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# RANDOM ITERATION OF RATIONAL MAPS

by  
Jesse Feller

A Dissertation Submitted in  
Partial Fulfillment of the  
Requirements for the Degree of

Doctor of Philosophy  
in Mathematics

at  
The University of Wisconsin-Milwaukee  
August 2015

ABSTRACT  
RANDOM ITERATION OF RATIONAL MAPS

by  
Jesse Feller

The University of Wisconsin-Milwaukee, 2015  
Under the Supervision of Professor Suzanne Boyd

Random and non-autonomous iteration has been a subject of interest in Mathematics that has received some attention in the last few decades. The earliest paper on random iteration in the complex setting was written by Fornæss and Sibony. They have shown that given a family of functions  $\{f_c\}_{c \in \mathcal{W}}$  where  $\mathcal{W}$  is a small open set, for almost every  $z$  the random iteration is stable on a subset of  $\mathcal{W}^{\mathbb{N}}$  of full probability measure. Later, Hiroki Sumi further extended these results to a more general situation using rational semigroups. We will show that the results of Fornæss and Sibony can be extended using the concept of non-generic points. Then we describe the connection between Sumi's kernel Julia set and non-generic points.

In the third chapter, we will look at seed iteration. This is where a function  $f(w, z)$  is composed in the second variable to get a function  $f^n(w, z)$  and then we set  $z = w$  to get a sequence of functions  $F_n(w)$ . We will study the properties of the corresponding Julia and Fatou sets of the sequence  $F_n(w)$ . Furthermore, we will look at evidence that there may be basins of attraction and sub-invariant domains contained inside the space of analytic functions over a domain  $U$ , similar to what we see in classical iteration theory.

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For Verna Feller

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## CHAPTER 1

**Introduction****1. Content Overview**

We begin with a polynomial function  $f(c, z)$  where  $c \in \mathcal{W} \subset \mathbb{C}$  and  $z \in \overline{\mathbb{C}}$ , that is a function of the form

$$(1) \quad f(c, z) = \sum_{i=0}^m \alpha_i(c) z^i,$$

where  $\alpha_i(c)$  ( $i=0,1,\dots,m$ ) are polynomials in  $c$  and at least one is nonconstant. A *rational function in two variables* is a function of the form

$$(2) \quad f(c, z) = \frac{\sum_{i=0}^{m_1} \alpha_i(c) z^i}{\sum_{j=0}^{m_2} \beta_j(c) z^j}$$

where at least one of the polynomial expressions  $\alpha_i(c), \beta_j(c)$  ( $i = 0, 1, \dots, m_1; j = 0, 1, \dots, m_2$ ) is nonconstant. The set  $\mathcal{W}$  is an open or finite set. We assume  $\alpha_{m_1}, \beta_{m_2} \neq 0$ . The maximum of  $m_1, m_2$  is called the *degree of  $f$*  which we define only when the polynomials in the numerator and denominator of (2) have no common factors. For a fixed  $c \in \mathcal{W}$ , we denote by  $f_c(z)$  the function  $z \mapsto f(c, z)$ , and the function  $c \mapsto f(c, z)$  by  $f(c)_z$ .

We define a *rational function in one variable* to be a function of the form  $R(z) = \frac{P(z)}{Q(z)}$  where  $P$  and  $Q$  are polynomials with no common factors. Thus a rational function  $R$  is a function defined on the entire Riemann sphere  $\overline{\mathbb{C}}$  and  $R(z) = \infty$  when  $Q(z) = 0$ . In the case where  $\mathcal{W}$  is open, we require that the function  $f : \mathcal{W} \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  defined by (2) be holomorphic in both variables, and  $f_c(z)$  be a rational function for each  $c \in \mathcal{W}$ . Thus we must choose the set  $\mathcal{W}$  so that for each  $c \in \mathcal{W}$ ,  $f_c(z)$  is defined for all  $z \in \overline{\mathbb{C}}$ . We demonstrate this with an example.

Let  $f(c, z) = \frac{z(z+1)(z-2)}{z+c}$ . Then  $f_c(z)$  is defined except when  $c = 0, 1, -2$ . So in this example we must require  $\mathcal{W}$  to be a subset of  $\mathbb{C} - \{0, 1, -2\}$ .

The expressions defined in (2) or (1) gives rise to a family of rational functions  $\{f_c(z)\}_{c \in \mathcal{W}}$  where  $\mathcal{W} \subset \mathbb{C}$  is open. We also examine the case where the family  $\{f_c\}_{c \in \mathcal{W}}$  is finite. In this case we index the family by the natural numbers as opposed to the parameters, hence  $\mathcal{W} = \{0, 1, \dots, n-1\}$ .

DEFINITION 1.1. Let  $\bar{c} \in \mathcal{W}^{\mathbb{N}}$  where  $\mathcal{W} \subset \mathbb{C}$ . The sequence  $f_{\bar{c}}^n(z) = f^n(\bar{c}, z) = f_{c_n} \circ f_{c_{n-1}} \circ \dots \circ f_{c_1}(z)$  is called a *non-autonomous iteration*. In the case where  $\bar{c}$  is a sequence of random variables, we refer to  $f_{\bar{c}}^n(z)$  as a *random iteration*.

DEFINITION 1.2.  $z \in \bar{\mathbb{C}}$  is  $\bar{c}$ -stable if  $f_{\bar{c}}^n(z)$  has a subsequence that converges uniformly in a neighborhood of  $z$ .

Now define the Fatou set ( $\mathcal{F}$ ) and Julia set ( $\mathcal{J}$ ) as follows.

$$\mathcal{F}(\bar{c}) = \{z : f_{\bar{c}}^n(z) \text{ has a subsequence that converges locally uniformly at } z\}$$

$$\mathcal{J}(\bar{c}) = \bar{\mathbb{C}} - \mathcal{F}(\bar{c})$$

This matches with the definition of Julia and Fatou set in classical iteration theory (where  $\bar{c}$  is a constant sequence). If  $\infty \in \mathcal{F}(\bar{c})$  and  $f_c(z)$  is a polynomial function as defined in (1), we define the *filled Julia set* by the following.

$$\mathcal{K}(\bar{c}) = \{z : f_{\bar{c}}^n(z) \text{ is bounded}\}$$

An example of a filled Julia set is seen in Figure 1.1. We use computers to draw pictures of the filled Julia set  $\mathcal{K}$  which are colored black in all of our figures. Other colors represent  $z$  values in the complex plane where the sequence of iterations is unbounded. Each color is determined by the amount of time it takes the sequence to end up in a predetermined neighborhood of  $\infty$ .

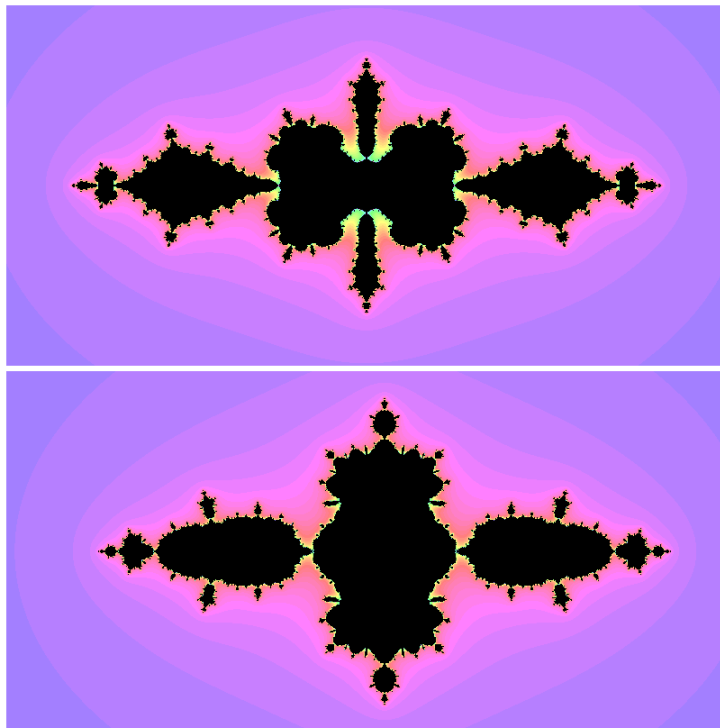


FIGURE 1.1. Filled Julia sets  $\mathcal{K}(\bar{c})$  for two non-autonomous iterations  $f_{\bar{c}}^n(z)$  where  $f_0 = z^2 - 1$  and  $f_1(z) = (z - 1/10)^2 - 1 + 1/10$ . Top and bottom represent two different sequences  $\bar{c}$  generated according to a Bernoulli distribution where  $\mathcal{P}(c_i = 0) = 1/2$ .

Random iteration has been studied in the case where  $z$  is a real number. For more information on this, see [BM07]. The first authors to write about random iteration in the complex setting were John Fornæss and Nessim Sibony [FS91], and their contribution is the main inspiration behind Chapter 2. Hiroki Sumi later extended their results to a more general setting. Several other authors have written about non-autonomous iteration in the complex setting [Brü00], [Büg98], [Büg97], [Com06], many of which focus on the properties of the Julia set  $\mathcal{J}(\bar{c})$ .

The following is an overview of the content of this paper. Chapter 2 covers random and non-autonomous iteration. In Section 1, we study the concept of a sub-invariant set, which is a set  $V$  such that  $f_c(V) \subset V$  for all  $c \in \mathcal{W}$ . Then in Section 2, we examine random iteration; thus  $\bar{c}$  is replaced by a sequence of independent and identically distributed random variables  $\mathbf{C}(\omega)$ . Suppose  $V$  is an invariant domain. Let  $E(z, V) = \{\bar{c} : f_{\bar{c}}^n(z) \in V \text{ for all sufficiently large } n\}$  and suppose  $\mathcal{P}$  is a probability measure on

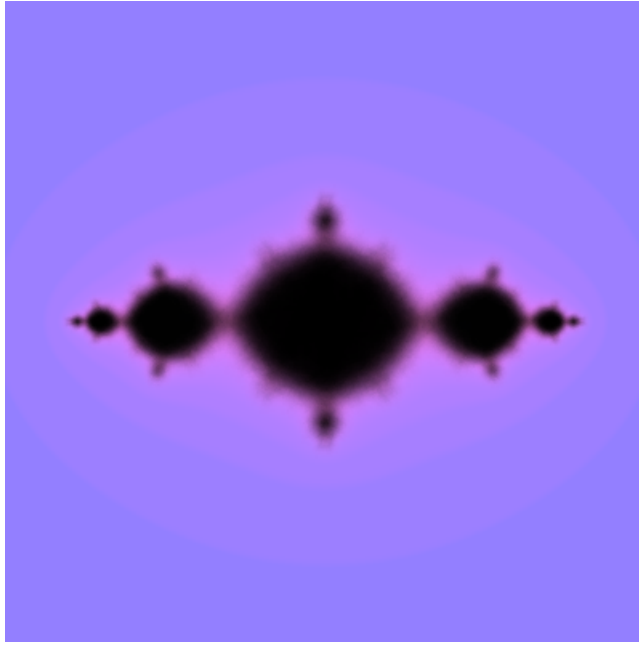


FIGURE 1.2. Pictorial representation of  $g(z)$  from Proposition 2.5. Here  $f(c, z) = z^2 + c$  and  $\mathcal{W} = \{c : |c - (-1)| < .075\}$  where  $\bar{c}_n$  is chosen uniformly and is independent and identically distributed.

$\mathcal{W}^{\mathbb{N}}$ . We denote the characteristic function of  $E(z, V)$  by  $\chi_{E(z, V)}(\bar{c})$ . Then by the strong law of large numbers, the following holds.

PROPOSITION (2.5). *Let  $S_n(z) = \frac{1}{n} \sum_{k=1}^n \chi_{E(z, V)}(\mathbf{C}(\omega)_k)$  where  $z \in \mathbb{C}$  is fixed. Then the sequence of averages  $S_n(z)$  converges to  $g(z) := \mathcal{P}(E(z, V))$  with probability one.*

Figure 1.2 contains a pictorial representation of the function  $g(z)$  from the previous proposition.

In Section 3, we study the function  $g(z) := \mathcal{P}(E(z, V))$ . Let  $c_0 \in \mathcal{W}$ . Define

$$\Omega = \{z : c \mapsto f(c, z) \text{ is constant}\}$$

$$\Omega_{\infty} = \{z : \bar{c} \mapsto f^n(\bar{c}, z) \text{ is constant for all } n\}$$

$$\Omega'_{\infty}(c_0) = \{z : z \in \Omega_{\infty} \text{ and } z \text{ is part of a nonattracting cycle of } f_{c_0}\}$$

If  $\Omega_{\infty}(c_0) = \emptyset$ , then we have the following result which is an extension of a theorem in [FS91].

THEOREM (2.8). *Suppose  $\mathcal{W} \subset B(c_0, \delta)$ . Let  $V_1, V_2, \dots, V_d$  be sub-invariant neighborhoods of the attracting cycles  $\gamma_1, \gamma_2, \dots, \gamma_d$  of  $f_{c_0}$  contained inside their respective attracting basins such that  $\partial V_j \cap f(\mathcal{W}, \Omega) = \emptyset$  for all  $j$ . Then there exists  $\delta_0 > 0$  such that for all  $\delta < \delta_0$  there are functions  $g_1, g_2, \dots, g_d$  continuous on  $\overline{\mathbb{C}}$  such that*

1. *The functions  $\{g_j(z)\}_{j=1}^d$  form a partition of unity, and*
2. *For  $z \in \overline{\mathbb{C}}$  there exists disjoint open sets  $E_j(z) \subset \mathcal{W}^{\mathbb{N}}$  such that  $\mathcal{P}(E_j(z)) = g_j(z)$  and for every  $\bar{c} \in E_j(z)$ ,  $f^n(\bar{c}, z) \in V_j$  for all sufficiently large  $n$ .*

Thus for every  $z \in \overline{\mathbb{C}}$ , we have that  $f^n(\bar{c}, z) \in V_j$  for some  $j$  with probability one, that is  $z$  is  $\bar{c}$ -stable with probability one. It follows that the Julia set of  $f_{\bar{c}}^n(z)$  is a set of Lebesgue measure zero for almost every  $\bar{c} \in \mathcal{W}^{\mathbb{N}}$ .

Section 4 begins with a summary of a few results by Hiroki Sumi who has independently studied the probability that  $z$  is  $\bar{c}$ -stable using semigroups. His result (Theorem 2.12) also generalizes a result by Fornæss and Sibony. Let  $\langle f_c \rangle_{c \in \mathcal{W}}$  be the semigroup generated by  $\{f_c\}_{c \in \mathcal{W}}$ . Define the *kernel Julia set* by

$$\mathcal{J}_{ker} \langle f_c \rangle_{c \in \mathcal{W}} := \bigcap_{g \in \langle f_c \rangle} g^{-1}(\mathcal{J} \langle f_c \rangle).$$

We prove the following characterization of Sumi's kernel Julia set.

