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Multi-Level Optimal Design Using Game Theory with Model Updating By Low Discrepancy Sampling

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MULTI-LEVEL OPTIMAL DESIGN USING GAME THEORY WITH MODEL UPDATING BY LOW DISCREPANCY SAMPLING

by

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ABSTRACT

MULTI-LEVEL OPTIMAL DESIGN USING GAME THEORY WITH MODEL UPDATING BY LOW DISCREPANCY SAMPLING

by

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Under the Supervision of Professor Anoop K. Dhingra

The Design of Experiment (DOE) based response surface methodology (RSM) is a commonly used technique for solving optimization problems. The traditional DOE method has some shortcomings when used to update the RSM model. This thesis aims to develop a new DOE technique to solve the model updating problems in design optimization. Toward this end, a new DOE based RSM method is proposed to solve this problem by using low-discrepancy sequence method to generate the additional data points needed to update the model to replace the traditional factor and level based DOE method.

Tested on a couple of numerical example problems, the low-discrepancy sequence method is seen to be effective not only in solving the model updating problem, but also more effective and convenient compared to the traditional DOE method.

The second part of this thesis deals with using game theory for solving multi-level design optimization problems. Based on three basic game modes, the Nash game (which is also considered as non-cooperative game), cooperative game, and Stackelberg game (a game between leaders and followers), two solution approaches for Stackelberg game with multiple leaders and followers are proposed: The Decentralized mode and the Hierarchical mode. During the research on these two game systems, solution approaches for a third system namely the Decentralized-Hierarchical model is also addressed in this thesis. It is

seen that the low discrepancy sampling based approaches proposed in this thesis are quite effective in solving multi-level optimization problems.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGEMENTS

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I would like to thank my parents as well. As the only child in my family, I get nothing but love from my parents. They worked hard to raise me and support my study overseas. I hope I can make them proud of me with the success of this thesis.

I would also say thank you to the faculty in the Mechanical Department who offered professional expertise to me. The knowledge that I learned from them was necessary to complete this thesis.

Finally, I want to thank my lab members. I will never forget the days we spent together. We helped each other not only in study, but also in life. I hope our friendship will never fade away even when we are in different corners of the world one day.

Chapter 1. Introduction

This thesis proposes a new idea to realize model updating while using the response surface method (RSM) to solve multi-level design optimization problems. The optimization problems are solved using a game theory based approach.

1.1 Literature Review

Optimal design is defined as one which satisfies all the design requirements and make the expenses the smallest. In another word, the optimal design is the best solution to the problem that can be achieved given design requirements.

In practice, to achieve the most effective result frequently requires multiple objectives. However, designs that make all the objective functions simultaneously minimum in a multi-objective problem rarely occur, because generally there exist conflicts among the multiple objectives present in the problem. Often, the decision makers have to choose one objective or several objectives that they are most concerned with. The multiple objectives are sometimes coordinated at multiple levels. Research about multi-level optimal design have been conducted since early 1970s, and many methods have been developed so far.

1.1.1 Global Criterion Method (GCM)

Faísca et al. (2006) proposed a global parametric programming optimization strategy for multi-level problems. One year later, Faísca et al. (2007) developed a global optimization approach for the solution of various classes of bi-level programming problems (BLPP).

Jose et al. (2012) mentioned that the Global Criterion Method was characterized as a strategy where the optimal solution is found by minimizing a preselected global criterion, F(x), such as the sum of the squares of relative deviations of individual objective functions from the feasible ideal solutions.

Several optimization algorithms can be applied to obtain the optimum solution using a GCM formulation. Genetic algorithm is one approach used to solve the global optimization problems. Dua and Pistikopoulos (2003) developed different algorithms for different objective function models.

1.1.2 Goal Attainment Method

In the Goal Attainment Method, goals are set as b_i for the objective function $f_I(X), i = 1, 2, \dots, k$. Also, a weight w_I is assigned to every objective function to denote the importance of the i^{th} objective function relative to other objective functions in meeting the goal b_i is considered as the overall objective function. Often the goal b_i is found by first solving the single objective optimization problem [Rao (2009)].

The Goal attainment method was first presented by Gembicki and Haimes (1975). This method overcame some of the limitations and disadvantages of methods available in early 1970s. It used vector optimization as a tool for analyzing static control problems with performance and parameter sensitivity indices.

1.1.3 Bounded Objective Function Method

In the Bounded Objective function method, the minimum and the maximum acceptable achievement levels for each objective function f_i are specified as L^i and U^I, respectively, for $i = 1, 2, \dots k$. Then the optimum solution x^* is found by minimizing the most important objective function [Rao (2009)].

In this approach, only the most important objective function is minimized and the other objective functions are considered as constraints. Lower and upper bounds on acceptable values are set for the other objective functions. Haimes et al. (1971) proposed the trade-off approach in which the lower bounds are excluded. Goicoechea et al. (1976), Cohon (1978) developed this approach to obtain feasible solutions.

In many cases, the solutions to multi-objective design problems are not a singleton. Many other methods such as the Utility Function Method, Inverted Utility Function Method, Lexicographic Method, Goal Programming Method have been used to find a single solution from a multitude of possible solutions to a multi-objective optimization problem.

The following solution approaches are considered methods in this thesis for solving multi-objective, multi-level problems.

1.2 Response Surface Method (RSM)

Response Surface Methodology is a collection of mathematical and statistical techniques for empirical model building. By careful use of design of experiments (DOE), the objective is to construct a response function that is influenced by several independent variables. The application of RSM for design optimization is aimed at reducing the cost of expensive analysis methods and their associated numerical noise. Generally, the structure of the relationship between the response and the independent variables is unknown. The first step in the RSM method is to find a suitable approximation to the true relationship.

The RSM has gained acceptance as a popular optimization methods in recent years. Anjum et al. (1997), Baş and Boyacı (2007), Bezerra et al. (2008) and have applied RSM as a tool to solve optimization problems in different fields.

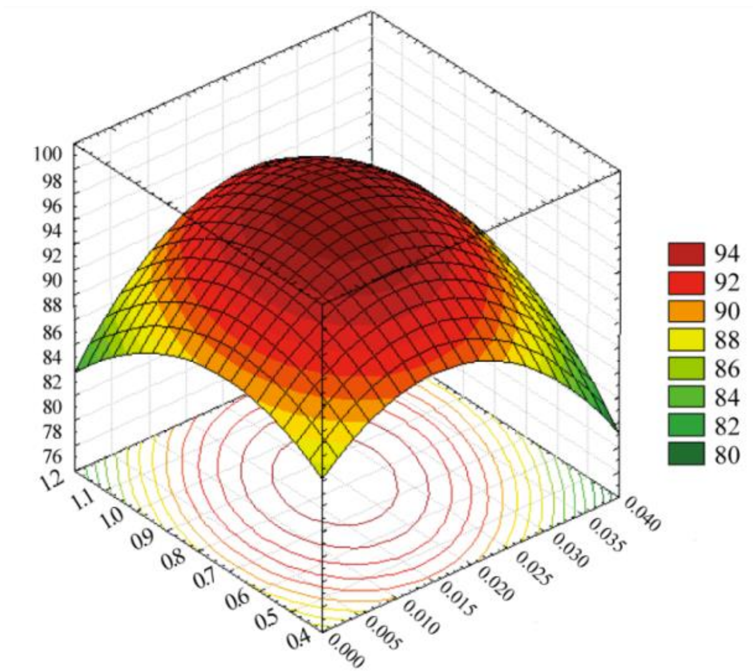

Figure 1.1: Response surface sketch

1.3 Low-discrepancy sequence

In statistics, low-discrepancy sequences can be applied as generating algorithms for testing randomly generated points for use with numerical methods (such as Monte Carlo simulation). Although these sequences are generated using prescribed relations, these sequences can be largely viewed as yielding randomly generated points. Pugazhendhi (2011) reported that the low-discrepancy sequences can be very effective for structural reliability estimation, and presented two algorithms generating randomly distributed test points.

1.3.1 Halton Sequence

The Halton sequence is constructed according to a deterministic method. For example, a 2D problem uses 2 prime numbers for example a and b corresponding to base points on X-axis and Y-axis. Both 2 axes generate points from the interval of (0,1). To generate a Halton sequence, let m be a prime number, and then any natural number k has a unique m-digit representation:

$$k = b_0 + b_1 m + b_2 m^2 + \cdots + b_r m^r \tag{1.1}$$

here $b_i \in \{0,1, \dots, m-1\}$ for $i = 0,1 \dots, r \ and \ m^r < m^{r+1}$. Define the base-m radical inverse function $\phi_m(k)$ as,

$$\phi_m(k) = b_0 m^{-1} + b_1 m^{-2} + \cdots + b_r m^{-(r+1)} \tag{1.2}$$

Note that for every k, $\phi_m(k) \in (0,1)$. Let p_i be s distinct prime numbers $i = 1, \dots, s$ and then the s-dimensional sequence P is called Halton sequence.

$$P = \{\phi_{p_1}(k), \phi_{p_2}(k), \dots, \phi_{p_s}(k)\} \tag{1.3}$$

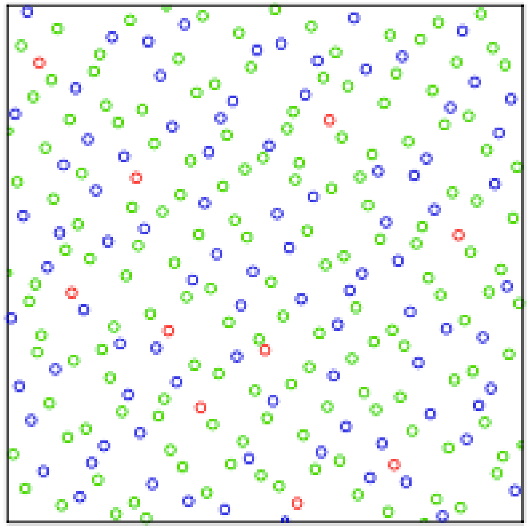

Figure 1.2: 2D Halton sequence of 256 points map

It can be seen from Figure 1.2 that the randomly generated points are distributed quite uniformly throughout the 2D space.

1.3.2 Hammersely Sequence

To generate a Hammersely sequence, let p_i be s distinct prime numbers $i = 1, \ldots, s$ and then the (s+1)-dimensional sequence P is called Hammersely sequence.

$$P = \{\left(\frac{2k-1}{2N}\right), \phi_{p_1}(k), \phi_{p_2}(k), \ldots, \phi_{p_s}(k)\} \tag{1.4}$$

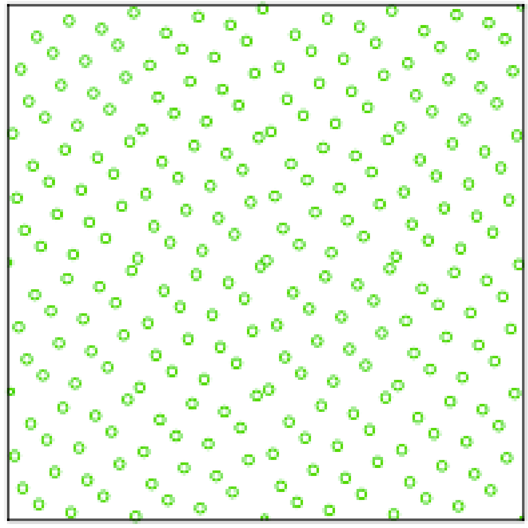

Figure 1.3: Hammersely sequence of 256 points map

Once again, Figure 1.3 shows that the Hammersely points are evenly distributed throughout the 2D space.

1.4 Traditional DOE Method

The DOE Method is a tool widely used in industry. By carefully applying DOE principles before volume production, the companies can save a lot of money and time. Many researchers have used DOE Methods to solve design optimization problems as well. Marston (2000) proposed a DOE based method for solving a pressure vessel optimal design problem. Ghotbi (2013) discussed this problem and proposed another DOE based method for solving a two bar-truss optimal design problem.

Traditional DOE Methods include full factorial design, OED (orthogonal experimental design), CCD (central composite deign), BBD (Box-Benhnken design) etc.

A full factorial experimental design means a design that takes all possible combinations of its levels across all such factors.

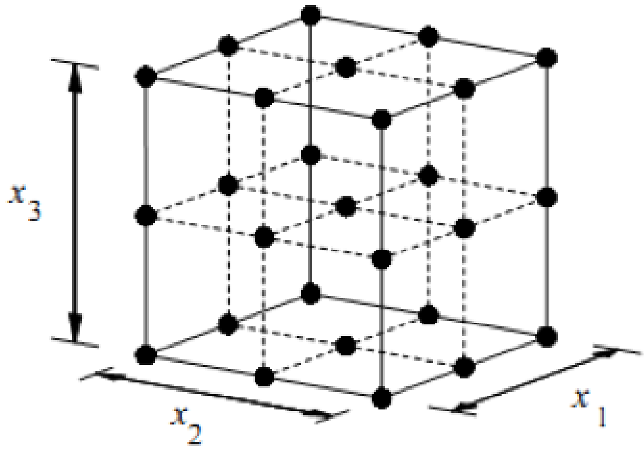

Figure 1.4: A 3 level full factorial design

A CCD or fractional factorial design with center points, augmented with a group of axial points (star points) let one estimate curvature of the response curve.

