frac{a}{2b}$, which is the only point at which Case 1(b) holds, w^*_3 reduces to w^*_3 = 0. So, the decisions in Table 2.1 can be reduced to:

Table 2.2: Summary of w^*_3 decisions

Case No.	Domain Conditions	w^*_3
Case (a)	$a-2bI_2 > w_3$	$w^*_3 = 0$
Case (b)	$a-2bI_2 \le w_3$	$w^*_3 = a$

In either case, the profit-function is zero, and hence we can say that $w^*_3 = a$ is the optimal solution, since, with all things being equal, the manufacturer would want to set his wholesale price at the higher point. (3-34)

2nd period retailer decision:

Given: (Assumptions)

$I_1 \ge 0$ (3-35)

$0 < w_2 < a, h > 0$ (3-36)

$0 < w_1 < a$ (3-37)

$a-bI_2 \ge 0$ (3-38)

$q_1 \ge 0$ (3-39)

Decision variables: q_2 , I_2

Requirement (constraints) on the decision variables:

$a-b\,(q_2 + I_1) \ge 0$ (3-40)

$q_2 \ge 0$ (3-41)

$I_2 \ge 0$ (3-42)

$a-b(I_2) \ge 0$ (3-43)

We can write the 2nd period problem for the retailer (to set q^*_2) as to maximize:

$\Pi R_2 + \Pi R_3 = (a - b(q_2 + I_1))(q_2 + I_1) - (w_2)(q_2 + I_2) - hI_2 \quad + 0$ (3-44)

$\frac{\partial(\Pi R_2 + \Pi R_3)}{\partial Q_2} = -2(q_2 + I_1)b + a - w_2$ (3-45)

$\frac{\partial^2(\Pi R_2 + \Pi R_3)}{\partial Q_2^2} = -2\mathrm{b}$ (3-46)

$\frac{\partial(\Pi R_2 + \Pi R_3)}{\partial I_2} = -(w_2 + h)$ (3-47)

$\frac{\partial(\Pi R_2 + \Pi R_3)}{\partial I_2^2} = 0$ (3-48)

(3-46) proves that (3-44) is concave in q_2.

Setting (3-45) to zero, we get the profit-maximizing q^o_2 value as:

$q^o_2 = \frac{a - w_2}{2\mathrm{b}} - I_1$ (3-49)

We now need that q^o_2(3-45) fulfill the constraints (3-40) and (3-41).

Rearranging (3-40) we get: $q_2 \leq \frac{a}{b} - I_1$ (3-50)

We see that q^o_2(3-49) readily fulfills (3-50) and hence (3-40).

For q^o_2(3-49) to fulfill (3-41),we have the following sub-cases:

Case (a): If q^o_2(3-49) ≤0, $q^*_2 = 0$ (3-51)

Case (b): If q^o_2(3-49)>0, $q^*_2 = q^o_2$(3-49) $= \frac{a - w_2}{2\mathrm{b}} - I_1$ (3-52)

As far as the I^*_2 decision, we see from (3-47) and (3-48) that the derivative of (3-44) with respect to I_2 is constant and hence the I^*_2 can be set at the maximum allowed by the domain conditions i.e., (3-47) and (3-48).

$\rightarrow I^*_2 = \frac{a}{b}$ (3-53)

2nd Period Manufacturer Decisions:

Given:

$I_1 \geq 0$ (3-54)

$a - bI_1 \geq 0$ (3-55)

$q_1 >= 0$ (3-56)

Decision variable: w_2

Requirement (constraints) for the decision variables.

$0 \leq w_2 \leq a$ (3-57)

The objective function is to maximize the sum of the manufacturer's 2nd and 3rd period profit given in (3-58)

$$\Pi M_2 + \Pi M_3 = w_2(q_2 + I_2) + w_3(q_3) \quad (3\text{-}58)$$

Case (1a): If $w_2 \geq a - 2bI_1$

Here, $q^*_2 = 0$, $I^*_2 = \frac{a}{b}$ (and $w^*_3 = a$, $q^*_3 = 0$)

hence the manufacturer's problem becomes to maximize:

$$\Pi M_2 + \Pi M_3 = w_2(q_2) + w_3(q_3) = w_2\left(\frac{a}{b}\right) \quad (3\text{-}59)$$

$$\frac{\partial(\Pi M_2 + \Pi M_3)}{\partial w_2} = \frac{a}{b} \quad (3\text{-}60)$$

$$\frac{\partial^2(\Pi R_2 + \Pi R_3)}{\partial w_2^2} = 0 \quad (3\text{-}61)$$

From (3-60) and (3-61), we see that the first derivative of (3-59) with respect to w_2 is a constant. Hence, w^*_2 here is set at the maximum allowed by the domain conditions i.e., $w_2 \geq a - 2bI_1$ AND $0 \leq w_2 \leq a$

So, $w^*_2 = a$ is the optimal decision here. (3-62)

Case 1(b): If $w_2<a-2bI_1$

Here, $q^*_2 = \frac{a-w_2}{2b} - I_1$ and hence the manufacturer's problem becomes to maximize:

$$\Pi M_2 + \Pi M_3 = w_2(q_2) + w_3(q_3) = w_2\left(\frac{a-w_2}{2b} - I_1\right) \quad (3\text{-}63)$$

$$\frac{\partial(\Pi M_2+\Pi M_3)}{\partial w_2} = \frac{(a-2w_2-2bI_1)}{2b} \quad (3\text{-}64)$$

$$\frac{\partial^2(\Pi R_2+\Pi R_3)}{\partial w_2^2} = -\frac{1}{b} \quad (3\text{-}65)$$

(3-64) and (3-65) prove that (3-63) is concave in w_2.

Setting (3-65) to zero, we get the profit-maximizing w^o_2 in this case as:

$$w^o_2 = \frac{a}{2} - bI_1 \quad (3\text{-}66)$$

Now, we need that w^o_2(3-66) fulfills the constraint (3-57) as well as the domain condition: $w_2<a-2bI_1$. We observe that w^o_2(3-66) always fulfills the constant $w_2 \le a-2bI_1$. We only need to check that w^o_2(3-66) fulfills the constraint (3-57) .

From (3-55) we observe that w^o_2(3-66) is always ≥0 and <=a

$w^*_2 = w^o_2(3\text{-}66) = = \frac{a}{2} - bI_1$ is the optimal w^*_2 decision in this area of the w^*_2 domain (3-67)

Table 2.3: Summary of 2nd period manufacturer wholesale price decision:

Case No.	Domain Conditions	w^*_2 value
Case 1(a)	If $w_2 \ge a-2bI_1$	$w^*_2 = a$
Case 1(b-i)	If $w_2<a-2bI_1$	$w^*_2 = \frac{a}{2} - bI_1$

Whether the manufacturer sets w^o_2 in the range $w_2 \ge a-2bI_1$ OR $w_2 < a-2bI_1$, depends on which part of the domain he/she makes a better profit.

$$\Pi M_2 + \Pi M_3(w^* = a) = w_2(q_2) + w_3(q_3) = a(a/b) = \frac{a^2}{b} \quad (3\text{-}68)$$

$\Pi M_2 + \Pi M_3(w* = \frac{a}{2} - bI_1)$

$=w_2(q_2) + w_3(q_3) = (\frac{a}{2} - bI_1)(\frac{a-w_2}{2b} - I_1) = 0.5bI_1^2 + \frac{0.125a^2}{b} - 0.5aI_1$ (3-69)

We know that the max. value of I_1 is a/b, since it needs to fulfill $a-bI_1>=0$.

Substituting $I_1=a/b$ in (3-69), we have:

$\Pi M_2 + \Pi M_3\left(w* = \frac{a}{2} - bI_1; I_1 = \frac{a}{b}\right) = \frac{0.125a^2}{b}$ (3-70)

So, we conclude that (3-68) is always less than (3-69), and hence:

$w*_2 = a$ is the optimal $w*_2$ decision here. (3-71)

1st period retailer decision:

Given: (Assumptions)

$0<w_1<a$ ` (3-72)

Decision variables: q_1 . I_1

Requirement (constraints) on the decision variables:

$a- b(q_1 + I_1)\geq 0$ (3-73)

$q_1\geq 0$ (3-74)

$a-bI_1\geq 0$ (3-75)

$I_1 \geq 0$ (3-76)

The 1st period problem for the retailer as to maximize: $\Pi R_1 + \Pi R_2 + \Pi R_3$

here,

$w^*_2 = a$; $q^*_2 = 0$, $I^*_2 =\frac{a}{b}$; $w^*_3 =a$, $q^*_3 = 0$

➔ $\Pi R_1 + \Pi R_2 + \Pi R_3$

$=(a-b(q_1))(q_1)-w_1(q_1+I_1)-hI_1 + (a-b(I_1))(I_1) - a(\frac{a}{b}) - hI_2 + (a-b(\frac{a}{b}))(\frac{a}{b})$ (3-77)

And

$$\frac{\partial(\Pi R_1+\Pi R_2+\Pi R_3)}{\partial q_1} = -2\mathrm{bq}_1 + a - w_1 \tag{3-78}$$

$$\frac{\partial^2(\Pi R_1+\Pi R_2+\Pi R_3)}{\partial q_1^2} = -2\mathrm{b} \tag{3-79}$$

$$\frac{\partial(\Pi R_1+\Pi R_2+\Pi R_3)}{\partial I_1} = -h - 2\mathrm{bI}_1 + a - w_1 \tag{3-80}$$

$$\frac{\partial^2(\Pi R_1+\Pi R_2+\Pi R_3)}{\partial I_1^2} = -2\mathrm{b} \tag{3-81}$$

(3-79) and (3-81) prove that (3-78) and (3-80) are concave in q_1 and I_1 respectively.

Setting (3-78) and (3-80) to zero respectively, we get the profit-maximizing q_1 and I_1 decisions as:

$\rightarrow q^o_1 = \frac{a-w_1}{2b}$ (3-82)

$I^o_1 = \frac{a-(w_1+h)}{2b}$ (3-83)

q_1(3-82) needs to fulfill (3-73) and (3-74). We know $w_1 \leq a$ (from (3-72)).

So, we can say that (3-82) always fulfills (3-73). We now check if (3-82) fulfills (3-74).

