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Analyzing State Attempts at Implementing the Common Core State Standards for High School Geometry: Case Studies of Utah and New York

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ANALYZING STATE ATTEMPTS AT IMPLEMENTING THE COMMON CORE STATE STANDARDS FOR HIGH SCHOOL GEOMETRY: CASE STUDIES OF UTAH AND NEW YORK

by

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ABSTRACT

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This study analyzes two state attempts at aligning curricula to the Common Core State Standards (CCSS) in secondary school geometry. The education departments of Utah and New York have approved curricula aimed at aligning to the Common Core State Standards: the Mathematics Vision Project (MVP) and EngageNY (ENY) respectively. This study measures the extent to which those curricula align with the content demands of the relevant Common Core Standards. The results indicate that, while the two curricula vary in structure and assumptions about learners, each one aligns well with the Common Core State Standards in secondary school geometry. We conclude with recommendations for individuals and entities concerned with aligning geometry curricula to the Common Core State Standards.

This Study is Dedicated

to the Memory of My Mother,

Patricia Steltenpohl

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LIST OF ABBREVIATIONS

CCSS/CCSSI- The Common Core State Standards Initiative and its standards

ENY- New York EngageNY

MVP- The Utah Mathematics Vision Project

NYCDE- New York City Department of Education

PISA- The Program for International Student Assessment

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“There is no royal road to geometry.”

– Euclid to Ptolemy I.

INTRODUCTION

It is simultaneously a demanding and exciting time to be a teacher of mathematics in the United States. The international comparison report released in 2012 by the Program for International Student Assessment (PISA) indicated that U.S. public school students are outranked in mathematics performance by 29 of 65 jurisdictions (PISA, 2012).[1] More troubling is the fact that this figure was up from 23 jurisdictions just three years prior. That poor ranking would seem to imply that there exist general shortcomings in the U.S. public education system in comparison to the higher performing countries. Indeed, if the U.S. hopes to become globally competitive in mathematics education, it needs to embrace substantive reforms that address the current shortcomings. To that end, the Common Core State Standards Initiative (CCSSI) provides internationally benchmarked standards which a large majority of U.S. states have already adopted.

Adopting the standards of the CCSSI (or other standards closely aligned to the CCSSI) is, of course, only a step in the right direction— adoption of the standards does not immediately imply the *action* of aligning curricula and teaching practices to the standards. Hence it is both an interesting and open question as to how, and by what means, U.S. states are actually implementing the CCSSI into their curricula. To answer this question, we will limit our analysis to the domain of geometry, a subtopic where students traditionally first encounter certain formalisms of mathematics such as "theorem" and "proof."

[1] PISA considers jurisdictions to be "countries and economies." The countries considered are those one would consider as being "developed."

There is a significant set of jurisdictions from which to do case studies. Forty-three U.S. states, the District of Columbia, the Department of Defense Education Activity, and four territories have adopted the Common Core State Standards. Alaska, Indiana, Minnesota, Nebraska, Oklahoma, Texas, and Virginia have not adopted the CCSSI mathematics standards (CCSSI, 2014).[2] Puerto Rico has also not adopted CCSSI standards.

A survey of the jurisdictions that have adopted CCSSI mathematics standards results in seeing some encouraging examples of efforts to actually integrate the standards into curricula. Unfortunately, there do exist states that have adopted the CCSSI mathematics standards but that do not demonstrate any significant evidence of really implementing the standards. We exclude such states from analysis and focus on states making real attempts at implementation. With that in mind, this paper will study two states that contrast well regarding the style of proposed Common Core-aligned geometry curricula as well as the structuring of mathematics courses: Utah and New York.

On the one hand, the Utah State Office of Education advocates for an "international/integrated" pathway that structures the standard courses as "Mathematics 1," "Mathematics 2," and "Mathematics 3" (USE, 2014). Each of those courses is comprised of a mixture of some degree of algebra, trigonometry, statistics, and geometry. Hence, this pathway aims to gradually develop students' mathematical capabilities through all subtopics during a wide timespan. That clearly contrasts to the more traditional U.S. structuring of courses: "Algebra 1," "Algebra 2," "Geometry,"

[2] Minnesota has, however, adopted the English and Language Arts (ELA) standards.

“Trigonometry,” and so forth. Additionally, the Utah State Office of Education supports curricula that is nearly exclusively discovery-based (i.e. the Mathematics Vision Project (MVP)). A discovery based curriculum has the defining feature that its topic sections almost everywhere begin with an exercise from which the authors hope learners will gain an intuitive basis for understanding the mathematics in the sections. The role of the teacher using solely discovery based material also becomes amplified to an extent because the teacher may be the single source for important information not provided in the written materials. For example, the teacher may need to provide precise definitions and statements of theorems.

On the other hand, the New York State Education Department advocates for the “traditional” pathway that structures the standard courses as already discussed (NYSED, 2014). Engage New York is supported by the New York State Education Department, and its most significant mission is to attempt to integrate CCSSI standards into curricula. We will see that the EngageNY efforts also utilize discovery based exercises like Utah’s MVP materials, but the developed material offer much more regarding formalisms such as definitions, theorems, and formal proof techniques (e.g. direct proof and proof by contradiction).

The first section of this paper is a review of the relevant literature. We begin by presenting the famous Van Hiele model that presents a framework for understanding how students learn geometry. That presentation will allow us to understand and explain some differences between the two curricula. Next, we provide a brief background of the Common Core and both Utah’s and New York’s adopted standards. Finally, we discuss techniques that have been used to analyze curricula alignment with the Common Core standards: the Wisconsin Center for Education Research's "Surveys of Enacted Curriculum" (SEC) and the New York City Department of Education's "Protocol for Math Performance Task Alignment."

The successive two sections conduct analyses of Utah (MVP) and New York (EngageNY) geometry curricula respectively that aim to align with the Common Core standards. We test the extent of alignment that each curriculum exhibits with the Common Core secondary school geometry standards. In that analysis, we use a scoring rubric (based on NYCDE's "Protocol for Math Performance Task Alignment") to assign alignment grades to each curriculum per relevant Common Core standard. We are then able to discuss the general alignment of each curriculum with the Common Core under the geometry domain.

In the final section, we summarize the strengths and weaknesses of each curriculum. Next, we emphasize that the structure of the curriculum had no effect on curriculum alignment with the Common Core. Finally, we offer suggestions to writers of geometry curricula so that their materials may be well-aligned (within the scope of our rubric) with the Common Core secondary school geometry standards.

LITERATURE REVIEW AND METHODOLOGY

Van Hiele Levels

Jean Piaget (1896-1980), a Swiss developmental psychologist and philosopher, pioneered work linking levels of understanding of mathematics to student age. Desiring a model representing mathematical learning more independent of age, Pierre van Hiele (1909-2010) and Dina van Hiele[3] went on to test the hypotheses of Piaget and formed what came to be known as Van Hiele "levels of abstraction in understanding mathematics" (Colignatus, 2014, p. 1-2). These so-called "Van Hiele levels" were, indeed, more independent of age than were the hypotheses of Piaget. In fact, adults who took a geometry course in secondary school could be at a lower Van Hiele level than an elementary school student— this depends on the degree of enrichment to which a person has been exposed. We focus here on what are often referred to as *the* Van Hiele levels: the theory that establishes a hierarchy of student geometric learning stages and ability levels. By reviewing what the Van Hiele levels are, we can establish a theoretical understanding regarding the extent to which curricula can feasibly assist students in developing geometric reasoning abilities.

Any discussion of the Van Hiele levels must begin with an understanding of the loaded word "abstraction." Colignatus (2014) argues that thought itself is "essentially abstract" and "mathematics is equivalent if not identical to abstraction." While Van Hiele doesn't like the word "abstraction," the Van Hiele levels can be interpreted as a gauge of the extent to which students can make connections between what they are convinced is

[3] Dina and Pierre were married, and each wrote dissertations at Utrecht University in the Netherlands.

true in a specific case and what might be the general case. Indeed, this is an important component of what a mathematician might call abstraction.

While there have been critics of the Van Hiele hypothesis (see e.g. Clements et al., 1999), it is nevertheless the case that the Van Hiele levels have influenced many sets of geometry standards including the Common Core Standards (CCSSI, 2014). Therefore, understanding the progression of the levels will help in a comparative analysis of curricula that aims to adhere to the Common Core Standards. In the remainder of this section, we detail each Van Hiele level and then make some important observations.

Level 0 – Visualization

Students at the 0 Van Hiele level begin recognizing geometric shapes based on gross characteristics and visual experiences (Burger, 1986). Such a student might say "this figure is a circle because it looks like the Sun." Thus, a prototype circle is envisioned in the student's mind which she/he uses as classifier. Additionally, the student can differentiate between her/his prototypical notions of shapes. For example, a level 0 student might say a square is not a circle because the square has corners. That is to say, in our example, a square does not look like the Sun. However, figures that vaguely resemble the student's prototype, but have some substantively different feature in comparison to her/his prototype, may or may not be classified as being a member of the prototype. For example, a student might look at the following triangle and say that it is too thin to be a triangle:

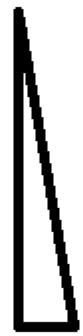


Figure 1- "Skinny triangle"

On the other hand, a figure might meet all features of a level 0 student's prototype, but nevertheless the figure is not an example of the shape on which the student's prototype is based. For example, the student might say an ellipse with eccentricity $0<m<1$ is a circle because it is completely round. Figure 2 shows another such case where a pseudo-triangular shape might be classified as a triangle:

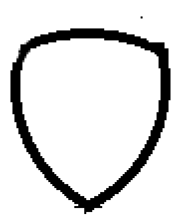


Figure 2-A student at VHL0 might call this a triangle.

Level 1- Analysis

At level 1, students are now able to recognize attributes of shapes. For example, she/he might say a square has four sides of equal length and four congruent angles (Burger, 1986). Properties thus become very important in classifying shapes— perhaps even more important than visual appearance. Nevertheless, there can still be imperfections in the students' analysis because she/he does not understand, for example, that a square is also a rectangle or a circle is also an ellipse. She/he will reason inductively from a plethora of examples. However, she/he does not reason deductively because she/he lacks an understanding of the relationship between geometric properties. Moreover, she/he does not understand necessary and sufficient conditions. Interestingly, many adults and even teachers reside in this level (Mayberry, 1983).

Level 2- Abstraction

Properties of geometric figures begin to exhibit order at level 2 (Burger, 1986). Students at level 2 will understand geometric properties and be able to connect them deductively. They begin noticing that geometric figures can be grouped based upon certain similarities, but nevertheless the figures may exhibit difference. That is to say, they realize, for example, that all squares are rectangles, but not all rectangles are squares. Furthermore, they will be able to explain claims they make regarding geometric figures. For example, a student might say "this is a right triangle because it has a 90-degree angle." Students at this level will also understand necessary and sufficient conditions and will be able to write out complex definitions. However, they will not be able to follow complex arguments nor understand the role of axioms. Even the role of definitions may seem obscure to a student at this level.

Level 3- Deduction

Students at level 3 can reason deductively much more significantly than they could at level 2 (Burger, 1986). They can now write (and understand) simple proofs, and they understand the roles of undefined terms, axioms, definitions, and theorems. However, all of this is limited to Euclidean geometry: an understanding of non-Euclidean geometry remains on the horizon.

Level 4- Rigor

Level 4 corresponds to the level of geometric understanding of a mathematician (Burger, 1986). Students at this level have a very high level of understanding of Euclidean geometry. They realize that definitions are arbitrary and that definitions do not have to refer to any “real world” realization to be legitimate. Most frequently, they reason deductively. Furthermore, they can study and understand non-Euclidean geometry (e.g. elliptic geometry). Finally, they understand, and can explain, the differences between geometry and other sub-disciplines in mathematics.

Properties of the Van Hiele Levels

The following observations help in understanding the Van Hiele Levels.

1) Students do not skip Van Hiele Levels (Mayberry, 1983). That is to say, each student must travel the developmental path from levels 0 to 4. However, students may move more quickly through the levels in comparison to their peers.
2) The levels are interconnected with succeeding levels capitalizing on nascent understandings from the earlier levels. Moreover, students may be at higher levels in certain topics than they are in other topics.
3) Each level has its own symbols and networks of relationships. Correctness at one level does not imply correctness at another level. For example, at level 0, a student might say that a circle is something that looks like the sun. This is a correct assertion at this level. However, in higher levels where precise definitions play a more salient role, saying a figure is a circle because it looks like the sun is not completely correct.

4) Teachers and students fall into different Van Hiele levels. Thus, communication and realistic expectations are necessary in the classroom. That is to say, in an introductory (high school) geometry course, expecting complete understanding at VHL4 is quite harsh. Furthermore, relating to students by emphasizing shared experiences and knowledge helps to establish a pathway to advancement in Van Hiele level. “Speaking at the students’ level” is key.
5) The Van Hieles made recommendations on how to advance students between levels based upon their findings. They found that, when dealing with new topics, teachers should provide the instruction and materials necessary for students to *discover* the nature of the topic. If possible, teachers should facilitate discussions about topics so that some synthesis may occur in student understanding. When discoveries are made by students, guided by the teacher, an attempt to introduce vocabulary should be made. Symbols can also be introduced so that students may link geometric ideas to the symbols. After students have sufficient background, more complex activities can be asked of them. The complex activities should involve a deeper level of contemplation and discovery. Finally, students should be asked to summarize what they have learned, and they should be encouraged to pose (and try to answer) new questions.

Conclusions Regarding the Van Hiele Levels

The Van Hiele levels offer an interesting theory as to the development of a learner’s geometric capabilities. The theory is quite encompassing— from the very basic notion that two figures share something in common, all the way to the level of understanding of a mathematician, the Van Hiele levels account for essentially every level of understanding. The levels can also help direct geometry teachers because there can be made, on each topic, a general consensus of where a class stands. Then the teacher can structure her/his instruction accordingly.

There are, of course, certain aspects of the levels that limit their usefulness. First, a student is never universally at level X. She/he may be level 3, for example, on parallel lines but at level 2 on properties of triangles. Second, curricula and courses have been developed based on the research of the Van Hieles (Guetierrez, 1998). Yet, American students generally score low within the Van Hiele levels. It is hypothesized that low Van Hiele levels are due to the fact that learning has simply been synonymous with memorization in many classrooms in America. High school geometry classes often assume readiness to move to Van Hiele level 3 across all subtopics, but the students may well still be back at level 0 or 1 on most of those subtopics. If students are at low levels of geometric understanding, they need to be worked up through the levels and not dropped into a course where topics and ideas are far advanced from where they stand. Thus, structuring courses around the Van Hiele levels, perhaps as well with any other sort of learning standard, needs to be done with extreme care.