THEOREM (2.13). *Let  $\{f_c\}_{c \in \mathcal{W}}$  be a holomorphic family of rational maps where  $\mathcal{W} \subset B(c_0, \delta)$  is open where  $c_0 \in \mathbb{C}$ . Then there exists a  $\delta$  such that either*

1.  *$\mathcal{J}_{ker} \langle f_c \rangle_{c \in \mathcal{W}} = \overline{\mathbb{C}}$  which implies that  $\mathcal{J} \langle f_c \rangle_{c \in \mathcal{W}} = \overline{\mathbb{C}}$  or*
2.  *$\mathcal{J}_{ker} \langle f_c \rangle_{c \in \mathcal{W}} \subset \Omega'_{\infty}(c_0)$ .*

In Chapter 3, the reader examines iteration of polynomial maps  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  where the composition takes place in the second variable. We refer to this as seed iteration, which is a concept of the author's own invention. More precisely, we define

$$f^n(w, z) = f(w, f(w, \dots f(w, f(w, z)) \dots)) \text{ (} n \text{ times)}$$

$$F_n(w) = f^n(w, w) \text{ (i.e. set } z = w \text{ in the previous line).}$$

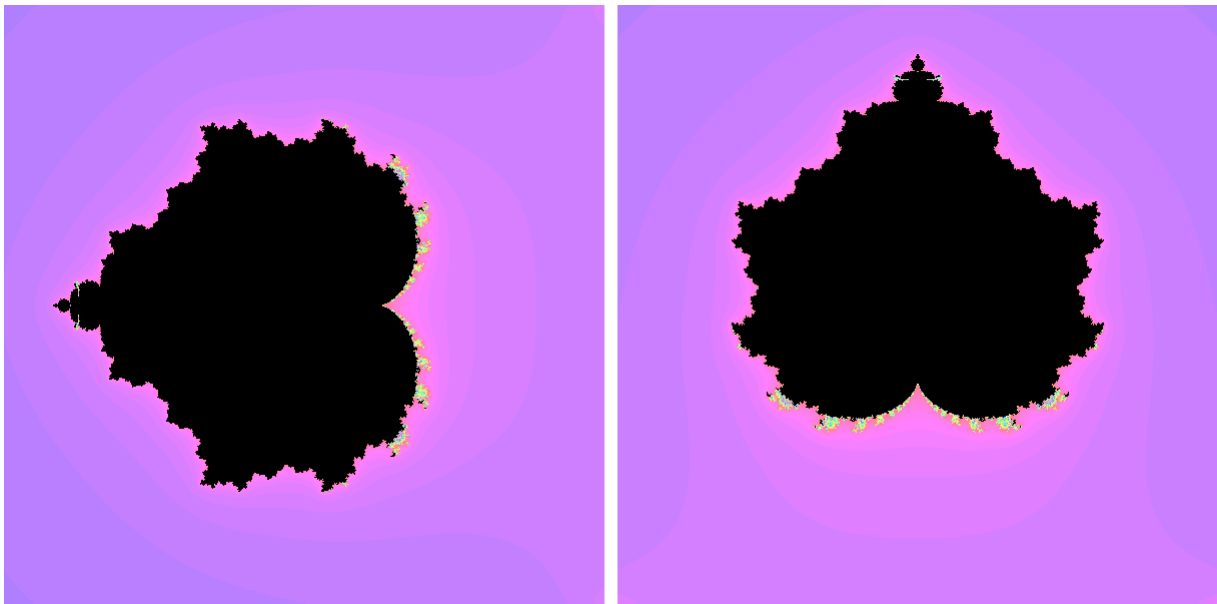


FIGURE 1.3.  $\mathcal{K}[f^n(w, -iw)]$  (left) and  $\mathcal{K}[f^n(iw, w)]$  (right) where  $f(w, z) = z^2 + w$ .

Now we define the Fatou set ( $\mathcal{F}$ ), Julia set ( $\mathcal{J}$ ) and filled Julia set ( $\mathcal{K}$ ) for seed iteration by the following.

$$\begin{aligned} \mathcal{F}(F_n) &= \{w : F_n(w) \text{ has a subsequence that converges locally uniformly at } w\} \\ (3) \quad \mathcal{J}(F_n) &= \mathbb{C} - \mathcal{F}(F_n) \\ \mathcal{K}(F_n) &= \{w : F_n(w) \text{ is bounded}\} \end{aligned}$$

Section 1 is an introduction to the seed iteration concept. In Section 2, we show that under certain assumptions  $\mathcal{J}(F_n) \neq \emptyset$  and  $\infty \in \mathcal{F}(F_n)$ . Section 3 is about affine properties of seed iteration. Recall that an affine function is a function of the form  $h(z) = az + b$  where  $a, b \in \overline{\mathbb{C}}$ . Readers will learn about the following (the image of  $A$  under  $h$  is denoted  $hA$  in this proposition).

PROPOSITION (3.4). *Suppose  $h(z)$  is an affine function. Then  $h\mathcal{F}[f^n(h(w), w)] = \mathcal{F}[f^n(w, h^{-1}(w))]$  and  $h\mathcal{J}[f^n(h(w), w)] = \mathcal{J}[f^n(w, h^{-1}(w))]$ .*

Figure 1.3 contains an example of an application of this proposition.

In Section 4 we use holomorphic motions to show that  $\mathcal{J}(F_n)$  has empty interior. In Section 5, we discuss some conjectures for seed iteration where the function  $f(w, z)$  is

allowed to change at each step of the iteration process according to a sequence of zeros and ones. Thus these conjectures blend the ideas of seed iteration with non-autonomous iteration. Our conjecture states that there may be basins of attraction and sub-invariant domains contained inside spaces of analytic functions which correspond to the analogous concepts found in classical and non-autonomous iteration theory.

Throughout this paper, we denote open balls by  $B_d(z, \delta) = \{z' : d(z', z) < \delta\}$  or  $B(z, \delta)$  when the metric  $d$  is clear. Unless otherwise stated, we use is the chordal metric  $\sigma$  on  $\overline{\mathbb{C}}$ . We have the following notation for the neighborhood of a set  $A$ :  $B_d(A, \epsilon) = \{z : d(z, A) < \epsilon\}$ . The boundary is denoted  $\partial A$  while the closure is  $\overline{A}$ . A neighborhood  $U$  is always an open set whose boundary has Lebesgue measure zero (this is used a few times in Section 3).  $\lambda$  always denotes Lebesgue measure on  $\mathbb{C}$ , and  $\frac{\partial f}{\partial z}$  is what we use to denote the partial derivative with respect to  $z$ . The word *domain* always means an open and connected set.

## 2. Background

Much of what is stated here can be found in many texts on holomorphic dynamics or iteration theory (see [Bea91] or [Mil06]). Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational map. We define the degree of  $f(z)$  to be the maximum of the polynomial degrees of the numerator and denominator. It is standard practice in (holomorphic) classical dynamics to assume the degree of  $f$  is always greater than one. Consider the family of maps  $\{f^n\}_{n \in \mathbb{N}}$  where  $f^n(z) = f \circ f \circ \dots \circ f(z)$  ( $n$  times). In classical iteration theory, we often study the behavior of the sequence  $(f^n(z))_{n \in \mathbb{N}}$  in terms of local uniform convergence of its subsequences. The sequence  $(f^n(z))_{n \in \mathbb{N}}$  is called the *orbit of  $z$* .  $z$  is called the *seed value*.

It is sometimes the case that the orbit of  $z$  repeats certain values as  $n \rightarrow \infty$ , and hence we have the following definition.

**DEFINITION 1.3.** An finite set  $\gamma = \{z_1, z_2, \dots, z_n\} \subset \overline{\mathbb{C}}$  is called a *periodic cycle* if  $f^n(z_i) = z_i$  for each  $i = 1, 2, \dots, n$ . The *period* of the cycle is the minimum integer  $n$  such that  $f^n(z_i) = z_i$ . An element  $z$  is *pre-periodic* if  $f^n(z)$  is periodic for some  $n > 1$ , but  $z$  itself is not periodic.

If  $\{z_1, z_2, \dots, z_n\}$  is a periodic cycle, it follows that  $f(z_1) = z_2, f(z_2) = z_3, \dots, f(z_n) = z_1$ .

DEFINITION 1.4. A periodic cycle is

1. *attracting* if  $|(f^n)'(z_1)| < 1$
2. *repelling* if  $|(f^n)'(z_1)| > 1$
3. *neutral* if  $|(f^n)'(z_1)| = 1$ .

(observe that from the chain rule  $(f^n)'(z_1) = (f^n)'(z_2) = \dots = (f^n)'(z_n)$ )

It is known that every attracting cycle of period  $n$  has a neighborhood where  $f^{jn}(z) \rightarrow z_i$  for some  $i$  as  $j \rightarrow \infty$  and this convergence occurs locally uniformly.

The term “chaos” is used when small changes in the seed value  $z$  cause dramatic changes in the long term behavior of the sequence  $f^n(z)$ . The set of points where this chaotic behavior occurs is called the Julia set. Formally, the *Fatou set* ( $\mathcal{F}$ ) and *Julia set* ( $\mathcal{J}$ ) are defined as stated below.

DEFINITION 1.5. Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational function.

$$\mathcal{F}(f) = \{z : f^n(z) \text{ has a subsequence which converges locally uniformly at } z\}$$

$$\mathcal{J}(f) = \overline{\mathbb{C}} - \mathcal{F}(f)$$

Recall that a family of functions is *normal* on  $U$  if every sequence of functions has a subsequence which converges uniformly on  $U$ . A set  $A$  is *completely invariant* if  $f(A) = A$  and  $f^{-1}(A) = A$ . Both  $\mathcal{J}(f)$  and  $\mathcal{F}(f)$  are completely invariant under the image of  $f$ . Notice every element of an attracting cycle  $\gamma$  is contained inside an open subset of  $\mathcal{F}(f)$ . We also know that the repelling periodic cycles are dense in  $\mathcal{J}(f)$ . When  $f(z)$  is a polynomial, we define the *filled Julia set* by  $\mathcal{K}(f) = \{z : f^n(z) \text{ is a bounded sequence}\}$ . It is well known that  $\partial\mathcal{K}(f) = \mathcal{J}(f)$ .

These sets have generated great interest to mathematicians for decades due to the beautiful fractal images they produce. With the help of computers, we can generate pictorial approximations of  $\mathcal{K}$ . Computers have allowed mathematicians studying holomorphic dynamics to make conjectures and visually demonstrate the properties of  $\mathcal{K}$ . Figure 1.4 contains a few examples of filled Julia sets.

A common endeavor in holomorphic dynamics is to explore the properties of the sets  $\mathcal{J}, \mathcal{F}$ , and  $\mathcal{K}$ . For example, in most cases we know that  $\mathcal{J}$  is a set of Lebesgue



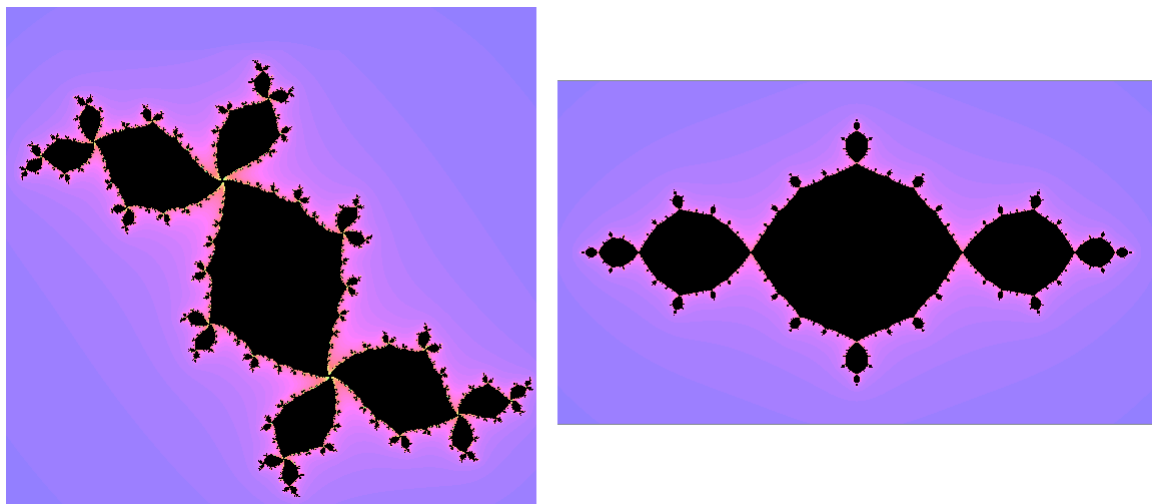


FIGURE 1.4. Filled Julia sets of the functions  $z \mapsto z^2 - .1 + .75i$  (left, “rabbit”) and  $z \mapsto z^2 - 1$  (right, “basilica”).

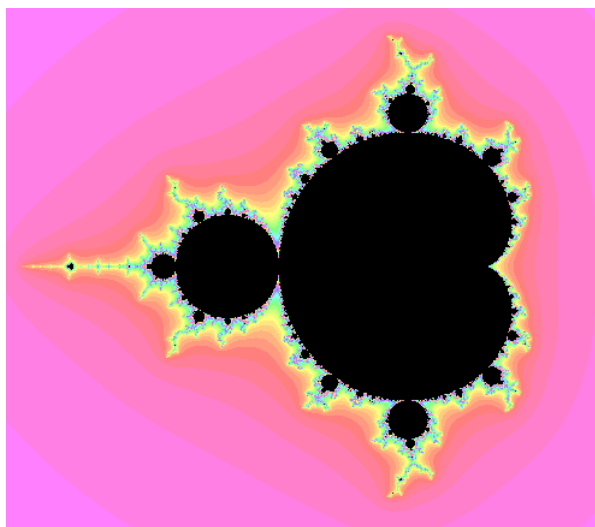


FIGURE 1.5. The Mandelbrot set.

measure zero, and is self-similar. Furthermore,  $\mathcal{J}$  is always nonempty and either  $\mathcal{J} = \overline{\mathbb{C}}$  or  $\mathcal{J}$  has empty interior. The most common examples in holomorphic dynamics come from the quadratic family of polynomials  $f_c(z) = z^2 + c$  where  $c \in \mathbb{C}$  is a parameter. The *Mandelbrot set* (see Figure 1.5) is the set of all  $c \in \mathbb{C}$  where the corresponding Julia set of  $z^2 + c$  is connected. It is well known that the Mandelbrot set is the same as the set  $\{c : f_c^n(0) \text{ is bounded}\}$ .

Now a set  $A$  is *forward invariant* if  $f^n(A) = A$  for some  $n$ . It can be shown that each forward invariant connected component of  $\mathcal{F}(f)$  can be classified into four categories which are defined as follows.

DEFINITION 1.6. Let  $\mathcal{B}$  be a forward invariant connected component of  $\mathcal{F}(f)$ .

1.  $\mathcal{B}$  is an *attracting component* if  $\mathcal{B}$  contains an element of an attracting cycle.
2.  $\mathcal{B}$  is an *parabolic component* if for each  $z \in \mathcal{B}$ ,  $f^n(z) \rightarrow z_0 \in \partial\mathcal{B}$  as  $n \rightarrow \infty$ .
3.  $\mathcal{B}$  is a *Siegel disk* if  $f : \mathcal{B} \rightarrow \mathcal{B}$  is analytically conjugate to an irrational rotation of the unit disk.
4.  $\mathcal{B}$  is a *Herman ring* if  $f : \mathcal{B} \rightarrow \mathcal{B}$  is analytically conjugate to an irrational rotation of an annulus.

Furthermore, every connected component of  $\mathcal{F}(f)$  can be mapped onto a forward invariant component of the Fatou set. Thus the previous definition completely describes the behavior of the sequence  $f^n(z)$  when  $z \in \mathcal{F}(f)$ . If  $\gamma$  is an attracting cycle of period  $n$ , then the components of  $\mathcal{F}(f)$  where  $f^{kn}$  converges to an element of  $\gamma$  as  $k \rightarrow \infty$  is called the *attracting basin of  $\gamma$* .

## CHAPTER 2

**Random Iteration****1. Sub-invariant Neighborhoods**

DEFINITION 2.1. If  $\Gamma$  is an index set and  $\{f_\gamma\}_{\gamma \in \Gamma}$  is a family of functions, we say that  $A$  is *sub-invariant with respect to*  $\{f_\gamma\}$  if  $f_\gamma(A) \subset A$  for all  $\gamma$ .

A family of degree  $m$  polynomials  $\{f_c\}_{c \in \mathcal{W}}$  which have the form given in (1) is called a *class  $\mathcal{B}$  family* if the following holds.

1. For some  $a > 0$ ,  $|\alpha_m(c)| \geq a$  for all  $c \in \mathcal{W}$ , and
2. for some  $A \geq 0$  we have  $|\alpha_j(c)| < A|\alpha_m(c)|$  for all  $j = 0, 1, \dots, m$  and all  $c \in \mathcal{W}$ .

The above definition is a subtle modification of the definition of a class  $\mathcal{B}$  sequence of functions found in [Büg97]. It can be shown that a class  $\mathcal{B}$  family, has a sub-invariant domain  $V$  at infinity and  $f_{\bar{c}}^n(z)$  converges locally uniformly to  $\infty$  on  $V$  for any  $\bar{c} \in \mathcal{W}^{\mathbb{N}}$  [Büg97].

Our first result deals with sub-invariant domains in the case where  $\mathcal{W}$  is finite.