Rearranging (3-74) we get: $q_1 < \frac{a}{b} - I_1$ (3-84)

We now need to check that (3-82) fulfills (3-84) .

For this, we have the following sub-cases:

Case (a): If $w_1 < a - 2bI_1$, $q^*_1 = q^o_1$(3-82) $= \frac{a-w_1}{2b}$ (3-85)

Case (b): If $w_1 \geq a - 2bI_1$, $q^*_1 = q^o_1$(3-82) $= a - 2bI_1$ (3-86)

Next, we check if (3-83) fulfills (3-75) and (3-76).

We see that (3-75) readily fulfills (3-75) and (3-76) since $w_1 < a$ always and $a >> h$

$I^*_1 = I^o_1 = \frac{a-(w_1+h)}{2b}$ is the optimal I^*_1 decision here. (3-87)

Table 2.4 Summary of 1st period retailer decisions –(q^*_1):

Case No.	Domain Conditions	q^*_1 decision
Case 2.5(a)	If $w_1 < a - 2bI_1$	$q^*_1 = \frac{a-w_1}{2b}$
Case 2.5(b)	If $w_1 \geq a - 2bI_1$	$q^*_1 = a - 2bI_1$

We know from (3-87) that $I^*_1 = \frac{a-(w_1+h)}{2b}$. Using this in Case (a) and (b), we see that the domain condition of case (a) reduces to h>0, which always holds. Also, we see that, using this result, the domain condition for case (b) reduces to h<0, which is infeasible.

Hence, we conclude that : $q^*_1 = \frac{a-w_1}{2b}$ is the only optimal q^*_1 decision in this case

1st Period Manufacturer Decisions:

Given:

$$a, b > 0 \tag{3-88}$$

Decision variable: w_1

Requirement (constraints) for the decision variables.

$$0 \le w_1 \le a-h \tag{3-89}$$

The manufacturer's profit function over the three periods becomes:

$$\Pi M_1 + \Pi M_2 + \Pi M_3 = w_1(q_1 + I_1) + w_2(q_2 + I_2) + w_3(q_3) \tag{3-90}$$

Here, $q^*_1 = \frac{a-w_1}{2b}$ and $I^*_1 = \frac{a-(w_1+h)}{2b}$, w^*_2 =a-h, q^*_2 = 0, $I^*_2 = \frac{a}{b}$, w^*_3 =a, q^*_3 = 0 (From previous sections)

Hence, the manufacturer's 1st period problem becomes:

$$\Pi M_1 + \Pi M_2 + \Pi M_3 = w_1\left(\frac{a-w_1}{2b} + \frac{a-(w_1+h)}{2b}\right) + (a-h)\left(\frac{a}{b}\right) \tag{3-91}$$

$$\frac{\partial(\Pi M_1 + \Pi M_2 + \Pi M_3)}{\partial w_1} = w_1\left(\frac{-1}{2b} + \frac{-1}{2b}\right) + \left(\frac{a-w_1}{2b} + \frac{a-(w_1+h)}{2b}\right) \tag{3-92}$$

$$\frac{\partial(\Pi M_1 + \Pi M_2 + \Pi M_3)}{\partial {w_1}^2} = \frac{-2}{b} < 0 \tag{3-93}$$

(3-93) provea that (3-91) is concave in w_1. Setting (3-92) to zero, we get the profit-maximizing w_1 in this case as:

$w_1 = \frac{a}{2} - \frac{h}{4}$ (3-94)

since a>>h, we can say that w_1(3-94) ≥0. We also observe that w_1(3-94) fulfills the RHS of (3-95)

Hence $w^*_1 = \frac{a}{2} - \frac{h}{4}$ is the optimal w^*_1 decision.

<u>SUMMARY – Anand's three period model:</u>

<u>1st period:</u>

$w^*_1 = \frac{a}{2} - \frac{h}{4}$; $q^*_1 = \frac{a-w_1}{2\text{b}} = \frac{a-(\frac{a}{2}-\frac{h}{4})}{2\text{b}} = \frac{a}{4\text{b}} + \frac{h}{8\text{b}}$; $I^*_1 = \frac{a-(w_1+h)}{2\text{b}}$

<u>2nd period:</u>

$w^*_2 = a$, $q^*_2 = 0$, $I^*_2 = \frac{a}{b}$

<u>3rd period:</u>

$w^*_3 = a$, $q^*_3 = 0$

4. Three period Cournot duopoly Model

Key Assumption: Inventory is carried only one period forward.

three period Game Structure:

1st period

- manufacturer announces w_1
- retailers announce q_{11}, q_{12} and I_1 (each)
- sell q_{11}, q_{12}, carry I_1 to period 2

2nd period:

- manufacturer announces w_2
- retailers announce q_{21}, q_{22}, I_2. (each)
- sell $q_{21} + I_1$, $q_{22} + I_1$ respectively.
- Carry I_2 to the 3rd period.

3rd period:

- manufacturer announces w_3
- retailers announce q_{31}, q_{32} respectively.
- sell $q_{31}+I_2$, $q_{32}+I_2$ respectively.

Retailer 1's third period decisions

Given: (Assumptions)

$$0< w_2< a \tag{4-1}$$

$$I_1, I_2 \geq 0, \ , h>0 \tag{4-2}$$

$$a-2bI_2 \geq 0 \tag{4-3}$$

Decision variables: q_{21}

Requirement (constraints) on the decision variables.

$$a\text{-}b(q_{31}+q_{32}+2I_2)\geq 0 \quad (4\text{-}4)$$

$$q_{21} \geq 0 \quad (4\text{-}5)$$

We can write the 3rd period profit function for retailer 2 as:

$$\Pi R_{31} = (a - b(q_{31} + q_{32} + 2\mathrm{I}_2))(q_{21} + I_2) - (w_3 q_{31}) \quad (4\text{-}6)$$

$$\frac{\partial \Pi R_{31}}{\partial q_{31}} = a - b(2\mathrm{q}_{31} + q_{32} + 3\mathrm{I}_2) - w_3 \quad (4\text{-}7)$$

$$\frac{\partial^2 \Pi_{31}}{\partial q_{21}^2} = -2\mathrm{b} < 0 \quad (4\text{-}8)$$

(4-8) shows that (4-6) is concave with respect to q_{31}

Setting (4-7) to zero, we get the profit-maximizing 3rd period order quantity for retailer 1

as:

$$q_{21}^o = \frac{a-w_2}{2\mathrm{b}} - \frac{q_{22}+3}{2} \quad (4\text{-}9)$$

(4-7) shows that (4-6) is concave with respect to q_{31}

Setting (4-6) to zero, we get the profit-maximizing 3rd period order quantity for retailer 1

as:

$$q_{31}^o = \frac{a-w_3}{2\mathrm{b}} - \frac{q_{32}+3\mathrm{I}_2}{2} \quad (4\text{-}10)$$

We need that (4-8) fulfills (4-3) and (4-4)

Re-arranging (4-3), we get:

$$q_{31} \leq \frac{a}{b} - (q_{32} + 2\mathrm{I}_2) \quad (4\text{-}11)$$

(4-8) can be re-written as:

$$q_{31}^o = \left(\frac{a}{2\mathrm{b}} - \frac{q_{32}}{2} - I_2\right) - \left(\frac{I_2}{2} + \frac{w_3}{2\mathrm{b}}\right) \leq \frac{a}{2\mathrm{b}} - \frac{1}{2}(q_{32} + 2\mathrm{I}_2) \leq \frac{a}{b} - (q_{32} + 2\mathrm{I}_2) \quad (4\text{-}12)$$

From (4-12), we can say that (4-9) always fulfills (4-3).

Next, we only need that (4-9) fulfills (4-4), which leads to the following sub-cases for the optimal q^*_{31} decision:

Case (a) If q^o_{31} (4-9) ≤ 0, $q^*_{31} = 0$ (4-13)

Case (b): If q^o_{31}(4-9) > 0, $q^*_{31} = q^o_{31}(4\text{-}9) = \frac{a-w_3}{2b} - \frac{q_{32}+3I_2}{2}$ (4-14

Retailer 2's 3rd period order quantity decisions

Given: (Assumptions)

$I_1, I_2 \geq 0,\ 0 < w_3 < a,\ h > 0$ (4-15)

$0 < w_2 < a,\ 0 < w_1 < a$ (4-16)

$a - 2bI_2 \geq 0$ (4-17)

Decision variables: q_{32}

Requirement (constraints) on the decision variables.

$a - b(q_{31}+q_{32}+2I_2) \geq 0$ (4-18)

$q_{32} \geq 0$ (4-19)

We can write the 3rd period profit function for retailer 2 as:

$$\Pi R_{32} = (a - b(q_{31} + q_{32} + 2I_2))(q_{32} + I_2) - (w_2 q_{32}) \quad (4\text{-}20)$$

Since retailer 2 is symmetrical to retailer 1, we can use a procedure similar to the one employed in Section 3.1 to derive retailer 2's 2nd period profit-maximizing order quantity as

$$q^o_{32} = \frac{a-w_3}{2b} - \frac{q_{31}+3I_2}{2} \quad (4\text{-}21)$$

For (4-22) to fulfill (4-16) and (4-17), we have the following two sub-cases, again, using a procedure similar to that in Section 3.1

Case (a) If q^o_{32}(4-21) ≤ 0, $q^*_{32} = 0$ (4-22)

Case (b): If $q^o_{32}(4\text{-}21) > 0$, $q^*_{32} = q^o_{32}(4\text{-}21) = \frac{a-w_3}{2b} - \frac{q_{31}+3I_2}{2}$ (4-23)

Combined equilibrium analysis – q_{31} and q_{32} decisions

Since the two retailers are identical in all respects, symmetrical, in Cournot competition with each other and take their decisions simultaneously, we can postulate that their equilibrium 3rd period order quantities are equal. In this case, there are only two possible equilibria:

Case (a): $q^o_{31}(4\text{-}13) \leq 0$ AND $q^o_{32}(4\text{-}22) \leq 0$ (Case 3.1(a) and Case 3.2(a))

Here, $q^*_{31} = q^*_{32} = 0$ (4-24)

Case (b): $q^o_{31}(4\text{-}13) > 0$ AND $q^o_{32}(4\text{-}22) > 0$ (Case 3.1(b) and Case 3.2(b))

Here, $q^*_{31} = \frac{a-w_3}{2b} - \frac{q_{32}+3I_2}{2}$ (4-25)

$q^*_{32} = \frac{a-w_3}{2b} - \frac{q_{31}+3I_2}{2}$ (4-26)

Solving (4-26) and (4-27) together, we obtain,

$q^*_{31} = q^*_{32} = \frac{a-w_3}{3b} - I_2$ (4-27)

Table 3.1: Summary of 3rd period retailer order quantity decisions

Case No.	Domain Condit	Equivalent Conditions	Equilibrium Decision (q^*_{31},q^*_{32})
3.3(a)	$q^o_{31}(4\text{-}13) \leq 0$ AND $q^o_{32}(4\text{-}22) \leq 0$	$\frac{a-w_3}{3b} - I_2 \leq 0$	0,0
3.3(b)	$q^o_{31}(4\text{-}13) > 0$ AND $q^o_{32}(4\text{-}22) > 0$	$\frac{a-w_3}{3b} - I_2 > 0$	$\frac{a-w_3}{3b} - I_2, \frac{a-w_3}{3b} - I_2$

3rd period manufacturer wholesale price decisions

Given:

$I_2 \geq 0$ (4-28)

$a\text{-}2bI_2 \geq 0$ (4-29)

$q_{21}, q_{22} > =0$ (4-30)

Decision variable: w_3

Requirement (constraints) for the decision variables.