Interesting questions to keep in mind while analyzing any geometry curriculum is a consideration of the following:

a) the degree to which the curricula have discovery-based activities aimed at getting students to develop their geometric intuition,
b) the extent to which definitions are given at each Van Hiele level,
c) the extent to which learners are forced to explain what they have learned,
d) the amount of recapitulation of previously learned definitions, theorems, and proofs in relevant Van Hiele levels, and

e) the amount of deductive and abstract reasoning that the material demands.

Overview of the Common Core State Standards Initiative and Utah/New York's Adopted Standards

The Common Core State Standards Initiative is a standards-driven educational initiative that is sponsored by the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) (CCSSI, 2014). Within its listed standards, it details what K-12 grade learners ought to know in English language arts and mathematics at the end of each grade. A significant impetus behind the development of the CCSSI came from reports (e.g. *Ready or Not: Creating a High School Diploma That Counts* by Achieve, Inc.) that demonstrated that high school graduates were not being provided with the requisite knowledge and skills needed in college and careers. In a significant stride to remedy that troubling situation, the NGA formed a group of people to develop the CCSSI: David Coleman (University of Arizona), William McCallum (University of Arizona), Phil Daro, Jason Zimba, and Susan Pimentel.

The CCSSI's stated purpose is to "provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them" (CCSSI, 2014). Moreover, the standards emphasize the coverage of "real world" examples within curricula. Indeed, as we will see in our analysis, certain standards emphasize applying mathematical knowledge and abilities to modeling real world phenomena.

The mathematical standards within the CCSSI include "Standards for Mathematical Practice and Standards for Mathematical Content" (CCSSI, 2014). These practices are based off of the five process standards from the National Council of Teachers of Mathematics and the five strands of proficient in the US National Research

Council's *Adding It Up* report. We list these eight principles of mathematical practice below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Furthermore, the CCSSI provides content standards that detail all of the content (together with "optional" content) that courses on the material ought to cover if learners are to be prepared for college and/or careers (CCSSI, 2014). At the secondary school level there are six conceptual categories in which the standards lie: Number and quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability. Each of those categories then contains subcategories. For example, the geometry category is split into several subcategories based upon topic: one subcategory, for instance, is "Congruence."

The CCSSI has been a popular topic for debate in politics and educational policy discussions. Professor Hung-Hsi Wu, a mathematician at Berkeley, argued in 2013 that many textbooks used in the United States operate under a set of mediocre, unwritten, set of national standards. He went on to say that the CCSSI levels the playing field for students because it establishes national standards that go beyond just addressing textbook content: the standards emphasize outcomes for learners.

In the introductory portion of this paper, we emphasized the jurisdictions that have formally adopted the Common Core State Standards. Upon adopting the CCSS, those states were allowed to make minor changes to the standards (CCSSI, 2014). For example, the New York's adoption of the standards often includes a delimiter that the content covered under each relevant standard can go above and beyond the demands of the standard. That is to say, the policy adopters in New York wanted instructors to not feel limited by the CCSSI. On the other hand, other states have adopted the CCSS nearly verbatim, like Utah.

Techniques Used to Measure Curriculum Alignment with the Common Core

Measuring the alignment of a curriculum with the Common Core can be a painstaking task because of the numerous complexities inherent in curriculum. These complexities of a curriculum include, but are not limited to, the assumptions that the curriculum makes about the learner, the extent to which the instructor is responsible for delivering on practice standards and content, and the length of time during which the learner is exposed to specific topics. The developers of rubrics that measure curriculum alignment both acknowledge and factor in (to a varying extent) those complexities.

As we saw when detailing the Van Hiele levels, assumptions made about learners are extremely important in curriculum to the extent that the learners need the prerequisite knowledge and abilities to truly advance their geometric understandings. For example, an obviously poor choice for a 9th grade geometry curriculum would be to demand proofs and understandings at the level of a mathematician (i.e. VHL4). Hence, if the Common Core demands a proof of some mathematical theorem, care must be taken to not necessarily require (within a scoring rubric) proofs at the level of a mathematician. Yet, a relevant proof still needs to be presented in some fashion in the curriculum. Moreover, examples of proofs most certainly need to be provided to the learner, lest she/he has no idea of the expectations to which she/he is being held.

The extent to which instructors are responsible for delivering on practice standards and content is also an important consideration for analyzing curriculum alignment. Discovery curriculum (for example the Mathematics Vision Project (MVP) of Utah) “textbook” material resembles a workbook with exercises aimed at developing intuition about the mathematics at play. An extremely significant responsibility is thus placed on the instructor who uses such material— the instructor is responsible for providing precise definitions and proofs and for orchestrating the exercises into a cohesive body of knowledge.

Depending to a degree on the structuring of the geometry course(s) in secondary school, an important question to address is whether there is enough instructional time to deliver on all content at the depth necessary to prepare a learner for college and/or a career. The authors of the Common Core, indeed, seem to have taken this potential issue into account: the CCSS frequently has "plused" (+) content standards that can be viewed as optional. The idea is that, by limiting some of the (perhaps) more advanced content, more time can be spent on the developmental content. Certainly the situation is changed when a course is geared toward more advanced students. Alignment techniques take these issues into account to varying degrees.

Since the Common Core State Standards are relatively new, the extant literature is limited with regard to in-depth analyses of the alignment of curricula to the Common Core. Heidi Ertl (2014) examined the relative alignment of the Common Core Standards for Mathematics with the standards used in Singapore. She found that there was a strong alignment between the two sets of standards. The main technique Heidi used to come to that conclusion is called the Survey of Enacted Curriculum (SEC, 2014). The SEC coding system emphasizes the amount of instructional time that is meant to be spent on specific content at varying levels of advancement in learning (memorization, performing activities, demonstrate understanding, conjecture/generalize/prove, and solve nonstandard problems).

The SEC coding system is an effective tool to comparatively analyze standards because it is generally quite clear the amount of time the standards want learners to spend on each topic. While the system could also be used to measure alignment between a specific curriculum and the Common Core, we find that it is a highly subjective enterprise to determine the amount of time the curriculum wants an instructor to spend on the relevant content. That is to say, content like the Pythagorean Theorem may well be used in a large portion of a curriculum, but the question is whether the Pythagorean Theorem was actually presented well enough in alignment with the Common Core in the first place.

The New York City Department of Education has produced another technique of analyzing curriculum alignment with the Common Core (NYCDE, 2014). It is a per-standard grading system that has a binary quality. That is to say, it asks the "grader" whether the content per standard is covered. However, there is a subjective quality as well because the "grader" needs to determine if performance related items are covered sufficiently well to meet the CCSS practice standards. We find that this technique has a significant benefit over the SEC alignment coding to the extent that time-spent analysis is removed in favor of whether material is covered sufficiently well. We omit the explicit NYCDE rubric in this section and provide it within the next section called Methodology.

Methodology

As discussed above, there are numerous considerations we must make in answering our main question. We find that the best approach in addressing the question is a presentation and analysis of a representative sample of the geometry curricula from Utah (MVP) and New York (Engage NY) from which we assign an alignment grade. The rubric we use follows this section, and it is inspired by the Common Core alignment rubric developed by the New York City Education Department (NYCED, 2014).

This rubric removes a significant amount of subjectivity in determining alignment grades. For example, the most significant portion that determines alignment grades has a binary character to it (i.e. does the material cover the content listed at the level expected by the Common Core?). It is important, however, to note that there is some subjectivity involved in determining whether performance items have sufficiently developed material allowing for learners to perform relevant actions like proving theorems. In every case, we are certain to point out any subjectivity that occurs in our analysis. The rubric is presented in the figure on the next page.

Rubric for Measuring CCSSI Alignment

Grade	Descriptor	Detail
3	Exceptional Alignment	Course material covers all content listed per standard and also all performance related items (e.g. prove theorems). Furthermore, all performance items have sufficiently developed material allowing the learner to actually perform the action.
2	Good Alignment	Course material covers at least one half of all content listed per standard and at least one half of performance related items. Material may lack a small portion of developmental material required to perform actions.
1	Weak Alignment	Course material covers less than one half of all content listed per standard or less than one half of all performance related items. Material may lack a significant portion of developmental material required to perform actions.
0	Minimal/No Alignment	Course material covers essentially no content listed per standard and no performance related items.

Table 1: CCSSI Alignment Rubric

ANALYZING THE ALIGNMENT OF THE UTAH MATHEMATICS VISION PROJECT (MVP) CURRICULUM WITH THE COMMON CORE

The Mathematics Vision Project (MVP) is an educator-driven educational initiative that aims to align well with the Common Core mathematics standards (MVP, 2014). As we have already discussed in our introduction and literature review, the MVP material is nearly exclusively discovery-based. It is an open source set of material that is available on the MVP Web Site. Our citation of the MVP material is as follows: (material version (S=student, T=Teacher), module number, page number). Moreover, the material is meant to be taken in grades 9 through 11.

Congruence

Congruence: Experiment with transformations in the plane

> CCSS.MATH CONTENT.HSG.COA 1: Know the precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

The undefined notions mentioned here are introduced in the MVP curricula via discovery exercise. For example, the relative positions of members of a high school drill team are presented, and the learner is asked to infer basic notions of distance (MVPS1, M7, p. 3). For instance, the learner is asked to compare distances given specific positioning of the drill team members. The positions of the drill team members are then changed and the learner is asked questions to surface differences between the new positions versus the old positions. This then leads to the following hypothetical situation:

> A student poses the following question to her class: “Hey, I wonder if there is a process that we could use like what we have been doing to find the distance between any two points on the grid…. I’m going to start with two points and draw

> the line between them that represents the distance that I'm looking for. Since two points could be anywhere, I named them A(x_1,y_1) and B(x_2,y_2). Hmmmmm… when I figured the length of the ribbons, what did I do next (MVPS1, M7, p. 5)?"

Hence, this material does lead the learner through a path of discovery of an undefined notion of distance, and it makes a point that the underlying notion of distance should apply in general. The learner, at this point, is expected to interpret distance as the number of grid marks from drill team member to other drill team member if they lie horizontally or vertically from each other. If the drill team members lie diagonally relative to each other on the grid, certainly the learner should be thinking back to her/his nascent understanding of the Pythagorean Theorem and/or the Euclidean distance between two points (see e.g. MVPS1, M6). However, neither of these are suggested in the material (until far into the exercises, see e.g. MVPS1, M7, p7 #12-15), and the student is left to his or her own devices, or the suggestion of the teacher, to form her/his understanding of "distance." Yet, COA.1 only demands an undefined notion/intuition of distance. There are, furthermore, discovery exercises aimed at developing the intuition behind the other notions COA.1 allows to be undefined: points, lines, distances along a lines (as exemplified), and distance around a circular arc. We conclude that the Utah MVP material aligns strongly with the second half of COA.1.

Moving to the more rigorous demands of COA.1, students should know "precise definitions of angle, circle, perpendicular line, parallel line, and line segment" based upon the undefined notions that the MVP material presents. It is important to emphasize, again, the nature of the MVP material: it is designed almost exclusively as a discovery-based textbook. Hence, while a student might at times be able to write out precise definitions based upon her/his discovery of them, there is often an obvious lack of precise definitions in the MVP material.

The MVP material does provide examples that may or may not lead to a learner forming her/his own precise definitions. For example, regarding perpendicular lines, the MVP material provides the example that: "horizontal and vertical lines are perpendicular" (MVPS1, M6, p.9). The learner is then asked to determine when two lines that are neither horizontal nor vertical are perpendicular with respect to each other. Then she/he is asked to prove that her/his observation is true, and exercises follow geared toward drawing perpendicular lines on pre-drawn graphs. No explicit definition is given of "perpendicular lines" in this treatment. We conclude the MVP material fails in both providing the precise definition of "perpendicular lines" demanded in COA.1 and in actually setting up a place for the learner to create her/his own definition.

In the subsequent module, perpendicular lines are reintroduced via discovery exercise, and a "theorem" that "slopes of perpendicular lines are negative reciprocals" is provided (MVPS1, M7, p. 9). While this is not a definition but a claim to be proven, this formulation is not entirely clear. The statement that "x and y have property A" can be understood to say that x and y are equivalent with respect to property A. Thus, stating that "the slopes of perpendicular lines are negative reciprocals" does not get at the heart of the matter that the slopes of such lines are negative reciprocals of each other. Moreover, this claim includes the object "slope," which is mentioned in the section discussed previously on the notion of "distance" (it is also dealt with algebraically in other modules):

"Triangle ABC is a slope triangle for the line segment AB where BC is the rise and AC is the run. Notice that the length of segment BC has a corresponding length on the y-axis and the length of AC has a corresponding length on the x-axis. The slope formula is written as $m=(y_2-y_1)/(x_2-x_1)$, where m is the slope."

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

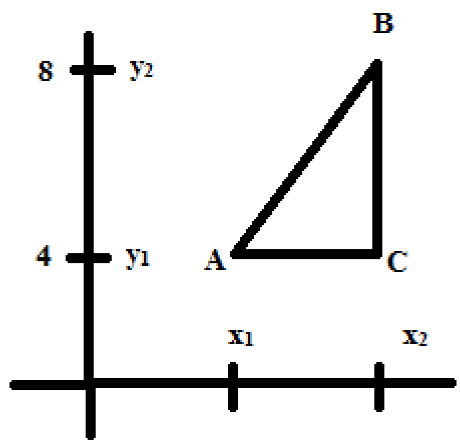


Figure 3- The "Slope Triangle" ABC (MVPS1, M7, p. 7)

While this definition of slope works in practice, the material does not clarify how the formula is derived and used. For example, is this quantity "m" independent of choice of points? Would, for example, taking the midpoint of AB and forming its slope triangle with A result in a different m? An engaged learner may well pose these questions to her/his instructor, but it also may well be the case that the learner walks away *not* knowing that slopes are independent of choice of points on a line. Hence, the MVP material leaves quite a bit to be desired regarding the objects it wants learners to use in the creation of precise definitions demanded in CO.A.1.

We conclude that the Utah MVP material covers all of the "undefined" notions in COA.1. to the extent that precise definition *could* be made of an "angle, circle, perpendicular line, parallel line, and [a] line segment." Yet, when definitions are actually presented, they are presented in a manner that does not indicate to the reader that they are definitions.[4] That is to say, definitions of "circle" and "angle" do actually exist (see footnote) in the MVP material, but the definitions are not provided when actually presenting the relevant material. And, as above, perpendicular lines are never precisely defined in the MVP materials. Nor are parallel lines and line segments precisely defined. Therefore we conclude that the Utah MVP material meets or exceeds the demands of COA.1 in six of nine of the requirements. Thus the Utah MVP material scores a 2 according to our rubric.

> CCSS.MATH CONTENT.HSG.COA 2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

[4] There is a "definition" of a circle given in a discovery activity, but the material makes no effort to emphasize that it is a definition: "A circle is the set of all points in a plane that are equidistant from a fixed point called the center of the circle" (MVPS1,M6,p 20). There is, similarly, a "definition" of "angle" in the activity. However, these definitions are provided passively— they are written within a calendar by a hypothetical student. The reader is asked to fill in a date on the calendar regarding definitions of transformations.