PROPOSITION 2.1. *Suppose  $\mathcal{W}$  is finite and for every  $c \in \mathcal{W}$ ,  $z_0$  is a finite attracting fixed point of  $f_c(z)$ . Then for any sequence  $\bar{c} \in \mathcal{W}^{\mathbb{N}}$ ,  $z_0$  is a fixed point of  $f_{\bar{c}}^n(z)$ . Furthermore, there exists  $\delta > 0$  such that  $B(z_0, \delta)$  is sub-invariant with respect to  $\{f_c\}_{c \in \mathcal{W}}$ , and for all  $z \in B(z_0, \delta)$ ,  $\lim_{n \rightarrow \infty} f_{\bar{c}}^n(z) = z_0$ .*

PROOF. The fact that  $z_0$  is a fixed point for  $f_c^n(z)$  is trivial. Let  $z \in B(z_0, \delta)$ . Using the Taylor expansion of  $f_c$  about  $z_0$ , we can choose for each  $c \in \mathcal{W}$  a number  $\alpha_c < 1$  such that  $|f_c(z) - z_0| < \alpha_c|z - z_0|$  for some sufficiently small  $\delta$ . Now choose a number  $\alpha$  such that  $\max\{\alpha_c\} \leq \alpha < 1$  where the maximum is taken over all  $c \in \mathcal{W}$ . Then  $|f_c(z) - f_c(z_0)| < \alpha|z - z_0| < \delta$  for every  $c \in \mathcal{W}$ . Thus  $f_c(B(z_0, \delta)) \subset B(z_0, \delta)$  for each  $c \in \mathcal{W}$

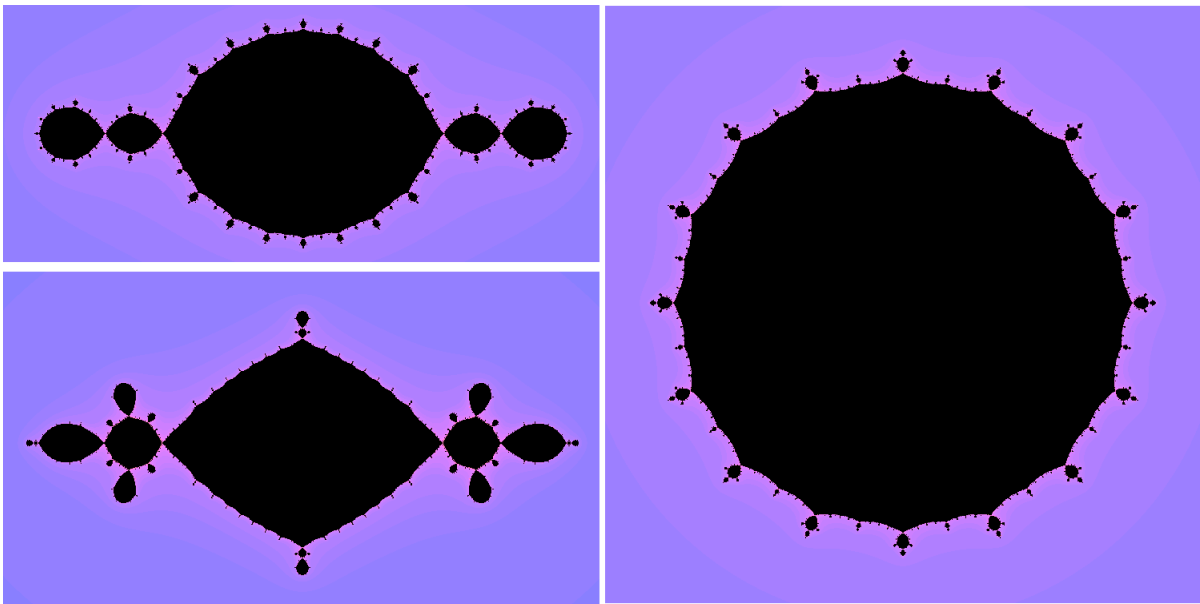


FIGURE 2.1. Filled Julia set of three non-autonomous iterations of  $f_0(z) = z^2$  and  $f_1(z) = z^4 - 2z^2$ , that is three sequences  $\bar{c} \in \{0, 1\}^{\mathbb{N}}$ .

Now assume  $|f_{\bar{c}}^{n-1}(z) - z_0| < \alpha^{n-1}|z - z_0|$ . Let  $w = f_{\bar{c}}^{n-1}(z)$ . Then  $w \in B(z_0, \delta)$ . Therefore,  $|f_{\bar{c}}^n(z) - z_0| = |f_{c_n}(w) - f_{c_n}(z_0)| < \alpha|w - z_0| < \alpha^n|z - z_0|$ , and the conclusion follows.  $\square$

EXAMPLE 2.1. Let  $f_0(z) = z^2$  and  $f_1(z) = (z^2 - 1)^2 - 1 = z^4 - 2z^2$ . Notice 0 is an attracting fixed point for both maps. In fact it is easy to check that  $B(0, 1/4)$  is sub-invariant. Figure 2.1 shows the filled Julia set for three different non-autonomous iterations. The origin is located in the center of all three pictures. We can see that there is clearly a neighborhood centered about 0 where the orbits are bounded, and in fact we know from Proposition 2.1 that  $f_{\bar{c}}^n(z)$  converges to 0 on  $B(0, 1/4)$  for any  $\bar{c} \in \{0, 1\}^{\mathbb{N}}$ .

Sub-invariant domains will be very useful later as they act as a “trapping region” for the orbit  $(f_{\bar{c}}^n(z))_{n=1}^{\infty}$ . Indeed, if  $z \in V$  where  $V$  is a sub-invariant domain, then  $f_{\bar{c}}^n(z) \in V$  for all  $n$ . Furthermore, for a sequence  $\bar{c} \in \mathcal{W}^{\mathbb{N}}$ ,  $f_{\bar{c}}^n(V) \subset V$  for all  $n$ . It follows from Montel’s theorem that  $(f_{\bar{c}}^n)$  is a normal family on  $V$ . Thus a sub-invariant domain is always contained in the Fatou set for any given  $\bar{c}$ .

The family  $\{f_c\}_{c \in \mathcal{W}}$  is an *analytic family* if the function  $f(c, z)$  is a holomorphic function in two variables.

**THEOREM 2.2.** [FS91] *Suppose  $\mathcal{W} = B(c_0, \delta)$  and let  $\{f_c(z)\}_{c \in B(c_0, \delta)}$  be an analytic family of rational maps. Suppose that  $f_{c_0}(z)$  has an attracting cycle  $\gamma$ . Then for a sufficiently small  $\delta > 0$ , there exists a sub-invariant neighborhood  $V$  containing  $\gamma$  such that  $V$  is contained inside the immediate attracting basin of  $\gamma$ .*

The above theorem came from an unjustified step of a proof written by Fornæss and Sibony in [FS91]. For the sake of completeness, we include a proof here. In the case where the family of maps consists of one function, a sort of topological proof can be found by using the open mapping theorem.

To prove Theorem 2.2, we need some facts regarding holomorphic functions of several variables. A polydisk of radius  $(r_0, r_1)$  is an open subset of  $\mathbb{C}^2$  of the form  $B(z, r_0) \times B(c, r_1)$ . A function  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  is *holomorphic* if it is continuous and if its first partial derivatives exist. It is *analytic at  $(c_0, z_0)$*  if it can be represented as a uniformly convergent power series in some open polydisk centered at  $(c_0, z_0)$ :

$$(4) \quad f(c, z) = \sum_{i,j=0}^{\infty} \alpha_{i,j} (z - z_0)^i (c - c_0)^j$$

$$z \in B(z_0, r_0), c \in B(c_0, r_1) \text{ for some } r_0, r_1 > 0.$$

Just as in the case of functions of a single variable, analytic and holomorphic are equivalent. In the power series expansion given above, we can let

$$(5) \quad \alpha_{i,j} = \frac{1}{i!j!} \frac{\partial f^{i+j}}{\partial z^i \partial c^j} (c_0, z_0).$$

For more details on the theory of functions of several complex variables, see [Nis01].

We need the following lemma to prove Theorem 2.2.

**LEMMA 2.3.** *Let  $M_1, M_2, \dots, M_n$  be positive real numbers and suppose  $\prod_{k=1}^n M_k < 1$ . Given a number  $\epsilon_1 > 0$  there exists positive numbers  $\epsilon_2, \dots, \epsilon_n$  such that*

1.  $\epsilon_k M_k < \epsilon_{(k \bmod n) + 1}$  for all  $k = 1, 2, \dots, n$ , and
2.  $\epsilon_k < \max\{\epsilon_1/M_n, \epsilon_1/M_n M_{n-1}, \dots, \epsilon_1/(M_n M_{n-1} \cdot \dots \cdot M_1)\}$  for all  $k = 2, \dots, n$ .

PROOF. Since  $\prod_{j=1}^n M_j < 1$  we have  $\epsilon_1 \prod_{j=1}^n M_j < \epsilon_1$ . We can then inductively define a finite sequence  $\epsilon_k$  as follows. For each  $k = 2, 3, \dots, n$  we have  $\epsilon_{k-1} \prod_{j=k-1}^n M_j < \epsilon_1$ . So choose  $\epsilon_k$  such that  $\epsilon_{k-1} \prod_{j=k-1}^n M_j < \epsilon_k \prod_{j=k}^n M_j < \epsilon_1$ . We then have

$$(6) \quad \epsilon_1 \prod_{j=1}^n M_j < \epsilon_2 \prod_{j=2}^n M_j < \dots < \epsilon_{n-1} M_{n-1} M_n < \epsilon_n M_n < \epsilon_1.$$

The inequalities from statement 1 also hold. This can be seen by taking each inequality in (6) and dividing out the appropriate factors. The bounds on the numbers  $\epsilon_k$  also follows from the inequalities in (6).  $\square$

*Proof of Theorem 2.2.* Let  $\gamma = \{z_1, z_2, \dots, z_n\}$  be an attracting cycle of  $f_{c_0}(z)$ . Since  $f(c, z)$  is continuous, we can choose  $\epsilon'$  and  $\delta'$  such that

1.  $f(B(c_0, \delta'), B(z_k, \epsilon'))$  is contained inside the immediate basin of attraction of  $\gamma$  for each  $k = 1, 2, \dots, n$ .
2.  $f(c, z)$  has a Taylor expansion in  $B(c_0, \delta') \times B(z_k, \epsilon')$ .

For each  $k$  let  $T_k(c, z) = \sum_{i \geq 1, j \geq 0} \alpha_{i,j} (z - z_k)^{i-1} (c - c_0)^j$  and  $U_k(c) = \sum_{j=1}^{\infty} \alpha_{0,j} (c - c_0)^j$  where  $\alpha_{i,j}$  are the coefficients of the power series expansion of  $f(c, z)$  around  $(c_0, z_k)$ . Then the power series expansion is given by  $f(c, z) = f(c_0, z_k) + (z - z_k)T_k(c, z) + U_k(c)$ .

Now since  $f(c, z)$  is holomorphic, and  $\gamma$  is an attracting cycle, we can choose  $\epsilon'' < \epsilon'$ ,  $\delta'' < \delta'$ , and  $\mu$  so that  $\prod_{k=1}^n |T_k(c, z)| < \mu < 1$  for  $z \in B(\gamma, \epsilon'')$  and  $c \in B(c_0, \delta'')$ . For each  $k = 0, 1, \dots, n$  let  $M_k = \sup\{|T_k(c, z)| : z \in B(z_k, \epsilon''), c \in B(c_0, \delta'')\}$ . Then  $\prod_{k=1}^n M_k \leq \mu < 1$ . Now choose  $\epsilon_1$  such that  $\max\{\epsilon_1, \epsilon_1 / (M_n M_{n-1} \dots M_k) : k = 1, 2, \dots, n\} < \epsilon''$ . Now we can choose for each  $j = 2, \dots, n$  an  $\epsilon_j$  such that the inequalities of Lemma 2.3 hold. Since  $f(c, z)$  is analytic in  $c$ ,  $U_k(c) \rightarrow 0$  as  $c \rightarrow c_0$ . So for each  $k = 1, \dots, n$  choose  $\delta_k$  such that for  $c \in B(c_0, \delta_k)$ ,  $\epsilon_k M_k + |U_k(c)| < \epsilon_{(k \bmod n)+1}$ .

Finally, let  $\delta = \min\{\delta_1, \delta_2, \dots, \delta_n, \delta', \delta''\}$  and  $V_k = B(z_k, \epsilon_k)$ . Suppose  $z \in \bigcup_{k=1}^n V_k$  and  $c \in B(c_0, \delta)$ . Then  $z \in V_k$  for some  $k$ . So  $|f(c, z) - z_{(k \bmod n)+1}| = |f(c, z) - f(c_0, z_k)| =$

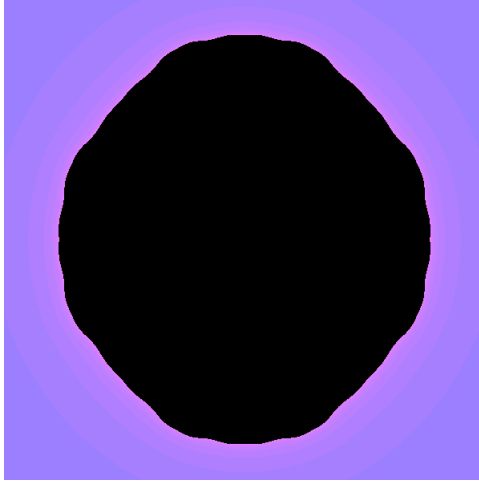


FIGURE 2.2. Filled Julia set for a non-autonomous iteration of  $f_0(z) = z^2$  and  $f_1(z) = z^2 - \frac{1}{5}z + \frac{11}{100}$ .

$|(z - z_k)T_k(c, z) + U_n(c)| \leq \epsilon_k M_k + |U_k(c)| < \epsilon_{(k \bmod n)+1}$ . Therefore,  $f_c(V_k) \subset V_{(k \bmod n)+1}$  which completes the proof.

□

PROOF. Let  $z \in \bigcup_{c \in \mathcal{W}} A_c$  and  $\bar{c} = (c_1, c_2, \dots) \in \mathcal{W}^{\mathbb{N}}$ . Then  $z \in A_d$  for some  $d \in \mathcal{W}$ . Suppose  $c_1 = d$ . Then by invariance  $f_{c_1}(z) \in A_d$ . Now suppose  $c_1 \neq d$ . By condition 2,  $f_{c_1}(z) \in A_c$  for all  $c \neq d$ . Therefore  $f_{c_1}(z) \in \bigcup_{c \in \mathcal{W}} A_c$  and  $\bigcup_{c \in \mathcal{W}} A_c$  is a sub-invariant set.

Now assume  $f_{\bar{c}}^{n-1}(z) = w \in \bigcup_{c \in \mathcal{W}} A_c$ . Then  $w \in A_d$  for some  $d \in \mathcal{W}$ . Using an argument similar to the previous paragraph, we can show that  $f_{c_n}(w) \in \bigcup_{c \in \mathcal{W}} A_c$ . Thus  $f_{\bar{c}}^n(z) \in \bigcup_{c \in \mathcal{W}} A_c$ . By condition 1 we have  $f_{\bar{c}}^n(z)$  is bounded. □

EXAMPLE 2.2. Let  $f_0(z) = z^2, g(z) = z + 1/10$  and  $f_1(z) = (g \circ f_0 \circ g^{-1})(z) = z^2 - \frac{1}{5}z + \frac{11}{100}$ . Then  $A_0 = B(0, 1/2)$  is sub-invariant under  $f_0$  and  $A_1 = B(1/10, 1/2)$  is sub-invariant under  $f_1$ . Notice  $A_0 \subset B(1/10, 3/5)$ . So  $f_1(A_0) \subset f_1(B(1/10, 3/5)) = B(1/10, 9/25) \subset A_1$ . A similar argument shows that  $f_0(A_1) \subset A_0$ . Therefore,  $B(0, 1/2) \cup B(1/10, 1/2)$  is invariant, and for any sequence  $\bar{c} \in \{0, 1\}^{\mathbb{N}}$ ,  $f_{\bar{c}}^n(z)$  is bounded for all  $z \in B(0, 1/2) \cup B(1/10, 1/2)$ . Figure 2.2 shows us the filled Julia set for a given  $\bar{c} \in \{0, 1\}^{\mathbb{N}}$ .