$0 \leq w_3 \leq a$ (4-31)

The objective function is to maximize the manufacturer's 2nd period profit given in (5-033)

$$\Pi M_2 = w_3(q_{31} + q_{32}) \tag{4-32}$$

Case (1): If $a\text{-}3bI_2 > w_3$

In this portion of the w^*_3 domain,from (4-27),

$q^*_{31} = q^*_{32} = \frac{a-w_3}{3b} - I_2$

Substituting q^*_{31} and q^*_{32} from (4-27) into (4-32), we get:

$$\Pi M_3 = (2\frac{a-w_3}{3b} - 2I_2)(w_3) \tag{4-33}$$

$$\frac{\partial \Pi M_3}{\partial w_3} = \frac{2a}{3b} - \frac{4w_3}{3b} - 2I_2 \tag{4-34}$$

$$\frac{\partial^2 \Pi M_3}{\partial w_3^2} = \frac{-4}{3b} < 0 \tag{4-35}$$

(4-35) shows that (4-33) is concave with respect to w_3.

Equating (4-34) to zero, we get the profit-maximizing 3rd period manufacturer wholesale price as

$$w^o_3 = \frac{a-3bI_2}{2} \tag{4-36}$$

Now, (4-36) needs to satisfy the constraint (4-31) as well as the domain conditions of Case 3.41 ($a\text{-}3bI_2 > w_3$)

We see that (4-36) always fulfills the RHS of (4-31) and also fulfills the domain condition,

$a\text{-}3bI_2 > w^o_3$, as long as $w^o_3>0$. This leads to the following sub-cases:

Case 1(a): If w^o_3(4-36) ≤ 0, $w^*_3 = 0$ (4-37)

Case 1(b): If w^o_3(4-37)>0, $w^*_3 = \frac{a-3bI_2}{2}$ (4-38)

<u>Case (2):</u> If $a\text{-}3bI \leq w_3$, $q^*_{31} = q^*_{32} = 0$ (neither retailer orders) and w^*_3 can be set at any value such that fulfills (4-31) as well as $w_3 \geq a\text{-}3bI_2$

This leads to the following two-cases for the optimal w^*_3 decision in this case:

Case 2(a): If w^o_3(4-36)≤ 0, $w^*_3 = 0$ (4-39)

Case 2(b): If w^o_3(4-36) > 0, $w^*_3 = a\text{-}3bI_2$ (4-40)

<u>Table 3.2: Summary of 3rd period manufacturer wholesale price decisions</u>

Case No.	Domain Conditions	w^*_3
3.41(a)	$I_2 \geq \frac{a}{3b}$	0
3.41(b)	$I_2 < \frac{a}{3b}$	$\frac{a-3bI_2}{2}$
3.42(a)	$a\text{-}3bI_2 \leq 0$ OR $I_2 \geq \frac{a}{3b}$	0
3.42(b)	$a\text{-}3bI_2 > 0$ OR $I_2 < \frac{a}{3b}$	$w^*_3 = a\text{-}3bI_2$

From Table 3.2, we see that when $I_2 \geq \frac{a}{3b}$, the equilibrium w^*_3 decision is always $w^*_3 = 0$.

When $I_2 < \frac{a}{3b}$, the manufacturer has a choice of either setting any $w^*_2 = a\text{-}3bI_2$ OR $w^*_3 = \frac{a-3bI_2}{2}$.

The equilibrium decision in this case will be $w^*_3 = \frac{a-3bI_2}{2}$ as this decision will lead to a higher profit for the manufacturer (both retailers order), versus when $a\text{-}3bI_2 \leq w^*_3 \leq a$ (neither retailer orders), in which case his profit is zero. This decision is consistent with the Cournot conjecture that each entity always acts to maximize it's own profit. Hence, there are two possible equilibrium values in this case, depending on the I_1 value from the 1st period:

Case (i): If $\frac{a}{3b} \leq I_2$, $w^*_3 = 0$ (4-41)

Case (ii): If $I_2 < \frac{a}{3b}$, w^*_3 =any w^*_3 that fulfills $a\text{-}3bI_1 <= w_2 <= a$ (4-42)

2nd period retailer's decisions:

Given: (Assumptions)

$I_1 \geq 0$, $h > 0$ (4-43)

$0 < w_2 < a$, (4-44)

$a - 2bI_1 \geq 0$ (4-45)

Decision variables: q_{21}, q_{22}, I_2

Requirement (constraints) on the decision variables.

$a - b(q_{21} + q_{22} + 2I_1) \geq 0$ (4-46)

$q_{21} \geq 0$ (4-47)

$a - 2bI_2 \geq 0$ (4-48)

$I_2 \geq 0$ (4-49)

We can write the 2nd period profit function for retailer 1 as:

$$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2I_1))(q_{21} + I_1) - w_2(q_{21} + I_2) - hI_2 \quad (4\text{-}50)$$

The 1st period problem for retailer 1 is to set a q_{21} and I_2 to maximize the sum of 2nd and 3rd period profits. i.e., retailer 1 needs to maximize:

$\Pi R_{21} + \Pi R_{31}$

There are two possible sub-cases here, based on the 2nd period wholesale price and order quantities (which determine 2nd period profits)

Case (1): In the part of the I^*_2 domain - $I_2 < \frac{a}{3b}$

Her,e from (4-41) and (4-42) $w^*_3 = \frac{a - 3bI_2}{2}$ and

$$q^*_{31} = q^*_{32} = \frac{a - w_3}{3b} - I_2 = \frac{a}{6b} - \frac{I_2}{2} \quad (4\text{-}51)$$

So, the 2nd period profit function becomes:

ΠR_{21} = (a-b($\frac{a}{3b} + I_2$))($\frac{a}{6b} + \frac{I_2}{2}$) -($\frac{a-3bI_2}{2}$)*($\frac{a}{6b} - \frac{I_2}{2}$) = ($\frac{2a}{3} - bI_2$)($\frac{a}{6b} + \frac{I_2}{2}$) $-$ ($\frac{a-3b\ _2}{2}$)($\frac{a}{6b} - \frac{I_2}{2}$)

(4-52)

Hence, retailer 2nd period problem becomes to max:

$\Pi R_{21} + \Pi R_{31} = (a - b(q_{21} + q_{22} + 2I_1))(q_{21} + I_1) - w_2(q_{21} + I_2) - hI_2 + (\frac{2a}{3} - bI_2)(\frac{a}{6b} + \frac{I_2}{2}) -$

$(\frac{a-3\ \ _2}{2})(\frac{a}{6b} - \frac{I_2}{2})$ (4-53)

$$\frac{\partial(\Pi R_{21} + \Pi R_{31})}{\partial q_{21}} = a - 2bq_{21} - bq_{22} - 3bI_1 - w_2 \quad (4\text{-}54)$$

$$\frac{\partial^2(\Pi R_{21} + \Pi R_{31})}{\partial q_{21}^2} = -2b < 0 \quad (4\text{-}55)$$

$$\frac{\partial(\Pi R_{21} + \Pi R_{31})}{\partial I_2} = -(w_2 + h) + (\frac{2a}{3}) - \frac{5b}{2} I_2 \quad (4\text{-}56)$$

$$\frac{\partial^2(\Pi R_{21} + \Pi R_{31})}{\partial I_2^2} = -\frac{5b}{2} < 0 \quad (4\text{-}57)$$

(4-55) and (4-57) show that (4-53) is concave in q_{21} and I_2 respectively.

Setting (4-54) and (4-56) to zero respectively, we get the profit-maximizing q_{21} and I_2 values for retailer 1 as:

$$q_{21}^o = \frac{a - w_2}{2b} - \frac{q_{22} + 3\ _1}{2} \quad (4\text{-}58)$$

$$I_2^o = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h) \quad (4\text{-}59)$$

Similarly, we can write for retailer 2:

$$q_{22}^o = \frac{a - w_2}{2b} - \frac{q_{21} + 3I_1}{2} \quad (4\text{-}60)$$

$$I_2^o = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h) \quad (4\text{-}61)$$

We note that the q^o_{22} values are independent of the I^o_2 values.

Solving (4-58) and (4-60) together, we get:

$$q^o_{21} = q^o_{22} = \frac{a - w_2}{3b} - I_1 \quad (4\text{-}62)$$

We now need that (4-62) fulfills (4-46) and (4-47) as well as the domain conditions for this sub-case i.e., $I_2 < \frac{a}{3b}$.

Rearranging (4-46) we get:

$$q_{21} \leq \frac{a}{b} - (q_{22} + 2I_1) \tag{4-63}$$

since $q_{21} = q_{22}$, we can write (4-63) as: $q_{21} \leq \frac{a}{2b} - I_1$. (4-64)

We observe that (4-62) always fulfills (4-64).