The MVP material introduces transformations in the plane by providing a real world example: computer graphics animators want to move the image of a two dimensional lizard without distorting its “size” or shape. The material indicates that there are three ways to achieve that: “translations (slides), reflections (flips), and rotations (turns)” (MVPS1, M6, p.4.). Furthermore, those procedures may be combined any number of times without distortion of size/shape of the lizard. The learner is then presented with a specific position of the lizard on the plane as follows:

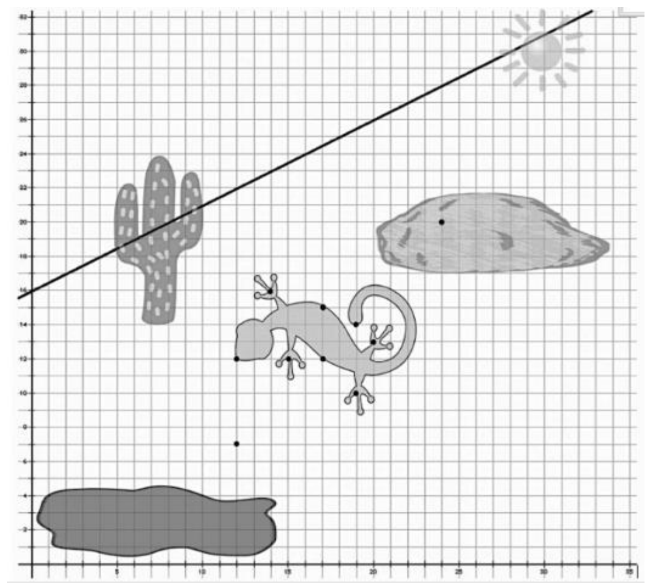


Figure 4- Transformation Activity (MVPS1, M6, p. 5)

The learner is then asked to “translate the original lizard so the point at the top of its nose is located at (24,20), making the lizard appear to be sunbathing on the rock” (MVPS1, M6, p4). There are, of course, several ways the learner could achieve that task. For example, a learner might suggest “moving the lizard diagonally so that the lizard’s nose is located at (24, 20) and such that it appears the lizard is sunbathing on the rock.” This discovery exercise is clear in scope, and it presents transformations in the plane using an alternate delivery technique that fits under COA.2.

The MVP material continues its treatment of transformations by asking learners to compose two or more transformations (MVPS1, M6, p. 7). For example, a point in the plane is provided, and the learner is asked to rotate that point about the origin by 90 degrees, then to reflect over the x-axis, and finally to apply the rule (x-2,y-5) to the point. Hence the material wants the learner to represent transformations as specific coordinate-based procedures, with some algebra involved. It is only natural, then, to describe transformations using functions, but the MVP material fails in this aspect, as discussed below.

The MVP material goes on to talk about transformations in terms of "preimages" and "images" — i.e. language associated with functions (MVPS1, M6, p. 15-17). However, talking about transformations by explicitly using the word "function" does not happen in the MVP materials. However, Section 4 in Module 7 talks about shifts of graphs of functions, e.g. if k is a real number, then $f(x) = x^2 + k$ represents a vertical shift of the "standard parabola" by k units up or down depending on whether k is positive or negative respectively. While the learner may be able to infer that such a procedure as "adding k" is a transformation, it is not emphasized in the MVP material. Furthermore, no significant effort is made to get the learner to understand that there are transformations that do not preserve distance: the emphasis is on rigid motions.

We can thus conclude that, while the MVP material does a very effective job at establishing the intuition behind transformations, and it talks about transformations using words associated with functions like "preimage" and "image," it does not explicitly emphasize that transformations can be represented as functions taking points as inputs to points as outputs. Nor does the material differentiate between rigid motions and transformations that do not preserve distance. Therefore, for standard COA.2., the MVP material receives a grade of 1 per our rubric.

> CCSS.MATH.CONTENT.HSG.CO.A.3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

The MVP material makes a strong effort within this standard by first defining "lines of symmetry" as "lines that reflect a figure onto itself" (MVPS1, M6, p.27). Moreover, figures that can be carried onto themselves via rotation exhibit rotational symmetry. The learner is then asked to determine the possible rotations and reflections that carry figures onto themselves: rectangles, parallelograms, rhombi, squares, trapezoids, and n-gons (3<=n<=10) (MVPS1, M6, p.27-33&48). Therefore, we conclude the Utah MVP material completely covers the demands of CO.A.3. So, for this standard, it receives a grade of 3 per our rubric.

> CCSS.MATH.CONTENT.HSG.CO.A.4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

The MVP material again uses discovery based exercises to attempt to deliver on the demands of CO.A4. For example, it provides the example of a wagon wheel and asks the learner "What fraction of a turn does the wagon wheel below need to turn in order to appear the very same as it does right now? How many degrees of rotation would that be? (MVPS1, M6, p. 38)"

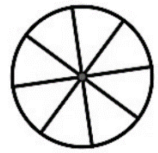


Figure 5- Wagon Wheel

The MVP material goes on to emphasize degree of rotation as the fundamental ingredient in the definition of a rotation. It also, as previously discussed, emphasizes the role that the center of rotation has under a rotation. Furthermore, it asks the learner explicitly to give the definition of rotation her/himself (see e.g. MVPS1, M6, p.21). Hence, the MVP material provides the ingredients for a definition of rotation (development), and it actually asks for that definition (execution). The formal definition is meant to be provided by the teacher.

Definitions of translations and reflections are handled similarly to rotations. Translations on the plane are talked about in terms of “rise” and “run” (MVPS1, M6, p. 15). Reflections are talked about in terms of lines of reflection. Learners are asked to determine, via positions of asymmetric figures (frogs) what transformations would be necessary to get from pre-image to any of the possible images. Since the MVP material covers all of the content and performance items under this standard, it earns a grade of 3 per our rubric for CO.A.4.

> CCSS.MATH.CONTENT.HSG.CO.A.5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

This content standard has largely already been addressed, necessarily, in the previous standards. The MVP material asks the learner to take geometric figures (and also figures like frogs, lizards, and houses) and draw rotations, reflections, or translations on graphs (see e.g. MVPS1, M6, p.50). Furthermore, in the image below, learners are asked to describe sequences of transformations carrying for example image 4 to image 5. We conclude that the MVP materials delivers on the demands of CO.A.5 completely, and therefore it earns a grade of 3 on this content standard.

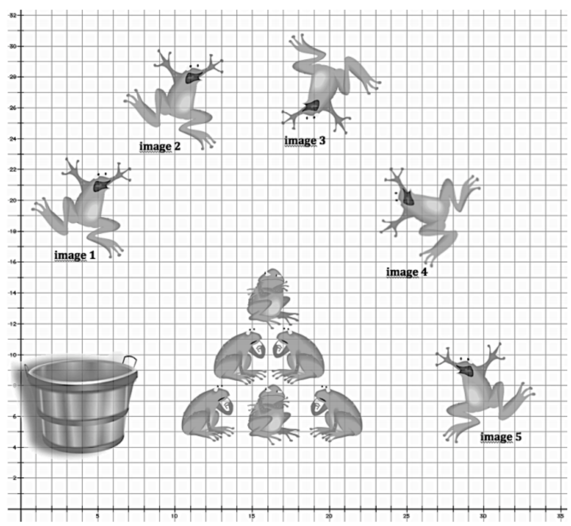


Figure 6- Transformation Activity (MVPS1, M6, p. 16)

Congruence: Understand congruence in terms of rigid motion

CCSS.MATH.CONTENT.HSG.CO.B.6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

The first part of this standard has already been addressed: the MVP material exhibits strong efforts to get learners to transform figures via rigid motions and to predict or infer the effects rigid motions will have on given figures. Turning toward the second portion of the standard, the MVP material at first asks the learner to define congruence her/himself (MVPS1, M6, p.46), and it sets up definitions of geometric figures like rhombi in terms of congruence of sides (MVPS1, M6, p. 60). The MVP material asks the learner the following questions: "What do you know about two figures if they are congruent?" and "What do you need to know about two figures to be convinced that the two figures are congruent?" While the flavor of developing a definition of congruence between two figures is there, a definition in terms of rigid motions is not given until the subsequent year of mathematics (MVPS2, M6, p.16).[5] Finally, the learner is asked to determine whether two figures are congruent and to justify her/his answers (MVPS1, M6, 57).

We find that the MVP material delivers strongly in having learners use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure. However, we notice a trend that the MVP material lacks significant strides to provide definitions to the learner in relevant modules even after discovery exercises. Under this standard the relevant definition is congruent. Alarming is the fact that the MVP material indeed delivers on the last requirement in CO.B.6, but that requirement uses a definition of congruence which is not defined until the next year for the student (unless the student creates her/his own precise definition and her/his teacher

[5]"Two figures are said to be congruent if the second can be obtained from the first by a sequence of rotations, reflections, and translations" (MVPS2, M6, p.16). Congruence is also defined in a more intuitive sense via the statement "two figures are congruent if they are the same size and shape."

ensures the correctness). Nevertheless, the MVP material does, in the end, deliver on both of the requirements of CO.B.6, and it earns a grade of 3 per our rubric.

> CCSS.MATH.CONTENT.HSG.CO.B.7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

This content standard is not explicitly covered in Mathematics One (MVP's first year material). The Mathematics One material goes directly to a claim that meeting the "Angle-Side-Angle" (ASA) criterion ensures that two triangles are congruent (MVPS1, M6, p.52). The discovery activity that follows that claim involves getting the learner to justify why the ASA criterion works. The second and third year material does, however, emphasize the definition of similarity of figures (and triangles specifically) (MVPS2, M6, p. 16).

After a brief treatment of similarity, the material asks the learner to infer what similarity entails for corresponding pairs of sides and angles. Furthermore, the learner is asked to explain the conjecture "two triangles are similar if their corresponding angles are congruent" (MVPS2, M6, p. 19). It is a natural consequence from this treatment that, given the definition of congruence using rigid motions, the learn should be able to show that two triangles are then congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. However, the MVP material does not capitalize on the situation. We assign a grade of 2 for this standard because all of the ingredients are sprinkled throughout the material, but the bi-conditional statement is never made, proven, or given as an exercise.

CCSS.MATH.CONTENT.HSG.CO.B.8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Again, we note that the definition of congruence in terms of rigid motions is not explicitly provided until Mathematics Two, while the content attempting to deliver on this standard lies in Mathematics One. ASA, SAS, and SSS are, however, explained via rigid motions. We present a worked exercise that attempts to get students to understand SSS in terms of transformations (this exercise appears in MVPS1, M6, p. 54-55):

“Zac and Sione are wondering about other criteria, such as SAS or SSS, or perhaps even AAA (which Zac immediately rejects because he thinks two triangles can have the same angle measures but be different sizes) (MVPS1, M6, p. 54).”

7. Draw two triangles that have SSS congruence. Be sure to mark your triangles to show which sides and which angles are congruent.

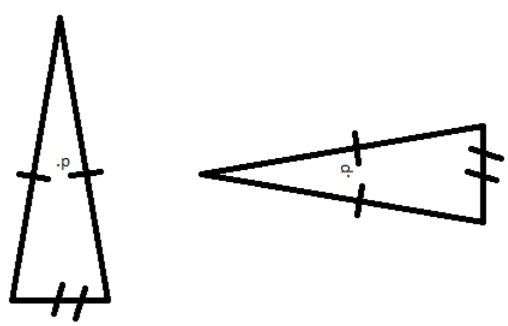


Figure 7- Congruence Activity

8. Write out a sequence of transformations to show that the two triangles potentially coincide.

* Rotate the left-most triangle 90 degrees about point p. Shift the left-most triangle to the right until the sides of both triangles coincide.

9. If Sione were to examine your work in #8, what question would he wonder about?

*Why does this procedure allow us to say these triangles are congruent?

10. How can you use SSS to resolve Simone's wonderings?

* The rigid motions we performed (rotation and translation) preserve distances and carry the left-most triangle onto the right-most. After the transformation, we know that the sides that now overlap coincide in distance because the sides were the same distance originally. Hence the triangles "have the same size and shape," and, furthermore, we showed that there exists a rigid transformation taking the plane to the plane that also takes the left-most triangle onto the right-most triangle. So the triangles are congruent.

This exercise is meant to be done for each of the congruence criteria. This activity, indeed, does an effective job at getting the learner to apply her/his nascent understanding of congruence to an understanding of SSS, SAS, and ASA. But, again, we must underscore the fact that very little is done prior to this point to establish a precise definition of congruence. So, if we assume the learner has that definition available—

even the intuitive definition regarding the “size” and “shape” of figures— then this activity achieves the demands of COA.B.8. Therefore we assign a grade of 3 to the MVP materials on COA.B.8.

Congruence: Prove geometric theorems

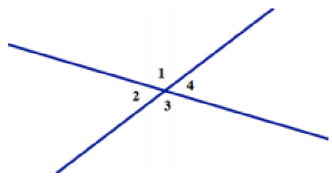
> CCSS.MATH.CONTENT.HSG.CO.C.9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

Before addressing whether the MVP materials lead learners through a path of proving the relevant theorems, we should first outline what the MVP material means by “conjectures,” “proofs,” and “theorems.” Though “conjecture” is never defined, the general character of questions involving conjectures is standard: conjectures are understood to be propositions that are unproven but that one has reason to believe, based on experience, might be true (see e.g. MVPS2, M5, p. 9). More specifically, learners are often asked in the MVP material to speculate about geometric properties that might be true *in general*. The MVP defines theorems as “statements that are supported by justification and proof” (MVPS2, M5, p. 23). There is an obvious problem with this definition to the extent that it does not emphasize the nature of the justification nor proof in this section. However, the MVP material does go on to emphasize that justification of statements should be based on prior established facts. Furthermore, it also emphasizes the difference between “property x being true sometimes” and “property x being true in general” (MVPS2, M5, p. 24).

Focusing now on the proofs demanded in CO.C.9, the MVP material asks the learner to conjecture about vertical angles. The exercise appears in the following figure:

Vertical Angles

When two lines intersect, the opposite angles formed at the point of intersection are called *vertical angles*. In the diagram below, ∠ 1 and ∠ 3 form a pair of vertical angles, and ∠ 2 and ∠ 4 form another pair of vertical angles.



Examine the tessellation diagram above, looking for places where vertical angles occur. (You may have to ignore some line segments and angles in order to focus on pairs of vertical angles. This is a skill we have to develop when trying to see specific images in geometric diagrams.)

Based on several examples of vertical angles in the diagram, write a conjecture about vertical angles.