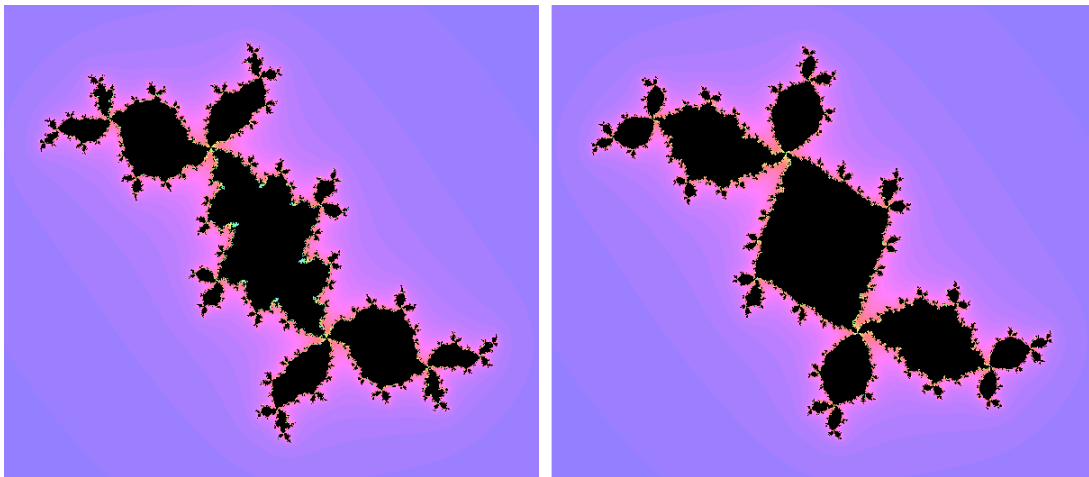


FIGURE 2.3. Filled Julia sets for non-autonomous iterations (i.e. two realized sequences  $\bar{c}$ ) of  $\{z^2 + c\}_{c \in \mathcal{W}}$ ,  $\bar{c} \in \mathcal{W} = B(-.1 + .75i, .04)$  is generated according to a uniform distribution.

## 2. Introduction to Random Iteration

We now introduce random iteration. We adopt the notation  $\mathbf{C}(\omega)$  for a sequence of random variables. We always assume that  $\mathbf{C}(\omega) = (C_1, C_2, \dots)$  is a sequence of independent and identically distributed (IID) random variables. In all of our examples, we use computers to generate a realized sequence  $\bar{c}$  according to a uniform distribution or, in the discrete case, a Bernoulli distribution where the probability that  $C_j = 0$  is  $1/2$ . Consider the following situations.

EXAMPLE 2.3. Suppose  $f_c(z) = z^2 + c$ . Let  $\mathcal{W} = B(-.1 + .75i, .04)$ . Recall that the filled Julia set of  $z^2 - .1 + .75i$  is a familiar set referred to as the “rabbit” in discrete dynamics. The filled Julia sets for two non-autonomous (realized) iterations of this family are shown in Figure 2.3.

EXAMPLE 2.4. Let  $f_c(z)$  and  $\mathcal{W}$  be defined the same as in the previous example. Now suppose that  $\bar{c}_z$  is a sequence dependent on  $z \in \bar{\mathcal{C}}$ . That is for each  $z$  a different sequence  $\bar{c}$  is generated according to a uniform distribution. Then we may get different pictures for the filled Julia set. An example of a non-autonomous iteration in this case is shown in Figure 2.4.



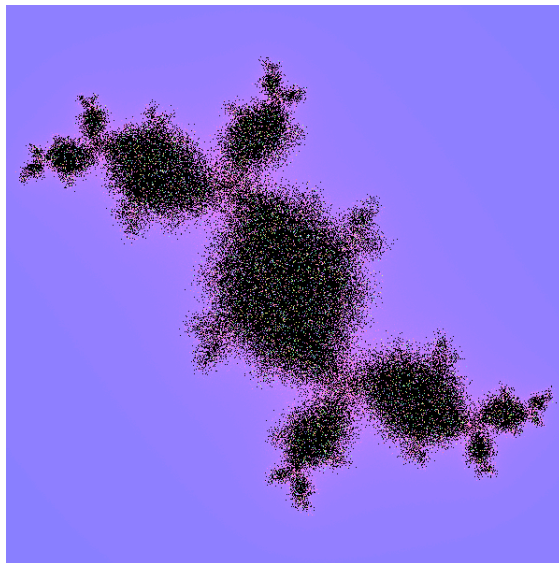


FIGURE 2.4. Filled Julia sets for a non-autonomous iteration of  $\{z^2 + c\}_{c \in \mathcal{W}}$ ,  $\mathcal{W} = B(-.1 + .75i, .04)$  where the realized sequences  $\bar{c}_z$  are functions of  $z$ .

We will see from Theorem 2.5 that the two processes shown in Figures 2.3 and 2.4 are the same “on the average”.

It is clear from Figures 2.1 and 2.3 that using different sequences  $\bar{c}$  may give us different filled Julia sets. We would like to know for a fixed  $z \in \bar{\mathbb{C}}$  the probability that  $f_{\mathbf{C}(\omega)}^n(z)$  ends up in an invariant domain for large  $n$ . In order to answer this question, we need to establish the existence of a probability measure on  $\mathcal{W}^{\mathbb{N}}$ . The development of such a measure is described by K.L. Chung in [Chu01]. Let’s review a few of those details.

Recall that the triple  $(\Omega, \mathcal{A}, \mathcal{P})$  where  $\Omega$  is a nonempty set and  $\mathcal{A}$  is a  $\sigma$ -algebra is called a *probability space* if  $\mathcal{P} : \mathcal{A} \rightarrow [0, 1]$  is a countably additive measure with  $\mathcal{P}(\Omega) = 1$ . Let  $(\Omega_n, \mathcal{A}_n, \bar{\mathcal{P}}_n)_{n=1}^{\infty}$  be a probability space for each  $n$  (for our purposes, we let  $\Omega_n = \mathcal{W}$  and the  $\bar{\mathcal{P}}_n$  will usually be the same measure for each  $n$ ). A subset  $A$  is called a *finite-product set* if

$$(7) \quad A = \times_{n=1}^{\infty} A_n$$

where  $A_n \in \mathcal{A}_n$ , and for all but finitely many  $n$   $A_n = \Omega_n$ . Now define  $\mathcal{A}_0$  to be the collection of all finite-product sets and  $\mathcal{A}$  to be the sigma algebra generated by  $\mathcal{A}_0$ .

Define a set function  $\mathcal{P}$  on  $\mathcal{A}_0$  by

$$(8) \quad \mathcal{P}(A) = \prod_{n=1}^{\infty} \bar{\mathcal{P}}_n(A_n)$$

where  $A$  has the form given in (7). And if  $B = \bigcup_{k=1}^n B_k$  where the  $B_k$ 's are (pairwise) disjoint finite-product sets, then

$$(9) \quad \mathcal{P}(B) = \sum_{k=1}^n \mathcal{P}(B_k)$$

It can be shown that  $\mathcal{P}$  is a probability measure on  $\mathcal{A}_0$  and can be extended uniquely to a probability measure on  $\mathcal{A}$  [Chu01].

A measure  $\mathcal{P} : \mathcal{A} \rightarrow [0, 1]$  is *absolutely continuous* with respect to Lebesgue measure if for every measurable set  $A$  such that  $\lambda(A) = 0$  we have  $\mathcal{P}(A) = 0$ . We adopt the convention that absolute continuity always refers to absolute continuity with respect to  $\lambda$ . Let  $X \subset \bar{\mathbb{C}}$  and  $\mathcal{B}(X)$  be the sigma algebra of all Borel subsets of  $X$ . A Borel probability measure  $\mathcal{P} : \mathcal{B}(X) \rightarrow [0, 1]$  is *regular* if both of the following conditions hold:

1. For each  $A \in \mathcal{B}(X)$ ,  $\mathcal{P}(A) = \inf\{\mathcal{P}(U) : A \subset U \text{ and } U \text{ is open}\}$
2. For each open  $U \subset X$ ,  $\mathcal{P}(U) = \sup\{\mathcal{P}(K) : K \subset U \text{ and } K \text{ is compact}\}$

A topological space  $X$  is *second countable* if there is a countable collection of open sets  $\mathcal{B}$  (called a basis) such that every open set  $U \subset X$  is a union of elements in  $\mathcal{B}$ . If  $X$  is a compact Hausdorff space, then  $X$  is second countable iff  $X$  is metrizable. Finally, if  $X$  is a second countable locally compact Hausdorff space with a Borel probability measure  $\mathcal{P}$ , then  $\mathcal{P}$  is regular [Coh80].

We need the following for a later proof.

LEMMA 2.4. *Let  $V$  be a Lebesgue measurable subset of  $\mathbb{C}$ . Let  $f : V \rightarrow \mathbb{C}$  be a nonconstant analytic map and suppose  $\lambda(A) = 0$  where  $A \subset \mathbb{C}$  is compact. Then*

1.  $f(A)$  is a set of Lebesgue measure zero.
2.  $f^{-1}(A)$  is also a set of Lebesgue measure zero.

PROOF. Let's begin with statement 1. We claim that there are finitely many critical points contained in  $A$ . Let  $\mathcal{C}$  denote the set of critical points in  $A$ . By contradiction

assume  $\mathcal{C}$  is infinite. Since  $A$  is compact we know  $\mathcal{C}$  has a limit point, and by continuity this limit point must also be a critical point. This contradicts the fact that the zeros of the analytic function  $f'$  are isolated. Thus our claim holds, and  $\mathcal{C}$  is a set of measure zero.

Now let  $A_n = A - B(\mathcal{C}, 1/n)$ . We will show that  $\lambda(f(A_n)) = 0$  for each  $n$ . For each  $z'$  in  $A_n$ , we can choose a  $\delta_{z'}$  such that  $f(z) \neq f(z')$  on  $B(z', \delta_{z'})$  and  $B(z', \delta_{z'})$  contains no critical points. Then  $\{B(z', \delta_{z'}) : z' \in A_n\}$  is an open cover of  $A_n$ . So we can choose a finite set  $z_1, z_2, \dots, z_m$  whose corresponding neighborhoods cover  $A_n$ . Let  $A_n^k = B(z_k, \delta_{z_k}) \cap A_n$  where  $k = 1, 2, \dots, m$ . Define  $f_k$  by restricting the domain of  $f$  to  $A_n^k$ . By the inverse function theorem [Gam01], for each  $k = 1, 2, \dots, m$  the functions  $f_k$  are  $C^1$  diffeomorphisms on  $B(z_k, \delta_{z_k})$ . Then

$$\begin{aligned} \lambda(f_k(A_n^k)) &= \int_{A_n^k} |f'(z)|^2 d\lambda \\ &\leq \sup\{|f'(z)|^2 : z \in A_n^k\} \lambda(A_n^k) \\ &= 0 \end{aligned}$$

(see [Coh80] page 171) It follows that  $\lambda(f(A_n)) \leq \lambda(\bigcup_{k=1}^m f(A_n^k)) = 0$ .

Now since  $A_n$  is an increasing sequence of sets,  $\lambda(\bigcup_{n=1}^{\infty} f(A_n)) = \lim \lambda(f(A_n)) = 0$ .

Since  $f(A) - f(\mathcal{C}) = \bigcup_{n=1}^{\infty} f(A_n)$  we have that  $f(A)$  is a set of measure zero.

Statement 2 can be shown from statement 1 using similar techniques.  $\square$

Now we adopt the notation  $\bar{\mathbf{c}}^n$  for a finite sequence of length  $n$ . For our next result, assume  $\{f_c\}_{c \in \mathcal{W}}$  has a sub-invariant domain  $V$  (for example, the family could be a collection of class  $\mathcal{B}$  maps), and note that for a fixed  $z$  the function defined by  $\bar{\mathbf{c}}^n \mapsto f^n(\bar{\mathbf{c}}^n, z)$  is  $\mathcal{A}^n$  measurable for each  $n \in \mathbb{N}$  ( $\mathcal{A}^n = \times_{k=1}^n \mathcal{A}$ ). We adopt the notation  $f^n(\bar{\mathbf{c}}^n)_z$  for this function.

Now, let

$$E(z, V) = \{\bar{\mathbf{c}} \in \mathcal{W}^{\mathbb{N}} : f^n(\bar{\mathbf{c}}, z) \in V \text{ for all sufficiently large } n\}$$

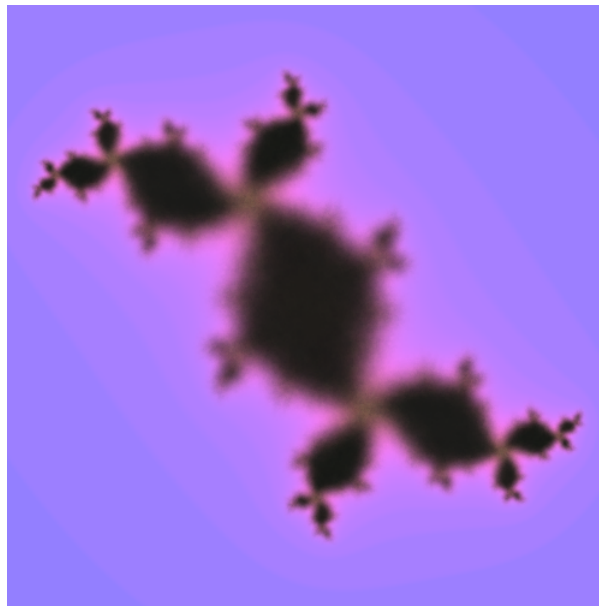


FIGURE 2.5. Probability of  $\bar{c}$ -stability for a non-autonomous iteration of  $\{z^2 + c\}_{c \in \mathcal{W}}$  where  $\mathcal{W} = B(-.1 + .75i, .04)$ .

where  $V$  is a sub-invariant domain. We will need to make sure that this set is measurable. For each  $n \in \mathbb{N}$  define, just as in [FS91],  $E^n(z, V) = \{\bar{c} \in \mathcal{W}^{\mathbb{N}} : f^n(\bar{c}, z) \in V\} = f^{-n}(V)_z \times \mathcal{W}^{\mathbb{N}}$ . Then  $E^n(z, V)$  is measurable. Notice that  $E^n(z, V)$  is a nested increasing sequence of sets, and we justify this as follows. Suppose  $\bar{c} \in E^n(z, V)$ . Then  $f^n(\bar{c}, z) \in V$ . Let  $y = f^n(\bar{c}, z)$ . Since  $V$  is a sub-invariant domain and  $y \in V$  we have that  $f_c(y) \in V$  for all  $c \in \mathcal{W}$ . Thus  $f^{n+1}(\bar{c}, z) \in V$  and we have that  $\bar{c} \in E^{n+1}(z, V)$ .

Now observe that

$$(10) \quad E(z, V) = \bigcup_{n=1}^{\infty} E^n(z, V)$$

So  $E(z, V)$  is measurable. It follows that for each  $z \in \mathbb{C}$ , the probability that  $f^n(\mathbf{C}(\omega), z)$  is contained in  $V$  is defined by  $g(z) := \mathcal{P}(E(z, V))$ .

Suppose  $\chi_E$  is the characteristic function. If we choose a sequence of independent and identically distributed random sequences  $\mathbf{C}_1(\omega), \mathbf{C}_2(\omega), \dots$ , then  $\chi_E(\mathbf{C}_1(\omega)), \chi_E(\mathbf{C}_2(\omega)), \dots$  is a sequence of discrete random variables which are independent and identically distributed. Now the expected value of  $\chi_E$  is equal to  $\mathcal{P}(E(z, V))$ . By the strong law of large numbers, we get the following result.

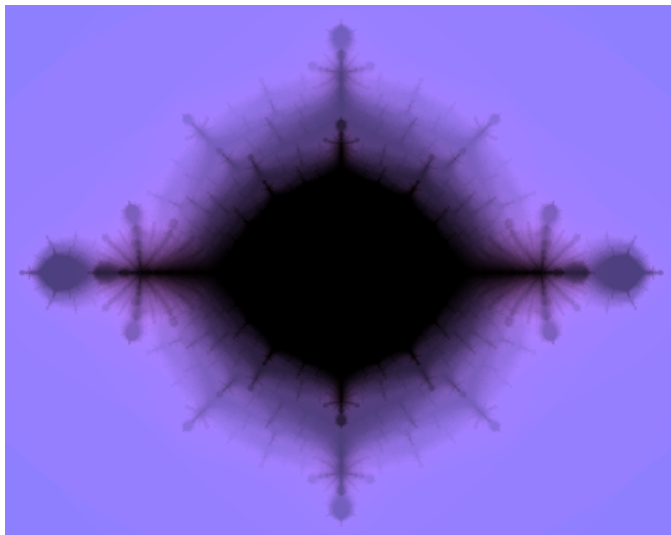


FIGURE 2.6. Probability of  $\bar{c}$ -Stability for random iteration over  $\{f_0, f_1\}$  where  $f_0(z) = z^2$  and  $f_1(z) = z^4 - 2z^2$ .