Now we only need to check that (4-62) fulfills (4-47), which leads to the following two-cases for the optimal q^*_{21} decision:

Case 1(a) If q^o_{21} (4-62)≤0, $q^*_{21} = 0$ (4-65)

Case 1(b): If $q^o_{21} > 0$, $q^*_{21} = q^o_{21}(4\text{-}62) = \frac{a-w_2}{3b} - I_1$ (4-66)

Further, we need that (4-60) (and (4-61) which is the identical value for retailer 2) fulfills (4-48), (4-49) as well as the domain conditions for this case i.e., $I_2 < \frac{a}{3b}$. From (4-47) and $I_2 < \frac{a}{3b}$ together, we observe that we need only $I_2 < \frac{a}{3b}$ and if this condition is satisfied, (4-47) is automatically satisfied.

We further observe that I^o_2(4-61) is always $< \frac{a}{3b}$. We only need to check that (4-61) fulfills (4-49), which leads to the following sub-cases for the optimal I^*_2 decision:

Case 1(c): If I^o_2(4-61) < 0, then $I^*_2 = 0$ (4-67)

Case 1(d): If I^o_2(4-61) > =0, $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$ (4-68)

We can obtain a similar 4 sub-cases for the q^*_{22} and I^*_2 decisions of retailer 2, exactly identical to those for retailer 1.i.e.,

Case 1(e) If q^o_{22} (4-62)≤0, $q^*_{22} = 0$ (4-69)

Case 1(f): If $q^o_{22}>0$, $q^*_{22} = q^o_{22}$(4-62) = $\frac{a-w_2}{3b} - I_1$ (4-70)

Case 1(g): If I^o_2(4-61) < 0, then $I^*_2 = 0$ (4-71)

Case1(h): If I^o_2(4-61) > =0, $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$ (4-72)

Case (2):When I_2 is in the range: $\frac{a}{3b}<I_2<\frac{a}{2b}$

In this case, $w^*_3 = 0$ (from 5-043) and $q^*_{31} = q^*_{32} = 0$ (from Table 3.1, Case 3.3(a)). Hence the 3rd period profit becomes:

$$\Pi R_{31} = (a-2bI_2)(I_2) \quad (4\text{-}73)$$

Hence, retailer 2nd period problem becomes to max:

$$\Pi R_{21} + \Pi R_{31} = (a - b(q_{21} + q_{22} + 2I_1))(q_{21} + I_1) - w_2(q_{21} + I_2) - hI_2 + (a-2bI_2)(I_2) \quad (4\text{-}74)$$

$$\frac{\partial(\Pi R_{21}+\Pi R_{31})}{\partial q_{21}} = a - 2bq_{21} - bq_{22} - 3bI_1 - w_2 \quad (4\text{-}75)$$

$$\frac{\partial^2(\Pi R_{21}+\Pi R_{31})}{\partial q_{21}^2} = -2b < 0 \quad (4\text{-}76)$$

$$\frac{\partial(\Pi R_{21}+\Pi R_{31})}{\partial I_2} = -(w_2 + h) + a - 4bI_2 \quad (4\text{-}77)$$

$$\frac{\partial^2(\Pi R_{21}+\Pi R_{31})}{\partial I_2^2} = -4b < 0 \quad (4\text{-}78)$$

(4-76) and (4-78) prove that (4-74) is concave in q_{21} and I_2 respectively. Setting (4-75) and (4-77) to zero respectively, we obtain the profit-maximizing q_{21} and I_2 respectively as:

$$q^o_{21} = \frac{a-w_2}{2b} - \frac{(q_{22}+3I_1)}{2} \quad (4\text{-}79)$$

$$I^o_2 = \frac{a-(w_2+h)}{4b} \quad (4\text{-}80)$$

Similarly, we can obtain for retailer 2:

$$q^o_{22} = \frac{a-w_2}{2b} - \frac{(q_{21}+3\ _1)}{2} \quad (4\text{-}81)$$

$I^{o}_{2} = \frac{a-(w_2+h)}{4b}$ (4-82)

Solving (4-79) and (4-81) together, we get: $q^{o}_{21} = q^{o}_{22} = \frac{a-w_2}{3b} - I_1$ (4-83)

We need that (4-80) (and 4-82) fulfill the constraints (4-48), (4-49) as well as the domain condition for this sub-case $\frac{a}{3b} \leq I_2 \leq \frac{a}{2b}$.

We also observe that (4-80) and (4-82) is always $\leq \frac{a}{3b}$.

Hence, $I^*_2 = \frac{a}{3b}$ is the optimal I^*_2 decision in this case. (4-84)

<u>Summary of q^*_{21} and I^*_2 decisions:</u>

Two cases for optimal q^*_{21} decisons:

Case 2-S(a): If $\frac{a-w_2}{3b} - I_1 \leq 0$, $q^*_{21} = 0$ ($w_2 \geq a-3bI_1$) (4-85)

Case 2-S(b): If $\frac{a-w_2}{3b} - I_1 > -0$, $q^*_{21} = \; = \frac{a-w_2}{3b} - I_1$ ($w_2 < a-3bI_1$) (4-86)

$I^*_2 = \frac{a}{3b}$ always.

We can similarly obtain the decisions for retailer 2

<u>Table 3.3: Summary of 2nd period retailer decisions – order quantity (q^*_{21}, q^*_{22})</u>

Case No.	Domain Conditions	Equilibrium values
q (S-i)	If $\frac{a-w_2}{3b} - I_1 \leq 0$	$q^*_{21} = q^*_{22} = 0$
q (S-ii)	If $\frac{a-w_2}{3b} - I_1 > 0$	$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I_1$

<u>Table 3.4: Summary of 2nd period retailer decisions – Inventory quantities (I^*_2)</u>

Case No.	Domain Conditions	Equilibrium values
I-(S-i)	If $I_2 < \frac{a}{3b}$ AND $\frac{4a}{15b} - \frac{2}{5b}(w_2+h) < 0$	$I^*_2 = 0$
I-(S-ii)	If $I_2 < \frac{a}{3b}$ AND $\frac{4a}{15b} - \frac{2}{5b}(w_2+h) \geq 0$	$I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2+h)$
I-(S-iii)	If $\frac{a}{3b} \leq I_2 < \frac{a}{2b}$	$I^*_2 = \frac{a}{3b}$

From Table 3.4 above, we can see that, the retailers can set I^*_2 either in the region $I_2 < \frac{a}{3b}$ (in which case, the options are: $I^*_2 = 0$ OR $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$) or in the region $I_2 \geq \frac{a}{3b}$ (in which case $I^*_2 = \frac{a}{3b}$). Which value is chosen ultimately depends on, in which region of the I^*_2 domain, the retailers make more profit. We now check the profit values for each of the I^*_2 options in the following (we use $q^*_{21} = q^*_{22} = 0$, for ease of computation, since the q^*_{21} and q^*_{22} values are independent of the I^*_2 values and hence it does not matter at which q^*_{21}, q^*_{22} level, we compare the profit-functions to find which I^*_2 value yields the most profit)

$\Pi R_{21} + \Pi R_{31}$ ($I^*_2 = 0$) $= (a - b(2I_1))(I_1) - w_2(I_2) - hI_2 + (\frac{2a}{3} - bI_2)(\frac{a}{6b} + \frac{I_2}{2}) - (\frac{a-3bI_2}{2})(\frac{a}{6b} - \frac{I_2}{2})$

$= (a - b(2I_1))(I_1) + (\frac{2a}{3})(\frac{a}{6b}) - (\frac{a}{2})(\frac{a}{6b}) = I_1(a - 2I_1 b) + \frac{a^2}{36b} \qquad = \frac{-(72b^2I_1^2 - 36I_1ab - a^2)}{36b}$

$$= 2bI_1^2 + \frac{(0.028a^2)}{b} + aI_1 \qquad (4\text{-}87)$$

$\Pi R_{21} + \Pi R_{31}$ ($I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$) $= (a - b(2I_1))(I_1) - w_2(I_2) - hI_2 + (\frac{2a}{3} - bI_2)(\frac{a}{6b} + \frac{I_2}{2}) - (\frac{a-3bI_2}{2})(\frac{a}{6b} - \frac{I_2}{2})$

$= \frac{108h^2 + h(216w_2 - 64a) - 600b^2I_1^2 + 300abI_1 + 3a^2 - 64aw_2 + 108w_2^2}{300b}$

$$= \frac{0.36h^2}{b} - \frac{0.2ah}{b} + \frac{0.72hw_2}{b} - 2I_1^2 b + \frac{0.01\;\;^2}{b} - \frac{0.21aw_2}{b} + \frac{0.36w_2^2}{b} + aI_1 \qquad (4\text{-}88)$$

$\Pi R_{21} + \Pi R_{31}$ ($I^*_2 = \frac{a}{3b}$) $= (a - b(q_{21} + q_{22} + 2I_1))(q_{21} + I_1) - w_2(q_{21} + I_2) - hI_2 + (a - 2bI_2)(I_2)$

$= (a - b(2I_1))(I_1) - w_2(I_2) - hI_2 + (a - 2bI_2)(I_2) = \frac{-3ah + 18b^2I_1^2 - 9abI_1 - a^2 + 3aw_2}{9b}$

$$= \frac{-0.33ah}{b} - 2I_1^2 b + \frac{0.11a^2}{b} - \frac{0.33aw_2}{b} + aI_1 \qquad (4\text{-}89)$$

The max value of I_1 is $I_1 = \frac{a}{2b}$

(4-87) at $I_1 = \frac{a}{2b}$ becomes: $\frac{(1.028a^2)}{b}$ (4-90)

(4-88) at $I_1=\frac{a}{2b}$become:$\frac{0.36h^2}{b}-\frac{(0.2ah)}{b}+\frac{(0.72w_2h)}{b}+\frac{(0.01\quad ^2)}{b}-\frac{(0.21\quad _2a)}{b}+\frac{(0.36w_2^2)}{b}$ (4-91)

(4-89) at $I_1=\frac{a}{2b}$become: $\frac{-(0.33ah)}{b}+\frac{(0.11\quad ^2)}{b}-\frac{(0.33w_2a)}{b}$ (4-92)

since a>>h, and a>w_1 we can ignore the terms in (4-90), (4-91) and (4-92), so now,

(4-90) becomes: $\frac{(1.02\quad ^2)}{b}$ (4-93)

(4-91) becomes: $\frac{(0.01a^2)}{b}-\frac{(0.21w_2a)}{b}+\frac{(0.36w_2^2)}{b}<<\frac{-(0.21w_2a)}{b}+\frac{(0.37\quad ^2)}{b}$ (4-94)

(4-92) becomes: $\frac{(0.11a^2)}{b}-\frac{(0.33w_2a)}{b}$ (4-95)

It is clear from (4-93), (4-94) and (4-95), that (4-93) and (4-94) are always greater than (4-95).