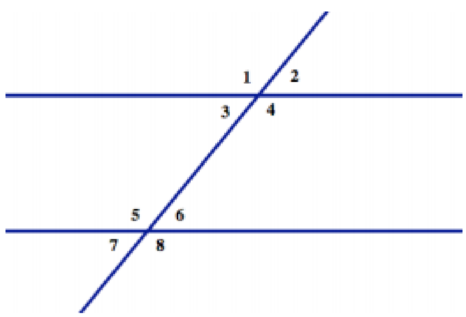
My conjecture:

Figure 8- Vertical Angles Conjecture (MVPS2, M5, p. 28)

From this activity, a learner could conjecture that “vertical angles are congruent.” However, no example is given on how to prove this is true, and it is rather asked that the learner prove the conjecture. The fact that the section in which the activity resides provides no example on how to prove a conjecture causes a bit of alarm regarding adherence to the content standard. Furthermore, the other proofs demanded in CO.C.9 are handled in a similar manner (see, e.g., “when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent” in the figure that follows this page). So, the MVP material does lead the learner through developing conjectures that it wants the learner to also go on to prove. The MVP material thus does cover the demands of the content of CO.C.9, but the fact that the learner is supposed to prove everything her/himself with no salient example leaves a lot to be desired here. We assign the MVP material a 2 because the action item "prove" is not handled at a level that provides us with confidence that the learner would be able to complete all the proofs her/himself.

Parallel Lines Cut By a Transversal

When a line intersects two or more other lines, the line is called a *transversal* line. When the other lines are parallel to each other, some special angle relationships are formed. To identify these relationships, we give names to particular pairs of angles formed when lines are crossed (or cut) by a transversal. In the diagram below, ∠ and ∠ are called *corresponding angles*, ∠ and ∠ are called *alternate interior angles*, and ∠ and ∠ are called *same side interior angles*.



Examine the tessellation diagram above, looking for places where parallel lines are crossed by a transversal line.

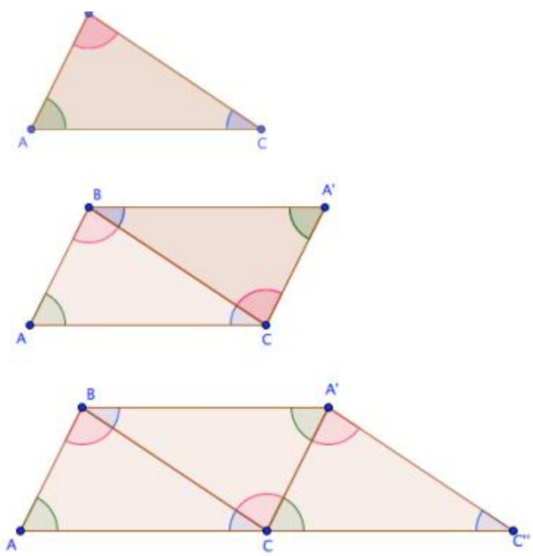
Based on several examples of parallel lines and transversals in the diagram, write some conjectures about corresponding angles, alternate interior angles and same side interior angles.

My conjectures:

Figure 9- Parallel Lines Cut by a Transversal (MVPS2, M5, p. 29)

CCSS.MATH.CONTENT.HSG.CO.C.10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

The MVP material introduces the theorem that measures of interior angles of a triangle sum to 180° by stating to the learner "you probably know that the sum of the interior angles of any triangle is 180°" (MVPS2, M5, p. 3). It then goes on to explain that, when someone makes such a claim, she/he ought to more formally describe why the claim is true. The learner is then asked to walk through a discovery exercise to get her/him to believe that the sum of the angles of a triangle is 180°. The exercise results in the following sequence of figures:



Here are some interesting questions we might ask about this diagram:

5. **Will the second figure in the sequence always be a parallelogram? Why or why not?**
6. **Will the last figure in the sequence always be a trapezoid? Why or why not?**

Figure 10- Sum of angle of triangles is 180 degrees activity (MVPS2, M5, p. 4)

Given its place in the curriculum, the MVP material wants the learner to be thinking back to her/his knowledge about the relationship between interior angles formed by parallel lines. In the subsequent section, linear pairs are introduced and certainly could be used in an attempt at an alternate proof of this section. Hence, the requisite knowledge required to produce a proof is in the MVP material. However, the MVP material never explicitly asks the learner for the proof.

The MVP material's treatment of the remainder of the content in CO.C.10 is similar to the above. That is to say, discovery exercises are provided from which the MVP material hopes the learner will be able to prove the theorems. In every case, the ingredients to a proof are either explicitly given or implied. However, like the above example with triangles, sometimes the material does not capitalize on the fact that is has provided the ingredients for proofs of the theorems in CO.C.10. We assign the MVP material a grade of 2 per our rubric for this section because the action item "prove" is not dealt with the explicitness CO.C.10 demands.

> CCSS.MATH.CONTENT.HSG.CO.C.11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

The MVP material allows the learner to conjecture about properties of parallelograms in Mathematics One. In Mathematics Two, the focus is more on proving the conjectures established in Mathematics One. Each theorem of CO.C.11 is presented as an exercise in proving. The following figure shows how the proofs are presented.

1. Explain how you would locate the center of rotation for the following parallelogram. What convinces you that the point you have located is the center of rotation?

2. If you haven't already, draw one or both of the diagonals in the above parallelogram. Use this diagram to prove this statement: *opposite sides of a parallelogram are congruent*

3. Use this diagram to prove this statement: *opposite angles of a parallelogram are congruent*

4. Use this diagram to prove this statement: *the diagonals of a parallelogram bisect each other*

Figure 11- CO.C. 11 Proofs (MVPS2, M5, p. 33)

Thinking to the previous year's Mathematics One, the learner should be able to use her/his nascent understanding of the requirements for segments to be congruent to prove (in particular by using rigid motions) e.g. opposite sides of a parallelogram are congruent. Furthermore, the components of each statement are provided prior to the proofs being requested (e.g. what does it mean for diagonals to bisect each other?). Therefore the MVP material earns a grade of 3 on CO.C.11 per our rubric.

Congruence: Make geometric constructions

CCSS.MATH.CONTENT.HSG.CO.D.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

The MVP material excels on this standard by providing a plethora of examples of how to use tools and methods to make formal geometric constructions. Compass and straightedge are used most frequently (see the exercise on bisecting angles in MVPS2, M5, p. 39). Furthermore, rope is used to trace out objects like circles and to compare distances. Paper folding is also employed in discovery exercises and in supportive material for forming conjectures (e.g. MVPS2, M5, p. 3-4). Hence, since all of the content is covered under this standard and all action items are covered (including "using software"), we assign the MVP material a grade of 3 on CO.C.12.

CCSS.MATH.CONTENT.HSG.CO.D.13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

The MVP material delivers well on the demands of CO.D.13 (MVPS2, M5, p.8-18 and MVPS2, M7). The presentation is relatively standard, but it does provide some nice exploration into comparing perimeters of inscribed figures within circles with the circumference of the circle— the figure below is illustrative. We assign the MVP materials a grade of 3 per our rubric for this standard.

9. Which of the polygons below would have an area and perimeter closest to the circle it is inscribed within? Why?

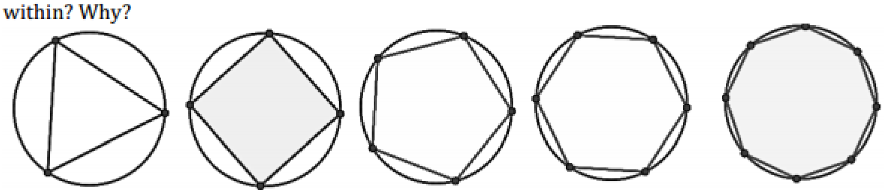


Figure 12- Regular n-gons inscribed inside a circle (MVPS2, M7, p. 30)

Summary of the MVP Alignment with Common Core Strand Congruence

Common Core Standard	Grade
CCSS.MATH.CONTENT.HSG.CO.A.1	2
CCSS.MATH.CONTENT.HSG.CO.A.2	1
CCSS.MATH.CONTENT.HSG.CO.A.3	3
CCSS.MATH.CONTENT.HSG.CO.A.4	3
CCSS.MATH.CONTENT.HSG.CO.A.5	3
CCSS.MATH.CONTENT.HSG.CO.B.6	3
CCSS.MATH.CONTENT.HSG.CO.B.7	2
CCSS.MATH.CONTENT.HSG.CO.B.8	3
CCSS.MATH.CONTENT.HSG.CO.C.9	2
CCSS.MATH.CONTENT.HSG.CO.C.10	2
CCSS.MATH.CONTENT.HSG.CO.C.11	3
CCSS.MATH.CONTENT.HSG.CO.D.12	3
CCSS.MATH.CONTENT.HSG.CO.D.13	3
Mean=	Approximately 2.54

Table 2. MVP Material Alignment Grade Summary (Congruence)

Under the Common Core Congruence Content Strand, the MVP material scores, on average, 2.54 out of 3 per our rubric. This corresponds to a good to excellent alignment with the content strand. As described, the MVP material does an overall excellent job at providing developmental and discovery based exercises. However, there was often a lack of examples of proofs, definitions, and restatements of relevant theorems. Additionally, it was often the case that, while the “ingredients” for each proof were provided in general, the proofs were neither requested of the learner nor supplied to the learner.

Similarity, Right Triangles, and Trigonometry

Understand Similarity in Terms of Similarity Transformations

CCSS.MATH.CONTENT.HSG.SRT.A.1. Verify experimentally the properties of dilations given by a center and a scale factor:

CCSS.MATH.CONTENT.HSG.SRT.A.1.A. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

CCSS.MATH.CONTENT.HSG.SRT.A.1.B. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

We analyze SRT.A.1 together with its subparts. The MVP material adequately provides the learner with experiments from which she/he can understand properties of dilations given by a center and a scale factor. For example, learners are asked to determine the scale factor that a pre-image was multiplied by to create a “larger” or “smaller” image (MVPS2, M6, p. 6). Furthermore, learners are asked to create scaled triangles based on criteria such as lengths of sides of triangles, scale factors, and centers of dilation (MVPS2, M6, p. 9). Finally, learners are also asked to infer when dilations result in “image” lines being parallel to the lines corresponding to them in the pre-image. The MVP material thus scores a 3 per our rubric on SRT.A.1.

CCSS.MATH.CONTENT.HSG.SRT.A.2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

The MVP material provides the learner with a definition of when two figures are similar: “Two figures are similar if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations” (MVPS2, M6, p.16). The learner is then walked through a discovery exercise toward the end of which it is conjectured that “two triangles are similar if their corresponding angles are congruent” (MVPS2, M6, p. 19). This establishes AAA similarity of triangles.

In the subsequent module, the proportionality of sides of similar figures is emphasized. However, there is only minimal effort, beyond its appearance in the definition of similarity, to emphasize the role of transformations in determining whether figures are similar. The “minimal” effort we mention occurs when the material tries to convince the learner that two rectangles (or triangles) are similar: “I can translate and rotate rectangle ABCD until vertex A coincides with vertex Q in rectangle QRST” (MVPS2, M6, p. 17). Hence, the action item in this content standard “explain using similarity transformations…” is not adequately detailed for the learner. We find that it would significantly improve this area of the curriculum if the authors were to explain why the components of similarity transformations are important. We therefore assign a grade of 2 for this content standard.

> CCSS.MATH.CONTENT.HSG.SRT.A.3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

After its treatment of AAA similarity (as discussed under SRT.A.4), the material leads the learner to the conjecture that "two triangles are similar if they have two pair of corresponding congruent angles" (MVPS2, M6, p. 20). Certainly, having established AAA similarity, the student should be able to use her/his understanding about the sum of the angles of any triangle to infer that "if two corresponding angles of two triangles are congruent, then the third angles are also congruent." Hence the learner can deductively obtain the AA criterion for similarity of two triangles, and therefore we assign a grade of 3 to the MVP materials for this content standard.

Prove theorems involving similarity

> CCSS.MATH.CONTENT.HSG.SRT.B.4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

The MVP material delivers well, developmentally, on this content standard. For example, learners are presented with exercises where a line parallel to one side of a specific triangle divides the other two proportionally (MVPS2, M6, p. 43). Learners are also asked to infer side length when a specific triangle is split into two similar triangles. However, regarding the action item "prove," the MVP material leaves a lot to be desired. While the observations that the material wants learners to make are highly intuitive, the material does not ask the learner to summarize what she/he found. Nor does it lead the learner through a path commensurate with conjecturing or proving example statements from this standard. We therefore assign the MVP material a grade of 1 per our rubric for this content standard.

> CCSS.MATH.CONTENT.HSG.SRT.B.5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

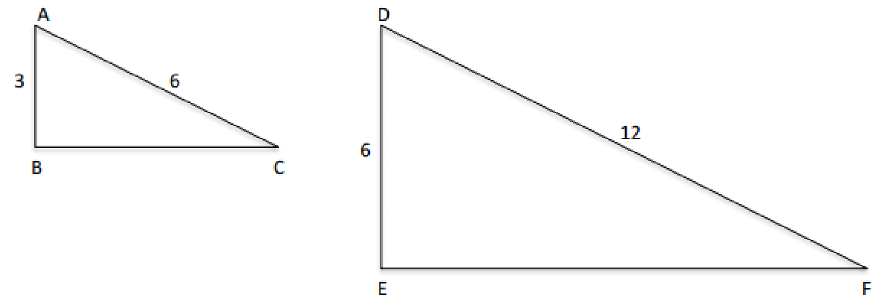
The MVP material delivers well on this content standard. For example, learners are asked to prove that certain polygons are similar using congruence and similarity criteria for triangles (MVPS2, M6, p.20-21). It furthermore uses congruence and similarity criteria for triangles in delivering on the previous content standard (SRT.B.4) (MVPS2, M6, p. 41-42). While it would likely do the material well to include far more exercises under this content standard, the material does meet the demands of the standard entirely. Hence, the MVP material earns a grade of 3 for this content standard.

Define trigonometric ratios and solve problems involving right triangles

> CCSS.MATH.CONTENT.HSG.SRT.C.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

We present the discovery exercise from the MVP material on this content standard in the figure below. This exercise gets the learner to capitalize on her/his understanding of similarity by noticing that side ratios in right triangles are properties of the angles in the triangle. Furthermore, the trigonometric ratios are developed without the (at first obscure) language of sine, cosine, and tangent.

4. In the right triangles below find the missing side length and then create the desired ratios based on the angle of reference (angle A and angle D).



List the ratios for ΔABC using angle A as the angle of reference.

$\frac{opposite\ side}{hypotenuse} =$

$\frac{adjacent\ side}{hypotenuse} =$

$\frac{opposite\ side}{adjacent\ side} =$

List the ratios for ΔDEF using angle D as the angle of reference.