PROPOSITION 2.5. Let  $S_n(z) = \frac{1}{n} \sum_{k=1}^n \chi_{E(z,V)}(\mathbf{C}_k(\omega))$  where  $z \in \bar{\mathbb{C}}$  is fixed. Then the sequence of averages  $S_n(z)$  converges to  $\mathcal{P}(E(z, V))$  with probability one.

This proposition means that if we average the images of the filled Julia sets of a family of random iterations, we get a pictorial representation of the function  $g(z) := \mathcal{P}(E(z, V))$ .

EXAMPLE 2.5. Let  $\mathcal{W} = B(-.1 + .75i, .04)$ . If we choose 500 random iterations of  $\{z^2 + c\}_{c \in \mathcal{W}}$  and average the images of their corresponding filled Julia sets, we get the image seen in Figure 2.5 where the invariant domain  $V$  is a neighborhood of  $\infty$ . The black represents where  $g(z) = \mathcal{P}(E(z, V))$  is close to zero, and blue represents where  $\mathcal{P}(E(z, V))$  is close to one. We will see in theorem 2.8 that  $g(z)$  is continuous when  $\mathcal{W}$  is contained inside a small neighborhood.

EXAMPLE 2.6. Recall in example 2.1 we let  $f_0(z) = z^2$  and  $f_1(z) = (z^2 - 1)^2 - 1$ , if we choose 500 random iterations of  $\{f_0, f_1\}$  and average the images of the corresponding filled Julia set together we get the image seen in Figure 2.6. The coloring scheme is the same as in the previous example.

### 3. Probability and $\bar{c}$ -Stability

In this section, we look at the properties of the probability functions  $g(z)$  introduced in Example 2.5 when  $\mathcal{W}$  is an open set of finite Lebesgue measure. The motivation behind this endeavor is based on a result by Fornæss and Sibony [FS91] which deals with stability of a random iteration when  $\mathcal{W} = B(c_0, \delta)$ . Given  $c_0 \in \mathbb{C}$  let  $\bar{\lambda}$  denote the normalized Lebesgue measure on  $\mathcal{W} = B(c_0, \delta)$  and  $\lambda$  denote the corresponding product measure on  $\mathcal{W}^{\mathbb{N}}$ .  $f(c)_z$  denotes the function  $c \mapsto f(c, z)$ .

A function  $f(c, z)$  is called *generic* if for every  $z \in \bar{\mathbb{C}}$  the function  $f(c)_z$  is nonconstant. Recall that a family  $\{f_c(z)\}$  is analytic if  $f(c, z)$  is a holomorphic function in two variables.

**THEOREM 2.6.** [FS91] *Let  $f : B(c_0, \delta) \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  be generic where the family  $\{f_c\}_{c \in \mathcal{W}}$  is an analytic family of rational maps of constant degree  $m$ . Suppose that  $f_{c_0}(z)$  has  $d \geq 1$  attracting cycles  $\gamma_1, \gamma_2, \dots, \gamma_d$ . For each  $1 \leq j \leq d$  let  $V_j$  be a neighborhood contained in the basin of attraction of  $\gamma_j$ . Then there exists a  $\delta_0$  s.t. for all  $\delta < \delta_0$  there are continuous functions  $g_1, g_2, \dots, g_d$  defined on  $\bar{\mathbb{C}}$  such that*

1.  $0 \leq g_j(z) \leq 1$  and  $\sum_{j=1}^d g_j(z) = 1$ , and

2. For  $z \in \bar{\mathbb{C}}$  there exists disjoint open sets  $E_j(z) \subset B(c_0, \delta)^{\mathbb{N}}$  such that  $\lambda(E_j(z)) = g_j(z)$ , and if  $\bar{c} \in E_j(z)$ , then for all sufficiently large  $n$ ,  $f^n(\bar{c}, z) \in V_j$ .

Hence for every  $z$ , the sequence  $f_{\bar{c}}^n(z)$  is  $\bar{c}$ -stable for almost every  $\bar{c}$ . The authors state (without proof), the above theorem is valid for more general probability measures other than  $\lambda$ . For the rest of this section, let  $\mathcal{P}^1$  be an absolutely continuous probability measure on the Borel subsets of  $\mathcal{W}$ . We assume that the support of  $\mathcal{P}^1$  is  $\mathcal{W}$ . Let  $\mathcal{P}^n$  be the  $n$ -dimensional product measure on  $(\mathcal{W}^n, \mathcal{B}(\mathcal{W}^n))$ .  $\mathcal{P}$  will be the product measure on  $(\mathcal{W}^{\mathbb{N}}, \mathcal{B}(\mathcal{W}^{\mathbb{N}}))$  as introduced in the previous section.

The requirement that  $f(c, z)$  be generic in Theorem 2.6 is very restrictive. Indeed, if  $f(c, z) = \frac{P(c, z)}{Q(c, z)}$  where  $P, Q$  are polynomials in  $z$  and  $c$ , then it is easy to see that if  $\deg(P_c) \neq \deg(Q_c)$  then  $f(c, z)$  will not be generic since  $f(c, \infty)$  is constant. So we would

like to find a way to remove this restriction, which we can achieve by assuming  $f(c, z)$  has the form given in (2).

DEFINITION 2.2. We call  $z_0$  a *non-generic point* if  $f(c)_{z_0}$  is constant for all  $c \in \mathcal{W}$ , and  $z_0$  is called a *generic point* if  $f(c)_{z_0}$  is nonconstant. The set of all non-generic points is denoted by  $\Omega$ .

If  $z_0$  is non-generic, then  $\frac{\partial f}{\partial c}(c, z_0) = 0$  for all  $c \in \mathcal{W}$ . Conversely, if for some fixed  $z_0$ ,  $\frac{\partial f}{\partial c}(c, z_0) = 0$  for all  $c \in \mathcal{W}$ , then  $f(c)_{z_0}$  is non-generic. So the partial derivative can be used to find the non-generic points of a family  $\{f_c(z)\}_{c \in \mathcal{W}}$ .

Now notice that the image of a non-generic point  $z_0$  (for any  $c \in \mathcal{W}$ ) may be another non-generic point. In fact, it may be the case that the orbit of a non-generic point may consist entirely of non-generic points (i.e.  $f^n(\mathcal{W}^n, z_0)$  consists of one point for all  $n$ ). So a non-generic point can be classified into two categories.

DEFINITION 2.3. Let  $z_0 \in \Omega$ .

1.  $z_0$  is *finitely non-generic* if for some  $n$   $f^n(\mathcal{W}^n, z_0)$  is open and we write  $z_0 \in \Omega_{<\infty}$ .
2.  $z_0$  is *infinitely non-generic* if  $f^n(\mathcal{W}^n, z_0)$  consists of a single point for all  $n$  and we write  $z_0 \in \Omega_\infty$ .

Both of these cases can occur. Consider the following examples. If  $f(c, z) = (z^2 - 4)c + z^2 - 2$ , it is easy to check that  $\Omega_\infty = \{2, -2, \infty\}$  and these are the only non-generic points. Now consider  $g(c, z) = (z^2 - 4)c$ . For this family,  $\Omega_\infty = \{\infty\}$  but  $\Omega_{<\infty} = \{2, -2\}$ .

First, we show that  $\Omega$  is finite.

PROPOSITION 2.7. *Suppose  $f(c, z)$  is rational in both variables and  $\{f_c(z)\}$  is of constant degree for all  $c \in \mathcal{W}$ . The family  $\{f_c(z)\}$  has only finitely many non-generic points.*

PROOF. We can write  $f(c, z)$  in the form

$$(11) \quad \frac{P(c, z)}{Q(c, z)} = \frac{\sum_{i=0}^{m_1} \alpha_i(z) c^i}{\sum_{j=0}^{m_2} \beta_j(z) c^j}$$

where the coefficient functions  $\alpha_i(z), \beta_j(z)$  are polynomials and  $\deg(P(c)_z) = m_1, \deg(Q(c)_z) = m_2$ .

Now fix  $z \in \overline{\mathbb{C}}$ .

Case 1: Suppose  $f(c)_z \equiv 0$ . Then  $P(c)_z \equiv 0$  which implies  $\alpha_i(z) = 0$  for all  $i = 0, 1, \dots, n$ . Since these polynomials have only finitely many zeros in common, we have finitely many non-generic points in this case. The case where  $f(c)_z \equiv \infty$  is similar.

Case 2: Suppose  $f(c)_z \equiv \kappa$  where  $\kappa \neq 0, \infty$ . Then  $f(c)_z$  has no zeros or poles. It follows that  $P(c)_z$  and  $Q(c)_z$  are constant functions. Hence  $\alpha_i(z) = 0$  for  $i = 1, 2, \dots, n$  and  $\beta_j(z) = 0$  for  $j = 1, 2, \dots, m$ . Since these have only finitely many zeros in common, we have shown our result.  $\square$

If  $z \in \Omega_\infty$ , since  $\Omega_\infty$  is finite there exists  $k, m$  such that  $f^k(\mathcal{W}^k, z) = f^m(\mathcal{W}^m, z)$ . So the elements of  $\Omega_\infty$  are pre-periodic or periodic (see definition 1.3), and  $\Omega_\infty$  is a sub-invariant set. We need the following definitions. Let  $V_j$  be an invariant domain containing an attracting cycle  $\gamma_j$  for  $f_{c_0}$ .

DEFINITION 2.4. Let  $c_0 \in \mathcal{W}$ .  $\Omega'_\infty(c_0)$  will denote the infinitely non-generic points not contained in any attracting basin of  $f_{c_0}$ .

DEFINITION 2.5.

$$\begin{aligned}
 E_j^n(z) &= \{\bar{c}^n \in \mathcal{W}^n : f^n(\bar{c}^n, z) \in V_j\} \text{ for } j = 1, 2, \dots, d \\
 E_j(z) &= \bigcup_{n=1}^{\infty} E_j^n(z) \\
 S_0^n(z) &= \{\bar{c}^n : f^n(\bar{c}^n, z) \in \overline{\mathbb{C}} - \overline{V}\} = f^{-n}(\overline{\mathbb{C}} - \overline{V})_z \\
 S^n(z) &= \{\bar{c}^n : f^n(\bar{c}^n, z) \in \overline{\mathbb{C}} - V\} \\
 g_j^n(z) &= \mathcal{P}^n(E_j^n(z)) \\
 h_j^n(z) &= \mathcal{P}^n(S_0^n(z))
 \end{aligned}
 \tag{12}$$

Observe that  $E_j^n(z)$  is a nested increasing sequence. With the knowledge that  $\Omega$  is finite, we can show that for  $z \in \overline{\mathbb{C}} - \Omega_\infty$  the event  $E_j(z)$  occurs with probability one.



**THEOREM 2.8.** *Suppose  $f : \mathcal{W} \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  has the form given in (2). Assume the family of rational maps  $\{f_c\}_{c \in \mathcal{W}}$  is of constant degree greater than 1, and assume  $\mathcal{W}$  is an open set where  $c_0 \in \mathcal{W} \subset B(c_0, \delta)$ , and  $\Omega'_\infty(c_0) = \emptyset$ . Let  $V_1, V_2, \dots, V_d$  be sub-invariant neighborhoods of the attracting cycles  $\gamma_1, \gamma_2, \dots, \gamma_d$  of  $f_{c_0}$  contained inside their respective attracting basins such that  $\partial V_j \cap f(\mathcal{W}, \Omega) = \emptyset$  for all  $j$ . Then there exists  $\delta_0 > 0$  such that for all  $\delta < \delta_0$  there are functions  $g_1, g_2, \dots, g_d$  continuous on  $\overline{\mathbb{C}}$  such that*

1. *The functions  $\{g_j(z)\}_{j=1}^d$  form a partition of unity, and*
2. *For  $z \in \overline{\mathbb{C}}$  there exists disjoint open sets  $E_j(z) \subset \mathcal{W}^{\mathbb{N}}$  such that  $\mathcal{P}(E_j(z)) = g_j(z)$  and for every  $\overline{c} \in E_j(z)$ ,  $f^n(\overline{c}, z) \in V_j$  for all sufficiently large  $n$ .*

The proof of this theorem requires a few lemmas. We continue to use the same notation as in Theorem 2.8.

**LEMMA 2.9.** *Fix  $\delta > 0$ . Assume  $\mathcal{W} \subset B(c_0, \delta)$ , and  $\Omega'_\infty(c_0) = \emptyset$ . Then for every  $z \in \overline{\mathbb{C}}$  there exists  $\overline{c} \in \mathcal{W}^{\mathbb{N}}$  such that  $f^n(\overline{c}, z) \in V = \bigcup_{i=1}^d V_i$  for large enough  $n$ .*

**PROOF.** We know that  $z \in \mathcal{J}_{c_0} \cup \mathcal{F}_{c_0}$ . If  $z \in \mathcal{J}_{c_0}$  then  $z$  is contained in the boundary of some attracting basin for  $f_{c_0}$  [Bea91]. If  $z$  is generic, then  $f(c)_z$  is an open map. Thus there exists a  $c_1 \in \mathcal{W}$  such that  $f(c_1, z)$  is in an attracting basin, and then for some  $n$ ,  $f^n(\overline{c}^n, z) \in V$  where  $\overline{c}^n = (c_1, c_0, \dots, c_0)$ . Now if  $z_0 \in \Omega_{<\infty}$  then there exists an  $n$  such that  $f^n(\overline{c}^n, z)$  is generic and then we use the previous argument. In the case where  $z \in \Omega_\infty$  then  $z_0$  is a preperiodic point of an attracting cycle, so the constant sequence  $\overline{c} = (c_0, c_0, \dots)$  will suffice. The rest of the proof is similar to the proof in [FS91].  $\square$

Let  $D \subset \overline{\mathbb{C}}$  be a domain. Recall that a function  $f : D \rightarrow \mathbb{R}$  is *lower semicontinuous* if for all  $z \in D$  and real numbers  $A < f(z)$  there exists a  $\delta > 0$  such that if  $\sigma(z', z) < \delta$  then  $A < f(z')$ .  $f$  is *upper semicontinuous* if  $-f$  is lower semicontinuous. It is well known that the sum of two lower(upper) semicontinuous functions is also lower(upper) semicontinuous. A function is continuous iff it is both upper and lower semicontinuous.

**LEMMA 2.10.** *Let  $\mathcal{P}^n$  be a Borel probability measure on  $\mathcal{W}^n$  where  $\mathcal{W}$  is open and bounded. For each  $n$  and  $j$ , the functions  $g_j^n(z)$ , and  $h_j^n(z)$  are lower semicontinuous on  $\overline{\mathbb{C}}$ .*

PROOF. Let  $z \in \overline{\mathbb{C}}$  and  $A < g_j^n(z)$ . For  $z, z' \in \overline{\mathbb{C}}$  and  $\overline{\mathbf{c}}^n, \overline{\mathbf{d}}^n \in \mathcal{W}^n$  let  $\sigma_\infty[(\overline{\mathbf{c}}^n, z), (\overline{\mathbf{d}}^n, z')] = \max\{\sigma(z, z'), |c_1 - d_1|, \dots, |c_n - d_n|\}$ . Then  $\sigma_\infty$  is a metric on  $\mathcal{W}^n \times \overline{\mathbb{C}}$ .

We claim that for any open set  $U \subset \mathcal{W}^n$  there exists a nested increasing sequence of compact sets  $K_m \subset \text{int}(K_{m+1})$  such that  $\bigcup_{m=1}^\infty K_m = U$ . Indeed, if we let  $K_m = \{\overline{\mathbf{c}}^n \in \mathcal{W}^n : d(\overline{\mathbf{c}}^n, \mathcal{W}^n - U) \geq \frac{1}{m}\}$  then the sequence of sets  $K_m$  does have that property [Con78]. Since  $\mathcal{P}^n$  is a Borel probability measure and  $\mathcal{W}^n$  is locally compact Hausdorff,  $\mathcal{P}^n$  is regular. Now  $E_j^n(z) = f^{-n}(V_j)_z$  is a nonempty open set by the continuity of  $f^n(\overline{\mathbf{c}}^n)_z$ . From our earlier claim with  $U = E_j^n(z)$ , there exists a sufficiently large  $m$  such that  $A < \mathcal{P}^n(K_m) \leq \mathcal{P}^n(E_j^n(z)) = g_j^n(z)$ .