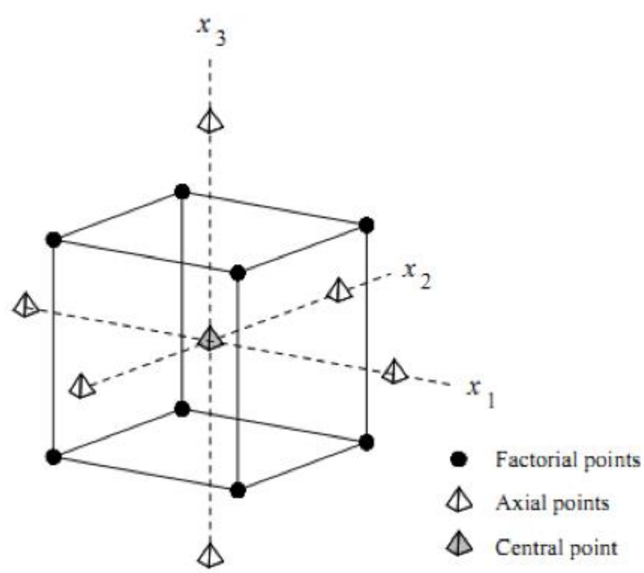

Figure 1.5: A 2 level 3 factor Central Composite Design

1.5 Game theory approaches in optimal design

In the game theory approach, a multi-level optimization problem is treated as a game where each player corresponds to an objective function being optimized. Many researchers (Rao (1987), Lewis and Mistree (1997), Liu (1998), et al.) have demonstrated the idea of using objective functions as players in a game. The players control a subset of design variables and seek to optimize their individual payoff.

1.5.1 Nash Game

In non-cooperative game theory, Nash equilibrium is a solution concept of a game involving two or more players where each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. If each player has chosen a strategy and no player can benefit by changing their strategy while the other players keep their unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium. Different algorithms to solve optimization design problems based on Nash game theory had been developed by many authors (Koskie and Gajic (2005), Liu (1998), Rao and Freiheit (1991)).

1.5.2 Stackelberg Game

In game theory, a Stackelberg game corresponds to a situation when there is a leader and a follower in the game. In a Stackelberg game, the leader makes its decisions first, the follower then makes its decisions according to the leader's decision to optimize its objective.

Although much research had been done about optimization design based on a

Stackelberg game, most of it has focused on how to model a Stackelberg game, with less effort devoted to developing algorithms for obtaining Stackelberg game based optimal solutions. Periaux et al. (2001) developed a genetic algorithm based approach to solve the Stackelberg game based optimization problems. Ghotbi (2013) developed a sensitivity approach for solving Stackelberg game based optimization problems.

1.5.3 Cooperative Game

In a cooperative game, all players or several groups of players cooperate with each other. The players have knowledge of the strategies chosen by other players and collaborate with each other to find a Pareto-optima solution.

Cooperative game theory has been widely applied to model multi-objective optimization design problems. Rao et al. (1988) applied cooperative game theory based approach to solve a vibration optimization design problem. Dhingra and Rao (1995) proposed a cooperative fuzzy game theoretic approach to multiple objective design optimization problems. Khan and Ahmad (2008) discussed an energy consumption optimization design problem based on the cooperative game theoretic concepts.

1.6 Motivation

Although many approaches have been proposed to solve design optimization problems, some methods still have limitations. For example, when using the traditional DOE Method, when a design problem has non-linear behavior, in this case the linear regression of the response surface would not match the true values, so that model updating may be needed. However, to get a more accurate regression model, more data points are needed which means more levels of experimental design are required. The number of required points increases exponentially as the number of levels is increased. For example, a three level three factor full factorial experiment requires $3^3 = 27$ points, while a four level experiment requires $3^4 = 81$ points. This increase in number of data points for model updating could be very large when the number of levels for a factor increases. This thesis proposes a new idea of using low discrepancy sequence to add arbitrary number of points to realize the model updating to improve the accuracy of the resulting RSM model.

The second half of this thesis discusses how to solve two game based scenarios-hierarchical and decentralized problems in the context of solution to multi-level optimal design problems.

1.7 Thesis organization

This thesis is divided into 3 main chapters.

Chapter 2 demonstrates how to utilize the low discrepancy sequence as a new DOE Method to solve optimal design problems that have been solved by using other approaches in the past. Comparisons between results obtained using different solution approaches are presented to demonstrate the feasibility and effectiveness of low discrepancy sequences as a viable tool for solving multi-level design optimization problems.

Chapter 3 demonstrates how to utilize low discrepancy sequence for model updating and to solve problems that could be computationally expensive and complicated using the traditional DOE Methods.

Chapter 4 discusses decentralized and hierarchical multi-level problems, two types of multi-level design optimization problems. In chapter 4, the thesis also presents application of sensitivity based approach, developed by Ghotbi (2013) for solving these two types of problems. Three numerical examples are presented in chapter 4 with respect to the two problem modes (Decentralized and Hierarchical) and a third mode which combines the two previous modes is also presented.

Chapter 5 summarizes the main achievements of this thesis and proposes future extension of the research work that has been done in this thesis.

Chapter 2. Application of low discrepancy sequences in design optimization

This chapter discusses how to utilize low discrepancy sequences as a way to generate data points for setting up the regression model and use the regression model to find a solution to optimal design problems.

2.1 Basic idea of multi-level design optimization problems

The design of practical systems involves a large number of elements or subsystems with multiple-load conditions and large number of design variables and constraints. The design optimization problem becomes unmanageably large, and the solution process becomes cumbersome and poses numerical difficulties. In such cases, the design optimization problem can be broken into a series of smaller problems using different strategies. The multi-level optimization is a decomposition technique in which the global problem is reformulated as several smaller sub-problems (one for each subsystem) and a coordination problem (at system level) to preserve the coupling among the sub-problems (subsystems).

2.2 Traditional DOE-RSM method

The DOE method has been connected with RSM to find solutions for many optimization problems arising in engineering. The basic idea of DOE-RSM method is to design an experiment where the leader design variables are parameters and the follower design variables are unknowns to generate the response surface to find the solution for the follower variables as a function of leader variable values, then substitute this result into the objective functions to solve the design optimization problem for the leader.

2.3 Low-discrepancy sequence method

The low-discrepancy sequence method in this thesis is considered to be a new way to generate the data points and build the response surface. The advantage of the low-discrepancy sequence compared to the traditional DOE method is the data points generated from the low-discrepancy sequence are evenly distributed throughout the solution space so that often less data points generated from low-discrepancy can build a more accurate response surface. Besides, the number of data points that traditional DOE-RSM methods require are fixed. For example, using a full factorial design to build a RS model for a 3 level and 3 factor experiment needs 27 data points, even if using a CCD it is reduced to at least 15 data points (not including the repeating points at the center). However, there is no limitation on the numbers of points need for the low-discrepancy sequence method, any number of data points that is proper can be used for the experiments.

In the process of studying low-discrepancy sequence method, a lot of testing work has been done, and it was proved that both the Hammersely and Halton sequence are good low-discrepancy sequences to generate sample data points. However, an interesting fact was found that Halton sequence has a great advantage comparing with the Hammersely sequence. See Table 2.1 and 2.2 for the difference between the sample points generated from Hammersely and Halton sequence (three columns of each) when 5 and 6 points are generated using these sequences.

Table 2.1: Three columns of sample points generated from Halton and Hammersely sequences of 5 points

Halton sequence (5 points)			Hamersely sequence (5 points)		
Column 1	Column 2	Column 3	Column 1	Column 2	Column 3
0.5000	0.3333	0.2000	0.1667	0.5000	0.3333
0.2500	0.6666	0.4000	0.3333	0.2500	0.6666
0.7500	0.1111	0.6000	0.5000	0.7500	0.1111
0.1250	0.4444	0.8000	0.6667	0.1250	0.4444
0.6250	0.7778	0.0400	0.8333	0.6250	0.7778

Table 2.2: Three columns of sample points generated from Halton and Hammersely sequences of 6 points

Halton sequence (6 points)			Hamersely sequence (6 points)		
Column 1	Column 2	Column 3	Column 1	Column 2	Column 3
0.5000	0.3333	0.2000	0.1429	0.5000	0.3333
0.2500	0.6666	0.4000	0.2857	0.2500	0.6666
0.7500	0.1111	0.6000	0.4286	0.7500	0.1111
0.1250	0.4444	0.8000	0.5714	0.1250	0.4444
0.6250	0.7778	0.0400	0.7143	0.6250	0.7778
0.3750	0.2222	0.2400	0.8571	0.3750	0.2222

It is found that when additional points are added, the former data points generated from Halton sequence remain the same, whereas the former data points generated from Hammersely sequence are changed. This means that the sample points generated from the Hammersely sequence are changed from one iteration to the next. Therefore, when solving model updating problems (which will be discussed in chapter 3), sample points generated by Halton sequence in a previous iteration can be reused, and only the newly generated sample points are considered to be added into the previous model. When using a Hammersely sequence method to solve model-updating problems, the model needs to be re-built because the data points in previous iterations are changed when new points are added.

In this chapter, the linear regression model that is generated from both traditional DOE method and the low-discrepancy sequence method is discussed in the contest of solution to a pressure vessel problem.

2.4 Numerical example

A numerical example is presented in this chapter to compare the results obtained using traditional DOE-RSM method and the low-discrepancy sequence data point generating-RSM method.

2.4.1 Pressure vessel problem

This problem had been used as a test problem in the literature by some researchers (Rao et al. 1997, Lewis and Mistree 1998, Marston 2000, Ghotbi 2013). Consider the pressure vessel in Fig 2.1, there are three design variables in this problem, the radius of the pressure vessel R, the length L and the wall thickness T.

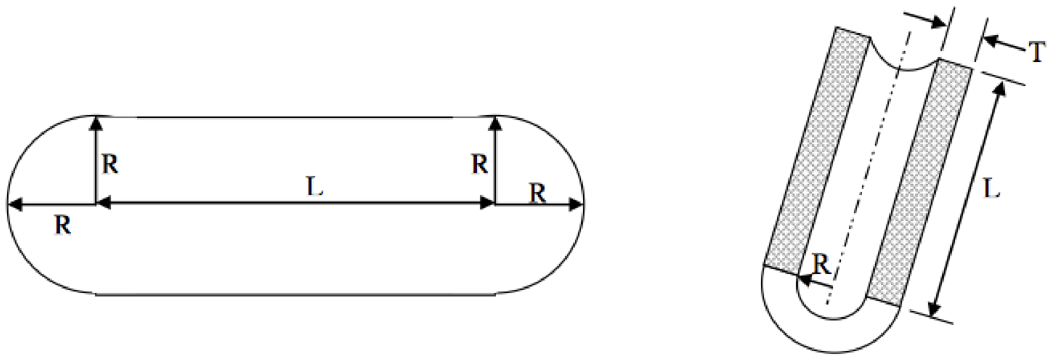

Figure 2.1: Pressure vessel design problem

Two objective functions are considered: maximizing the volume (VOL) and minimizing the weight (WGT) of the vessel. Player 1 (VOL) wishes to maximize the volume by controlling variables R and L whereas player 2 (WGT) wishes to minimize the weight with control over variable T. The vessel is under internal pressure P. The problem constraints include: (i) the circumferential stress in the wall should not exceed the tensile stress, and (ii) some additional geometric constraints due to space limitations. These constraints are given in Eqns. (2.1)- (2.4).

$$\sigma_{circ} = \frac{PR}{T} \leq S_t \tag{2.1}$$

$$5T - R \leq 0 \tag{2.2}$$

$$R + T - 40 \leq 0 \tag{2.3}$$

$$L + 2R + 2T - 150 \leq 0 \tag{2.4}$$

The objective functions of the problems for players VOL and WGT are given in Eqns.

(2.5) and (2.8) respectively.

For Player VOL:

$$Min\ f_1 = -V(R, L) = -\rho\left[\frac{4}{3}\pi R^3 + \pi R^2 L\right] \tag{2.5}$$

by varying R, L

subject to Eqns. (2.1)-(2.4)

$$R_l \le R \le R_u \tag{2.6}$$

$$L_l \le L \le L_u \tag{2.7}$$

For player WGT:

$$Min\ f_2 = W(R, T, L) = \rho\left[\frac{4}{3}\pi(R+T)^3 + \pi(R+T)^2 L - \left(\frac{4}{3}\pi R^3 + \pi R^2 L\right)\right] \tag{2.8}$$

by varying T

subject to Eqns. (2.1)- (2.4)

$$T_l \le T \le T_u \tag{2.9}$$

where ρ is the cylinder material density and $R_l, R_u, L_l, L_u, T_l, T_u$ denote the lower and upper bounds on radius, length and thickness of the pressure vessel respectively. The problem parameters are given in Table 2.3.

Table 2.3: Parameters of the pressure vessel

$P(lb)$	$S_t(lb)$	$\rho(\frac{lbs}{in^3})$	$L_l(in)$	$L_u(in)$	$R_l(in)$	$R_u(in)$	$T_l(in)$	$T_u(in)$
3890	35000	0.283	0.1	140	0.1	36	0.5	6

The Nash solution for this problem was derived analytically by Rao et al. (1997) and is given by Eqns. (2.10)- (2.12).

$$\frac{S_t(150 - L_u)}{2(P + S_t)} \leq R^N \leq \frac{40S_t}{P + S_t} \tag{2.10}$$

$$L^N = 150 - 2R^N(\frac{P}{S} + 1) \tag{2.11}$$

$$T^N = \frac{PR^N}{S_t} \tag{2.12}$$

where R^N, L^N, T^N denote the Nash solutions of the radius, length and thickness of this problem respectively.

Ghotbi (2013) mentioned that changing the initial point for the radius resulted in a different Nash solution which means that there are multiple Nash solutions for this problem. The traditional RSM-DOE based method is unable to provide all Nash solutions to this problem. However, in this chapter it is demonstrated how to low-discrepancy sequence based RSM is able to generate all possible Nash solutions.

According to Marston (2000), 3 level 2 factor (leader level design variables R, L) CCD for follower player WGT, 9 basic data points with 5 repeating points at center (20.25, 55) (See 9 CCD points map in Fig 2.2).