$\frac{opposite\ side}{hypotenuse} =$

$\frac{adjacent\ side}{hypotenuse} =$

$\frac{opposite\ side}{adjacent\ side} =$

5. What do you notice about the ratios from the two given triangles? How do these ratios compare to the ratios from the triangle you made on the previous page?

6. What can you infer about the angle measures of ΔABC and ΔDEF ? Explain?

7. Why do the relationships you have noticed occur?

8. What can you conclude about the ratio of sides in a right triangle that has a 60^o? Would you think that right triangles with other angle measures would have a relationship among there ratios?

Figure 13- SRT.C.6. Exercise (MVPS2, M6, p. 45)

In the subsequent section, the MVP material talks about trigonometric ratios using the traditional vocabulary of sine, cosine, and tangent (MVPS2, M6, p.49). We find this to be a very good pedagogical—and mathematical— choice of the material. That is to say, getting a learner to first understand the interplay of ratios of sides ought to make defining “sine,” “cosine,” and “tangent” natural. Given these considerations and the fact that the MVP material delivers on all of the demands of SRT.C.6., we assign a grade of 3 per our rubric for this content standard.

CCSS.MATH.CONTENT.HSG.SRT.C.7. Explain and use the relationship between the sine and cosine of complementary angles.

Learners are asked, within this standard, to conjecture about the relationship between sine and cosine (MVPS2, M6, p.50). For example, given a general right triangle with a non-right angle a, the learner is asked whether cos(a)=sin(a). The definitions of the quantities cos(a) and sin(a) are given in terms of ratios, as previously discussed, in prior sections. Hence it is reasonable to expect a learner would conjecture that “cos(a)=sin(a) if and only if the ratio “(the length of the ‘adjacent side’)/(the length of the hypotenuse)” is equal to”(the length of ‘the opposite side’)/(the length of the hypotenuse).” Then, since the length of the hypotenuse is constant, it is natural to say “cos(a)=sin(a) if and only if “the length of the ‘opposite side’” is equal to “the length of the ‘adjacent side.’” It seems effective that a learner should make this observation prior to the consideration of the relationship between sine and cosine of complementary angles.

The MVP material asks the learner to conjecture about other quantities, including whether sin(a)=cos(90°-a) and cos(a)=sin(90°-a) (MVPS2, M6, p. 50). Again, it is natural for the student, at this point in the curriculum, to be able to infer the conditional truths of those statements. An explanation that a learner may come up with after learning from this material is that “the trigonometric ratios sine and cosine of a specific angle only differ in the numerator of the ratio. When we change our consideration to the other non-right angle (which is precisely b=90°-a) in the triangle, the formerly adjacent side is now the opposite side and vice versa." Furthermore, in subsequent sections, learners are asked to use this relationship between sine and cosine of complementary angles in exercises. With all of these considerations made, we find that the MVP material score a 3 per our rubric for this content standard.

> CCSS.MATH.CONTENT.HSG.SRT.C.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

The MVP material asks the learner to use trigonometric ratios and the Pythagorean Theorem in a plethora of examples (see, e.g. MVPS2, M6, p. 53-55). The applications range from simply finding missing measures of quantities of specific angles to real world applications of angles of elevation/depression. This content standard is extremely straightforward, and the MVP material delivers a substantial amount of content to which it is relevant. We therefore assign the MVP material a grade of 3 per our rubric for this content standard.

Apply trigonometry to general triangles

> CCSS.MATH.CONTENT.HSG.SRT.D.9. Derive the formula A = 1/2 ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

The figure below represents the only attempt in the MVP materials to get the learner to derive the formula A=1/2 ab sin(C). While it is admirable that the material wants the learner to really apply her/his knowledge to the activity, we find that the setup is far too vague. Even adding some guiding framework such as "You will now prove the following formula A=1/2 ab sin(C), with the relevant quantities as they appear in the following figure" would improve the alignment. The ingredients to a derivation (or even proof) of the relevant formula have been previously provided in the material. Nevertheless, we must assign the MVP material a grade of 1 per our rubric for this standard because very little guidance is provided in the attempt to get the learner to "derive" the relevant formula.

Find the area of the following two triangles using the strategies and procedures you have developed in the past few tasks. For example, draw an altitude as an auxiliary line, use right triangle trigonometry, use the Pythagorean theorem, or use the Law of Sines or the Law of Cosines to find needed information.

2. Find the area of this triangle.

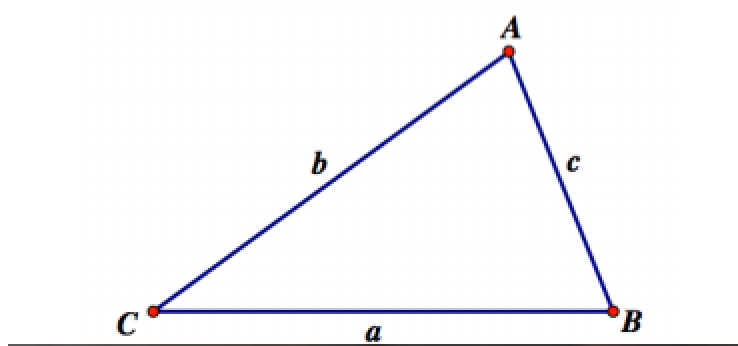


Figure 14- Area of a triangle activity (MVPS3, M5, p. 44).

CCSS.MATH.CONTENT.HSG.SRT.D.10. Prove the Laws of Sines and Cosines and use them to solve problems.

The MVP material aims to help learners develop the geometric and algebraic ideas involved in the Law of Sines and the Law of Cosines (MVPT, M5, p. 32-33) before proving them. The relevant discovery activities involve the learner finding missing quantities of non-right triangles. The learner is then asked to "find as many relationships as you can" between the quantities that describe an obtuse triangle (MVPS3, M5, p.32). The hope here, of course, is that the learner conjectures the Law of Sines simply based on the specific quantities she/he computes. Eventually the material asks the student to prove the Law of Sines given the structure of the proof (MVPS, M5, p. 40).

In a seemingly rare instance for the material, the MVP material provides proof of the Law of Cosines to the learner (MVPS2, M5, p. 35-40). The given proof is based off a proof by Don McConnell in *The Illustrated Law of Cosines* (McConnell, 2014). The proof seems relatively standard: Consider an arbitrary triangle and the areas of the squares formed by the sides of the triangles, with the squares drawn outward as in the following figure:

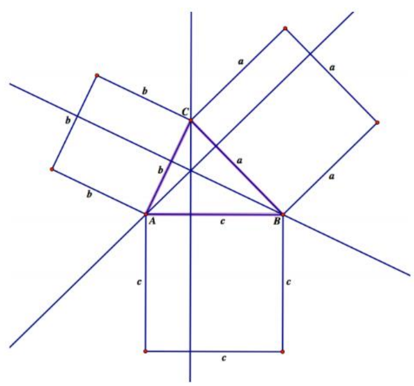


Figure 15- Law of Cosines Proof Setup (MVPS3, M5, p. 38)

After some algebraic considerations of the setup, the material leads the learner to the point where they state $a^2 = b^2 + c^2$-2bc cos(A) together with similar formulations for b^2 and c^2 (MVPS3, M5, p. 37-40 & MVPT3, M5, p.37-41). Additionally, the material asks the learner to use the Laws of Sines/Cosines in example problems. We conclude that the MVP material covers the Laws of Sines/Cosines very well— in fact, it does an objectively better job in providing the learner with the structure of a proof before demanding a similar proof. Thus, we assign the MVP material a grade of 3 per our rubric for this content standard.

> CCSS.MATH.CONTENT.HSG.SRT.D.11. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

As already discussed with the analysis of the alignment of the MVP material with SRT.D.10, the MVP material provides a plethora of examples to which learners can apply their nascent understandings of the Laws of Sines/Cosines. To illustrate further, we present an activity in the figure below (appears on the next page) that the MVP material provides the learner. We assign the MVP material a grade of 3 per our rubric for this standard because the material meets all action items (“understand”/”apply”) via various activities for the learner.

Jumal and Jabari are helping Jumal’s father with a construction project. He needs to build a triangular frame as a component of the project, but he has not been given all the information he needs to cut and assemble the pieces of the frame. He is even having a hard time envisioning the shape of the triangle from the information he has been given.

Here is the information about the triangle that Jumal’s father has been given.

- Side a = 10.00 meters
- Side b = 15.00 meters
- Angle A = 40.0°

Jumal’s father has asked Jumal and Jabari to help him find the measure of the other two angles and the missing side of this triangle.

Carry out each student’s strategy as described below. Then draw a diagram showing the shape and dimensions of the triangle that Jumal’s father should construct. (Note: To provide as accurate information as possible, Jumal and Jarbari decide to round all calculated sides to the nearest cm—that is, to the nearest hundredth of a meter—and all angle measures to the nearest tenth of a degree.)

Figure 16- Application Problem Using Laws of Cosine/Sine (MVPS3, M5, p.45)

Summary of MVP Alignment with Common Core Stand Similarity/Trigonometry

Common Core Standard	Grade
CCSS.MATH.CONTENT.HSG.SRT.A.1	3
CCSS.MATH.CONTENT.HSG.SRT.A.2	2
CCSS.MATH.CONTENT.HSG.SRT.A.3	3
CCSS.MATH.CONTENT.HSG.SRT.B.4	1
CCSS.MATH.CONTENT.HSG.SRT.B.5	3
CCSS.MATH.CONTENT.HSG.SRT.C.6	3
CCSS.MATH.CONTENT.HSG.SRT.C.7	3
CCSS.MATH.CONTENT.HSG.SRT.C.8	3
CCSS.MATH.CONTENT.HSG.SRT.D.9	1
CCSS.MATH.CONTENT.HSG.SRT.D.10	3
CCSS.MATH.CONTENT.HSG.SRT.D.11	3
Mean=	Approximately 2.55

Table 3. MVP Material Alignment Grade Summary (Similarity/Trigonometry)

On average, the MVP material scored a 2.55 per our rubric within the Similarity/Trigonometry content strand. That corresponds to a good to excellent alignment with the Common Core Standards. Furthermore, a score of 2.55 represents slightly better alignment in this strand as compared with the MVP alignment score of 2.46 within the congruence content strand. Qualitatively, the MVP material did an effective job, overall, in providing the learner with a strong basis for understanding the relevant algebra and geometry via discovery exercise. Moreover, the MVP material actually provided the majority of a proof of the Law of Cosines— a feature that is relatively rare in the MVP material.

To address its drawbacks, we note that the MVP material omitted necessary direction in getting learners to think about the Pythagorean Theorem in terms of similarity. Moreover, the lack of a structured discussion about the formula for the area of an arbitrary triangle left a lot to be desired in establishing alignment under the relevant content standard. Finally, we observe that the MVP material is still rather inconsistent in providing all the necessary ingredients/structure to learners before asking them for proofs.

Circles

Understand and apply theorems about circles

CCSS.MATH.CONTENT.HSG.C.A.1. Prove that all circles are similar.

The MVP material delivers strongly on this content standard. It provides guidance through a standard proof of the fact that all circles are similar: using translation and dilation (MVPS2, M7, p.9). Furthermore, guidance through an alternate proof focusing on finding the center of dilation that maps pre-image points on one circle to image points on another circle. Since the MVP material covers the content in this standard at a level that gives us confidence that learners can, at minimum, understand the ingredients of the proof, but likely they could even reproduce the proof themselves, we assign the MVP material a grade of 3 for this content standard.

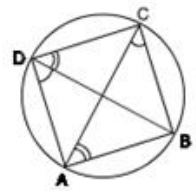
CCSS.MATH.CONTENT.HSG.C.A.2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

The MVP material walks learners through conjecturing statements of the relevant properties/theorems of this standard, and it also helps learners setup proofs of their conjectures (MVPS2, M7, p.14-21). We present an example of the manner in which the MVP material presents content under this standard: relationships among inscribed angles. In particular, the material asks the learner to conjecture about the measure of the angles of a cyclic[6] quadrilateral. The learner, based on previous work, would notice that any such cyclic quadrilateral can be sectioned into four triangles by drawing the two line segments that connect points through the cyclic quadrilateral. This is exemplified in the figure below. From that point, the learner can do some algebra, using her/his knowledge of inscribed angles, to obtain the proof that the sum of a cyclic quadrilateral's opposite angles is 180 degrees. We present the proof within the figure below. Since all the ingredients are provided for describing, conjecturing, and proving all specific content under this standard, we assign the MVP material a grade of 3 for this content standard.

[6] The MVP material defines cyclic polygons as polygons all of whose vertices lie on a specific circle.

CLAIM: The sum of the measures of the opposite angles in a cyclic quadrilateral is 180 degrees.

Proof. **Consider the figure to the right, and note the following equalities hold since the respective angles are in the same segment.**



$$\angle ACB = \angle ADB$$

and $\angle BAC = \angle BDC$

Now add these two equations to obtain:

$$\angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$$

Add $\angle ABC$ **to both sides:**

$$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$$

However, $\angle ACB + \angle BAC + \angle ABC = 180°$
since those three angles form a triangle, and the sum of the angles of any triangle is 180 deg.

Hence $\angle ADC + \angle ABC = 180°$

and thus $\angle BAD + \angle BCD = 360° - (\angle ADC + \angle ABC) = 180°.$ ■

Figure 17- PF that the sum of the opposite angles of a cyclic quadrilateral is 180 degrees.

CCSS.MATH.CONTENT.HSG.C.A.3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

The MVP material asks the learner to construct inscribed and circumscribed circles of a triangle with straightedge and compass (MVPS2, M7, p. 17). The learner, indeed, has been convinced via discovery exercise that there exists a point that is equidistant from each of the three sides of any triangle (MVPS2, M5, task 8). Hence the MVP material delivers on the first half of C.A.3. Moreover, we have already seen (as in C.A.2.) that the MVP material asks students to conjecture and prove properties of angles for quadrilaterals inscribed in circles. Therefore the MVP material receives a grade of 3 per our rubric for content standard C.A.3.

CCSS.MATH.CONTENT.HSG.C.A.4. Construct a tangent line from a point outside a given circle to the circle.

Learners are asked, in an exercise in conjecturing then proving that the measure of a tangential angle is equal to 180° minus the measure of the central angle that forms the same arc, to notice how to construct a tangent line from a point outside a given circle to the circle (MVPS2, M7, p. 18). The learner is then asked to write out a procedure to construct a tangent line from a point outside a given circle to the circle. Hence, a specific example in an exercise leads to the learner reasoning more abstractly. Since the MVP material covers all of the demands of C.A.4, we assign it a grade of 3 per our rubric.