Choose  $\epsilon > 0$  so that  $B_\sigma(f^n(K_m, z), \epsilon) \subset V_j$ . Now,  $f^n$  is uniformly continuous on  $\overline{\mathcal{W}^n} \times \overline{\mathbb{C}}$ . So choose  $\delta > 0$  such that if  $(\overline{\mathbf{c}}^n, z), (\overline{\mathbf{d}}^n, z') \in \mathcal{W}^n \times \overline{\mathbb{C}}$  and  $\sigma_\infty[(\overline{\mathbf{c}}^n, z), (\overline{\mathbf{d}}^n, z')] < \delta$ , then  $\sigma(f^n(\overline{\mathbf{c}}^n, z), f^n(\overline{\mathbf{d}}^n, z')) < \epsilon$ . Now assume  $\sigma(z', z) < \delta$ . We will show  $K_m \subset E_j^n(z')$ .

Let  $\overline{\mathbf{c}}^n \in K_m$ . Then  $\sigma_\infty[(\overline{\mathbf{c}}^n, z), (\overline{\mathbf{c}}^n, z')] < \delta$ . Thus  $\sigma(f^n(\overline{\mathbf{c}}^n, z), f^n(\overline{\mathbf{c}}^n, z')) < \epsilon$ , that is  $f^n(\overline{\mathbf{c}}^n, z') \in B_\sigma(f^n(K_m, z), \epsilon) \subset V_j$ . Therefore  $\overline{\mathbf{c}}^n \in E_j^n(z')$ , that is  $K_m \subset E_j^n(z')$ . We now have  $A < \mathcal{P}^n(K_m) \leq \mathcal{P}^n(E_j^n(z')) = g_j^n(z')$ .

The proof that  $h_j^n(z)$  is lower semicontinuous is similar.  $\square$

We are now ready to prove Theorem 2.8. It is similar to the proof of Theorem 2.6 found in [FS91]. We include it here with more details for the sake of completeness.

*Proof of Theorem 2.8.* From Theorem 2.2 we can choose  $\delta_0$  such that there are open sub-invariant sets  $V_j$  which are neighborhoods of attracting cycles  $\gamma_j$ . Furthermore, suppose  $f(\mathcal{W}, \Omega) \cap \partial V = \emptyset$ . For our first step, we show that for each  $j = 1, 2, \dots, d$ ,  $g_j^n(z)$  is continuous on  $\overline{\mathbb{C}}$  for every  $n$ . We then show that  $g_j^n$  converges uniformly to complete the proof.

We claim that for fixed  $\overline{\mathbf{c}}^{n-1}$ ,  $\mathcal{P}^1\{c : f^n((\overline{\mathbf{c}}^{n-1}, c), z) \in \partial V\} = 0$ . To prove this, notice that  $\partial V$  is a set of Lebesgue measure zero since it is a finite union of neighborhoods of the attracting cycles. So by Lemma 2.4,  $\{c : f^n((\overline{\mathbf{c}}^{n-1}, c), z) \in \partial V\}$  is a set of Lebesgue measure zero in  $\mathcal{W}$ . Thus the claim holds by the absolute continuity of  $\mathcal{P}^1$ .

As a result of our last claim,

$$\mathcal{P}^n(S^n(z) - S_0^n(z)) = \int_{\overline{\mathbf{c}}^{n-1} \in \mathcal{W}^{n-1}} \mathcal{P}^1\{c : f^n((\overline{\mathbf{c}}^{n-1}, c), z) \in \partial V\} d\mathcal{P}^{n-1} = 0$$

which implies  $\mathcal{P}^n(S_0^n(z)) = \mathcal{P}^n(S^n(z))$ .

Since  $E_j^n(z) \times \mathcal{W} \subset E_j^{n+1}(z)$ , we have  $g_j^n(z) \leq g_j^{n+1}(z)$ . Also, in Lemma 2.10, it was shown that  $g_j^n(z)$  and  $\mathcal{P}^n(S_0^n(z)) = \mathcal{P}^n(S^n(z))$  are lower semicontinuous. Now  $g_j^n(z) = 1 - \sum_{i \neq j} g_i^n(z) - \mathcal{P}^n(S^n(z))$  which implies  $g_j^n(z)$  is upper semicontinuous. So  $g_j^n(z)$  is continuous on  $\overline{\mathbb{C}}$ .

For each  $z \in \overline{\mathbb{C}}$  choose a sequence  $\overline{\mathbf{c}}_z$  such that  $f^n(\overline{\mathbf{c}}_z, z) \in V$  for all sufficiently large  $n$  (Lemma 2.9). Then for each  $z$  we can choose a  $\delta_z$  such that  $f^n(\overline{\mathbf{c}}_z, B(z, \delta_z)) \subset V$ . So  $\{B(z, \delta_z)\}$  forms an open cover of  $\overline{\mathbb{C}}$ . Thus we can choose a finite set  $\{z_1, z_2, \dots, z_k\}$  such that  $f^n(\overline{\mathbf{c}}_{z_i}, B(z_i, \delta_i)) \subset V$ . Thus there exists an  $N$  such that for all  $n \geq N$ ,  $f^n(\overline{\mathbf{c}}, z) \in V$  for some  $\overline{\mathbf{c}}$ . It follows that,  $\bigcup_{j=1}^d E_j^n(z)$  is a nonempty open set with positive measure.

For  $n \geq N$  there exists an  $M \in (0, 1)$  such that  $\sum g_j^n(z) > M$  and  $\mathcal{P}^n(S^n(z)) = 1 - \sum g_j^n(z) \leq 1 - M$  for all  $z \in \overline{\mathbb{C}}$ . We then have for all  $n \in \mathbb{N}$ ,

$$\mathcal{P}^{n+N}(S^{n+N}(z)) = \int_{\overline{\mathbf{c}}^n \in S^n(z)} \mathcal{P}^N(S^N(f^n(\overline{\mathbf{c}}^n, z))) d\mathcal{P}^n \leq (1 - M)\mathcal{P}^n(S^n(z)).$$

We can write  $n = qN + r$  where  $0 \leq r < N$ . By induction on  $q$  we get

$$\mathcal{P}^{qN+r}(S^{qN+r}(z)) \leq (1 - M)^q \mathcal{P}^r(S^r(z)).$$

Let  $C = \max\{\mathcal{P}^r(S^r(z)) : 1 \leq r < N, z \in \overline{\mathbb{C}}\}$ . Then

$$\begin{aligned} \mathcal{P}^{n+N}(S^{n+N}(z)) &= \mathcal{P}^{(q+1)N+r}(S^{(q+1)N+r}(z)) \\ &\leq (1 - M)^{q+1} \mathcal{P}^r(S^r(z)) \\ &\leq C(1 - M)^{n/N}. \end{aligned}$$

Thus  $\mathcal{P}^n(S^n(z)) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly. Now assume  $n, m \geq N$  and without loss of generality assume  $m \geq n$ . We have

$$\begin{aligned} g_j^m(z) - g_j^n(z) &\leq g_j^{n+m}(z) - g_j^n(z) \\ &= g_j^n(z) + \int_{\overline{\mathbf{c}}^n \in S^n(z)} \mathcal{P}^m(E_j^m(f^n(\overline{\mathbf{c}}^n, z))) d\mathcal{P}^n - g_j^n(z) \\ &\leq \mathcal{P}^n(S^n(z)). \end{aligned}$$

So  $g_j^n(z)$  converges uniformly to some continuous function  $g_j(z)$ .

□

From Theorem 2.8 we still get the following corollary just as the authors did in [FS91].

**COROLLARY 2.11.** *Let  $f$  be a rational map with the same properties as stated in Theorem 2.8. Then there is a set  $\mathcal{E} \subset \mathcal{W}^{\mathbb{N}}$  of full measure such that if  $\bar{c} \in \mathcal{E}$ ,  $f^n(\bar{c}, z) \in V = \bigcup_{j=1}^d V_j$  for almost every  $z \in \bar{\mathbb{C}}$  and all sufficiently large  $n$ . In particular,  $\mathcal{J}(\bar{c})$  is of Lebesgue measure zero with probability one.*

**PROOF.** Proof is the same as in [FS91].

□

#### 4. Random Iteration and Semigroups

The theory of random iteration is closely related to the dynamics of semigroups of rational maps. The dynamics of semigroups and their relation to random iteration has been extensively studied by Hiroki Sumi. In particular, Sumi discovered several generalizations of Theorem 2.6 in [Sum13] using what is called the kernel Julia set ( $\mathcal{J}_{ker}$ ). We state one of these results and show its connection with infinitely non-generic points (see definition 2.2).

A *rational semigroup*  $\mathcal{G}$  is a semigroup generated by a set of rational functions  $S$  where the operation on this set is function composition. This set is denoted by  $\langle S \rangle$ . The *Fatou set* of the rational semigroup  $\mathcal{F}(\mathcal{G}) \subset \bar{\mathbb{C}}$  is the set of all points where the rational semigroup forms a normal family. The *Julia set* is  $\mathcal{J}(\mathcal{G}) := \bar{\mathbb{C}} - \mathcal{F}(\mathcal{G})$ . Notice by Montel's theorem if  $U$  is a sub-invariant domain (with respect to  $S$ ), then  $U \cap \mathcal{J}(\mathcal{G}) = \emptyset$ , where  $\langle S \rangle = \mathcal{G}$ .

Rational semigroups were first studied by Hinkkanen and Martin. They have shown that several ideas from classical iteration theory carry over to dynamics of semigroups [HM96]. In particular, it is known that  $\mathcal{J}(\mathcal{G}) = \overline{\bigcup_{g \in \mathcal{G}} \mathcal{J}(g)}$ , and  $g^{-1}(\mathcal{J}(\mathcal{G})) \subset \mathcal{J}(\mathcal{G})$  for all  $g \in \mathcal{G}$ .

We need a few definitions.

**DEFINITION 2.6.** Let  $K$  be a sub-invariant compact subset and  $\mathcal{G}$  be a semigroup. We say that  $K$  is a *minimal set* if

1.  $K$  is a sub-invariant compact set with respect to  $\mathcal{G}$ , and
2. If  $K' \subset K$  is another sub-invariant compact set, then  $K' = K$ .

Thus  $K$  is a smallest compact sub-invariant subset with respect to the semigroup  $\mathcal{G}$ .

DEFINITION 2.7. **[Sum13]** If  $\mathcal{G}$  is a rational semigroup, we define the *kernel Julia set* by

$$\mathcal{J}_{ker}(\mathcal{G}) := \bigcap_{g \in \mathcal{G}} g^{-1}(\mathcal{J}(\mathcal{G}))$$

When the semigroup is clear from context, we may simply write  $\mathcal{J}_{ker}$ . It is easy to check that  $\mathcal{J}_{ker}(\mathcal{G})$  is a sub-invariant set which is compact in  $\overline{\mathbb{C}}$ , and  $\mathcal{J}_{ker}(\mathcal{G}) \subset \mathcal{J}(\mathcal{G})$ .

Now let  $\mathcal{P}$  be a Borel probability measure over the set of rational functions on  $\overline{\mathbb{C}}$ . We endow the set of rational functions on  $\overline{\mathbb{C}}$  with the topology induced by the metric  $d(f, g) = \sup\{\sigma(f(z), g(z)) : z \in \overline{\mathbb{C}}\}$ . Let  $\Gamma_{\mathcal{P}}$  be the support of  $\mathcal{P}$ . Suppose that  $\mathbf{G} = (g_1, g_2, \dots) \in \Gamma_{\mathcal{P}}^{\mathbb{N}}$ . Let  $G_n(z) = g_n \circ g_{n-1} \circ \dots \circ g_1(z)$  and  $\bar{\mathcal{P}} = \times_{i=1}^{\infty} \mathcal{P}$ .  $\mathcal{J}(G_n)$  denotes the Julia set of the random iteration  $G_n(z)$ . Then we have the following results due to Sumi.

THEOREM 2.12. **[Sum13]** (*Cooperation Principle II*) Let  $\mathcal{P}$  be a Borel probability measure over the rational functions on  $\overline{\mathbb{C}}$  where  $\Gamma_{\mathcal{P}}$  is compact and let  $S$  be the union of all compact sets that are minimal with respect to the group  $\langle \Gamma_{\mathcal{P}} \rangle$ . If  $\mathcal{J}_{ker}\langle \Gamma_{\mathcal{P}} \rangle = \emptyset$  and  $\mathcal{J}\langle \Gamma_{\mathcal{P}} \rangle \neq \emptyset$  then

1.  $\mathcal{J}(G_n)$  is a set of Lebesgue measure zero with probability one
2. For  $z \in \overline{\mathbb{C}}$  there exists a borel subset  $U_z$  where  $\bar{\mathcal{P}}(U_z) = 1$  such that for every  $G \in U_z$  we have  $d(G_n(z), S) \rightarrow 0$  as  $n \rightarrow \infty$ .

We make the connection between  $\mathcal{J}_{ker}(\mathcal{G})$  and infinitely non-generic points (see Definitions 2.3, 2.4). We define  $\text{Exc}(g) = \{z : g^{-n}(z) \text{ is finite}\}$  which is commonly referred to as the *exceptional set*. It is known that the exceptional set contains at most 2 elements.

THEOREM 2.13. Let  $\{f_c\}_{c \in \mathcal{W}}$  be a holomorphic family of rational maps. Then either

1.  $\mathcal{J}_{ker}\langle f_c \rangle_{c \in \mathcal{W}} = \overline{\mathbb{C}}$  which implies that  $\mathcal{J}\langle f_c \rangle_{c \in \mathcal{W}} = \overline{\mathbb{C}}$  or
2.  $\mathcal{J}_{ker}\langle f_c \rangle_{c \in \mathcal{W}} \subset \Omega_{\infty}$

PROOF. Let  $z \in \mathcal{J}_{ker}$ . Suppose  $z$  is a generic point. Then  $f(\mathcal{W})_z \subset \mathcal{J}(\mathcal{G})$  is an open set, and  $f(\mathcal{W})_z \cap \mathcal{J}(g) \neq \emptyset$  for some  $g \in \langle f_c \rangle_{c \in \mathcal{W}}$ . Thus  $\overline{\mathbb{C}} - \text{Exc}(g) = \bigcup_{n=1}^{\infty} g^n(f(\mathcal{W})_z)$  [Bea91] and  $\bigcup_{n=1}^{\infty} g^n(f(\mathcal{W})_z) \subset \mathcal{J}_{ker}$ . Since  $\mathcal{J}_{ker}$  is compact, we must have that  $\mathcal{J}_{ker} = \overline{\mathbb{C}}$  which gives us statement 1.

The other possibility is  $z$  and in fact every element of  $\mathcal{J}_{ker}$  is non-generic. Now if  $z \in \mathcal{J}_{ker}$  is finitely non-generic, then we can choose an  $n$  such that  $f^n(\mathcal{W}^n, z)$  is a generic point. But since  $\mathcal{J}_{ker}$  is forward sub-invariant, we have  $f^n(\mathcal{W}^n, z) \in \mathcal{J}_{ker}$  which is a contradiction. Thus  $z$  must be infinitely non-generic.  $\square$

## CHAPTER 3

**Seed Iteration****1. Introduction to Seed Iteration**

In classical iteration theory, we study the behavior of the sequence

$$z_{n+1} = f(z_n)$$

where  $f(z)$  is a rational function. The sequence  $\{z_0, z_1, \dots\} = \{f^n(z_0)\}_{n \in \mathbb{N}}$  is called the *orbit of  $z_0$*  and is denoted by  $O^+(z_0)$ . Notice that if  $z \in O^+(z_0)$  then  $O^+(z) \subset O^+(z_0)$ . In this section, we let  $f(w, z)$  be a polynomial function and study the behavior of the sequence

$$(13) \quad z_{n+1} = f(z_0, z_n).$$

For this type of iteration, it is not necessarily true that  $O^+(z) \subset O^+(z_0)$  for  $z \in O^+(z_0)$ . We define the function  $f(w, z)$  more precisely below.

In this chapter, we assume  $f$  is a polynomial function in  $z$  where  $\alpha_i$  are polynomial functions. So  $f$  has the form given in (1) where  $c$  is replaced by  $w$ . The formula is provided again here for easy reference.

$$(1) \quad f(w, z) = \sum_{j=0}^m \alpha_j(w) z^j$$

We call the  $\alpha_i$  the *coefficient functions*, and at least one of them is always be nonconstant. Of course we assume  $\alpha_m \not\equiv 0$  and the *degree*  $m \geq 1$ .