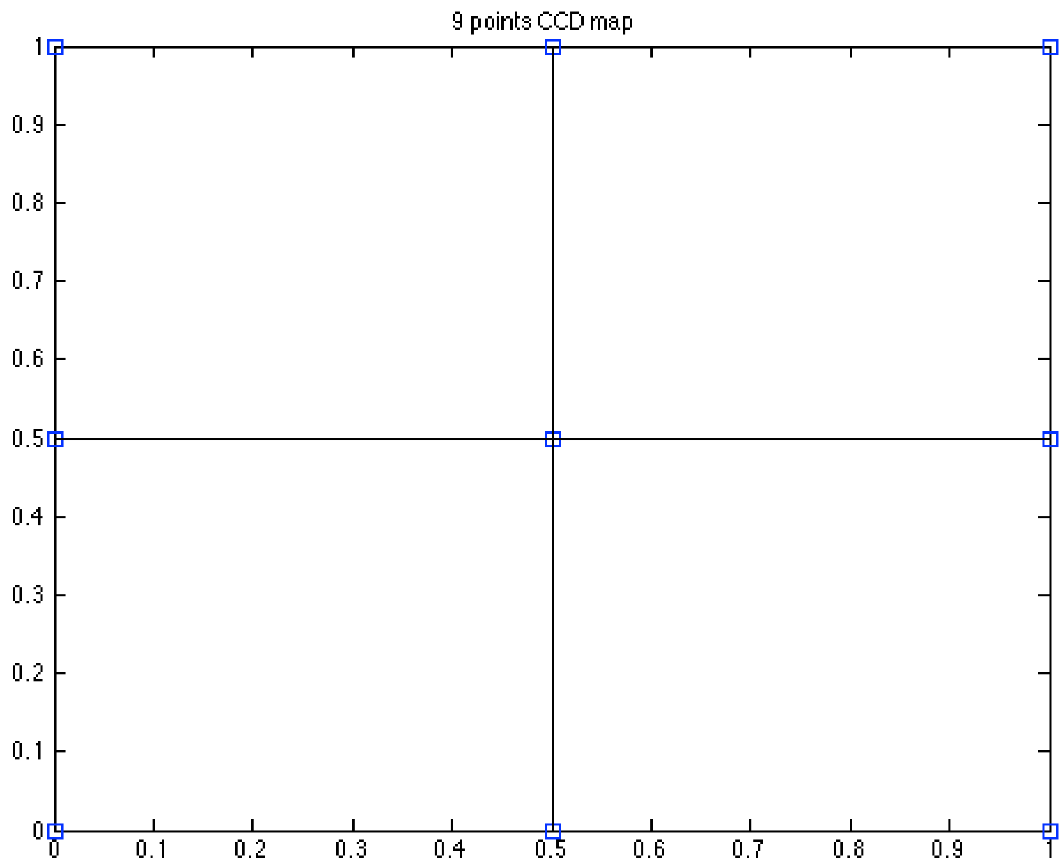

Figure 2.2. 9 point CCD map

The application of low-discrepancy sequence method started from 14 randomly generated points. Table 2.4 and Table 2.5 shows the 14 data points generated from the Halton sequence and Hammersely sequence respectively.

Table 2.4: R, L computed according to 2 columns of 14 Halton points

Generated points		R	L	T
0.5	0.333333333	20.25	40	2.250642857
0.25	0.666666667	12.375	70	1.375392857
0.75	0.111111111	28.125	20	3.125892857
0.125	0.444444444	8.4375	50	0.937767857
0.625	0.777777778	24.1875	80	2.688267857
0.375	0.222222222	16.3125	30	1.813017857
0.875	0.555555556	32.0625	60	3.563517857
0.0625	0.888888889	6.46875	90	0.718955357
0.5625	0.037037037	22.21875	13.33333333	2.469455357
0.3125	0.37037037	14.34375	43.33333333	1.594205357
0.8125	0.703703704	30.09375	73.33333333	3.344705357
0.1875	0.148148148	10.40625	23.33333333	1.156580357
0.6875	0.481481481	26.15625	53.33333333	2.907080357
0.4375	0.814814815	18.28125	83.33333333	2.031830357

Table 2.5: R, L computed according to 2 columns of 14 Hammersely points

Generated Points		R	L	T
0.066666667	0.5	6.6	55	0.733542857
0.133333333	0.25	8.7	32.5	0.966942857
0.2	0.75	10.8	77.5	1.200342857
0.266666667	0.125	12.9	21.25	1.433742857
0.333333333	0.625	15	66.25	1.667142857
0.4	0.375	17.1	43.75	1.900542857
0.466666667	0.875	19.2	88.75	2.133942857
0.533333333	0.0625	21.3	15.625	2.367342857
0.6	0.5625	23.4	60.625	2.600742857
0.666666667	0.3125	25.5	38.125	2.834142857
0.733333333	0.8125	27.6	83.125	3.067542857
0.8	0.1875	29.7	26.875	3.300942857
0.866666667	0.6875	31.8	71.875	3.534342857
0.933333333	0.4375	33.9	49.375	3.767742857

To make a comparison between the distribution of points used to generate the response surface using CCD and low-discrepancy method, Fig 2.3 and Fig 2.4 show the 14 points generated using the Hammersely method and the 14 points generated using the Halton method respectively.

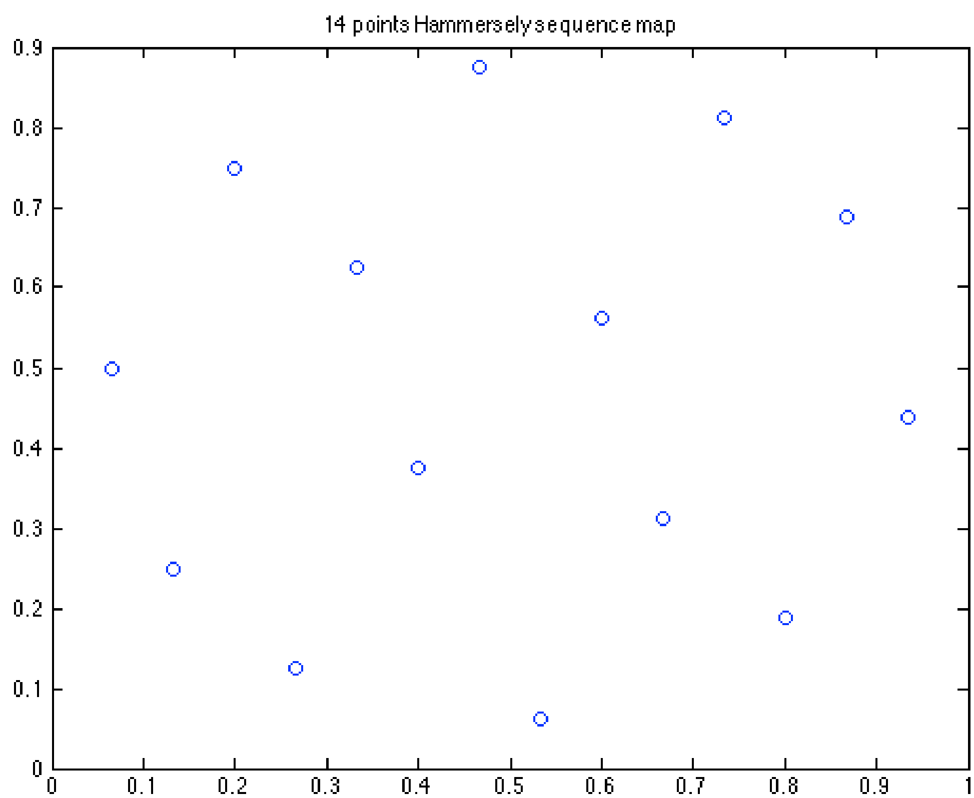

Figure 2.3: 14 points Hammersely sequence map

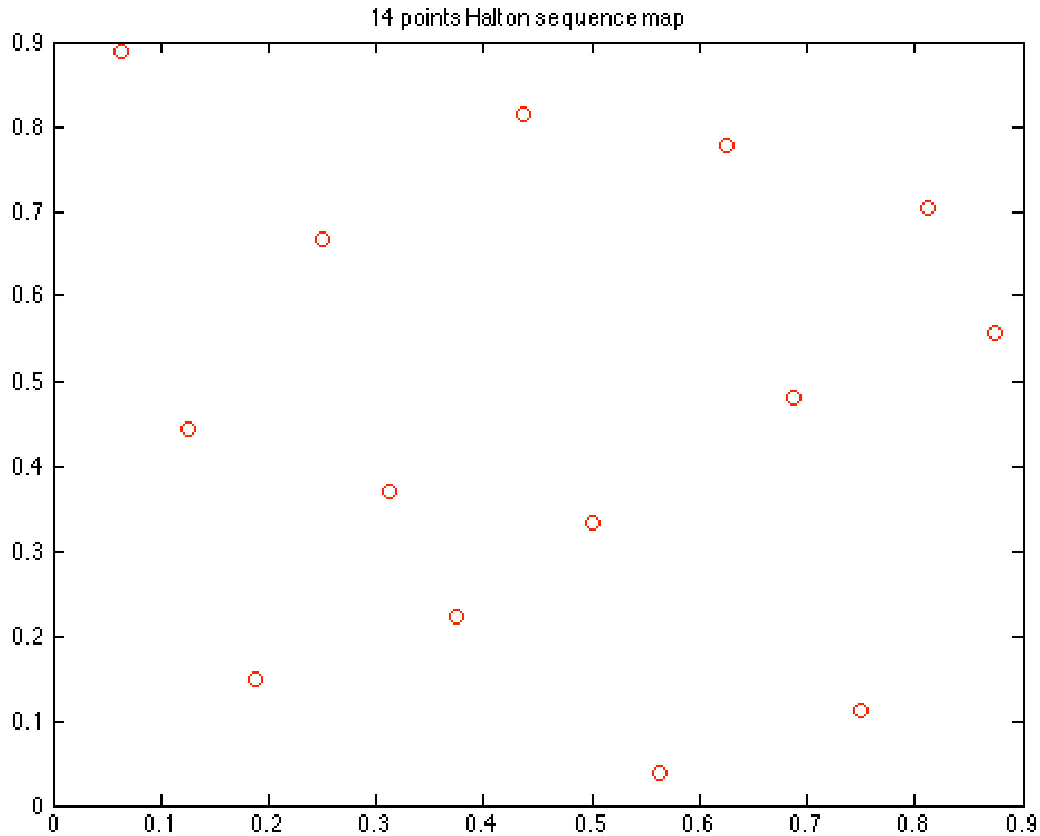

Figure 2.4: 14 points Halton sequence map

In the lower level (or the follower problem), the radius R and the length L are considered as parameters. Thus R and L values in Table 2.4 and 2.5 are computed by Eqns. (2.13) and (2.14) as:

$$R = R_L + (R_U - R_L)x_{1i} \tag{2.13}$$

$$L = L_L + (L_U - L_L)x_{2i} \tag{2.14}$$

where $L_U = 100\ in, R_U = 36\ in, L_L = 10\ in, R_L = 4.5\ in$ denote the highest and lowest value selected for the length and radius in the experiment, x_{1i}, x_{2i} denote the generated points in each column.

The fifth column in both Table 2.4 and Table 2.5 shows the optimum solution for T from each combination of (R, L) according to equation (2.12).

The linear regression for T(R, L) according to 14 point Hammersely sequence is obtained as Eqn. (2.15)

$$T(R, L) = 0.111142857142857R - 0.000000000142855 \tag{2.15}$$

Since the discrepancy in coefficients between each set of experiment is very small, while computing the response surface equation, the coefficients in Eqn. (2.15) are retained in long format.

Here, T(R, L) approximated the optimum vector of WGT problem for varying values of R and L. Repeating the above steps for the VOL problem yields the Rational Reaction Set (RRS) for variables R and L as follows:

$$R(T) = 8.9974293049469337T + 0.000000002402726 \tag{2.16}$$

$$L(T) = -19.99485861730061T + 150.0000000136055 \tag{2.17}$$

Comparing this result with the one reported by Marston (2000) $T(R, L) = -0.00021 + 0.1112R, R(T) = 9T, L(T) = 150 - 20T$, there are only some very small perturbations in the coefficients. However, the Nash solutions: $R = 24.4496, L = 95.6859, T = 2.7164$, obtained based on the three RRS computed in this chapter is quite a bit different compared to $R = 28.44, L = 86.8\ T = 3.16$ reported by Marston (2000).

Since it was already known from Ghotbi (2013) that the Nash solution for this pressure vessel problem is not unique, and DOE based RSM at that time was not able to find all Nash solutions. To verify if this is one of the possible Nash solutions and if the new DOE method can be applied to find all Nash solutions, the study continued.

By reducing the number of initial experiment data points from 14 to 9, repeating the steps above, 6 sets of Nash solutions were obtained by Hammersely sequence method and

6 sets of Nash solutions were obtained by Halton sequence method. They are shown in Table 2.6 and Table 2.7 respectively.

Table 2.6: 6 sets of Nash solutions obtained by Hammersely sequence method

Number of data points	T (in)	R (in)	L (in)
14	1.1581	10.4203	126.8430
13	2.4152	21.7306	101.7083
12	2.6796	24.1099	96.4210
11	1.2942	11.6448	124.1219
10	3.8795	34.9051	72.4308
9	1.6417	14.7707	117.1753

Table 2.7: 6 sets of Nash solutions obtained by Halton sequence method

Number of data points	T (in)	R (in)	L (in)
14	2.7164	24.4406	95.6859
13	1.4540	13.0826	120.9266
12	2.0281	18.2480	109.4476
11	4.3221	38.8875	63.5809
10	2.9451	26.4987	91.1124
9	3.1781	28.5945	86.4549

The results given in Table 2.6 and Table 2.7 are plotted in 4 charts shown below.