Hence, we can say that the retailers will always set their I^*_2 in the range $I_2<\frac{a}{3b}$, in which case, the equilibrim I^*_2 decision is taken as per the following two subcases:

Table 3.5: Summary of I^*_2 decisions

I-(S-i)	If $\frac{4a}{15b}-\frac{2}{5b}(w_2+h)<0$	$I^*_2 = 0$
I-(S-ii)	If $\frac{4a}{15b}-\frac{2}{5b}(w_2+h)\geq 0$	$I^*_2 = \frac{4a}{15b}-\frac{2}{5b}(w_2+h)$

$\frac{4a}{15b}-\frac{2}{5b}(w_2+h)<0 \rightarrow \frac{2a}{3}<(w_2+h) \rightarrow w_2>\frac{2a}{3}-h$

Manufacturer's 2^{nd} period decisions:

Given:

$I_1 \geq 0$ (4-96)

$a\text{-}2bI_1 \geq 0$ (4-97)

$a\text{-}b(q_{21}+q_{22}+2I_1)\geq 0$ (4-98)

$a\text{-}b(q_{21}+q_{22})\geq 0$ (4-99)

$q_{21}\ q_{22} > =0$ (4-100)

Decision variable: w_2

Requirement (constraints) for the decision variables.

$0\leq w_2 \leq a$ (4-101)

The objective function is to maximize the sum of manufacturer's 2nd period and 3^{rd} period profit profit given in (5-096)

$$\Pi M_2+\Pi M_3= w_2(q_{21}+q_{22}+2\mathrm{I}_2)+w_3(q_{31}+q_{32}) \quad (4\text{-}102)$$

There are two main subcases here:

Case (1): If $I_1 > \frac{a}{9\mathrm{b}}+\frac{h}{3\mathrm{b}}$

Here, from Tables 3.3, 3.4 and 3.5:

$q^*_{21} = q^*_{22} = 0, I^*_2 = 0,$ (4-103)

Substituting q^*_{21}, q^*_{22} and I^*_2 into (5-096) from (5-097) we get:

$$\Pi M_2 + \Pi M_3 = w^*_2(0)+(\frac{a-3bI_2}{2})(2*\frac{a-w_3}{3b}-I_2) \quad (4\text{-}104)$$

$$\frac{\partial(\Pi M_1+\Pi M_2)}{\partial w_2}=0 \quad (4\text{-}105)$$

$$\frac{\partial^2(\Pi \mathrm{M}_1+\Pi \mathrm{M}_2)}{\partial \mathrm{w}_2^2}=0 \quad (4\text{-}106)$$

(4-104) and (4-105) prove that (4-098) is constant with respect to w_2.

So, w^*_2 here is set at the maximum permissible by the domain conditions i.e.,

$I_1 > \frac{a}{9b} + \frac{h}{3b}$AAND $0 \le w_2 \le a$ (4-107)

$\rightarrow$ $w^*_2 = a$ is the optimal w^*_2 decision in this case. (4-108)

<u>Case (2): If</u> $I_1 \le \frac{a}{9b} + \frac{h}{3b}$

Here, from Tables 3.3 and 3.4:

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I_1$AND $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$, (I^*_2 is $< \frac{a}{3b}$, hence, $w^*_3 = \frac{a-3bI_2}{2}$)

(4-109)

Substituting q^*_{21}, q^*_{22} and I^*_2 into (5-096) from (5-116) we get:

$\Pi M_2 + \Pi M_3 = w^*_2(q^*_{21}+q^*_{22}+2I^*_2) + w^*_3\,(q^*_{31}+q^*_{32})$ (4-110)

$\Pi M_2 + \Pi M_3 = 2(\frac{a-w_2}{3b} - I_1 + \frac{4a}{15} - \frac{2}{5b}(w_2 + h))(w_2) + \quad (\frac{a-3bI_2}{2})(2^*\,(\frac{a-w_3}{3b} - I_2))$

$= 2(\frac{a-w_2}{3b} - I_1 + \frac{4a}{15} - \frac{2}{5b}(w_2 + h))(w_2) + \quad (\frac{(6h+a+6\quad {}_2)}{10})(2^*\,\frac{a-w_3}{3b} - I_2)$ (4-111)

$= 2(\frac{a-w_2}{3b} - I_1 + \frac{4a}{15b} - \frac{2}{5b}(w_2 + h))(w_2) + (\frac{(6h+a+6\quad {}_2)}{10})(\frac{(6h+a+6w_2)}{15b})$ (4-112)

$\frac{\partial(\Pi M_1+\Pi M_2)}{\partial w_2} = \frac{-(24h-96a+184\quad {}_2)}{75b}$ (4-113)

$\frac{\partial^2(\Pi M_1+\Pi M_2)}{\partial w_2^2} = \frac{-184}{75b}$ (4-114)

(4-128) proves that (4-126) is concave in w_2.

Setting (5-127) to zero, we get the profit-maximizing w^*_2 as :

$w^o_2 = \frac{12a-3}{23}$ (4-115)

we now need to check that w^o_2(4-115) fulfills the domain conditions for this case i.e.,

$I_1 \leq \frac{a}{9b} + \frac{h}{3b}$ AND $0 \leq w_2 \leq a$

We see that (4-115) easily fulfills these constraints and hence.

$w^*_2 = w^o_2(4\text{-}115) = \frac{12a-}{23}$ (4-116)

Table 3.6: Summary of w^*_2 decisions:

Case No.	**Constraint**	**Optimal w^*_2 value:**
Case S-I:	If $I_1 \leq \frac{a}{9b} + \frac{h}{3b}$	$w_2^* = \frac{12a-3h}{23}$
Case S-II:	$I_1 > \frac{a}{9b} + \frac{h}{3b}$	$w^*_2 = a$

<u>1^{st} period retailer 1 decisions:</u>

Given: (Assumptions)

$h>0$ (4-117)

$0< w_1< a$ ` (4-118)

Decision variables: $q_{11}, q_{12} . I_1$

Requirement (constraints) on the decision variables.

$a- b(q_{11}+q_{12})\geq 0$ (4-119)

$q_{11} \geq 0$ (4-120)

$a-2bI_1 \geq 0$ (4-121)

$I_1 \geq 0$ (4-122)

We can write the 1^{st} period profit function for retailer 1 as:

$$\Pi R_{11} =(a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I_1) - hI_1 + (a - b(q_{21} + q_{22} + 2I_1))(q_{21} + I_2)$$

$$-w_2(q_{21} + I_2) - hI_2 + (a - b(q_{31} + q_{32} + 2I_2))(q_{31} + I_2) - w_3 q_{31} \quad (4\text{-}123)$$

The 1^{st} period problem for retailer 1 is to set a q_{11} and I_1 to maximize the sum of 1^{st}, 2^{nd} and 3rd period profits. i.e., retailer 1 needs to maximize:

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31}$ (4-124)

Case (1): If $I_1 > \frac{a}{9b} + \frac{h}{3b}$

Here, $q^*_{21} = q^*_{22} = 0$, $I^*_2 = 0$, $w^*_2 = a$ and $w^*_3 = \frac{a-3bI_2}{2}$, $q^*_{31} = q^*_{32} = 0$ (4-125)

Substituting the values from (4-125) into the retailer profit function over the three periods, we get:

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31} =(a-b(q_{11}+q_{12}))(q_{11}) - w_1(q_{11}+I_1)-hI_1 + (a-b(q_{21}+q_{22}+2I_1))(q_{21}+I_1) -$

$w_2(q_{21}+I_2)-hI_2+ (a-b(q_{31}+q_{32}+2I_2))(q_{31}+I_2) -w_3q_{31}$ (4-126)

$= (a-b(q_{11}+q_{12}))(q_{11}) - w_1(q_{11}+I_1)-hI_1 + (a-2bI_1)(I_1)$ (4-127)

$$\frac{\partial(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial q_{11}} = a - 2\mathrm{b}\mathrm{q}_{11} - bq_{12} - w_1 \tag{4-128}$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial q_{11}^2} = -2\mathrm{b} < 0 \tag{4-129}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial I_1} = a - h - 4\mathrm{b}\mathrm{I}_1 - w_1 \tag{4-130}$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial I_1^2} = -4\mathrm{b} < 0 \tag{4-131}$$

(4-129) and (4-131) prove that (4-128) Is concave in q_{11} and I_1 respectively.

Setting (4-128) and (4-130) to zero, we get the profit-maximizing q_{11} and I_1 as:

$$q^o_{11} = \frac{a-w_1}{2\mathrm{b}} - \frac{q_{12}}{2} \tag{4-132}$$

$$I^o_1 = \frac{a-(w_1+h)}{4\mathrm{b}} \tag{4-133}$$

Similarly, we can write the profit maximizing quantities for retailer 2 as:

$$q^o_{12} = \frac{a-w_1}{2\mathrm{b}} - \frac{q_{11}}{2} \tag{4-134}$$

$$I^o_1 = \frac{a-(w_1+h)}{4\mathrm{b}} \tag{4-135}$$

Solving (4-132) and (4-134) together, we get:

$$q^o_{11} = q^o_{12} = \frac{a-w_1}{3\mathrm{b}} \tag{4-136}$$

and the I^o_1 decision for either retailer is: $I^o_1 = \frac{a-(w_1+h)}{4\mathrm{b}}$ (4-137)

We now need (4-136) and (4-137) to fulfill the constraints (4-119), (4-120), (4-121) and (4-122).

as well as $I_1 > \frac{a}{9\mathrm{b}} + \frac{h}{3\mathrm{b}}$. (4-138)

since the order quantities for the two retailers are equal, (4-119) can be written as:

$a\text{-}2\mathrm{b}q_{11} \geq 0$

---> $q_{11} \leq \frac{a}{2b}$ (4-139)

We see that (4-142) always fulfills (4-144). We now only need to check that (5-135) fulfills $q^o_{11} > 0$, which leads to the following sub-cases:

Case 1(a): If q^o_{11}(4-141) < = 0, $q^*_{11} = 0$ (4-140)

Case 1(b): If q^o_{11}(4-141) > 0, $q^*_{11} = q^o_{11} = \frac{a-w_1}{3b}$ (4-141)

Similarly, we need that (4-142) fulfills the constraints (4-126) and (4-127).