Now define  $f^n : \mathbb{C}^2 \rightarrow \mathbb{C}$  by  $f^n(w, z) = f(w, f(w, \dots f(w, f(w, z)) \dots))$  ( $n$  times) . So the expression  $f(w, z)$  is composed in the  $z$  variable  $n$  times to get a function  $f^n(w, z)$ . Then we let  $z = w$  to get another function denoted by  $F_n(w) := f^n(w, w)$ . Thus the variables  $w$  in (1) remain unchanged in the composition process. For this reason,  $w$  is

called the *seed* variable, and we refer to this family of compositions as *seed iteration*. We need to ensure that the the highest degree terms in (1) do not vanish when we replace  $z$  with  $w$ . Hence we require that the degree of the term  $\alpha_m(w)w^m$  is strictly greater than the degrees of  $\alpha_j(w)w^j, j = 0, 1, \dots, m - 1$ . (the author thanks Dr. Hans Volkmer for drawing our attetion to this problem in an earlier draft of this paper)

Seed iteration is a concpet of the author's own invention. We will be studying the family of functions  $\{F_n(w)\}_{n \in \mathbb{N}}$ . The notation in this chapter becomes very complex, so we may use the following conventions:

- $fg = f \circ g$
- $fA = f(A)$  where  $A \subset \mathbb{C}$
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $\mathcal{J}_w = \{z : \text{no subsequence of } f_w^n(z) \text{ converges locally uniformly at } z\}$

Thus  $\mathcal{J}_w$  denotes the Julia set from classical iteration theory. We still define the Fatou and Julia set in the same way, with slight notational modifications:

$$\mathcal{F}(F_n) = \{w : F_n(w) \text{ has a subsequence that converges locally uniformly at } w\}$$

$$\mathcal{J}(F_n) = \mathbb{C} - \mathcal{F}(F_n)$$

$$\mathcal{K}(F_n) = \{w : F_n(w) \text{ is bounded } \}$$

Observe that we are working in the metric space  $(\mathbb{C}, | \cdot |)$ , so we define  $\infty \in \mathcal{F}(F_n)$  if the family of maps diverges locally uniformly as defined in [Mil06]. Even though  $w$  is not a parameter in the traditional sense, it is still useful to use what we know about the family  $\{f_w\}$  when  $w$  is a parameter. Thus we still make use of the notation  $f_w(z)$  for the function  $w \mapsto f(w, z)$ .

EXAMPLE 3.1. Let  $f(w, z) = z^2 + w$  and note that the Mandelbrot set  $\mathcal{M}$  is the filled Julia set  $\mathcal{K}(F_n)$ . In this case  $\mathcal{J}(F_n) = \partial\mathcal{M}$  (this fact is a corollary of Theorem 4.2 in [McM94]).

EXAMPLE 3.2. Now let  $f(w, z) = z^2 + w - 1$ . Then  $F_2(w) = (w^2 + w - 1)^2 + w - 1$ . The filled Julia set for this sequence of compositions is seen in Figure 3.1.



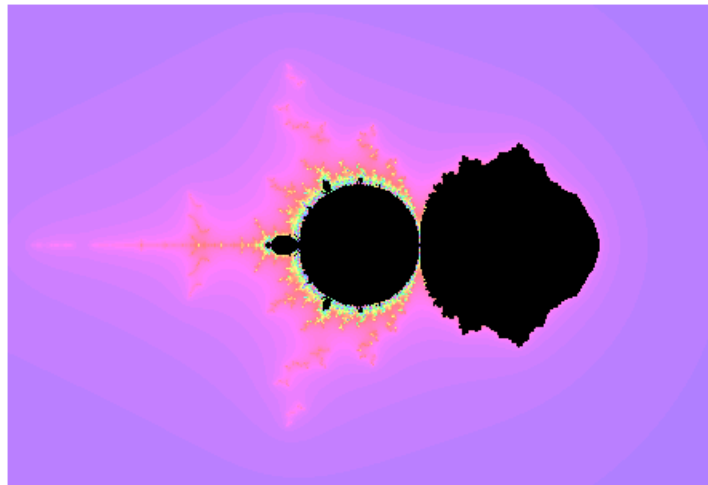


FIGURE 3.1. Filled Julia set for  $F_n(z)$  where  $f(w, z) = z^2 + w - 1$ .

## 2. Properties of Seed Iteration

The first question we answer is whether or not  $\mathcal{J}$  is nonempty for this type of iteration, and we will see that this depends on the chosen coefficient functions. Recall that Hurwitz's Theorem says the following.

**THEOREM 3.1.** *Let  $\{f_n(z)\}$  be a sequence of analytic functions which converges uniformly to  $f(z)$  on  $U$  and suppose  $f(z)$  has a zero of multiplicity  $N$  at  $z_0$ . Then there exists a  $\delta > 0$  such that  $f_n(z)$  has  $N$  zeros counting multiplicity in  $B(z_0, \delta)$ . [Gam01]*

We will use this in the next theorem. Recall that  $\deg(P \circ Q) = \deg(P) \deg(Q)$

**PROPOSITION 3.2.** *Suppose  $f(w, z)$  has the form given in (1) where  $m \geq 2$  and the coefficient functions  $\alpha_i$  are all polynomials. Then  $\mathcal{J}(F_n) \neq \emptyset$ .*

**PROOF.** From equation (1) letting  $z = F_n(w)$  we know that the degree of each term is  $\deg(\alpha_j) + j \deg(F_n)$  (we define  $\deg(F_0) := 1$ ). Thus

$$\deg(F_{n+1}) = \max_{j=0,1,\dots,m} \{\deg(\alpha_j) + j \deg(F_n)\} = \deg(\alpha_m) + m \deg(F_n).$$

Observe that

$$\deg(\alpha_m) + m > \deg(\alpha_j) + j \text{ for all } j = 0, 1, \dots, m-1 \implies$$

$$\deg(\alpha_m) - \deg(\alpha_j) > (j - m) > (j - m) \deg(F_n) \implies$$

$$\deg(\alpha_m) - \deg(\alpha_j) > (j - m) \deg(F_n) \implies$$

$$\deg(\alpha_m) + m \deg(F_n) > \deg(\alpha_j) + j \deg(F_n)$$

for all  $j$ . Hence  $\deg(F_{n+1}) = \deg(\alpha_m) + m \deg(F_n)$  which is a strictly increasing sequence (since  $\deg(F_n) > 1$  for all  $n$ ).

By contradiction suppose some subsequence  $G_k$  of  $F_n$  converges locally uniformly on  $\overline{\mathbb{C}}$  to  $G(w)$ . For each zero  $\{w_i\}_{i=1}^d$  of  $G(w)$  whose corresponding multiplicities are  $\{M_i\}_{i=1}^d$  we can choose a  $\delta_i$  such that  $B(w_i, \delta_i)$  contains  $M_i$  zeros of  $G_k$  for all large enough  $k$ . Hence the degree of  $G_k$  is constant for all large enough  $k$  which is a contradiction as  $\deg(G_k)$  is a strictly increasing sequence.  $\square$

Now at the beginning of this section, we defined the Filled Julia set  $\mathcal{K}(F_n)$ . So naturally, we would like to know if  $\infty \in \mathcal{F}(F_n)$ . In [Büg97] Büger showed that  $\infty \in \mathcal{F}(f_{\overline{c}}^n)$  where  $f_{\overline{c}}^n$  is a non-autonomous iteration of class  $\mathcal{B}$  functions (see Section 1 for definition). We use techniques similar to his proof to show the same thing for seed iteration.

**THEOREM 3.3.** *If  $f(w, z)$  has the form given in (1) with polynomial coefficient functions and  $\min\{\deg(f(w)_z), \deg(f_w(z))\} \geq 2$ , then  $\infty \in \mathcal{F}(F_n)$ .*

**PROOF.** Since  $f(w, z)$  is a polynomial function in  $z$  and  $w$  with  $\deg(f_w(z)) \geq 2$ , we can choose an  $R > 0$  such that  $\frac{|f(w, z)|}{|z|} \geq 2$  for  $|z| > R$  and  $|w| > R$ . We now show that  $|f^n(w, z)| > 2^n |z|$  if  $|z|, |w| > R$ .

We already have that the formula holds for  $n = 1$ . Now suppose  $|f^{n-1}(w, z)| > 2^{n-1}|z|$  if  $|z|, |w| > R$ . Then  $|f^n(w, z)| = |f(w, f^{n-1}(w, z))| > 2|f^{n-1}(w, z)| > 2^n|z|$ . Thus the claim holds.

Now setting  $z = w$  we get  $|F_n(w)| > 2^n|w|$ . It follows that the family  $\{F_n\}$  omits 3 points at infinity. Therefore, by Montel's Theorem  $\infty \in \mathcal{F}(F^n)$ .  $\square$

### 3. Affine Properties

In general iteration theory, it is known that  $\mathcal{F}(f^n h) = h\mathcal{F}(f^n)$  [Bea91] (in fact this holds when  $f, h$  are rational functions). A similar result holds for seed iteration. We prove this and then show an example. Observe that

$$\begin{aligned} f^n(w, h(z)) &= f(w, f(w, \dots f(w, f(w, h(z)))) \dots) \text{ and} \\ f^n(h(w), z) &= f(h(w), f(h(w), \dots f(h(w), f(h(w), z)))) \dots) \end{aligned}$$

**PROPOSITION 3.4.** *Suppose  $h(z)$  is a nonconstant affine transformation,  $f(w, z)$  has the form given in (1). Then  $h\mathcal{F}[f^n(h(w), w)] = \mathcal{F}[f^n(w, h^{-1}(w))]$  and  $h\mathcal{J}[f^n(h(w), w)] = \mathcal{J}[f^n(w, h^{-1}(w))]$ .*

**PROOF.** Let  $w_0 \in h\mathcal{F}[f^n(w, h(w))]$ . Then there exists  $z_0 \in \mathcal{F}[f^n(w, h(w))]$  such that  $h(z_0) = w_0$ . Choose a neighborhood  $V$  of  $z_0$  such that  $f^n(z, h(z))$  has a subsequence which converges uniformly on  $V$ . Since  $h(z)$  is analytic and nonconstant,  $h(V)$  is a neighborhood of  $w_0$ .

We claim that  $f^n(h^{-1}(w), w)$  has a subsequence which converges uniformly on  $h(V)$ . First let  $\epsilon > 0$  and choose a subsequence  $f^{n_k}(w, h(w))$  and an  $N$  such that for all  $k, l \geq N$ ,  $|f^{n_k}(w, h(w)) - f^{n_l}(w, h(w))| < \epsilon$  for every  $w \in V$ . Now let  $w \in h(V)$  and let  $z = h^{-1}(w)$ . Then  $z \in V$  so if  $k, l \geq N$ ,  $|f^{n_k}(h^{-1}(w), w) - f^{n_l}(h^{-1}(w), w)| = |f^{n_k}(z, h(z)) - f^{n_l}(z, h(z))| < \epsilon$ .

We have  $h\mathcal{F}[f^n(w, h(w))] \subset \mathcal{F}[f^n(h^{-1}(w), w)]$ . Showing containment in the other direction is similar. Taking complements of both sides of  $h\mathcal{F}[f^n(w, h(w))] = \mathcal{F}[f^n(h^{-1}(w), w)]$  produces the second statement of the conclusion (note  $h$  is one-to-one so  $h(\mathbb{C} - A) = \mathbb{C} - h(A)$ ).  $\square$

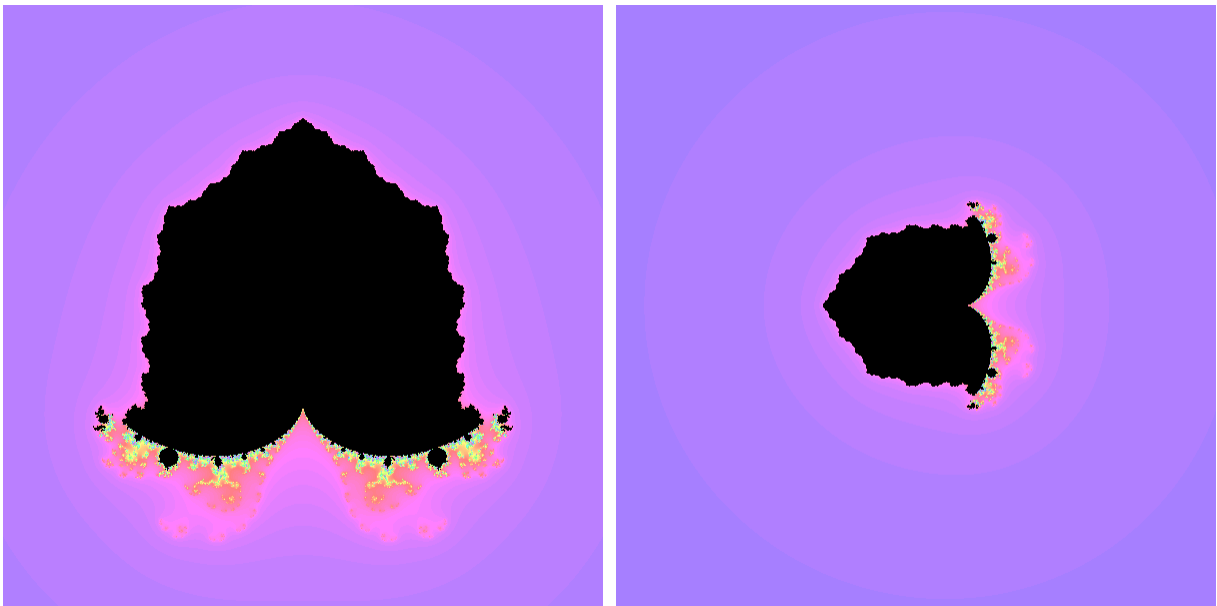


FIGURE 3.2.  $\mathcal{K}[f^n(\frac{i}{2}w, w)]$  (left) and  $\mathcal{K}[f^n(w, -2iw)]$  (right) where  $f(w, z) = z^2 + w$ .

EXAMPLE 3.3. Let  $f(w, z) = z^2 + w$ . Then from Proposition 3.4,  $\mathcal{J}[f^n(w, -2iw)]$  is a rotation by  $\pi/2$  followed by a dilation of  $1/2$  of the set  $\mathcal{J}[f^n(\frac{i}{2}w, w)]$ . Pictures of the filled Julia Set of these sequences are shown in Figure 3.2 which motivates this theorem. It is important to note that we have yet to show that  $\partial\mathcal{K} = \mathcal{J}$  for seed iteration.

#### 4. Holomorphic Motions and Stability

Our next theorems looks at the structure of  $\mathcal{J}(F_n)$ , and we will use the properties of the classical Julia sets  $\mathcal{J}_w$  to do so. In particular, we examine when the Julia set  $\mathcal{J}(F_n)$  has empty interior.

We need the concepts of a holomorphic motion and stability found in [McM94].

DEFINITION 3.1. Let  $\mathcal{W} \subset \mathbb{C}$  be connected and choose a basepoint  $x \in \mathcal{W}$ . A *holomorphic motion* of a set  $E \subset \overline{\mathbb{C}}$  is a family of injections  $\{\phi_w\}_{w \in \mathcal{W}}$  which map  $E$  into  $\overline{\mathbb{C}}$  such that

1. For a fixed  $z \in E$ ,  $\phi(w, z) = \phi(w)_z$  is a holomorphic function of  $w$ , and
2.  $\phi_x(z)$  is the identity function.

The map  $\phi_w(z)$  can be extended to a quasiconformal map on the Riemann sphere; hence,  $\phi_w$  is a homeomorphism.

Given an analytic family of rational functions  $\{f_w\}_{w \in \mathcal{W}}$  we say that the Julia sets *move holomorphically* if there exists a holomorphic motion  $\phi : \mathcal{W} \times \mathcal{J}_x \rightarrow \overline{\mathbb{C}}$  such that  $\phi_w(\mathcal{J}_x) = \mathcal{J}_w$  and  $\phi_w \circ f_x(z) = f_w \circ \phi_w(z)$ . Define the *stable set* by  $\mathcal{S} = \{w : \mathcal{J}_w \text{ moves holomorphically}\}$ . It is known that for the family  $z^2 + w$ ,  $\mathcal{S} = \mathbb{C} - \partial\mathcal{M}$  [McM94].