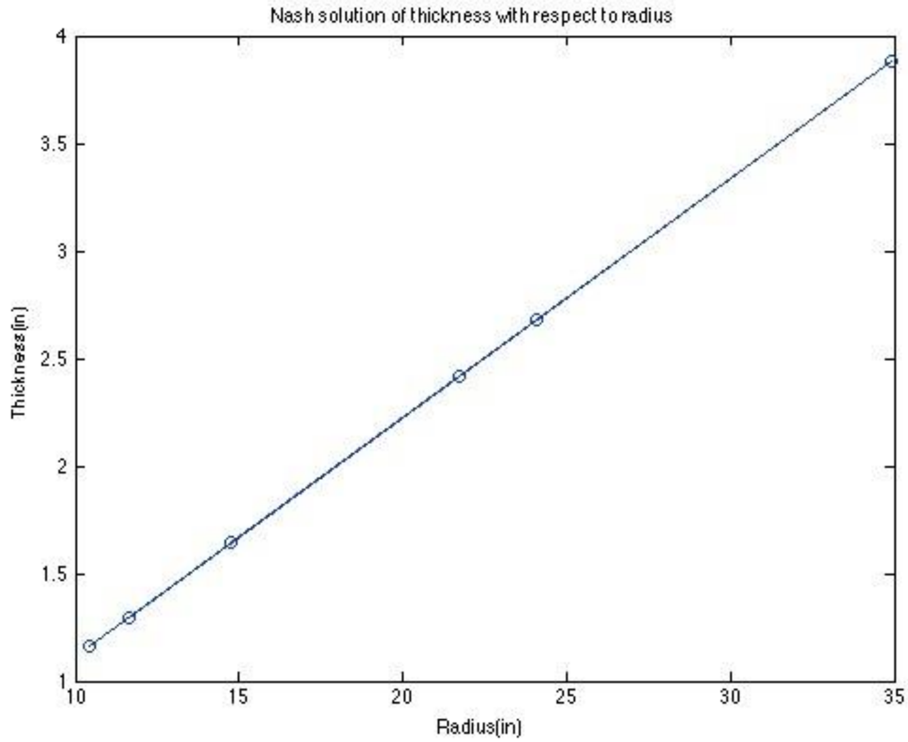

Figure 2.5: Thickness versus radius using Hammersely sequence

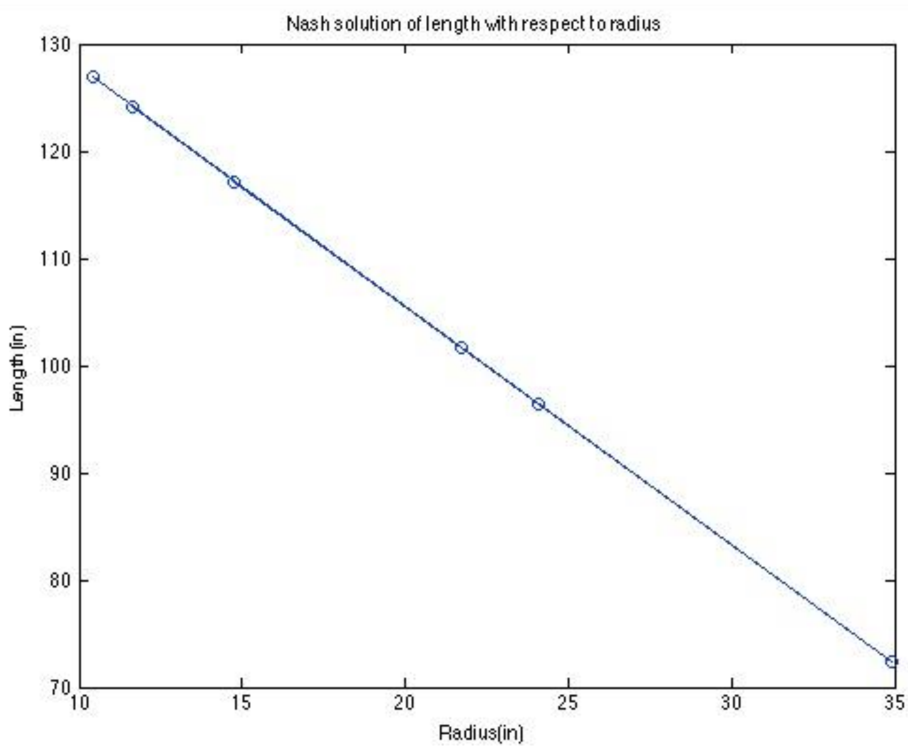

Figure 2.6: Length versus radius using Hammersely sequence

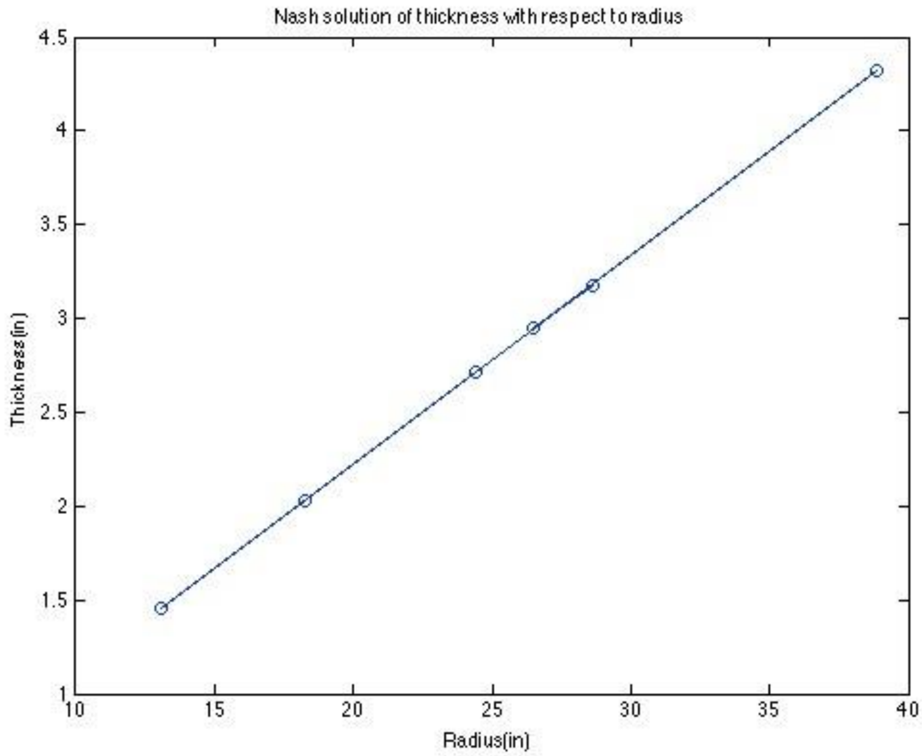

Figure 2.7: Thickness versus radius using Halton sequence

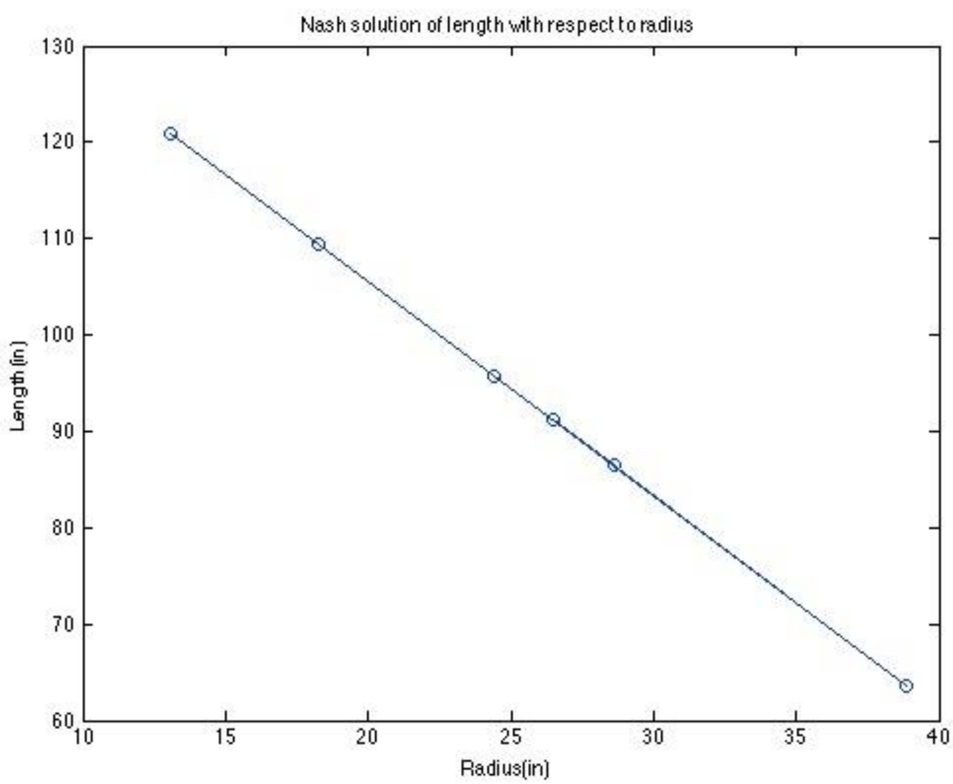

Figure 2.8: Length versus radius using Halton sequence

Ghotbi (2013) proposed a sensitivity based approach to solve this problem and successfully found all Nash solutions as shown in Fig 2.9 and Fig 2.10.

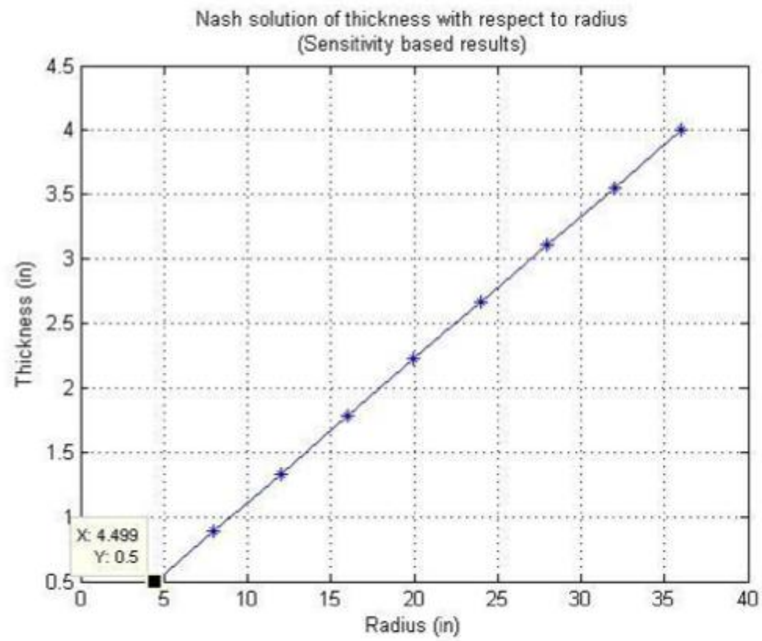

Figure 2.9: Nash solution found by Ghotbi (thickness versus radius)

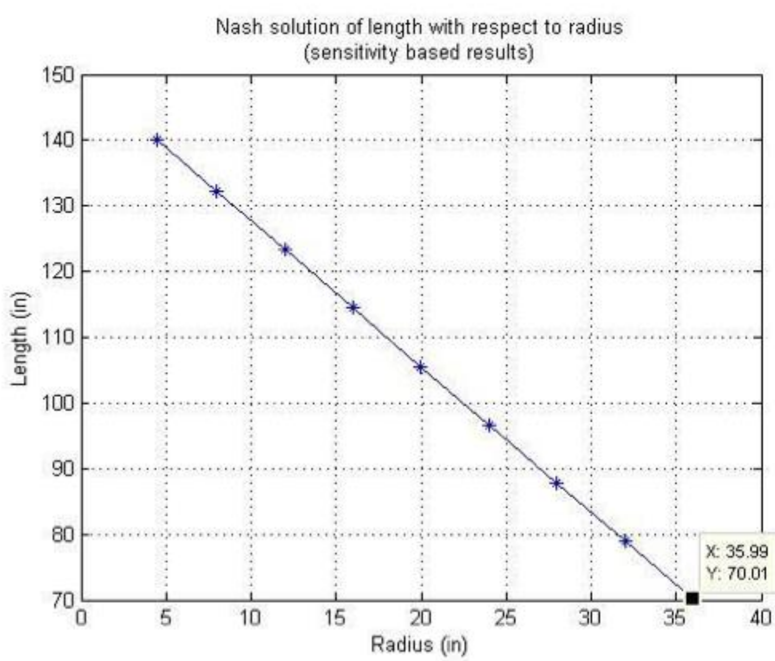

Figure 2.10: Nash solution found by Ghotbi (length versus radius)

Comparing the 2 sets of figures in this chapter with the 2 figures plotted by Ghotbi (2013), a conclusion was drawn that by varying the numbers of data points of the low-discrepancy sequence, all Nash solutions of the pressure vessel problem can be found.

2.5 Conclusions

This chapter demonstrated how to apply low-discrepancy sequence method based RSM to find Nash solution for a design optimization problems. From the results of the numerical example in this chapter, it is obvious that the low-discrepancy sequence method has a lot of advantages compared to the traditional DOE method.

Firstly, less data points are needed to build a relatively accurate response surface model by using low-discrepancy sequence method.

Secondly, the results of the numerical example shows that it is possible to find all Nash solutions for the design optimization problems when the Nash solution is not a singleton. Similarly, by adding or reducing the repeated points at the center of the CCD experiment dose not help to find all Nash solutions for the pressure vessel problem.

Thirdly, unlike the traditional DOE method where the number of data points is fixed by the number of factors and their levels in the experiment, the low-discrepancy sequence method offers the convenience that the experiment can be started from any proper number of data points.

Since the Nash solutions for the pressure vessel problem were analytically available, there was no need to do model updating. To further illustrate the efficiency of the low-discrepancy sampling method, the next chapter discusses how to realize model updating using this method.

Chapter 3. Model updating using low discrepancy sampling

In the pressure vessel design problem discussed in previous chapter, a linear numerical regression model was used to approximate the response surface. In this chapter, another numerical example is discussed to demonstrate utilization of low-discrepancy sequence method based RSM to realize model updating and solve the design optimization problem.