Find arc lengths and areas of sectors of circles

> CCSS.MATH.CONTENT.HSG.C.B.5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Each of the action items under this content standard is addressed explicitly by the MVP material at a level where learners should be able to persevere in learning the content (MVPS2, M7, tasks 7-9). As an example, we consider the MVP's attempt to define the radian measure of an angle as the constant of proportionality. The MVP material asks the learner to observe (via exercises) that there is another way to measure angles: "all arcs in the same sector have the same degree measurement, and all arcs in the same sector have the same for the ratio of arc length to radius… [the] new numbers for measuring angles in terms of the ratio of the arc length to the radius are know (sic) as *radians* and that they make the rules of calculus much easier than if angles are measure in degrees" (MVPS2, M7, p. 43-44). Forgiving the typo and the irrelevant (to the learner) information about calculus, this definition is in the flavor of what the Common Core demands. Moreover, the discovery exercises really do lead learners through effective examples to get them to see that radian measures really do behave the "same way" that degree measurements behave. With all of these considerations made— and noting that the rest of the content under this standard is handled at a similar level— we assign the MVP material a grade of 3 for this content standard.

Summary of MVP Alignment with Common Core Strand Circles

The MVP material earned a "perfect" 3 under each content standard within the circle strand (and thus averages 3 for the strand— we omit a table breaking down the scores per standard). This corresponds to an excellent alignment with the Common Core Standards. There are very few criticisms we can make of the MVP treatment of circles since it adequately develops the foundational ideas, leads learners through a plethora of general and specific examples to underscore the ideas, and delivers on all action and content items. There was a case where the authors might consider removing extraneous information about calculus (see the discussion of C.B.5)— it is easy to imagine the learner reading this material and wondering what calculus is and why she/he should care whether radians make it "easier." Of course that is an extremely minor criticism, and it has very little to do with alignment of the material to the Common Core.

Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

> CCSS.MATH.CONTENT.HSG.GPE.A.1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

The MVP material delivers well on all of the action/content items in GPE.A.1. To exemplify that, let us consider its discovery exercise for deriving the equation of a circle of a given center and radius using the Pythagorean Theorem. Learners are asked to cut out four congruent triangles from paper with the condition that the hypotenuse is 6” long (MVPS2, M8, p. 3). The learner is then asked to place the triangles in the same fashion as they are placed in the following figure.

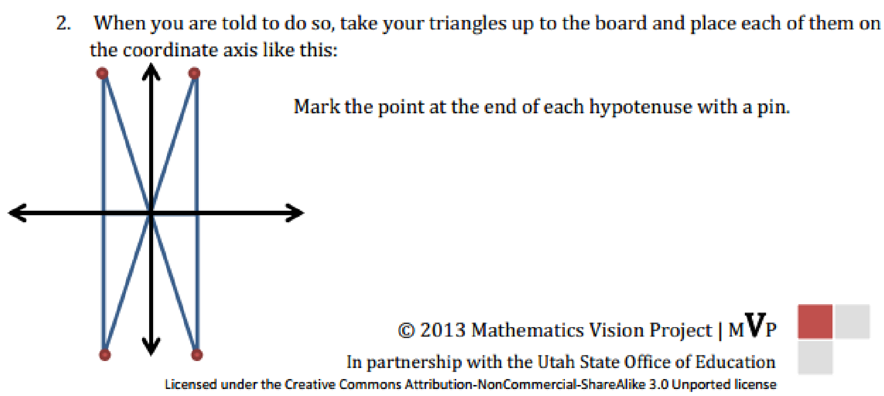


Figure 18- Placement of triangles (MVPS2, M8, p.3.).

The material expects the teacher to guide her/his class into arranging the triangles to form the following figure.

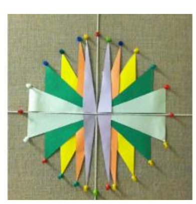


Figure 19- SS triangle placement (MVPT2, M8, p.6)

Following that setup, the learner would identify that all of the triangles taken together appear to be “filling in” a circle of radius r=6. The learner is then asked to notice that $x^2 + y^2 = 6^2$ by the Pythagorean Theorem and to consider all of the pin points on the figure. The material then leads the learner to the conclusion that the equation of the circle of radius 6 centered at the origin is given by $x^2 + y^2 = 36$. To reinforce the concepts and to generalize further, the material asks how the equation might change if the circle was shifted up 2 units and to the right 6 units. Using their understanding of shifts of graphs, it is clear the student would be able to come up with the answer and therefore a general equation for a circle centered at (h,k) of radius r. Furthermore, the material goes on to use the technique of completing the square within example equations to rewrite them in the general form of a circle which they have developed. We conclude the MVP material excels in every area of this content standard, and therefore it earns a grade of 3 per our rubric for this content standard.

CCSS.MATH.CONTENT.HSG.GPE.A.2. Derive the equation of a parabola given a focus and directrix.

The MVP material leads the learner through a derivation of the equation of a parabola given a focus and directix using string (MVPS2, M8, p. 17-18). The learner is first asked to construct a single parabola by positioning a piece of string such that the midpoint is equidistant from the line called the directrix and the focus point provided. The learner should have a figure resembling the following figure.

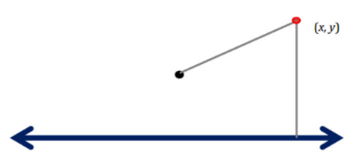


Figure 20- Step in the construction of a parabola (MVPS2, M8, p. 17).

The learner is then asked to repeat the above procedure with another piece of string that is equal in length to the string used above (obtaining the pinpoint corresponding to the reflection of the point (x,y) about the vertical line of symmetry through the focus). Next, the learner is asked to deduce what would happen if she/he used strings of different lengths in the same procedure. This leads the learner to a definition of “parabola” which is explicitly provided and then to the formulation of the equation of a general parabola. Further, questions are posed to the learner regarding what happens to the equation when the directrix and/or focus is changed. Thus, the MVP material delivers on the demand of GPE.A.2 completely, and we assign it a grade of 3 per our rubric for GPE.A.2.

CCSS.MATH.CONTENT.HSG.GPE.A.3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

The MVP material does not make any effort to deliver on this standard explicitly. We conclude that the MVP material’s grade for this standard is 0, however the grade is not applicable regarding the general alignment because the content standard is plused.

Use coordinates to prove simple geometric theorems algebraically

CCSS.MATH.CONTENT.HSG.GPE.B.4. Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).

This content standard is very general, and the MVP material does provide some discovery exercises geared toward using coordinates to prove simple geometric theorems algebraically. For example, the MVP material asks the learner whether given figures are squares, rectangles, or rhombi (MVPS1, M7, p.17-20). The learner is then asked to justify her/his thoughts (i.e. “prove”). Additional activities exist throughout the curriculum relevant to this standard as well. We therefore assign the MVP material a grade of 3 for GP.E.B.4.

> CCSS.MATH.CONTENT.HSG.GPE.B.5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

We have largely discussed the MVP material related to this standard within the congruence content strand. Learners are led through activities from which they learn to justify the slope criteria for parallel and perpendicular lines (MVPS1, M7, p. 9-13). Furthermore, activities exist in the relevant module to find equations of lines parallel or perpendicular to given lines passing through given points. Hence the MVP material covers all content and action items within this content standard so we assign it a grade of 3 per our rubric for GPE.B.5.

> CCSS.MATH.CONTENT.HSG.GPE.B.6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

The MVP material delivers on this content standard by first asking the learner how she/he might find the “midpoint” of a directed segment (MVPS2, M6, p. 33). The material explains that the midpoint of a directed segment divides the segment in two equal length segments. Thus, if the endpoints of a line segment are a=(x,y) and b=(x',y'), then the midpoint must lie on the segment and be exactly half the distance relative to the horizontal distance between a and b and exactly half the distance relative to the vertical distance between a and b. The learner is then asked to find points on directed line segments between two given points that partitions the segments in given ratios. For example, the learner is walked through an exercise where she/he is asked where to place a "stake" in a garden so that a row proportioned in a 2:3 ratio (using given coordinates). We therefore find that the MVP material delivers completely on the action item of this standard, and we assign it a grade of 3 per our rubric for this standard.

> CCSS.MATH.CONTENT.HSG.GPE.B.7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

The MVP material again delivers on the content of this standard very well. For example, learners are asked to find the perimeter of the following polygon using the distance formula:

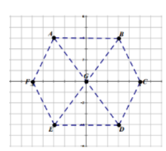


Figure 21- Find the perimeter of this figure (MVPS1, M7, p. 4)

Furthermore, there are a plethora of examples throughout the curriculum where learners are asked to compute areas of geometric figures. The methods of computation range from using the Euclidean distance formula to using trigonometric formulations for the area of a general triangle (as already discussed in the standards in the SRT strand). We therefore conclude that the MVP material earned a grade of 3 per our rubric for GPE.B.7.

Summary of MVP Alignment with Common Core Strand Expressing Geometric Properties with Equations

The MVP material averaged a “perfect” 3 for the GPE strand alignment, so we omit a table that breaks down the scores. This corresponds to an excellent alignment with the Common Core. Very generally speaking, the standards within the GPE strand are very exercise-oriented. And, as we have seen, the MVP material excels at providing discovery exercises that are highly explanatory and that lead “successful” learners to the formation of a strong intuition for the mathematics at play. It is natural, then that the MVP material would score extremely well on this strand.

Geometric Measurement & Dimension

Explain volume formulas and use them to solve problems

> CCSS.MATH.CONTENT.HSG.GMD.A.1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

As previously discussed in standard CO.D.3, the MVP material does an effective job at developing the intuition behind the formula for the circumference of a circle by looking at perimeters of regular n-gons inscribed in circles (as n increases, i.e. it is an informal limit argument). Furthermore, the MVP material gives the learner discovery exercises aimed at arguing formulae for areas of circles, volumes of cylinders, pyramids, and cones. However, the MVP material lacks any stride to explain Cavalieri’s principle.

We therefore assign the MVP material a grade of 2 per our rubric for this content standard.

> CCSS.MATH.CONTENT.HSG.GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

This standard is a “plused” standard, and the MVP material, as described in GMD.A.1, lacks any effort to explain Cavalieri’s principle, and it does not use the principle either. We assign a grade of 0 for this standard, but note that the fact it is a “plused” standard means the grade will not be factored into MVP’s mean for this strand.

> CCSS.MATH.CONTENT.HSG.GMD.A.3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

The MVP material leads learners through the use of volume formulas for cylinders, pyramids, cones, and spheres (MVPS3, M5, p. 1-20). The practice problems range from computing volumes of a relevant solids given dimensions to application problems like finding the volume of the Washington Monument (learners first find the volume of the pyramidal “tip” and then the volume of frustum). Since there is a plethora of examples to which learners can apply the relevant volume formulae, we assign the MVP material a grade of 3 per our rubric for GMD.A.3.

Visualize relationships between two-dimensional and three-dimensional objects

CCSS.MATH.CONTENT.HSG.GMD.B.4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

The demands of this standard are very straightforward, and he MVP material delivers completely by providing discovery exercises tailored to each sub-standard. We provide an example where students are asked to identify the three-dimension object generated by rotation of a two-dimensional object below (the 3d figure will look like a short barbell). We assign the MVP material a grade of 3 per our rubric for GMD.B.4.

14. Draw a sketch of the three-dimensional object formed by rotating the figure about the *x*-axis.

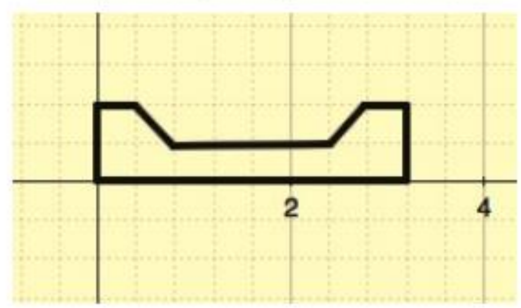


Figure 22- 3-d rotation of a 2-d figure (MVPS3, M5, p. 18)

Summary of MVP Alignment with Common Core Strand GMD

Standard	Grade
CCSS.MATH.CONTENT.HSG.GMD.A.1	2
CCSS.MATH.CONTENT.HSG.GMD.A.2	0 but "PLUSED"
CCSS.MATH.CONTENT.HSG.GMD.A.3	3
CCSS.MATH.CONTENT.HSG.GMD.B.4	3
Mean=	Approximately 2.67

Table 4. MVP GMD Alignment

The MVP material earned an average of approximately 2.67 for the "Geometric Measurement & Dimension" content strand. This corresponds to a good to excellent alignment with the Common Core. We note that the MVP material does an excellent job at providing a plethora of examples from which students can learn material relevant to each content standard. However, the MVP material spends no discernable time on describing Cavalieri's principle. We note that the intuition behind Cavalieri's principle is simple to describe: stack 10 U.S. quarters vertically in line, and then stack 10 U.S. quarters so that a portion of each is not in line with the next (as in the figure below). Certainly each stack of quarters is of the same height and volume. It would do the MVP material well to provide this or a similar example to the learner.

Figure 23- Cavalieri's Principle "real world" example (Chap, 2013)

Modeling with Geometry

Apply geometric concepts in modeling situations

CCSS.MATH.CONTENT.HSG.MG.A.1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

The MVP material offers the learner many exercises wherein she/he uses geometric shapes, their measures, and their properties to describe real world objects. We have already seen such an example in the prior content strand: the MVP material helps the learner model the volume of the Washington Monument using a pyramid and frustum. Another example occurs when the MVP material asks the learner to model a snowman, barrel, and flashlight using 2-d geometric shapes (MVPS3, M5, p.12) and their revolutions about axes. We therefore assign the MVP material a grade of 3 per our rubric for MG.A.1.

CCSS.MATH.CONTENT.HSG.MG.A.2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

The MVP material again delivers on the demands of this standard by providing numerous examples to the learner (MVPS3, M5, p. 23). For example, consider the figure that follows this paragraph. The activity is explanatory and gets the learner to apply concepts of density in modeling situations. We therefore assign the MVP material a grade of 3 for standard MG.A.2.

9. Density relates to the degree of compactness of a substance. A cubic inch of gold weighs a great deal more than a cubic inch of wood because gold is more dense than wood. The density of grains also varies. Use the information below to calculate how many tons of each grain can be stored in one silo. (1 ton = 2000 lbs.)

1 bushel of oats weighs 32 pounds

1 bushel of barley weighs 48 pounds

1 bushel of wheat weighs 60 pounds

Figure 24- Density problem (MVPS2, M5, p. 23)

CCSS.MATH.CONTENT.HSG.MG.A.3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

The MVP material delivers well on this standard as well. For example, learners are presented with constraints on physical dimensions of a pickup truck's bed, and then they are asked to determine if a certain volume of material will fit in the bed of the truck (MVPS3, M5, p. 24). Given the MVP material covers this standard with a plethora of examples, we assign it a grade of 3 per our rubric for MG.A.3.

Summary of MVP Alignment with Common Core Strand Modeling with Geometry

The MVP material aligns with a “perfect” 3 score across all standards in this strand. That is, of course, expected since we know that the MVP material provides numerous examples and discovery activities for learners. Perhaps the material could be improved by further diversifying activity types: often one activity’s setup will span multiple pages in the material.