We observe that (4-142) always fulfills (4-126). Now, for (4-142) to fulfill (4-127), we have the following sub-cases:

Case 1(c): If I^o_1(4-140)≤0, $I^*_1 = 0$ (4-142)

Case 1(d): If I^o_1(4-140)>0, $I^*_1 = I^o_1$(5-136)= $\frac{a-(w_1+h)}{4b}$ (4-143)

Case (2): If $I_1 <= \frac{a}{9b} + \frac{h}{3b}$

Here, $q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I_1$, $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$, $w^*_2 = \frac{12a-}{23}$

and $w^*_3 = \frac{a-3bI_2}{2}$, $q^*_{31} = q^*_{32} = 0$ (4-144)

Substituting the values from (4-149) into the retailer profit function over the three periods, we get:

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31} = (a-b(q_{11}+q_{12}))(q_{11}) - w_1(q_{11}+I_1) - hI_1 +$

$$\frac{(297h^2+(-207bI_1-582a)h-75I_1ab+22 \quad ^2)}{4761b} \tag{4-145}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial q_{11}} = a - 2bq_{11} - bq_{12} - w_1 \tag{4-146}$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial q_{11}^2} = -2b < 0 \tag{4-147}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial I_1} = \frac{-207bh-759ab}{4761b} - w_1 \tag{4-148}$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21}+\Pi R_{31})}{\partial I_1^2} = 0 \tag{4-149}$$

(4-154) proves that (4-152) is concave in q_{11} and I_1 respectively.

Setting (4-153) to zero, we get the profit-maximizing q^*_{11} quantity as:

$$q^o_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2} \tag{4-150}$$

We see from (4-155) and (4-156) that (4-155) is constant in I_1. Thus,

$$I_1^o = + \text{Infinity} \tag{4-151}$$

Similarly, we can write for retailer 2:

$$q^o_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2} \tag{4-152}$$

$$I_1^o = + \text{Infinity} \tag{4-153}$$

Solving (4-150) and (4-152) together we get:

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ (4-154)

$I_1^o =$ +Infinity (4-155)

Now, we need that (4-154) and (4-155) fulfill the constraints (4-119) through (4-122) as well as the domain condition: $I_1 < \frac{a}{9b} + \frac{h}{3b}$ (4-156)

Since the order quantities for the two retailers are equal, (4-119) can be written as:

$a-2bq_{11} \geq 0$

---> $q_{11} \leq \frac{a}{2b}$ (4-157)

We see that (4-154) always fulfills (4-157). We now only need to check that (4-154) fulfills $q^o_{11} > 0$, which leads to the following sub-cases:

Case 2(a): If q^o_{11}(4-154) < = 0 , q^o_{12}(5-135) < = 0 , $q^*_{11} = q^*_{12} = 0$ (4-158)

Case 2(b): If q^o_{11}(4-154) > 0, $q^*_{11} = q^o_{11} = \frac{a-w_1}{3b}$ (4-159)

Further, since I^o_1=+Infinity, the I^*_1 that fulfills constraints (4-119) through (4-122) is the maximum allowed by the domain conditions, i.e.,

$I_1 \leq \frac{a}{9b} + \frac{h}{3b}$ and $0 <= I_1 <= \frac{a}{2b}$

➔ $I^*_1 = \frac{a}{9b} + \frac{h}{3b}$ (4-160)

Summary of 1st period retailer q^*_{11} and q^*_{12} decisions:

Case No.	Constraints	Optimal Value
R-S-i	$\frac{a-w_1}{3b} <= 0$	$q^*_{11} = q^*_{12} = 0$
R-S-ii	$\frac{a-w_1}{3b} > 0$	$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$

We can further see: $\frac{a-w_1}{3b} <= 0$ is infeasible since we know $w_1 <= a$, always. So, $q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ is the only optimal w^*_1 decision. (4-160a)

As for the I^*_1 decision, the two possible cases are:
In the range of the I_1 domain: $I_1 > \frac{a}{9b} + \frac{h}{3b}$;

If $\frac{a-(w_1+h)}{4b} \leq 0; I_1^* = 0$ (4-161)

If $\frac{a-(w_1+h)}{4b} > 0$; $I^*_1 = \frac{a-(w_1+h)}{4b}$ (4-162)

And in the range of the I_1 domain: $I_1 <= \frac{a}{9b} + \frac{h}{3b}$; $I^*_1 = \frac{a}{9b} + \frac{h}{3b}$

The final I^*_1 value decision depends on, in which range of the I^*_1 domain, the retailer makes a higher profit (over all the 3 selling seasons).

Note: In the profit functions below: we only check the I^*_1 values since the q^*_{11} values are independent of the I^*_1 domain and will not change with the I^*_1 domain.

Here: $q^*_{21} = q^*_{22} = 0$, $I^*_2 = 0$, $w^*_2 = a$ and $w^*_3 = \frac{a-3bI_2}{2}$, $q^*_{31} = q^*_{32} = 0$

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31}$ $(I^*1 = 0) = -(w_1+h)I_1 + a-b(2I_1)(I_1) = 0$ (4-163)

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31}$ $(I^*_1 = \frac{a-(w_1+h)}{4b}) = (h^2/(4*b) - (a*h)/(4*b) + (w[1]*h)/(2*b) + h/2 - (w[1]*a)/(4*b) + w[1]^2/(4*b) + a/2 + w[1]/2)$ (4-164)

When $I^*_1 = \frac{a}{9b} + \frac{h}{3b}$; $q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I_1$, $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h)$, $w^*_2 = \frac{12a-3h}{23}$

$\Pi R_{11} + \Pi R_{21} + \Pi R_{31}$ $(I^*_1 = \frac{a}{9b} + \frac{h}{3b}) =$

$(\frac{47}{69} - \frac{2h}{23})(\frac{h}{23b} + \frac{11a}{69b}) + (\frac{16h}{23} + \frac{61a}{69})(\frac{4a}{69} - \frac{8h}{23b}) - (109h^2)/1587b + (845*a*h)/(4761*b) - (w[1]*h)/(3*b) - (88*a^2)/(1587*b) - (w[1]*a)/(9*b)$ (4-165)

Comparing profits in (4-163), (4-164) and (4-165), we see that, the retailer makes more profit always with I*1 in the range: $I_1 <= \frac{a}{9b} + \frac{h}{3b}$;

➔ $I^*_1 = \frac{a}{9b} + \frac{h}{3b}$ is the optimal I^*_1 decision here. (4-166)

1st period manufacturer decisions

Given:

a, b, > 0 (4-180)

q_{11}, q_{12}, I ≥0 (4-181)

Decision variable: w_1

Requirement (constraints) for the decision variables.

$0 \le w_1 \le a$ (4-182)

The manufacturer's profit function over the three periods becomes:

$$\Pi M_1 + \Pi M_2 + \Pi M_3 = w_1(q_{11} + q_{12} + 2\mathrm{I}_1) + w_2(q_{21} + q_{22} + 2\mathrm{I}_2) + w_3(q_{31} + q_{32}) \quad (4\text{-}183)$$

There is only one optimal combination of w^*_2, q^*_{21},q^*_{22}, I^*_2, w^*_3 and q^*_{31},q^*_{32} here

Which is: $I^*_1=\frac{a}{9\mathrm{b}} + \frac{h}{3\mathrm{b}}$, $q^*_{11}=q^*_{12} = \frac{a-w_1}{3\mathrm{b}}$, $q^*_{21} = q^*_{22} = \frac{a-w_2}{3\mathrm{b}} - I_1$, $I^*_2 = \frac{4\mathrm{a}}{15\mathrm{b}} - \frac{2}{5\mathrm{b}}(w_2 + h)$,

$w^*_2 = \frac{12\mathrm{a}-3\mathrm{h}}{23}$; and $w^*_3 = \frac{a-3\mathrm{bI}_2}{2}$, $q^*_{31} = q^*_{32} = 0$ (4-184)

substituting (4-184) into (4-183), we get:

$$\Pi M_1 + \Pi M_2 + \Pi M_3 = w_1(q_{11} + q_{12} + 2\mathrm{I}_1) + \frac{(36h^2 + (36bI_1 - 12a)h - 144I_1ab + 47a^2)}{138b}$$

=

$$(36h^2 + h\left(36b\left(\frac{h}{3b} + \frac{a}{9b}\right) - 12a\right) - 144ab\left(\frac{h}{3b} + \frac{a}{9b}\right) + 47a^2)/(138b) + w_1\left(2\left(\frac{h}{3b} + \frac{a}{9b}\right) + \frac{2(a-w_1)}{3b}\right) \quad (4\text{-}184)$$

$\frac{\partial(\Pi M_1+\Pi M_2)}{\partial w_2}$ =2*(h/(3*b)+a/(9*b))+(2*(a-w[1]))/(3*b)-(2*w[1])/(3*b) (4-185)

$\frac{\partial^2(\Pi M_1+\Pi M_2)}{\partial w_2^2}$= -4/3b <0 (4-186)

(4-186) shows that (4-184) is concave in w_1. Setting (4-185) to zero, we get the profit-maximizing $w^o{}_1$ decision as:

$w^o{}_1 = \frac{4a+3h}{6}$ (4-187)

we readily see that (4-187) is >a, i.e., > than the RHS of the constraint on the w*1 value i.e., (4-182).

Hence, w^*_1 = a is the optimal w^*_1 decision here.