3.1 Basic concept of model updating

Generally in the DOE method, the study of numerical regression starts from the first order model in the form of Eqn. (3.1).

$$\tilde{y} = a_0 + \sum_{j=1}^{n} a_j x_j \tag{3.1}$$

To make the generated response surface more accurate and closer to the real case, the model may need to be updated. The model updated should be as simple as possible while giving reasonably accurate results. Two types of second order models, the pure quadratic model and the full quadratic model in the form of Eqn. (3.2) and Eqn. (3.3) respectively are used frequently.

$$\tilde{y} = a_0 + \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{n} a_{jj} x_j^2 \tag{3.2}$$

$$\tilde{y} = a_0 + \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i x_j \tag{3.3}$$

Generally, the more accurate the model, the better is the optimal solution obtained, however, this requires more data points in the DOE formulation.

As discussed before, because of the exponential relationship between the number of data points and the number of levels used in the experimental design, sometimes it can be very hard to realize model updating using traditional DOE based methods. Even if it is possible to conduct a numerical experiment and add more data points to realize model updating, it may be difficult to implement in many problems because of an exponential increase in number of trials required when a level is added and the additional experiments can cost lot of money and take a lot of time.

3.2 Model updating using low-discrepancy sequence method

In section 2.4, several advantages of the low-discrepancy sequence method were reported. This chapter introduces one more advantage of low-discrepancy sequence method that when trying to update the regression model, any number of data points can be added into the model.

The basic idea of utilizing low-discrepancy sequence method for model updating is to take advantage of its flexibility. If the original model fails to meet the desired accuracy expected of the response surface, it is easy to go back to the data point generating step, and add one more data point to the experiment, and then check if the new regression model fits the actual model better. Repeat these steps until the response surface model fits the actual process accurately. Using this model to find the solution for the design optimization problem is expected to yield improved solutions. These steps are summarized in the flowchart given in Figure 3.1.

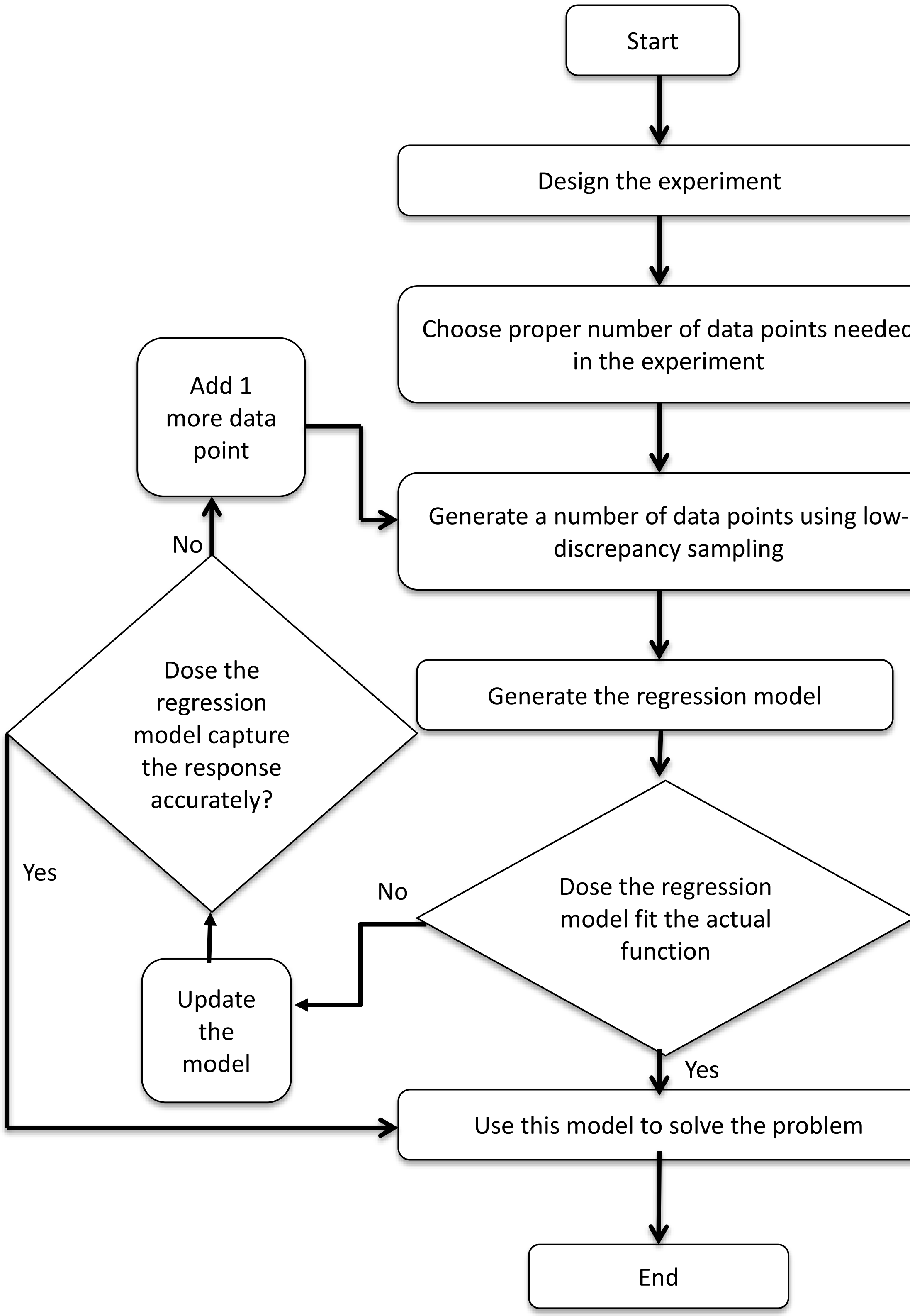

Figure 3.1: Flow chart of model updating using low-discrepancy sequence method

3.3 Numerical example

A numerical example is presented in this chapter to demonstrate how to apply low-discrepancy sampling method to update the RSM model and solve the design optimization problem under consideration.

3.3.1 Two-bar truss design problem

Azarm and Li (1990) discussed the bi-level optimization problem which is shown in Fig. 3.2.

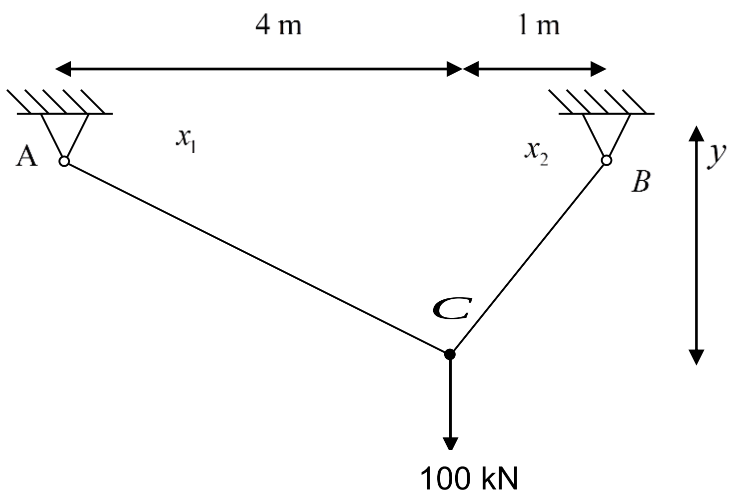

Figure 3.2: Two-bar truss design problem

A vertical load of 100 KN is applied at point C. The design variables are the cross-sectional areas of the bars x_1, x_2, and the y-coordinate of joint C. The problem constraints include limitations on the stress in the elements, which should not exceed $100000\ kN/m^2$, and the boundary conditions on vertical coordinate (y). The objective function is to minimize the total volume of the two members as shown in Eqn. (3.4).

$$Minimize\ f(x_1, x_2, y) = x_1(16 + y^2)^{0.5} + x_2(1 + y^2)^{0.5}$$

subject to:

$$\begin{gathered} 20(16 + y^2)^{0.5} - 100000yx_1 \leq 0 \\ 80(1 + y^2)^{0.5} - 100000yx_2 \leq 0 \\ 1 \leq y \leq 3 \\ x_1, x_2 > 0 \end{gathered} \tag{3.4}$$

Azarm and Li (1990) decomposed this problem into two levels. Level 1 is the follower problem, with two players, Player 1 and Player 2, who have control over variables x_1 and x_2 respectively. Level 2 is the leader problem with Player 3 who has control over variable y.

The follower level problems are given as:

$$\begin{aligned} &\underset{x_1}{minimize}\ f_1(x_1, y) = x_1(16 + y^2)^{0.5} \\ &subject\ to: \\ &\qquad 20(16 + y^2)^{0.5} - 100000yx_1 \leq 0 \\ &\qquad x_1 > 0 \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\underset{x_2}{minimize}\ f_2(x_2, y) = x_2(1 + y^2)^{0.5} \\ &subject\ to: \\ &\qquad 80(1 + y^2)^{0.5} - 100000yx_2 \leq 0 \\ &\qquad x_2 > 0 \end{aligned} \tag{3.6}$$

The leader problem is given as:

$$\begin{aligned} &\underset{y}{minimize}\ \ f(x_1, x_2, y) = f_1(x_1, y) + f_2(x_2, y) \\ &\text{subject to:} \\ &\qquad 1 \leq y \leq 3 \end{aligned} \tag{3.7}$$

This problem can be modeled as a Stackelberg game with two players in the follower level. Using the principles of monotonicity analysis, it can be verified that the constraints are active at optimum solution of the follower problems when they are optimized individually. The optimum solutions of follower problems are as follows:

$$x_1^*(y) = 20(16 + y^2)^{0.5} / (100,000y) \tag{3.8}$$

$$x_2^*(y) = 80(1 + y^2)^{0.5} / (100,000y) \tag{3.9}$$

where x_1^*, x_2^* are the optimal solutions for the follower problems which are the closed-form expressions of RRS for the followers. By substituting these RRS in the leader level, the optimum solution of leader problem can be obtained.

Since there is a single factor y in the leader problem, the design of experiment became simple. The traditional DOE based RSM method can just evenly divide the interval of y [1,3], and use the same concept to update the model and solve the problem. The results of three different types of models (linear, quadratic and cubic model) computed by both Halton sequence and Hammersely sequence are shown in Table 3.1.

Table 3.1: Solutions obtained from low-discrepancy sequence method

Type / Variables	Linear model		Quadratic model		Cubic model	
	Halton	**Hammersely**	**Halton**	**Hammersely**	**Halton**	**Hammersely**
x_1	$2.5870e^{-4}$	$2.5870e^{-4}$	$4.1647e^{-4}$	$4.1284e^{-4}$	$4.4979e^{-4}$	$4.4548e^{-4}$
x_2	$7.9300e^{-4}$	$2.5870e^{-4}$	$8.7582e^{-4}$	$8.7657e^{-4}$	$8.9768e^{-4}$	$8.9840e^{-4}$
y	3	3	2.1500	2.1621	1.9673	1.9973

Comparing the solution reported by Azarm and Li (2000) ($x_1 = 4.48e^{-4}, x_2 = 8.96e^{-4}, y = 2$) and the solution reported by Ghotbi (2013) ($x_1 = 4.49e^{-4}, x_2 = 8.95e^{-4}, y = 1.9981$), with the cubic model solution in this thesis, it is seen that cubic solution given in Table 3.1 is the most accurate one among the three pairs of the solutions obtained from the low-discrepancy sequence method. This simple example shows that the low-discrepancy sequence method can be considered as an effective way to update regression models.

3.3.2 Enhanced version of the Two- bar truss problem

Since there was only one leader design variable in problem 3.3.1, only one factor was considered while designing the experiment. When updating the model, it can be easily set up by using the traditional DOE method which involves dividing y into evenly spaced intervals and reducing the interval size to accommodate more data points to the experiment. Thus, an additional problem is considered here, by adding two more leader design variables y_1, y_2 into the problem (See Fig. 3.3).

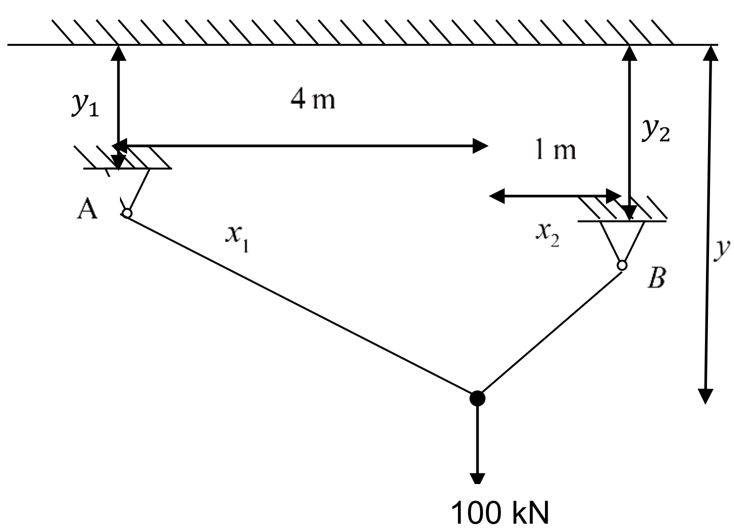

Figure 3.3: Enhanced two-bar truss problem

The new objective function becomes as follows:

$$Minimize\ f(x_1, x_2, y_1, y_2, y) = x_1[16 + (y - y_1)^2]^{0.5} + x_2[1 + (y - y_2)^2]^{0.5}$$

subject to:

$$\begin{gathered} 20(16 + (y - y_1)^2)^{0.5} - 100000(y - y_1)x_1 \le 0 \\ 80(1 + (y - y_2)^2)^{0.5} - 100000(y - y_2)x_2 \le 0 \\ 1 \le y \le 3 \\ 0 \le y_1, y_2 \le 0.5 \\ x_1, x_2 > 0 \end{gathered} \tag{3.10}$$

So the optimum solution for $x_1$and x_2 change to:

$$x_1^*(y_1, y) = 20[16 + (y - y_1)^2]^{0.5}/100{,}000(y - y_1) \tag{3.11}$$

$$x_x^*(y_2, y) = 80[1 + (y - y_2)^2]^{0.5}/100{,}000(y - y_2) \tag{3.12}$$

The response surface according to the optimum solution x_1 and x_2 are shown in Fig 3.4 and Fig 3.5 respectively.