General Conclusions Regarding the MVP Material Alignment with the Common Core Geometry Domain

As detailed above, the MVP material aligns at a good to excellent or excellent level with the Common Core across all geometry domain strands. The strengths of the MVP material lie in providing numerous examples that aim to get learners to understand the underlying mathematics. It furthermore provides, either via teacher or in text, most of the ingredients that a learner would need to produce a relevant proof or definition. On the other hand, occasionally the MVP material omits important information (such as Cavalieri’s Principle), or it reserves capitalization on concepts until the successive year in mathematics (such as definition of congruence in terms of rigid motion). Indeed, the fact that the MVP material breaks apart the geometry curriculum into three sections has both pros and cons. We reserve further discussion to that end until after we have presented an analysis of the alignment of EngageNY’s geometry curriculum to the Common Core geometry domain.

ANALYZING THE ALIGNMENT OF THE NEW YORK ENGAGENY SECONDARY SCHOOL GEOMETRY CURRICULUM WITH THE COMMON CORE

EngageNY is an educational initiative that aims to produce material that aligns well with the Common Core mathematics standards (ENY, 2014). As we have already discussed in our introduction and literature review, the EngageNY material has a significant amount of discovery-based material, but it also has more traditional content like in-depth explanations within the material.. It is an open source set of material that is available on the EngageNY Web Site. Our citation of the EngageNY material is as follows: (material version (S=student, T=Teacher), module number, topic, lesson number, page number).

Congruence

Congruence: Experiment with transformations in the plane

> CCSS.MATH CONTENT.HSG.COA 1: Know the precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

The EngageNY material develops each of the undefined notions mentioned in this standard through discovery exercise (ENY, M1)[7]. For example, learners are presented with a hypothetical situation where people are playing catch. The learner is asked how to position the three players such that the distance is the same between each player. Thus this portion of the material asks the learner to gain some intuition about distance. The material then goes on to more precisely define the distance between two points: “The length of the segment $\overline{AB}$ is the distance from A to B denoted AB. Thus AB=dist(A,B)”

[7] Our citation of the EngageNY material is as follows: (EngageNY material version (S=student, T= teacher, no suffix= both), module number, topic section, lesson number, and pages when relevant).

(ENYS, M1, TA, L1, p. 5). The other undefined notions within this standard are handled similarly. Hence, the EngageNY material actually goes a step further from simply developing the undefined notions— it actually defines them.

Moving toward the first demands in this standard, we note that the EngageNY material asks the learner for precise definitions of circles, angle, and line segment. For example, the learner is asked to “fill in the blank” for a definition of a circle (ENYS, M1, TA, L1, p.1). Parallel and perpendicular lines are meant to be defined by the instructor utilizing this material, and both of those objects are used in proofs in a manner where the definition is used explicitly. Explicit definitions of parallel and perpendicular lines are also asked of the learner far into the material (e.g. ENYS, M1, TC, L17, p.1). We therefore assign the EngageNY material a grade of 3 for this content standard, as it covers every item here at the level expected by the standard.

> CCSS.MATH CONTENT.HSG.COA 2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

The learner is presented with examples of transformations in the plane graphically in the EngageNY material, and instructors are meant to show their students representations of transformations in the plane using technology (ENYT, M1, TC, L12). Transformations are, furthermore, described as functions that take each point p to a point f(p). The material also emphasizes the difference between transformations that preserve distance and angle and those that do not (through developing a definition of dilation).

Hence, the EngageNY material delivers on each of the demands of this content standard and earns a grade of 3 per our rubric for this standard.

CCSS.MATH.CONTENT.HSG.CO.A.3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

The EngageNY material provides an example in determining the rotations and reflections that carry a figure onto itself (ENYS, M1, TC, L15). It then asks the learner to describe the rotations and reflections that carry a figure to itself, starting with the more simple examples. We present an exercise offered the learner in the figure below, and conclude that the EngageNY material delivers on the demands of this content standard fully. So it earns a grade of 3 per our rubric for this standard.

Exercises 1–3

Use Figure 1 to answer the questions below.

1. Draw all lines of symmetry. Locate the center of rotational symmetry.

2. Describe all symmetries explicitly.
   a. What kinds are there?
   b. How many are rotations? (Include a "360° rotational symmetry," i.e., the identity symmetry.)
   c. How many are reflections?

Figure 1

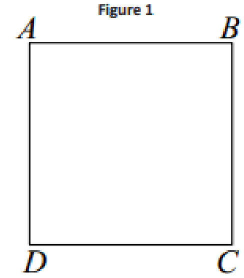
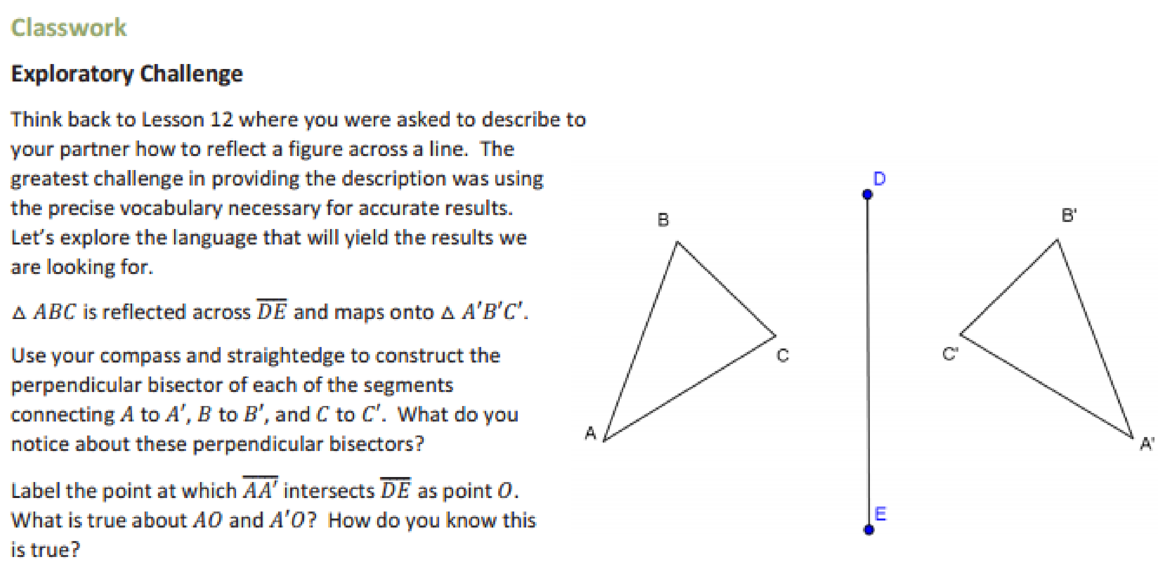


Figure 25- CO.A.3. Exercise (ENYS, M1, TC, L15, p. 3).

CCSS.MATH.CONTENT.HSG.CO.A.4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

The EngageNY material both develops and gives definitions of rotations, reflections, and translations (ENYS, M1, TC, L13-16). As an example of the development of the definitions, consider the figure that follows this paragraph. It presents an example of reflecting a triangle about a line. As the exercise tries to get the learner to notice, reflections are defined using lines of reflections. Finally, the EngageNY material provides a definition of a reflection: A reflection across line l in the plane is a transformation r of the plane such that a) for any point P on the line l, r(P)=P, and b) for any point p not on the line l, r(P) is the point Q so that l is the perpendicular bisector of the segment PQ. EngageNY's treatment of reflections and translations is similar. We therefore conclude that EngageNY delivers on everything demanded in this standard. Thus, it earns a grade of 3 per our rubric for this content standard.



Classwork

Exploratory Challenge

Think back to Lesson 12 where you were asked to describe to your partner how to reflect a figure across a line. The greatest challenge in providing the description was using the precise vocabulary necessary for accurate results. Let's explore the language that will yield the results we are looking for.

$\triangle ABC$ is reflected across $\overline{DE}$ and maps onto $\triangle A'B'C'$.

Use your compass and straightedge to construct the perpendicular bisector of each of the segments connecting A to A', B to B', and C to C'. What do you notice about these perpendicular bisectors?

Label the point at which $\overline{AA'}$ intersects $\overline{DE}$ as point O. What is true about AO and $A'O$? How do you know this is true?

Figure 26- Reflecting a triangle about a line (ENYS, M1, TC, L14, p. 1).

CCSS.MATH.CONTENT.HSG.CO.A.5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

The EngageNY material asks the learner to draw transformed figures using graph paper and WordArt in Microsoft Word (e.g. ENYS, M1, TC, L13, p. 8). Furthermore, learners are asked on many occasions to determine sequences of transformations that will carry given figures to others (e.g. ENYS, M1, TC, L17 and L19). Hence, EngageNY earns a grade of 3 per our rubric for this content standard.

Congruence: Understand congruence in terms of rigid motion

CCSS.MATH.CONTENT.HSG.CO.B.6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

The learner is told by the EngageNY material that rotations, translations, and reflections are all rigid motions— i.e. they preserve the lengths of segments and the measures of angles (ENYS, M1, TC, L19, p.1). The material goes on to refresh for the learner the notion of congruence she/he had learned in grade 8, namely that congruent figures have the "same size and shape." The material emphasizes to the learner that such a description of congruence is not a precise definition of congruence. It then tells the learner that two figures are congruent with respect to each other if each can be obtained from the other by a sequence of rotations, reflections, and translations. The following figure is relevant.

2. Use the figures below to answer each question:

   a. $\triangle ABD \cong \triangle CDB$. What rigid motion(s) maps $\overline{CD}$ onto $\overline{AB}$? Find two possible solutions.

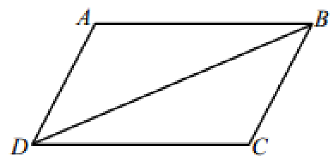


Figure 27- Congruence activity (ENYS, M1, TC, L19)

The EngageNY material goes on to ask the learner to predict effects of given rigid motions on given figures (ENYS, M1, TC, L19, p. 3). The learner is then expected to use the above definition of congruence in terms of rigid motions to decide whether two figures are congruent. We note, however, that the EngageNY material lacks a significant number of exercises that it wants the learner to go on and do. Nevertheless, the EngageNY material delivers on all of the demands of this content standard, and therefore it earns a grade of 3 for this content standard.

> CCSS.MATH.CONTENT.HSG.CO.B.7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

The EngageNY material refreshes for the learner the definition of congruence: two figures are congruent if there exists a rigid motion that maps the first onto the second (ENY, M1, TC, L20, p1). It then explains that rigid motions produce one-to-one correspondences between points in a figure and points in the image. Also, rigid motions map each part of a figure to a corresponding part of the image. Therefore, corresponding parts of congruent figures are congruent. The learner is then asked to apply this knowledge to triangles. Hence, the EngageNY material provides the basic structure of a proof of the bi-conditional statement in this standard, and it asks the learner to apply it to a general triangle.

We conclude that the EngageNY material delivers on this standard, but we note that it never formulates the content explicitly as it is in the standard— i.e. there is no explicit proof asked or given that “two triangles are congruent iff corresponding pairs of sides and corresponding pairs of angles are congruent.” Yet, as described above, we can be confident that a learner could prove this statement as necessary. Therefore the EngageNY material receives a grade of 3 per our rubric on this content standard.

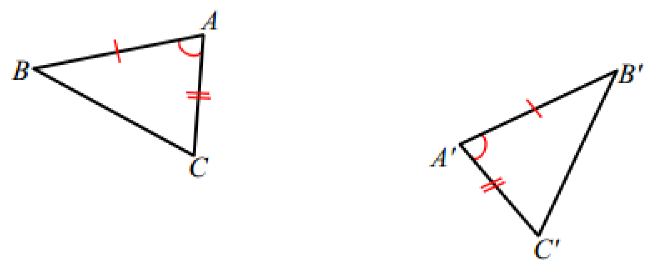
> CCSS.MATH.CONTENT.HSG.CO.B.8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

The EngageNY material states the criteria for triangle congruence and provides proofs of them in terms of definition of congruence in terms of rigid motions (ENY, M1, TD, L22-25). We provide the example provided by EngageNY of SAS triangle congruence below. Note that the “proof” uses exactly what the Common Core standard wants (i.e. the use of definition of congruence in terms of rigid motions). We assign a grade of 3 to EngageNY per our rubric for this standard.

Side-Angle-Side Triangle Congruence Criteria (SAS): Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $\mathrm{m}\angle A = \mathrm{m}\angle A'$ (Angle), $AC = A'C'$ (Side). Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. It is important to understand that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfies the SAS criteria

Proof: Provided the two distinct triangles below, assume $AB = A'B'$ (Side), $\mathrm{m}\angle A = \mathrm{m}\angle A'$ (Angle), $AC = A'C'$ (Side).



By our definition of congruence, we will have to find a composition of rigid motions will map $\triangle A'B'C'$ to $\triangle ABC$. We must find a congruence F so that $F(\triangle A'B'C') = \triangle ABC$. First, use a translation T to map a common vertex.

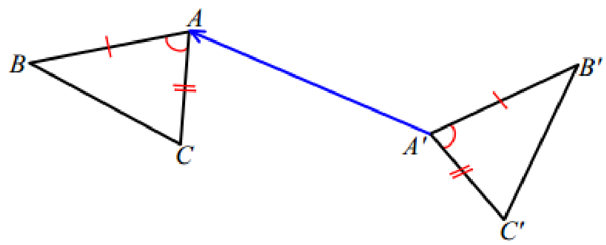
Which two points determine the appropriate vector?

Can any other pair of points be used? ________ Why or why not?

____

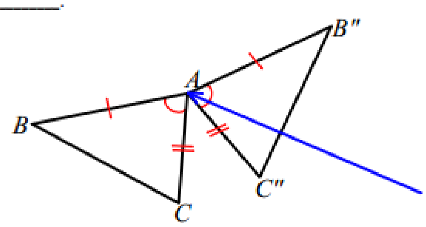
State the vector in the picture below that can be used to translate $\triangle A'B'C'$: ____________

Using a dotted line, draw an intermediate position of $\triangle A'B'C'$ as it moves along the vector:

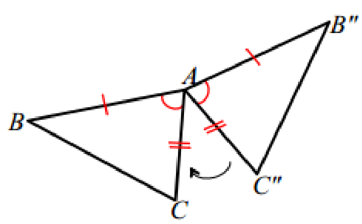


After the translation (below), $T_{\overrightarrow{A'A}}(\triangle A'B'C')$ shares one vertex with $\triangle ABC$, A. In fact, we can say

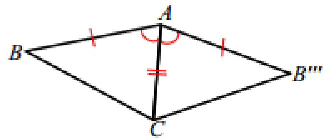
T_____($\triangle$ __________) = $\triangle$ __________.