<u>Summary of decisions - three period Cournot model:</u>

<u>1st period:</u>

$w^*_1 = a$; $I^*_1 = \frac{a}{9b} + \frac{h}{3b}$, $q^*_{11} = q^*_{12} = 0$

<u>2nd period:</u>

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I_1 = \frac{10a-60h}{207b}$

$w^*_2 = \frac{12a-3}{23}$; $I^*_2 = \frac{4a}{15b} - \frac{2}{5b}(w_2 + h) = \frac{4a-24h}{69b}$

<u>3rd period:</u>

$w^*_3 = \frac{a-3bI_2}{2}$, $q^*_{31} = q^*_{32} = 0$

* * *

5. 2 period Cournot duopoly with strategic inventories and vertical control

Analysis under *Commitment contract*

5.1 Retailer 1, 2^{nd} period retailer decisions:

Given:

$a > 0, b>0, h>0, I>0$ (5-001)

$0<w_2<=a, 0<w_1<=a$ (5-002)

$q_{11} >=0, q_{12}>=0$ (5-003

$q_{22}>=0$ (5-004)

$a-b(q_{22}+2I)>=0$ (5-005)

$a-2bI >=0$ (5-006)

Constraints:

$q_{21}>=0$ (5-007)

$a-b(q_{21}+q_{22}+2I)>=0$ (5-008)

The 2nd period profit function for retailer 1 can be written as:

$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2\mathrm{I}))(q_{21} + I) - (w_2 q_{21})$ (5-009)

$\frac{\partial \Pi R_{21}}{\partial q_{21}} = a - b(2\mathrm{q}_{21} + q_{22} + 3\mathrm{I}) - w_2$ (5-010)

$\frac{\partial^2 \Pi R_{21}}{\partial q_{21}^2} = -2\mathrm{b} < 0$ (5-011)

(5-011) shows that (5-009) is concave with respect to q_{21}

Setting (5-010) to zero, we get the profit-maximizing 2^{nd} period order quantity for retailer 1 as:

$q_{21}^{o} = \frac{a-w_2}{2\mathrm{b}} - \frac{q_{22}+3\mathrm{I}}{2}$ (5-012)

We need that (5-012) fulfills (5-007) and (5-008) .

Re-arranging (5-008), we get:

$q_{21} <= \frac{a}{b} - (q_{22} + 2I)$ (5-013)

We observe that (5-012) always fulfills (5-013) and hence (5-008).

So, we now only need to check that (5-012) fulfills (5-007), which leads to the following sub-cases for the optimal q^o_{21} decision:

Case 4.1(a): If q^o_{21}(5-013) <=0, $q^*_{21} = 0$ (5-014)

Case 4.1(b): If q^o_{21}(5-013) > 0, $q^*_{21} = q^o_{21}$(3-3009) = $\frac{a-w_2}{2b} - \frac{q_{22}+3I}{2}$ (5-015)

<u>4.2: Retailer 2's 2nd period order quantity decisions</u>

Given: (Assumptions)

$I_1, I_2 >= 0,\ 0 < w_2 < a,\ h > 0$ (5-016)

$a-b\,(I_1+I_2) >= 0$ OR $a-2bI >= 0$ (5-017)

$a-b(q_{21}+I) >= 0$ (5-018)

Decision variables: q_{22}

Requirement (constraints) on the decision variables.

$a-b\,(q_{21}+I_1+q_{22}+I_2) >= 0$ OR $a-b(q_{21}+q_{22}+2I) >= 0$ (5-019)

$q_{22} >= 0$ (5-020)

We can write the 2nd period profit function for retailer 1 as:

$\Pi R_{21} = (a - b(q_{21} + q_{22} + 2I))(q_{22} + I) - (w_2 q_{22})$ (5-021)

Since retailer 2 is symmetrical to retailer 1, we can use a procedure similar to the one employed in Section 3.11 to derive retailer 2's 2nd period profit-maximizing order quantity as $q^o_{22} = \frac{a-w_2}{2b} - \frac{q_{21}+3I}{2}$ (5-022)

For (5-022) to fulfill (5-019) and (5-020), we have the following two sub-cases, again, using a procedure similar to that in Section 4.1

Case 4.2(a) If q^o_{22}(5-022) <=0, $q^*_{22} = 0$ (5-023)

Case 4.2(b): If q^o_{22}(5-022) > 0, $q^*_{22} = q^o_{22}$(5-022) = $\frac{a-w_2}{2b} - \frac{q_{21}+3I}{2}$ (5-024)

<u>4.3: Combined equilibrium analysis – q_{21} and q_{22} decisions</u>

Since the two retailers are identical in all respects, symmetrical, in Cournot competition with each other and take their decisions simultaneously, we can postulate that their equilibrium 2nd period order quantities are equal. In this case, there are only two possible equilibria:

Case 4.3 (a): q^o_{21}(5-013) <=0 AND q^o_{22}(5-022) <=0 (Case 4.1(a) and Case 4.2(a))

Here, $q^*_{21} = q^*_{22} = 0$ (From (5-014) and (5-023)) (5-025)

Case 4.3 (b): q^o_{21}(5-013) >0 AND q^o_{22}(5-022) >0 (Case 4.1(b) and Case 4.2(b))

Here, $q^*_{21} = \frac{a-w_2}{2b} - \frac{q_{22}+3I}{2}$ From (5-014) (5-026)

$q^*_{22} = \frac{a-w_2}{2b} - \frac{q_{21}+3I}{2}$From (5-023) (5-027)

Solving (5-026) and (5-027) together, we obtain,

$q^*_{21} = q^*_{22} = \frac{a-w_2}{3b} - I$ (5-028)

Table 1. Summary of 2nd period retailer order quantity decisions

Case No.	Domain Conditions	Equivalent Conditions	Equilibrium Decision (q^*_{21},q^*_{22})
4.3(a)	q^o_{21}(5-013) <=0 AND q^o_{22}(5-022) <=0	$\frac{a-w_2}{3b} - I$<= 0	0,0

4.3(b)	q^o_{21}(5-013) >0 AND q^o_{22}(5-022) >0	$\frac{a-w_2}{3b} - I > 0$	$\frac{a-w_2}{3b} - I$, $\frac{a-w_2}{3b} - I$

Since w_1, and w_2 are decided at the beginning of the 1st period itself, the next decision we analyze, using the backward induction framework are the 1st period order quantity decisions for either retailer.

<u>4.4: 1st period retailer 1 order quantity decisions for retailer 1:</u>

Given: (Assumptions)

a, b, h >0

(5-029)

$0 <= w_1+h < a,\ 0 <= w_2 < a$ (5-030)

$q_{12} >= 0$ (5-031)

$a-b(q_{12}) >= 0$ (5-032)

$a-bI >= 0$ (5-033)

Decision variables: q_{11}, I

Requirement (constraints) on the decision variables.

$q_{11} >= 0$ (5-034)

$a-b\ (q_{11}+q_{12}) >= 0$ (5-035)

$I >= 0$ (5-036)

$a-2bI >= 0$ (5-037)

We can write the 1st period profit function for retailer 1 as:

$\Pi R_{11} = (a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I) - hI$ (5-038)

The 1st period problem for retailer 1 is to set a q_{11} and I to maximize the sum of 1st and 2nd period profits. i.e., retailer 1 needs to maximize:

$$\Pi R_{11} + \Pi R_{21} \tag{5-039}$$

From Table 1, we can observe that there are two possible sub-cases here, since ΠR_{21} is different depending on whether (q^*_{21},q^*_{22}) is given by Case 4.3(a) or Case 4.3(b). Hence, we have the following two sub-cases:

Case 4.4(a): $\frac{a-w_2}{3b} - I <= 0$ OR $I >= \frac{a-w_2}{3b}$

In this case, from Table 1, $q^*_{21} = q^*_{22} = 0$ and hence

$$\Pi R_{21} = (a-b(2I))(I) \tag{5-040}$$

The sum of 1st and 2nd period profits for retailer 1 hence becomes:

$$\Pi R_{11} + \Pi R_{21} = (a - b(q_{11} + q_{12}))(q_{11}) - w_1(q_{11} + I) - hI + (a-b(2I))(I) \tag{5-041}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}} = a - 2bq_{11} - bq_{12} - w_1 \tag{5-042}$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}^2} = -2b \tag{5-043}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial I} = -(w_1 + h) + a - 4bI \tag{5-044}$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial I} = -4b \tag{5-045}$$

(5-043) and (5-045) show that (5-041) is concave in q_{11} and I respectively.

Setting (5-042) and (5-044) to zero respectively, we get the profit-maximizing q_{11} and I values as:

$$q^o_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2} \tag{5-046}$$

$$I^o = \frac{a-(w_1+h)}{4b} \tag{5-047}$$

We now need (5-046) and (5-047) to fulfill the constraints (5-034) ~ (5-037).

Rearranging (5-035), we have:

$q_{11} <= \frac{a}{b} - q_{12}$ (5-048)

Similarly, from (5-037), we can write:

$I <= \frac{a}{2b}$ (5-049)

We see that (5-046) and (5-047) fulfill (5-048) and (5-049) respectively.

As such, We have the following four sub-cases for the optimal q^*_{11} decision:

Case 4.4(a-i): if q^o_{11}(5-046) <0, $q^*_{11} = 0$ (5-050)

Case 4.4(a-ii): If q^o_{11}(5-046)>=0, $q^*_{11} = q^o_{11}$(5-046) = $\frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (5-051)

We observe that (5-047) needs to fulfill (5-036) as well as the domain condition for this case i.e.,

$\mathbf{I >=} \frac{a-w_2}{3b}$ (5-052)

We see if $\frac{a-w_2}{3b} <= 0 \rightarrow w_2 >= a$.

From (5-030), we see that w_2 is always < a.