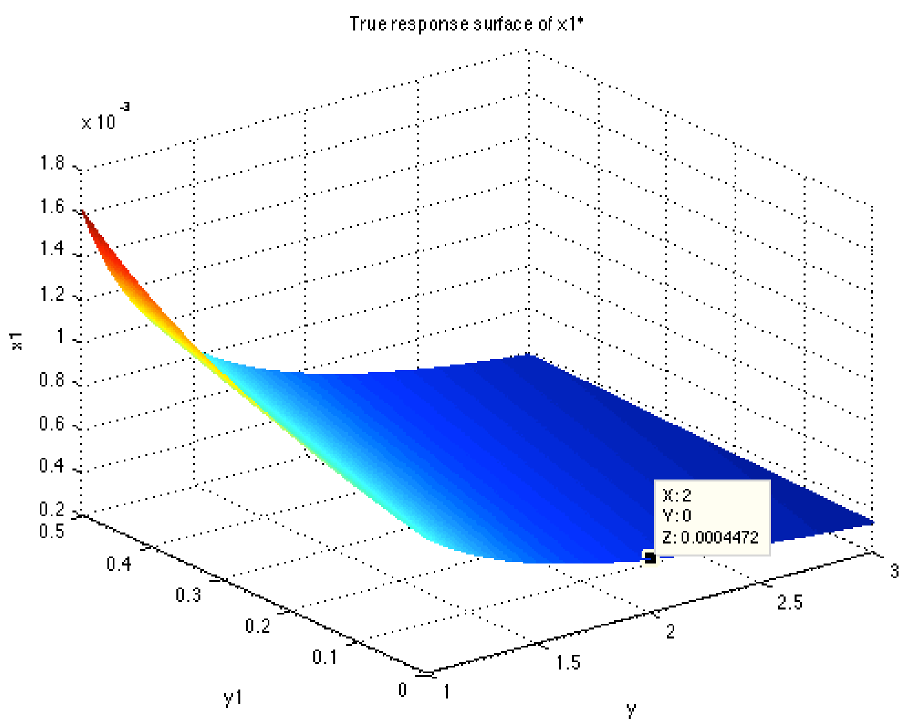

Figure 3.4: Response surface of $x_1^*(y_1, y)$

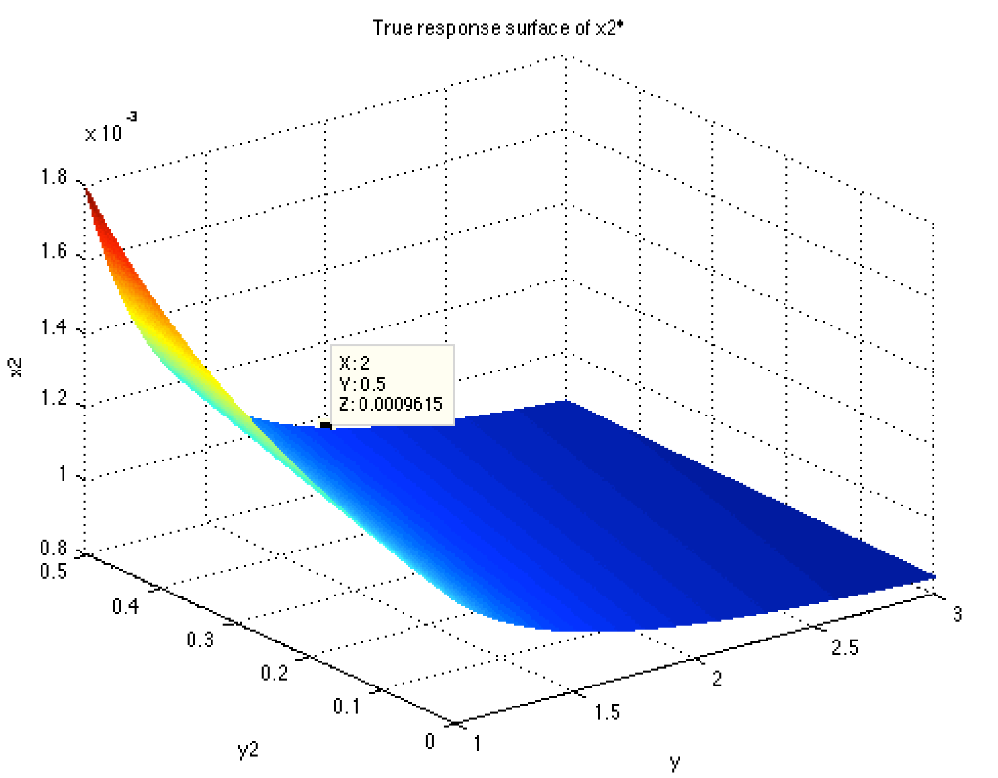

Figure 3.5: Response surface of $x_2^*(y_2, y)$

An 8 point Hammersely sequence was generated to solve this problem. The linear regression model of the follower level were computed as follow:

$$x_1 = 1.0408e^{-3} + 3.5073e^{-4}y_1 - 2.9518e^{-4}y \tag{3.13}$$

$$x_2 = 1.2843e^{-3} + 1.8183e^{-4}y_2 - 1.8186e^{-4}y \tag{3.14}$$

Substitute this solution into the leader problem. The solution of the global design problem $y = 3, y_1 = 0, y_2 = 0, x_1 = 1.5527e^{-4}, x_2 = 7.3872e^{-4}$ was obtained.

To verify if the solution is indeed the correct solution to this problem, a sensitivity based approach was applied, and $y = 2.0029, y_1 = 0, y_2 = 0.5, x_1 = 4.4670e^{-4}, x_2 = 9.6092e^{-4}$ was obtained as the solution to this problem.

The solution obtained from the first order model is not the same as the solution obtained with the sensitivity based approach, which means the model needs to be updated.

Since it is unknown if additional data points are necessary, the Hammersely sequence was retained in the form of 8 points for the continued study of model updating. The solutions obtained from full quadratic model and full cubic model are shown in Table 3.2.

Table 3.2: Solutions obtained from 8 point Hammersely sequence

Model / Variables	Quadratic model	Cubic model
y	2.1501	2.1003
y_1	0	0
y_2	0.3291	0.4938
x_1	$4.1468e^{-4}$	$4.4574e^{-4}$
x_2	$9.1327e^{-4}$	$9.3734e^{-4}$

As is shown in the Table 3.2, the solution obtained from the cubic model is very close to the solution obtained from the sensitivity based approach. If more accurate solutions are desired, 2 methods can be applied: 1) increase the model order, 2) add more data points to the full cubic model. Considering the number of computations needed in formulating a 4th order regression equation, the second method was adopted here.

When the number of data points was increased to be 11, a set of solution that met the tolerance requirement of $\varepsilon \leq 0.001$ was obtained, this solution can be viewed as the optimum solution for this problem (See Table 3.3)

Table 3.3: Hammersely and Halton 11 point solution versus sensitivity based approach

Method / Variables	Hammersely 11 points cubic model	Halton 11 points cubic model	Sensitivity based approach
y	2.0023	2.0026	2.0029
y_1	0	0	0
y_2	0.4998	0.4999	0.5
x_1	$4.4670e^{-4}$	$4.4670e^{-4}$	$4.4670e^{-4}$
x_2	$9.5991e^{-4}$	$9.6000e^{-4}$	$9.6092e^{-4}$

According to the result obtained from the 11 point Hammersely and Halton sequence, the low-discrepancy sequence method is an effective tool in model updating study, and can be considered as an effective method in solving design optimization problems.

The traditional DOE based RSM was also considered in this problem. A 3 factor 2 level experiment with 8 points was designed to solve this problem. However, the full cubic model result ($y = 2.4378, y_1 = 0, y_2 = 0.1844, x_1 = 7.8850e^{-4}, x_2 = 10.9114e^{-4}$) was far away from the exact solution. The reason that this result cannot be trusted is that the 8 points in a 2 level traditional design experiment are way too dispersed compared to the low-discrepancy sequence method. Thus, more data points are required in the traditional DOE approach. To update the numerical model using the traditional DOE method, 1 more level is required which means the data points need to be increased from 8 points to 27 points. On the other hand, an 11 point low-discrepancy sequence can give a very accurate solution.

3.4 Conclusions

In this chapter, the two-bar truss design problem was considered as a test problem to demonstrate one factor experiment-model updating. The second numerical example, which has three leader design variables, is a modified version of the two-bar truss problem. In this case, the traditional DOE method was not able to realize model updating by adding data points to the experiment one by one.

When the factors in the experiment were increased to be 3 or more, the traditional DOE method was inapplicable in this case. From the result of the second numerical example, it is seen that the low-discrepancy sequence method is a fast and effective way to update the regression model when dealing with multiple factors. Since the sample points generated using low-discrepancy sequences are uniformly distributed throughout the solution space, the underlying response surface is reasonably accurate.

Chapter 4. Multi-level optimum design based on Decentralized and Hierarchical models

This chapter presents an application of sensitivity based approach to solving multi-level design optimization problems. Two types of multi-level problems are considered in this chapter. The first one is the decentralized case, wherein the multiple objective functions are considered as two groups, with one (or more) objectives in the leader group and one (or more) objectives in the follower group. Two solution scenarios are considered. In the first scenario, the behavior in each group is considered as a cooperative game, and the interaction between the leader and follower group is considered as a Stackelberg Game. In the second scenario, at the lower level, the interactions between the players are considered as a non-cooperative Nash game while at the leader level the interactions between the players remains a cooperative game. The interaction between the two levels is still considered as a Stackelberg game.

In the second case, called the hierarchical mode, multiple objective functions are considered as multiple levels from the highest to the lowest. This means there is a hierarchical order of the leaders and followers in this mode. To find the solution to this situation, the interaction between each set of players is considered as a Stackelberg game.

4.1 Decentralized mode

Consider a bi-level decentralized game with 4 players in the game for example (See Fig 4.1). The 4 players represent their own objective functions f_1, f_2, f_3, f_4 respectively. Player 1 and player 2 are considered as leader players in Level 1, player 3 and player 4 are considered as follower players in Level 2.

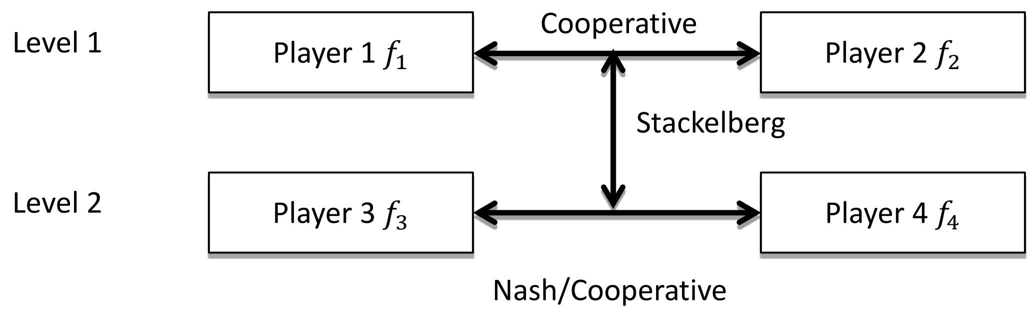

Figure 4.1: Bi-level 4 player decentralized model

The design optimization problem for the players is modeled as:

Leader level:

For player 1

$$Min\ f_1(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.1)
subject to $g_j^1(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^1$

For player 2

$$Min\ f_2(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.2)
subject to $g_j^2(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^2$

Follower level:

For player 3

$$Min\ f_3(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the follower design variables (4.3)
subject to $g_j^3(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^3$

For player 4

$$Min\ f_4(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the follower design variables (4.4)

subject to $g_j^4(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^4$

There are two possible cases in a decentralized system, so two scenarios are considered in this chapter, 1) the interactions between the objectives of both leader level and follower level is considered as a cooperative game, the interaction between the leader level and the follower level is considered as a Stackelberg game; 2) the interaction at the follower level is considered as a Nash (Non-cooperative) game, the interaction at the leader level is considered as a cooperative game, and the interaction between the two levels is a Stackelberg game.

4.2 Hierarchical mode

Considering a three level with 3 players (See Fig 4.2). The 3 players represent their own objective functions f_1, f_2, f_3 respectively. The objective function f_1 is considered as the only leader player in the first level. The objective function f_2 is considered as the first follower player in the second level. The objective function f_3 is considered as the second follower player in the third level.

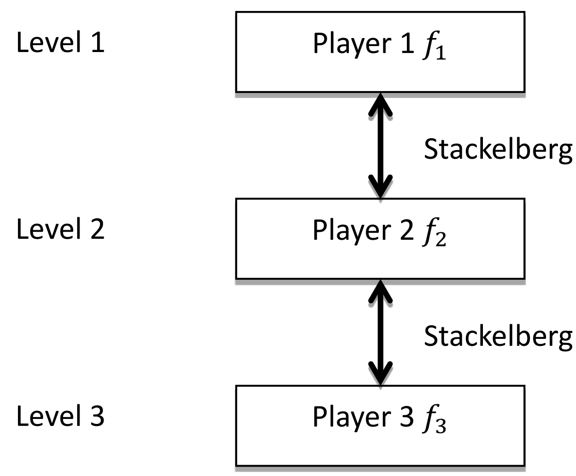

Figure 4.2: Tri-level 3 player Hierarchical model

The design optimization problem for the players is modeled as:

First level:

For player 1

$$Min\ f_1(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.5)
subject to $g_j^1(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^1$

Second level:

For player 2

$$Min\ f_2(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.6)
subject to $g_j^2(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^2$

Third level:

For player 3

$$Min\ f_3(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the follower design variables (4.7)
subject to $g_j^3(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^3$

Since in a hierarchical mode, all interactions between the neighboring levels are considered as Stackelberg games, the first step to solve this kind of problem is to find the Stackelberg solution for the lowest level (level 2 and level 3) as a function of higher levels design variables, and then substitute this solution progressively, level by level, into higher levels until we arrive at leader level for player 1. Finally, the complete solution to the problem can be found by combining rational reaction sets of all lower level Stackelberg problems.