**THEOREM 3.5.** *Let  $f(w, z)$  be a polynomial function and suppose  $\{w : w \in \mathcal{J}_w\} \cup \{w : f_w(z) \text{ has a neutral periodic cycle}\}$  has empty interior. Then  $\mathcal{J}(F_n) \subset \{w : w \in \mathcal{J}_w\} \cup \{w : f_w(z) \text{ has a neutral periodic cycle}\}$  and thus  $\mathcal{J}(F_n)$  has empty interior.*

**PROOF.** Let  $U$  be an open set disjoint from  $\{w : w \in \mathcal{J}_w\} \cup \{w : f_w(z) \text{ has a neutral periodic cycle}\}$ . Now let  $w_0 \in U$ . Then the subsequential limits of  $F_n(w_0) = f_{w_0}^n(w_0)$  converge towards some attracting cycle  $\gamma$ . Choose  $\delta_1$  such that  $\{f_w\}_{w \in B(w_0, \delta_1)}$  has a sub-invariant domain  $V$  containing  $\gamma$  and  $B(w_0, \delta_1) \subset U$ . Now choose  $N$  such that  $F_N(w_0) \in V$ . Since  $F_N(w)$  is continuous, we can choose a  $\delta_2$  such that  $F_N(B(w_0, \delta_2)) \subset V$ . Since  $V$  is a sub-invariant domain,  $f_w^n(F_N(w)) = F_{n+N}(w) \in V$  for all  $n \in \mathbb{N}$  and  $w \in B(w_0, \delta_2)$ . Now let  $\delta = \min\{\delta_1, \delta_2\}$ . Then  $F_n(B(w_0, \delta)) \subset V$  for all  $n \geq N$ . It follows from Montel's theorem that  $F_n$  is a normal family at  $w_0$  and on  $U$ . Thus  $\mathcal{J}(F_n) \subset \{w : w \in \mathcal{J}_w\} \cup \{w : f_w(z) \text{ has a neutral periodic cycle}\}$   $\square$

The previous theorem leads one to consider when the set  $\{w : w \in \mathcal{J}_w\}$  has nonempty interior. This is a question we will attempt to answer next.

Observe that a holomorphic motion can be used to partition a Julia set  $\mathcal{J}_w$  into equivalence classes. Suppose the Julia sets of  $\{f_w(z)\}_{w \in \mathcal{W}}$  move holomorphically and  $x \in \mathcal{W}$ . If  $z \in \mathcal{J}_c$  and  $z' \in \mathcal{J}_d$  define  $(c, z) \sim (d, z')$  if there exists a  $y \in \mathcal{J}_x$  such that  $\phi(c, y) = z$  and  $\phi(d, y) = z'$ . It is easy to check that this defines an equivalence relation.

It is these tools that may allow one to improve the hypothesis in Theorem 3.5 as well as shed some light on some open questions regarding hyperbolicity and stability from classical iteration theory (see [McM94] for details). One question that has eluded the author for some time is the following.

**QUESTION 3.1.** Suppose  $\{f_w\}_{w \in \mathcal{W}}$  is a family of polynomial functions where  $\mathcal{W}$  is a connected open set and  $f(w, z)$  has the form given in (1). Assume

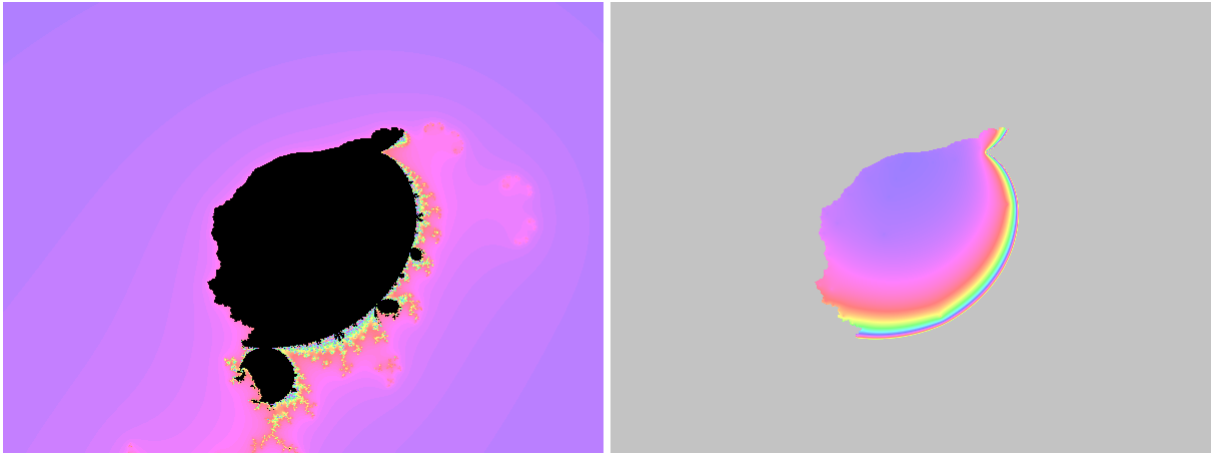


FIGURE 3.3. Filled Julia set  $\mathcal{K}(F_n(w))$  where  $f(w, z) = z^2 + w - i$  (left). The basin of attraction for the function  $\gamma(w) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(w - i)}$  (right).

1.  $\mathcal{J}_w$  moves holomorphically for  $w \in \mathcal{W}$ , and
2.  $\mathcal{J}_w$  is totally disconnected for all  $w \in \mathcal{W}$ .

If  $\{w : w \in \mathcal{J}_w\} = \mathcal{W}$ , then what can we say about the structure of  $\mathcal{J}(F_n)$ ? And how “often” does one run across the situation where  $w \in J_w$  for all  $w \in \mathcal{W}$ ?

### 5. Function Spaces and “Fixed Points” of Seed Iteration

Now we examine the limit functions for seed variable iteration. We start with an example.

EXAMPLE 3.4. Let  $f(w, z) = z^2 + w - i$ . Then  $\mathcal{K}(F_n)$  for this iteration is shown on the left of Figure 3.3. Now consider the function  $f_w(z)$ . We know that the attracting fixed point of this function is  $\gamma(w) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(w - i)}$ . Then the function  $\gamma$  is in a sense “fixed” as  $f(w, \gamma(w)) = \gamma(w)$ . Furthermore, the iterates of  $F_n(w)$  converge to the function  $\gamma(w)$  on some subset of  $\mathcal{K}(F_n)$  which we can think of as the basin of attraction for  $\gamma$ . Indeed, that subset is shown in Figure 3.3 on the right. The colors change depending on the amount of time it took for  $F_n(w)$  to get “close” to  $\gamma(w)$ .

So some of the limit functions of seed variable iteration are the functions  $\gamma(w)$  which map  $w$  to a periodic attracting cycle of  $f_w$ , and these functions are in a sense “fixed points” themselves.

Let  $\mathcal{C}(U)$  be the set of mappings continuous on the domain  $U \subset \mathbb{C}$ . Recall that there is a natural topology on  $\mathcal{C}(U)$ . We define a neighborhood of a function  $\phi \in \mathcal{C}(U)$

by  $N(\phi|_K, \delta) = \{\psi \in \mathcal{C}(U) : |\phi(z) - \psi(z)| \text{ for all } z \in K\}$  where  $K \subset U$  is compact. The topology generated by these neighborhoods is called the *topology of locally uniform convergence*. A sequence  $\phi_n$  converges in this topology if the maps converge locally uniformly on  $U$  [Mil06].

Define  $\mathcal{O}^n : \mathcal{C}(U) \rightarrow \mathcal{C}(U)$  to be the operator induced by  $f^n(w, z)$ , that is  $\mathcal{O}^n(\gamma)(w) = f^n(w, \gamma(w))$ . If  $\mathcal{O}^n(\gamma) = \gamma$  and  $|\frac{\partial f^n}{\partial z}(w, \gamma(w))| < 1$  then  $\gamma(w)$  is an attracting fixed point of  $f_w$ . It is easy to check that  $\gamma$  is analytic (see [Gam01] page 234 for details). So the function  $\gamma$  is a fixed point of the operator  $\mathcal{O}$ . A natural question to ask is: does  $\mathcal{O}^n(\phi) \rightarrow \gamma$  as  $n \rightarrow \infty$  for  $\phi$  in some neighborhood of  $\gamma$ ?

**PROPOSITION 3.6.** *Let  $\gamma(w)$  be an attracting fixed point for  $z \mapsto f(w, z)$  and  $U = \{w : |\frac{\partial f}{\partial z}(w, \gamma(w))| < 1\}$ . Suppose  $\mathcal{O} : \mathcal{C}(U) \rightarrow \mathcal{C}(U)$  is the operator induced by  $f(w, z)$ . Let  $K \subset U$  be compact. Then there exists a  $\delta > 0$  such that if  $\phi \in B(\gamma|_K, \delta)$  then  $\mathcal{O}^n(\phi) \rightarrow \gamma$  as  $n \rightarrow \infty$ .*

**PROOF.** Choose a compact subset  $K \subset U$ . Suppose the degree of  $z$  in  $f(w, z)$  is  $n$ . Let  $\alpha_j = \max_{w \in U} |\frac{\partial^j f}{\partial z^j}(w, \gamma(w))|$  for  $j = 1, 2, \dots, n$ . Now choose  $\delta > 0$  such that  $\alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + \dots + \alpha_n\delta^{n-1} \leq \alpha < 1$  where  $\alpha > 0$  is any number greater than  $\alpha_1$ . Now suppose  $\phi \in B_K(\gamma, \delta)$ . Then, using the Taylor expansion of  $f(w, z)$  about  $\gamma(w)$ , we have

$$\begin{aligned} |\mathcal{O}(\phi)(w) - \mathcal{O}(\gamma)(w)| &= |f(w, \phi(w)) - \gamma(w)| = \\ &= \left| \frac{\partial f}{\partial z}(w, \gamma(w))(\phi(w) - \gamma(w)) + \dots + \frac{\partial^n f}{\partial z^n}(w, \gamma(w))(\phi(w) - \gamma(w))^n \right| \leq \\ &= |\phi(w) - \gamma(w)| |\alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + \dots + \alpha_n\delta^{n-1}| < \alpha |\phi(w) - \gamma(w)| \end{aligned}$$

Thus  $\mathcal{O}$  is a contraction on  $B(\gamma|_K, \delta)$ . □

This motivates the following definition.

**DEFINITION 3.2.** Let  $\mathcal{O}$  be the operator induced by  $f(w, z)$  and let  $\mathcal{O}^n$  denote its  $n$ -th iterate. A continuous function  $\phi$  is *periodic* with period  $n$  if  $\mathcal{O}^n(\phi) = \phi$  and  $n$  is the smallest natural number with this property.

1.  $\gamma$  is an *attracting function* on  $U$  if  $|\frac{\partial f^n}{\partial z}(w, \gamma(w))| < 1$  for all  $w \in U$

2.  $\gamma$  is a *repelling function* on  $U$  if  $|\frac{\partial f^n}{\partial z}(w, \gamma(w))| > 1$  for all  $w \in U$

Now if  $\gamma$  is an attracting function on  $U$ , then define

$$\mathcal{B}(\gamma) = \{\phi \in \mathcal{C}(U) : \mathcal{O}^n(\phi) \text{ converges to } \gamma\}$$

This  $\mathcal{B}(\gamma)$  of course corresponds to the basin of attraction that we are familiar with in classical iteration theory. Now, this theory of seed iteration can be tied in with non-autonomous or random iteration as the following example illustrates.

EXAMPLE 3.5. Let  $\bar{c} \in \{0, 1\}$ . Define  $f_{\bar{c}}^n(w, z) = f_{c_n}(w, f_{c_{n-1}}(w, \dots f_{c_2}(w, f_{c_1}(w, z)) \dots))$ , and  $F_{\bar{c}}^n(w) = f_{\bar{c}}^n(w, w)$ . We call this *random seed iteration*. Now suppose we let  $f_0(w, z) = z^2 + w$  and  $f_1(w, z) = z^2 - 1$ . Then the filled Julia set for  $F_{\bar{c}}^N(w)$  is pictured in Figure 3.4 (left).

Now we know that the function  $z \mapsto z^2 - 1$  has  $\{0, -1\}$  as an attracting cycle. So the constant function  $\gamma_1(w) = -1$  is the attracting function on a neighborhood of  $-1$  for  $f_1(z, w)$ . Furthermore, we know that  $f_0(w, z) = z^2 + w$  probably has an attracting function  $\gamma_0(w)$  of period 2 defined on a small neighborhood about  $-1$ . Thus the functions  $\gamma_0$  and  $\gamma_1$  are probably close (and in fact equal at  $-1$ ) in the topology of locally uniform convergence. This motivates the upcoming conjecture.

Let  $\{f_c(w, z)\}_{c \in \Gamma}$  be a family of functions. These induce a family of operators  $\{\mathcal{O}_c\}_{c \in \Gamma}$ . Define  $f_{\bar{c}}^n(w, z)$  as in Example 3.5. Does there exist a similar concept of invariant domains for the operators  $\mathcal{O}_c$ ?

CONJECTURE 3.7. *In example 3.5 there exists a  $\delta > 0$  and a set  $\mathcal{U} \subset \mathcal{C}(B(-1, \delta))$  such that  $\mathcal{O}_c(\mathcal{U}) \subset \mathcal{U}$  and  $F_{\bar{c}}^n(w)$  converges locally uniformly on  $B(-1, \delta)$ .*

EXAMPLE 3.6. On the right of Figure 3.4 is the filled Julia set for a non-autonomous seed iteration of  $f_0(w, z) = z^2 + w$  and  $f_1(w, z) = z^2 - .1 + .75i$ .



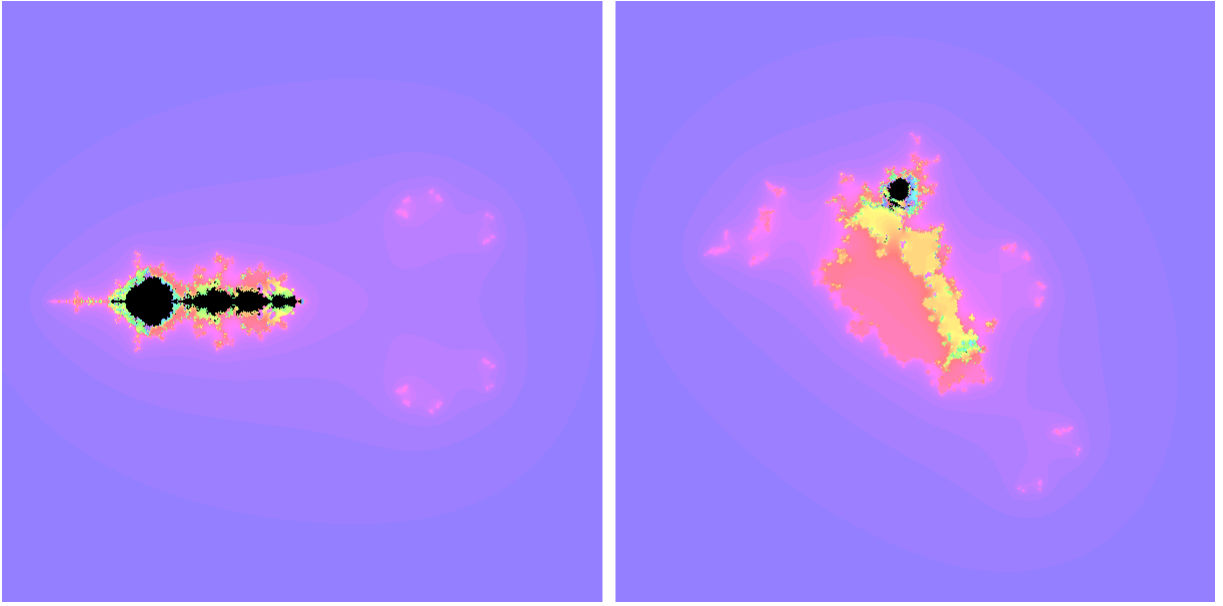


FIGURE 3.4. Filled Julia set for non-autonomous seed iteration of  $f_0(w, z) = z^2 + w$  and  $f_1(w, z) = z^2 - 1$  (left). Filled Julia set for random seed iteration of  $f_0(w, z) = z^2 + w$  and  $f_1(w, z) = z^2 - .1 + .75i$  (right).

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