$\rightarrow \frac{a-w_2}{3b}$ is always >0 (5-053)

Also, we see that $\frac{a-(w_1+h)}{4b} <= 0 \rightarrow w_1 >= a$, which again contradicts (5-030). As such, we can also write: $\frac{a-(w_1+h)}{4b} > 0$ always (5- 054)

From all of the above, we have the following sub-cases for the optimal I* decision:

Case 4.4(a-iv): If I^o(5-047)<=$\frac{a-w_2}{3b}$, $I^* = \frac{a-w_2}{3b}$ (5-055)

Case 4.4(a-v): If I^o(5-047) >$\frac{a-w_2}{3b}$, $I^* = I^o$(5-047) = $\frac{a-(w_1+h)}{4b}$ (5-056)

Table 2: Summary of Retailer 1's 1st period decisions:

Case No.	Domain Conditions	Decision
4.4(a-i)	$\frac{a-w_1}{2b}-\frac{q_{12}}{2}<0$	$q^*_{11}=0$
4.4(a-ii)	$\frac{a-w_1}{2b}-\frac{q_{12}}{2}>=0$	$q^*_{11}=\frac{a-w_1}{2b}-\frac{q_{12}}{2}$
4.4(a-iii)	$\frac{a-(w_1+h)}{4b}<=\frac{a-w_2}{3b}$	$I^*=\frac{a-w_2}{3b}$
4.4(a-iv)	$\frac{a-(w_1+h)}{4b}>\frac{a-w_2}{3b}$	$I^*=\frac{a-(w_1+h)}{4b}$

Case 4.4(b): $I<=\frac{a-w_2}{3b}$

In this case, we know from Table 1 that $q^*_{21}=q^*_{22}=\frac{a-w_2}{3b}-I$ and hence

$$\Pi R_{21}=(a-b(2*\frac{a-w_2}{3b}))(\frac{a-w_2}{3b})-w_2(\frac{a-w_2}{3b}-I) \qquad (5\text{-}057)$$

The sum of 1st and 2nd period profits for retailer 1 hence becomes:

$$\Pi R_{11}+\Pi R_{21}=(a-b(q_{11}+q_{12}))(q_{11})-w_1(q_{11}+I)-hI+\frac{(a-w_2)^2}{9b}+w_2 I \qquad (5\text{-}058)$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}}=a-2bq_{11}-bq_{12}-w_1 \qquad (5\text{-}059)$$

$$\frac{\partial^2(\Pi R_{11}+\Pi R_{21})}{\partial q_{11}^2}=-2b \qquad (5\text{-}060)$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial I}=-(w_1+h)+w_2 \qquad (5\text{-}061)$$

$$\frac{\partial(\Pi R_{11}+\Pi R_{21})}{\partial I}=0 \qquad (5\text{-}062)$$

(5-060) shows that (5-058) is concave in q_{11}

Setting (5-059) to zero we get the profit-maximizing q_{11} as:

$$q^o_{11}=\frac{a-w_1}{2b}-\frac{q_{12}}{2} \qquad (5\text{-}063)$$

(5-061) and (5-062) show that (5-058) increases linearly with increase in I^o. Hence, the profit-maximizing I^o = + Infinity.

$$I^o = +\text{ Infinity} \qquad (5\text{-}064)$$

We however need that (5-063) and (5-064) fulfill the conditions (5-034)~(5-037). We see readily that (5-063) always fulfills (5-035). Hence, we only need to check that (5-063) fulfills (5-034), which leads to the following two cases for the optimal q^*_{11} decision:

Case 4.4(b-i): if q^o_{11}(5-063) <0, $q^*_{11} = 0$ (5-065)

Case 4.4(b-ii): If q^o_{11}(5-063)>=0, $q^*_{11} = q^o_{11}$(5-063) = $\frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (5-066)

The optimal I* decision is the max. allowed by the domain conditions i.e.,

$I >= 0$ AND $I <= \frac{a}{2b}$ AND $I <= \frac{a-w_2}{3b}$

$\rightarrow I^* = \frac{a-w_2}{3b}$ (5-067)

In summary, retailer 1's 1st period decisions can be written as:

<u>q^*_{11} decisions:</u>

Case 4.5-S(i): If $\frac{a-w_1}{2b} - \frac{q_{12}}{2} <= 0$, $q^*_{11} = 0$ (5-068)

Case 4.5-S(ii): If $\frac{a-w_1}{2b} - \frac{q_{12}}{2} > 0$, $q^*_{11} = \frac{a-w_1}{2b} - \frac{q_{12}}{2}$ (5-069)

<u>I* decisions:</u>

If $I >= \frac{a-w_2}{3b}$ AND $\frac{a-(w_1+h)}{4b} <= \frac{a-w_2}{3b}$, $I^* = \frac{a-w_2}{3b}$ (5-070)

If $I >= \frac{a-w_2}{3b}$ AND $\frac{a-(w_1+h)}{4b} > \frac{a-w_2}{3b}$, $I^* = \frac{a-(w_1+h)}{4b}$ (5-071)

If $I < \frac{a-w_2}{3b}$, $I^* = \frac{a-w_2}{3b}$ (5-072)

We observe that: $\frac{a-(w_1+h)}{4b} <= \frac{a-w_2}{3b} \rightarrow 4w_2 - 3w_1 <= a+3h$ (5-073)

We know from (5-030) that:

$0 <= w_2 < a$ (5-074)

$0 <= w_1 <= a$ (5-075)

4*(5-075) – 3*(5-074) yields

$4w_2$-$3w_1$<=a+3h, which now always holds, since (5-074) and (5-075) are given conditions.

$\rightarrow \frac{a-(w_1+h)}{4b} <= \frac{a-w_2}{3b}$ holds always (5-076)

Thus, we can say that $\frac{a-(w_1+h)}{4b} > \frac{a-w_2}{3b}$ is impossible and hence (5-071) is impossible.

This leads to the fact that I* = $\frac{a-w_2}{3b}$ is the only profit-maximizing I* decision that holds always, and hence, this is the only I* decision possible. (5-077)

4.5 Retailer 2's 1st period decisions:

From symmetry to section 4.4, we can write:

Case 4.5-S(i): If $\frac{a-w_1}{2b} - \frac{q_{11}}{2} <= 0$, $q^*_{12} = 0$ (5-078)

Case 4.5-S(ii): If $\frac{a-w_1}{2b} - \frac{q_{11}}{2} > 0$, $q^*_{12} = \frac{a-w_1}{2b} - \frac{q_{11}}{2}$ (5-079)

and I* = $\frac{a-w_2}{3b}$ (5-080)

4.6: Combined equilibrium analysis - 1st period retailer decisions:

From (5-068) and (5-078), we can write: if $\frac{a-w_1}{2b} <= 0$, $q^*_{11} = q^*_{12} = 0$ (5-081)

However, we see that, $\frac{a-w_1}{2b} <= 0 \rightarrow a <= w_1$ (5-082)

From (5-030), we can see that (5-082) is impossible and hence the only possible q^*_{11}, q^*_{12} decision pair is: (From (5-069) and (5-079))

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b}$ (5-083)

and the only possible I* decision for either retailer is I* = $\frac{a-w_2}{3b}$ (5-084)

Substituting these back into Table 1, we see that the only optimal (q^*_{21}, q^*_{22}) decision is:

$q^*_{21} = q^*_{22} = 0$ (5-085)

This shows that the two retailers do not order anything in the 2nd period, under a Commitment contract. (and it follows that a Commitment contract structure is unable to prevent strategic inventory carriage by the retailers).

<u>Manufacturer's 1st and 2nd period wholesale price decisions:</u>

Given:

a, b, > 0 (5-086)

q_{11}, q_{12}, I, q_{21}, q_{22}>=0 (5-087)

Decision variable: w_1, w_2

Requirement (constraints) for the decision variables.

$0 <= w_1 + h < a$ (5-088)

$0 <= w_2 < a$ (5-089)

The objective of the manufacturer in the 1st period is to maximize the sum of 1st and 2nd period profits i.e., maximize:

$$\Pi M_1 + \Pi M_2 = w_1(q_{11}+q_{12}+2I^*)+w_2(q^*_{21}+q^*_{22}) \quad (5\text{-}089)$$

$$= w_1(2*\frac{a-w_1}{3b}+2*\frac{a-w_2}{3b}) \quad (5\text{-}090)$$

$$\frac{\partial(\Pi M_1+\Pi M_2)}{\partial w_1} = \frac{2a}{3b} - \frac{4}{3b}w_1 \quad (5\text{-}091)$$

$$\frac{\partial^2(\Pi M_1+\Pi M_2)}{\partial w_1^2} = \frac{-4}{3b} \leq 0 \quad (5\text{-}092)$$

$$\frac{\partial(\Pi M_1+\Pi M_2)}{\partial w_2} = \frac{2a}{3b} - \frac{4}{3b}w_2 \quad (5\text{-}093)$$

$$\frac{\partial^2(\Pi M_1+\Pi M_2)}{\partial w_2^2} = \frac{-4}{3b} \leq 0 \quad (5\text{-}094)$$

(5-092) and (5-094) prove that (5-089) is concave in w_1 and w_2 respectively. Setting (5-091) and (5-093) to zero respectively, we get the profit-maximizing $w^o{}_1$ and $w^o{}_2$ decisions for this case as:

$$w^o{}_1 = w^o{}_2 = \frac{a}{2} > 0 \qquad (5\text{-}095)$$

We observe that (5-095) fulfills (5-089)

For (5-095) to fulfill (5-088), we need $\frac{a}{2}$<=a OR a>=h, which usually holds. Hence, (5-095) mostly fulfills (5-088) also.

$$\rightarrow w^*{}_1 = w^*{}_2 = \frac{a}{2} \qquad (5\text{-}096)$$

Summary:

$w^*_1 = w^*_2 = \frac{a}{2}$ (5-S001)

$q^*_{11} = q^*_{12} = \frac{a-w_1}{3b} = \frac{a}{6b}$ (5-S002)

$q^*_{21} = q^*_{22} = 0$ (5-S003)

$I^* = \frac{a-w_2}{3b} = \frac{a}{6b}$ (5-S004)

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Curriculum Vitae

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Jang, J., and **Viswanathan, V.,** Strategic Inventories under a Commitment contract in a supply chain with downstream Cournot duopoly competition, Accepted to 4th Annual International Conference on Industrial, Systems and Design Engineering, June 2016, Athens, Greece.

Viswanathan, V., and Jang, J. , Impact of Cournot competition and Commitment contract on Strategic Inventory in a one-manufacturer, two-retailer supply chain. Proceedings of the 2012 Annual Meeting of the Production and Operations Society, Chicago, IL.

Viswanathan. V, and Jang, J., Strategic inventories in a two-period Stackelberg duopoly with vertical control, Proceedings of the 2010 Annual Meeting of the Western Decision Sciences Society, April 2010, Lake Tahoe, NV.

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- Graduate Student Travel Award to attend WDSI'10 – Spring 2010, UW-Milwaukee.
- Graduate Student Travel Award to attend POMS'09– Spring 2009, UW Milwaukee.
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Select Service:

- Reviewer, 2010 Academy of Management Annual Meeting, Chicago, IL Reviewer, 2010 Western DSI Annual Meeting, Lake Tahoe, NV
- Session Chair, supply chain Strategy, POMS Annual Meeting '09, Orlando, FL
- Reviewer, 2009 Academy of Management Annual meeting, Chicago, IL
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