4.3 Combined Decentralized and Hierarchical mode problem

In reality, a design problem may be more complicated where more factors in the design need to be considered with more objectives conflicting with each other. For example, imagine a design optimization problem that has 8 objectives to be optimized. More game modes should be considered to find an optimum solution, which cannot be classified simply as a decentralized mode or a hierarchical mode.

This thesis presents a solution for a new game structure which is a combination of the decentralized and the hierarchical modes. Consider a design optimization problem with 4 players where player 1 in the first level is the leader for players 2, 3, 4, player 2 in the second level is the leader for players 3, 4, players 3, 4 who stay at the third level are the followers.

This problem can be modeled as a three-level game in which the interactions in each level can be considered as either a Nash game or a cooperative game, the interaction of each level to its preceding level is a Stackelberg game (See Fig 4.3).

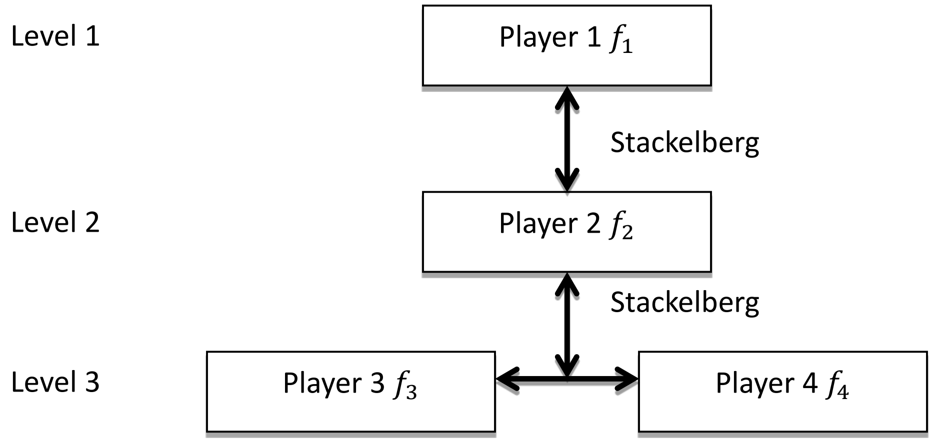

Figure 4.3: Tri-level Decentralized-Hierarchical model

The design optimization problem can be formulated as:

First level:

For player 1

$$Min\ f_1(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.8)
subject to $g_j^1(x_1, x_2, \dots\ x_i) \le 0\ j = 1,2, \dots n_g^1$

Second level

For player 2

$$Min\ f_2(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.9)
subject to $g_j^2(x_1, x_2, \dots\ x_i) \le 0\ j = 1,2, \dots n_g^2$

Third level:

For player 3

$$Min\ f_3(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.10)
subject to $g_j^3(x_1, x_2, \dots\ x_i) \le 0\ j = 1,2, \dots n_g^3$

For player 4

$$Min\ f_4(x_1, x_2, \dots\ x_i)\ x \in R^n$$

by varying the leader design variables (4.11)
subject to $g_j^4(x_1, x_2, \dots\ x_i) \leq 0\ j = 1,2, \dots n_g^4$

To solve this problem, first step is to find the solution at the third level with the leader design variables as parameters, then substitute the rational reaction set of the solutions into the second level and finally, repeat above steps until the global solution to the problem can be obtained.

4.4 Bargaining equation

The bargaining equation was applied to capture the cooperative behavior between the players in the same level in a decentralized model. The bargaining equation is expressed as:

$$f_B = \frac{(f_{w1} - f_1)(f_{w2} - f_2) \dots (f_{wi} - f_i)}{(f_{w1} - f_{b1})(f_{w2} - f_{b2}) \dots (f_{wi} - f_{bi})} \tag{4.12}$$

where f_B denotes the bargaining function, f_i are the values of the objective functions, f_{wi} is the worst solution for the f_i and f_{bi} denotes the best solution for f_i. It may be noted that the f_{bi} is the optimized solution for the objective function f_i and f_{wi} is the negative optimized solution for $-f_i$ (Assuming all the objective functions are in the standard form that the target is to minimize the function).

4.5 Sensitivity based approach

The sensitivity based approach is considered as a fast and effective method for finding the solutions to multi-level optimization design problems. The basic concept of the sensitivity based approach is to find how does the leader design variable change when varying the follower design variable according to the sensitivity $\frac{dx_f^*}{dx_l}$. Use the sensitivity to find the expression for the follower level solution:

$$x_f = x_f^{*k} + \frac{dx_f^{*k}}{dx_l}(x_l - x_l^k) \tag{4.13}$$

Substitute Eqn. (4.13) into the leader problem solve for x_l^* in each iteration until the convergence tolerance $\left|\frac{x_l^* - x_l^k}{x_l^k}\right|$ meets the requirement yielding the solution to the problem x_l^*, x_f^*.

Here x^* denotes the solution for the design variables, x_l and x_f are the leader design variables and follower design variables respectively, k is the iteration counter.

Hou et al. (2004) showed the general algorithm for an application of the sensitivity based approach. Ghotbi (2013) demonstrated how to use sensitivity based approach to obtain Stackelberg and Nash solution for optimization problems. Fig. 4.4 is the flow chart of the procedure to obtain Stackelberg solution using sensitivity based approach.

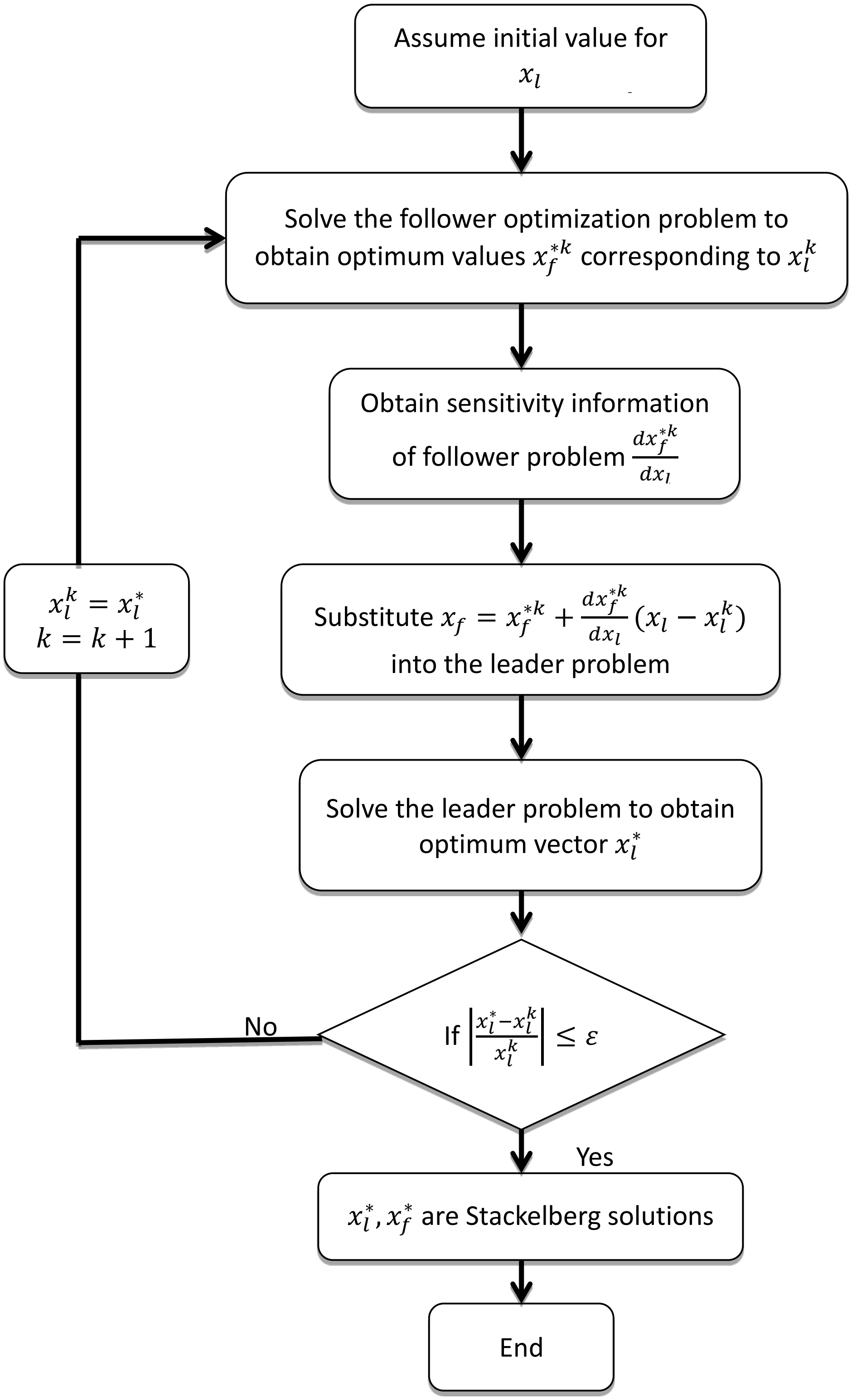

Figure 4.4: Flowchart of sensitivity based approach to obtain Stackelberg solution

4.6 Numerical example

In this chapter, three numerical examples are presented to solve three types of multi-level problems. The first one is a decentralized type of optimization problem, the second example is a hierarchical type of optimization problem, the third example is a combination of decentralized and hierarchical optimization problem.

4.6.1 Decentralized optimization problem

To demonstrate the application of the sensitivity based approach to solving a decentralized bi-level optimization problem, we consider a test problem which has been solved by Liu (1998) using genetic algorithm.

The bi-level programming is formulated as follows:

For the leader level:

$$\begin{aligned} min\ F(x, y_1, y_2) \\ &= (y_{11} + y_{21} - 200)(y_{11} + y_{21}) + (y_{12} + y_{22} \\ &\quad - 160)(y_{12} + y_{22}) \end{aligned} \tag{4.14}$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 40$$

$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 15, 0 \leq x_4 \leq 20$$

For the follower level:

$$min\ f_3(y_1) = (y_{11} - 4)^2 + (y_{12} - 13)^2$$

subject to

$$0.4y_{11} + 0.7y_{12} \leq x_1 \tag{4.15}$$

$$0.6y_{11} + 0.3y_{12} \leq x_2$$

$$0 \leq y_{11}, y_{12} \leq 20$$

$$min\ f_4(y_2) = (y_{21} - 35)^2 + (y_{22} - 2)^2$$

subject to

$$0.4y_{21} + 0.7y_{22} \leq x_3 \tag{4.16}$$

$$0.6y_{21} + 0.3y_{22} \leq x_4$$

$$0 \leq y_{11}, y_{12} \leq 40$$

The leader problem was decomposed into two levels with player 1 $f_1(x, y_1)$ and player 2 $f_2(x, y_2)$, and the objective functions in the leader level can be formulated as:

Player 1

$$min\ f_1(x, y_1, y_2) = (y_{11} + y_{21} - 200)(y_{11} + y_{21})$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 40 \tag{4.17}$$

$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 15, 0 \leq x_4 \leq 20$$

Player 2

$$min\ f_2(x, y_1, y_2) = (y_{12} + y_{22} - 160)(y_{12} + y_{22})$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 40 \tag{4.18}$$

$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 15, 0 \leq x_4 \leq 20$$

Therefore, the overall problem was modeled as a decentralized system.

Consider this problem in the first scenario of a decentralized mode problem with player 1 and 2 in the leader level, player 3 and 4 in the follower level. In the first scenario, the interaction between the players in a same level is considered as a cooperative game. Thus, the bargaining equation are applied (The best and worst values for each objective function are shown in Table 4.1):

For level 1:

$$f_{B1} = \frac{(0 - f_1)(0 - f_2)}{\big(0 - (-2775)\big)(0 - (-4375))} \tag{4.19}$$

For level 2:

$$f_{B2} = \frac{(185 - f_3)(1229 - f_4)}{(185 - 3.7556)(1229 - 5.6889)} \tag{4.20}$$

Table 4.1: Best and worst values for the objective function optimized individually

Objective function	Best value	Worst value
f_1	-2775	0
f_2	-4375	0
f_3	3.7556	185
f_4	5.6889	1229

Thus the optimization problem can be re-written as:

Level 1 min $-f_{B1}$

Level 2 min $-f_{B2}$

Subject to:

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 \leq 40 \\
& 0 \leq x_1 \leq 10, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 15, 0 \leq x_4 \leq 20 \\
& 0.4y_{11} + 0.7y_{12} \leq x_1 \\
& 0.6y_{11} + 0.3y_{12} \leq x_2 \\
& 0 \leq y_{11}, y_{12} \leq 20 \\
& 0.4y_{21} + 0.7y_{22} \leq x_3 \\
& 0.6y_{21} + 0.3y_{22} \leq x_4 \\
& 0 \leq y_{11}, y_{12} \leq 40
\end{aligned}
\tag{4.21}
$$

Then, the interaction between level 1 and level 2 is considered as a Stackelberg game. To solve this problem, the DOE-RSM approach was considered first, however, due to the constraints in this problem, an analytical solutions for the follower design variable in terms of the leader design variables was not possible. Thus, it is impossible to design an experiment of the follower level to generate the response surface. Therefore, the sensitivity based approach that was introduced in this chapter was applied to solve this problem (See results in Table 4.2).

Table 4.2: Solutions to the Decentralized mode example

Design variables and objective functions	**Stackelberg-Cooperative system result**	**Liu (2000) Nash equilibrium result**
x_1	9.1114	7.05
x_2	5	3.13
x_3	10.3554	11.93
x_4	15.5331	17.89
(y_{11}, y_{12})	(2.5552,11.5562)	(0.26,9.92)
(y_{21}, y_{22})	(25.8885,0)	(29.82,0)
$F = f_1 + f_2$	-6095.1078	-5814.3352
f_3	4.1719	23.474
f_4	87.0185	30.8324

It can be seen from the table that the decentralized mode found a better optimum solution for F and f_3, but worse solution for f_4. Considering the leader objective function is the main target to optimize, the cooperative mode in this problem can be viewed as a better strategy than Nash equilibrium mode.

Now consider this problem in the second scenario such that the interaction at the lower level is a Nash game, the interaction at the leader level remains a cooperative game. Thus, the first step to solve is to find the Nash solution for player 3 and 4, then substitute this solution into the bargaining equation f_{b1} in the leader level to find the Stackelberg solution for the global problem (Table 4.3 shows the solution of this problem for the second scenario in decentralized mode).

Table 4.3: Solution to the second scenario of the decentralized problem

Design variable and objective function	Stackelberg-Nash-Cooperative result
x_1	10
x_2	5
x_3	10
x_4	15
(y_{11}, y_{12})	(1.6667,13.333)
(y_{21}, y_{22})	(25,0)
$F = f_1 + f_2$	-6150
f_3	5.5556
f_4	104

Comparing the result from Table 4.3 with the result from Table 4.2, it can be seen that the optimum value of the leader objective function obtained from the second decentralized scenario is better optimized than the optimum value of the leader objective function obtained from both the first decentralized scenario and Nash equilibrium by Liu (2000).

4.6.2 Hierarchical mode example

The second numerical example is also a test problem discussed by Liu (2000). This problem was modeled with one leader player who has a control over three variables (x_1, x_2, x_3) and three followers who have control vectors $y = (y_{i1}, y_{i2}), i = 1,2,3$. The problem was formulated as:

Level 1:

$$min\ F\,(x, y_1, y_2) = -(y_{11}y_{12}sinx_1 + y_{21}y_{22}sinx_2 + y_{31}y_{32}sinx_3)$$

subject to (4.22)

$$x_1 + x_2 + x_3 \leq 10,\ x_1, x_2, x_3 \geq 0$$

Level 2:

$$min\ f_1 = -(y_{11}siny_{12} + y_{12}siny_{11})$$

subject to (4.23)

$$y_{11} + y_{12} - x_1 \leq 0, y_{11}, y_{12} \geq 0$$

Level 3:

$$min\ f_2 = -(y_{21}siny_{22} + y_{22}siny_{21})$$

subject to (4.24)

$$y_{21} + y_{22} - x_2 \leq 0, y_{21}, y_{22} \geq 0$$

Level 4:

$$min\ f_3 = -(y_{31} sin y_{32} + y_{32} sin y_{31})$$

subject to (4.25)

$$y_{31} + y_{32} - x_3 \leq 0, y_{31}, y_{32} \geq 0$$

In the hierarchical system mode, player 1 is the leader for all the rest players, player 2 is the leader for player 3 and player 4, and player 3 is the leader for player 4. The interaction between each level by each level was considered as a Stackelberg game. The problem was solved using sensitivity based approach starting from level 4.

Liu (2000) considered this problem in a bi-level Nash-Stackerberg game mode that the interactions between player 2, player 3, and player 4 were non-cooperative games in the follower level, and the interaction between the leader player 1 and follower level was considered as a Stackelberg game. Liu (2000) mentioned the Nash solution in this problem is not unique. Thus, two of them were selected to make a comparison with the results from the one obtained from the hierarchical mode (See Table 4.4).

Table 4.4: Solutions to the Hierarchical mode example

Variables and objectives	Hierarchical mode result	Nash-Stakelberg result from Liu(1)	Nash-Stackelberg result from Liu(2)
$x(x_1, x_{2,} x_3)$	(1.9368,8.0632,0)	(1.946,8.054,0)	(8.054,1.946,0)
$y_1(y_{11} y_{12})$	(1.3132,6.7500)	(0.973,0.973)	(1.315,6.793)
$y_2(y_{21} y_{22})$	(0.9684,0.9684)	(1.315,6.793)	(0.973,0.973)
$y_3(y_{31} y_{32})$	(0,0)	(0,0)	(0,0)
F	-9.5649	-9.566	-9.566
f_1	-7.1182	-1.609	-7.099
f_2	-1.5959	-7.099	-1.609
f_3	0	0	0

From the results shown in the table, it is found that although the Nash-Stackelberg mode does not have a unique solution, the leader design variable x have a control over the leader objective function, the solution to the leader objective function is unique. Also, it is found that in this problem, the result obtained from the hierarchical mode is very close to the results from Nash-Stackelberg mode, which can be viewed as one set of Nash-Stackelberg solution.

4.6.3 Decentralized-Hierarchical mode example

Consider a decentralized-hierarchical mode problem with player 1 f_1 with control over $x = (x_1, x_2)$ in level 1, player 2 f_2 with control over $y_1 = (y_{11}, y_{12})$ in level 2, player

3 f_3 and player 4 f_4 who have control over $y_2 = (y_{21}, y_{22})$ and $y_3 = (y_{31}, y_{32})$ respectively in level 3. The problem is formulated as:

Level 1 (Player 1)

$$min\ f_1(x, y_1, y_2, y_3) = \frac{3(y_{11} + y_{12})^2 + 5(y_{21} + y_{22})^2 + 3(y_{31} + y_{32})^2}{2x_1^2 + x_2^2 + 3x_1x_2} \tag{4.26}$$

subject to $x_1 + 2x_2 \leq 10$

$x_1, x_2 > 0$

Level 2 (Player 2)

$$min\ f_2(y_1) = y_{11}^2 + y_{12}^2$$

$$\text{subject to} \quad y_{11} + y_{21} + y_{31} - x_1 \leq 0 \tag{4.27}$$

$$y_{12} + y_{22} + y_{32} - x_2 \leq 0$$

$$y_{11} \geq 1, y_{12} \geq 2$$

Level 3(Player 3 and Player 4)

$$\min f_3(y_2) = y_{21} + y_{22} + \frac{y_{11}}{y_{21}} + \frac{y_{12}}{y_{22}} \tag{4.28}$$

subject to $y_{21}, y_{22} > 0$

$$\min f_4(y_3) = \frac{(y_{31} - y_{21})^2}{y_{31}} + \frac{(y_{32} - y_{22})^2}{y_{32}}$$

$$\text{subject to} \quad 2y_{31} + 3y_{32} = 5 \tag{4.29}$$

$y_{31}, y_{32} > 0$

This problem was modeled as a bi-level Stackelberg-Nash game by Liu (1998) and solved by genetic algorithm. It was also considered as a 4-level hierarchical mode problem by Ghotbi (2013) and solved by sensitivity based approach.

Now this problem is modeled as a Decentralized-Hierarchical problem. Consider a cooperative game in the third level between player 3 and player 4, Stackelberg game between level 3, level 2 and level 1. A cooperative game solution for the third level was obtained by the bargaining equation first, and then the Stackelberg solutions for level 2 and level 1 were obtained by sensitivity based approach. Table 4.5 showed the solution in this thesis comparing with the Stackelberg-Nash solution from Liu (1998) and Hierarchical solution from Ghotbi (2013).

Table 4.5: Decentralized-Hierarchical solution versus Decentralized solution and Stackelberg-Nash solution

	Decentralized-Hierarchical solution	**Liu (1998) solution**	**Ghotbi (2013) solution**
f_1	1.5019	1.510	1.5831
f_2	5.6882	12.323	5
f_3	5.2436	6.225	5.335
f_4	0.7928	0.835	0.8736
$x = (x_1, x_2)$	(4.3815,2.8092)	(5.768,2.116)	(4.3007,2.8497)
$y_1 = (y_{11}, y_{12})$	(1.2916,2.0000)	(2.885,2.000)	(1.000,2.000)
$y_1 = (y_{11}, y_{12})$	(1.6159,1.4140)	(1.699,1.414)	(2.0068,1.4142)
$y_1 = (y_{11}, y_{12})$	(1.4740,0.6840)	(1.183,0.878)	(0.8736,1.0843)

From Table 4.5, it is seen that when comparing this solution with the solution reported by Liu (1998), all four objective functions are better optimized. When comparing the solution obtained in this thesis with the solution reported by Ghotbi (2013), all but f_2 are better optimized. Therefore, the new game based model proposed in this thesis here can be viewed a better approach solving this multi-level problem.

Chapter 5. Conclusions

The main objectives of this thesis can be classified into three areas: (1) To study different methodologies to solve multi-objective design optimization problems including the DOE-RSM based approaches as well as game theory based methods, (2) To develop a new DOE based response surface method to solve the model updating problem in design optimization, which uses low-discrepancy sequence to generate additional data points for numerical regression, (3) To study two models for solving multi-level optimization problems (Decentralized mode and Hierarchical mode) and solve the multi-level optimization problems for these two different cases.

5.1 Model updating by low-discrepancy sequence method

Strictly speaking, the low-discrepancy sequence method based RSM is still a type of DOE-RSM. The only thing distinguishing it from those traditional DOE method is the way it generates the data points to establish the response surface. Using low-discrepancy sequence method tactfully avoids the inconvenience of the restrictions on the number of data points needed in conventional DOE based methods. Any proper number of data points can be applied in a low-discrepancy sequence designed experiment, so if the regression model does not fit the response surface, the model can be easily updated by adding any number of data points to the existing total.

The advantages of the low-discrepancy sequence method based RSM compared with the traditional DOE-RSM include:

1). The low-discrepancy sequence method can be used to update regression model by adding as little as one data point to the initial model. The traditional DOE methods add additional points by adding levels, which means large number of data points may be needed. In reality, the difficulty and cost of an experiment would increase significantly as additional levels are added.

2). Although the low-discrepancy sequences are generated using a deterministic numerical method, the points in the sequences can largely be viewed as random points. Thus, the data points generated from low-discrepancy sequences are more representative as samples distributed throughout the solution space, such that less data points generated from low-discrepancy sequences would give more accurate solution to the problems compared to data points generated from some other DOE method.

3). The initial number of data points to build a response surface model is generally fixed by the number of the levels and factors in a traditionally designed experiment. However, if applying a low-discrepancy sequence method to design the experiment it can be started from any proper number of data points.

4). In chapter 2, for the pressure vessel design problem, it is found that sometimes the Nash solution to a problem may not be singleton, and it is hard to use traditional DOE method to find all Nash solutions. By easily changing the initial number of data points in a low-discrepancy sequence based experiment, all Nash solutions can be obtained.

There may be additional advantages of this new DOE method that have not been found yet. But also a deficiency of this method should be mentioned here. Same as all other DOE methods so far, in some certain cases, the low-discrepancy sequence method is not able to find solutions for the multi-level optimization design problems. The basic idea of applying DOE-RSM to solve multi-level optimization design problems is to find the analytic solution for the lower level problem, and design the experiment based on this solution to find the global solutions to the problem. If the analytic solutions cannot be obtained from the constraints, it would be difficult to design an experiment to find the global solutions. Besides, since the highest and lowest value of the factors in an experiment are decided by the boundary conditions, it is necessary that the boundary conditions are given in the problem. The solutions obtained from DOE-RSM sometimes turn out to violate the constraints if the experiment is designed without proper consideration being given to boundary conditions.

5.2 Game theory based multi-level optimization design problems

Different strategies are decided by different game modes in multi-level optimization design problems. Roughly speaking, the game modes decide the weight of all the objective functions in a multi-level design optimization problems. Thus, different solutions could be obtained based on different strategies applied to a same problem.

Basically, there are three types of game modes that have been frequently used in the literature review: Cooperative game, Non-cooperative (Nash) game and Stackelberg game. Different combinations of these three game modes in a multi-level optimization design problem make different systems. Two of them were discussed in chapter 4: Decentralized mode and Hierarchical mode. One new combination was also proposed in chapter 4.

From the results of the first two numerical examples in chapter 4, it can be seen that the solution to a same problem can be different when applying different game strategies. Because different game strategies focus on different levels (or objective functions) in a problem, it is difficult to tell which strategy is better when the objectives conflicting each other. When applying another game strategy to a same multi-level optimization problem, some of the objective function values are better optimized while the rest may worse off. To see if a game strategy is better than another for a same multilevel optimization problem, one simple criteria that can be used is to check if the leader objective function value is better optimized, meanwhile the follower objective function values should not exceed certain percentage range of the solution obtained from the previous game strategy.

From the results obtained from the third numerical example, the new proposed decentralized-hierarchical game mode is seen to be a better approach mode compared with the previous ones used to solve that problem.

5.3 Scope of future work

The low-discrepancy sequence method discussed in this thesis has been proved to solve optimization problems effeciently and effectively. One aspect of the future work could be applying this method to design a real experiment where the objective function is unknown and the range of the variables is the only given information. By generating the sampling from the low-discrepancy sequence, response surface model can be built to find the optimum solution.

On another aspect, as summarized, the applicability of the low-discrepancy sequence method for multi-level optimization design problem greatly depends on the given constraints and boundary conditions. Therefore, developing a new approach to use DOE-RSM in general cases regardless of the boundary conditions and the constraints is important before this new method is generalized.

The multi-level design optimization problems discussed in this thesis were mostly bi-level or tri-level, such that the game combinations are very limited. To find more regularities and better strategies for multi-level optimization design problems, more numerical examples need to be tested. Although the new game system was proposed in this thesis seem to be advanced, more numerical examples are required to test it. Unfortunately, the most majority of the numerical examples reported by previous researches are bi-level problems. Thus, a third aspect of the future work of this thesis could be doing researches on optimization problems with 4 or more levels based on hierarchical, decentralized systems, or other systems that are comprised of one or more combinations of game theories.

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