Next, use a clockwise rotation $R_{\angle CAC''}$ to bring the sides $\overline{AC''}$ to $\overline{AC}$ (or counterclockwise rotation to bring $\overline{AB''}$ to $\overline{AB}$).

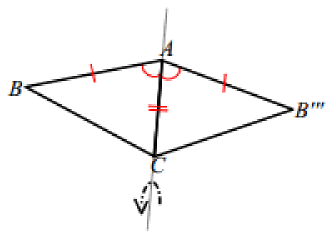


A rotation of appropriate measure will map $\overrightarrow{AC''}$ to $\overrightarrow{AC}$, but how can we be sure that vertex C'' maps to C? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps C'' to C. ($AC = AC''$; the translation performed is a rigid motion, and thereby did not alter the length when $\overline{AC'}$ became $\overline{AC''}$.)



After the rotation $R_{\angle CAC''}(\triangle AB''C'')$, a total of two vertices are shared with $\triangle ABC$, A and C. Therefore,

Finally, if B''' and B are on opposite sides of the line that joins AC, a reflection $r_{\overline{AC}}$ brings B''' to the same side as B.



Since a reflection is a rigid motion and it preserves angle measures, we know that $m\angle B'''AC = m\angle BAC$ and so $\overrightarrow{AB'''}$ maps to $\overrightarrow{AB}$. If, however, $\overrightarrow{AB'''}$ coincides with $\overrightarrow{AB}$, can we be certain that B''' actually maps to B? We can, because not only are we certain that the rays coincide but also by our assumption that $AB = AB'''$. (Our assumption began as $AB = A'B'$, but the translation and rotation have preserved this length now as AB'''.) Taken together, these two pieces of information ensure that the reflection over $\overline{AC}$ brings B''' to B.

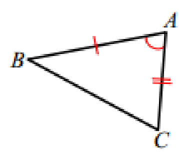
Another way to visually confirm this is to draw the marks of the ____________________ construction for $\overline{AC}$.

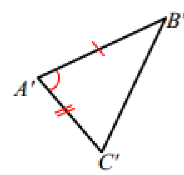
Write the transformations used to correctly notate the congruence (the composition of transformations) that take $\triangle A'B'C' \cong \triangle ABC$:

F ________

G ________

H ________





We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two distinct triangles, we could perform a similar proof. There is another situation when the triangles are not distinct, where a modified proof will be needed to show that the triangles map onto each other. Examine these below. Note that when using the Side-Angle-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and "SAS."

Figure 28- SAS congruence proof (continued) (ENY, M1, TD, L22, p. 1-3)

Congruence: Prove geometric theorems

> CCSS.MATH.CONTENT.HSG.CO.C.9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

We first describe how EngageNY describes "theorems" and "proofs." Proofs are said to be detailed explanations of how statements follow logically from other statements already accepted as being true (ENYS, M1, TB, L11). Theorems, on the other hand, are mathematical statements with proofs. Hence, when we analyze EngageNY's attempts to deliver on this content standard, we should note the extent to which proofs that are asked for or given are based on statements already accepted as being true (in the present material or prior (grade 8) material).

After providing examples to get the learner to recognize the relevant objects in this standard (e.g. "vertical angles"), the EngageNY material goes on to provide an example of proving a statement, and then it asks the student to prove the relevant theorems under this standard (e.g. ENY, M1, TB, L11). For example, given two parallel segments and a transversal, the learner is asked to prove the sum of interior angles of the same side of the transversal sums to 180 degrees. All of the other proofs under this standard are either provided or requested, and, moreover, the proofs can be (or are) based upon formerly established properties/theorems. The material even discusses converses of some of the statements. With all of these considerations made, we assign the EngageNY material a grade of 3 per our rubric for this content standard.

> CCSS.MATH.CONTENT.HSG.CO.C.10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

Again, EngageNY delivers well on this content standard. It either gives or requests a proof of the relevant theorems using formerly established properties (ENY, M1, TE, L29). For example, the learner is asked that, if she/he knows $\overline{XY}$ is a midsegment of triangle ABC, to prove that $\overline{XY} \| \overline{BC}$ and XY=(1/2)(BC). The material walks the learner through a formal proof of that statement, asking the learner to "explain" after everything done in the proof. Since all of the content under this standard is handled in a similar manner, we assign EngageNY a grade of 3 per this content standard.

> CCSS.MATH.CONTENT.HSG.CO.C.11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

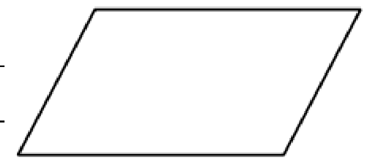
The EngageNY material asks the learner to prove each of the statements under this standard. For example, the learner is asked to prove that opposite sides are congruent in parallelograms (see the figure below). Since all of the content under this standard is covered at the level expected by the Common Core, we assign the EngageNY material a grade of 3 per our rubric for this content standard.

Example 1

If a quadrilateral is a parallelogram, then its opposite sides and angles are equal in measure. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why opposite sides and angles of a parallelogram are congruent.

Given: ______________________________

Prove: ______________________________



Construction: Label the quadrilateral $ABCD$, and mark opposite sides as parallel. Draw diagonal $\overline{BD}$.

Figure 29- Opposite sides and angles are equal in parallelograms (ENYS, M1, TE, L28, p. 2)

Congruence: Make geometric constructions

CCSS.MATH.CONTENT.HSG.CO.D.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

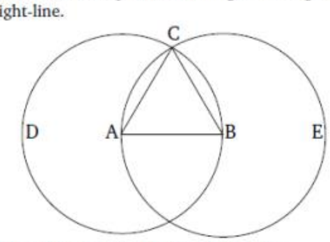
The EngageNY material often has learners making formal geometric constructions with compass and straightedge (e.g. ENYS, M1, TA, L5). Learners are, indeed, asked to copy segments and angles, bisect segments and angles, construct perpendicular lines, and construct a line parallel to a given line through a point not on the line. Furthermore, basic constructions, such as constructing a pair of “equidistant points,” are carried out in exercises. Since the material covers everything within this standard at the level expected by the Common Core, we assign the EngageNY a grade of 3 per our rubric for this standard.

> CCSS.MATH.CONTENT.HSG.CO.D.13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

The EngageNY material, again, covers every relevant content item under this standard. For example, in the construction of an equilateral triangle, the EngageNY material provides a construction proposition by Euclid as demonstrated in the figure below. Similarly, learners are asked to construct squares and regular n-gons inscribed in circles. We therefore assign EngageNY a grade of 3 per our rubric for this standard.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Figure 30- The construction of an equilateral triangle using circles (ENYS, M1, TA, L1, p.3)

Summary of the EngageNY Alignment with Common Core Strand Congruence

Under the Common Core Congruence Content Strand, the EngageNY Geometry Curriculum Map scores a “perfect” 3. That score corresponds to an excellent alignment with the Common Core. EngageNY’s strengths under this strand lie in providing all requisite definitions, guiding learners through all demanded proofs, and explaining differences between specific cases and general cases. The only significant weakness we found was that some sections lacked significant numbers of practice problems for the learner. However, our rubric makes little consideration for number of exercises in determining alignment. Nevertheless, it is an important consideration that we should make, especially when we compare New York’s EngageNY materials to Utah’s MVP materials.

Similarity, Right Triangles, and Trigonometry

Understand Similarity in Terms of Similarity Transformations

CCSS.MATH.CONTENT.HSG.SRT.A.1

Verify experimentally the properties of dilations given by a center and a scale factor:

CCSS.MATH.CONTENT.HSG.SRT.A.1.A

A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

CCSS.MATH.CONTENT.HSG.SRT.A.1.B.

The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

We analyze SRT.A.1 together with its subparts. The EngageNY material presents the learner with content covering everything within this standard at a highly explanatory level. For example, it proves to the learner that scale drawings of figures may be produced using either the “ratio” or “parallel” method (ENYS, M2, TA, L4). The material then goes on to demonstrate graphically that the learner need only consider the dilation of segments to prove the equivalence of those methods. Furthermore, for all content under this standard, the EngageNY material develops graphical explanations. Hence all content under this standard is covered by the EngageNY from the very basic provision of graphical explanations to utilizing facts, saliently, in a proof. We therefore assign the EngageNY material a grade of 3 per our rubric for SRT.A.1.

CCSS.MATH.CONTENT.HSG.SRT.A.2

Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

In order to use the definition of similarity in terms of similarity transformations, the EngageNY material first provides the learner with the definition:

Definition:

A ________________ ________________ (or ________________) is a composition of a finite number of dilations or basic rigid motions. The *scale factor* of a similarity transformation is the product of the scale factors of the dilations in the composition; if there are no dilations in the composition, the scale factor is defined to be 1.

Two figures in a plane are ____________________ if there exists a similarity transformation taking one figure onto the other figure.

Figure 31- SRT.A.2. Definitions (ENYS, M2, TC, L12, p. 1)

The material then goes on to ask the learner to determine which similarity transformations carry one figure onto another, and in another case to determine why no similarity transformation exists. However, there is no significant effort in the EngageNY material to have a learner explicitly understand the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. There are a few activities, as above, that ask the learner which transformations carry one triangle onto another. However, we find that the material does not go as far as the standard demands (when analyzing all sections that EngageNY claims cover SRT.A.2). Since it covers half of the content and action items of SRT.A.2, we assign EngageNY a grade of 2 per our rubric for SRT.A.2.

CCSS.MATH.CONTENT.HSG.SRT.A.3

Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

The EngageNY material asks the learner, in a discovery exercise, to use a protractor to draw two triangles of different size but with two angles of each triangle being equal in measure (ENYS, M2, TC, L15, p.1). The learner is then asked to measure the side lengths of one of the triangles and compare them to the side lengths of the other triangle. From this the learner can claim that the two triangles are similar, and, indeed, she/he should be able to identify the similarity transformation taking one triangle onto the other. The learner is then led to the AA conjecture based on these observations. The material does not provide the simple proof of the conjecture, but this content standard does not demand a proof. We therefore assign EngageNY a grade of 3 per our rubric for SRT.A.3.

Prove theorems involving similarity

CCSS.MATH.CONTENT.HSG.SRT.B.4

Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

The EngageNY material does, indeed, both prove theorems about triangles and have learners prove theorems about triangles. For example, learners are walked through exercises that provide examples tailored to understanding the Pythagorean Theorem using triangle similarity (ENYS, M2, TD, L21). The instructor is left to provide the formal proof to the learner (ENYT, M2, TD, L21). The other explicitly written theorem under this standard is also addressed in the EngageNY material. With these considerations having been made, we assign the EngageNY material a grade of 3 per our rubric for SRT.B.4.

CCSS.MATH.CONTENT.HSG.SRT.B.5

Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

The EngageNY material offers a plethora of exercises tailored toward getting learners to use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. We present a worked example below. EngageNY earns a grade of 3 per our rubric for this standard since it covers the action items completely, and learners certainly (as described in the above standards) have all the tools necessary to solve them.

Example 1

Given $\Delta ABC \sim \Delta A'B'C'$, find the missing side lengths.

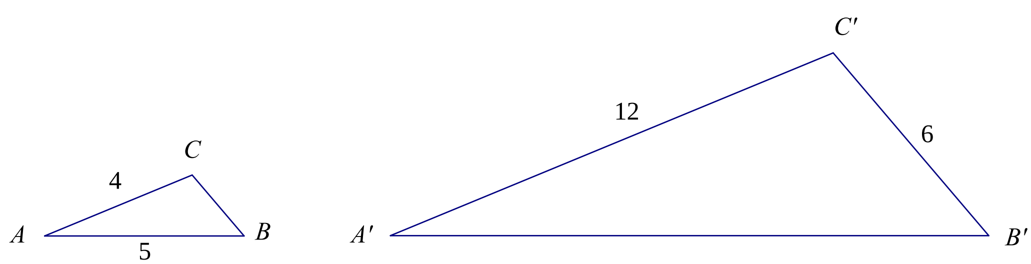


Figure 32- SRT.B.5 Exercise (NYSE, M2, TC, L16, p.1)

A learner would complete this exercise by using her/his knowledge about similarity of triangles. Since $\Delta ABC \sim \Delta A'B'C'$, and A’C’=12=3AC, we can conclude that the unknown side of $\Delta A'B'C$’ is 3 times the length of the corresponding side in ΔABC. Hence A’B’=5*3=15. Similarly, CB=6/3=2.

Define trigonometric ratios and solve problems involving right triangles

CCSS.MATH.CONTENT.HSG.SRT.C.6

Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

The EngageNY material effectively delivers on this standard. For example, similar triangles are presented, and then the material underscores ratios of corresponding sides between similar triangles are equal (ENYS, M2, TE, L25). It then talks about those ratios using the traditional vocabulary of “opposite,” “adjacent,” and hypotenuse. In that introductory section, no mention of sine, cosine, and tangent exists. As we commented within this standard for Utah’s MVP materials, we find this to be a highly effective approach. In subsequent sections those trigonometric terms are defined. We therefore assign the EngageNY material a grade of 3 per our rubric for SRT.C.6.

CCSS.MATH.CONTENT.HSG.SRT.C.7

Explain and use the relationship between the sine and cosine of complementary angles.

In its attempt to deliver on this content standard, the EngageNY material provides the following introductory exercise to the learner:

1. Consider the right triangle ABC so that $\angle C$ is a right angle, and the degree measures of $\angle A$ and $\angle B$ are α and β, respectively.

   a. Find $\alpha + \beta$.

   b. Use trigonometric ratios to describe $\frac{BC}{AB}$ two different ways.

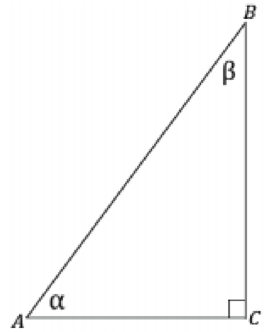


Figure 33- SRT.C.7 Activity (ENYS, M2, TE, L27, p. 1).

Clearly this activity wants the learner to develop an understanding of the relationship between the angles alpha and beta in the given triangle, i.e. complementary angles. The material tailors further exercises at getting the learner to understand more about the interplay of the non-right angles within right triangles. The material, furthermore, wants the instructor to reinforce that, as in the above example, the sum of the angle measures of alpha and beta must be 90 degrees since the sum of the measures of the angles of any triangle is 180 degrees (ENYT, M2, TE, L27). Therefore one can always represent alpha as the difference of 90 degrees and the measure of beta and likewise with beta as the difference of 90 degrees and the measure of alpha. Simple substitution then makes it clear to the learner the relationship between the sine and cosine of complementary angles. Since the EngageNY material completely covers the content under this standard at the level demanded by the Common Core, we assign it a grade of 3 per our rubric for SRT.C.7.

CCSS.MATH.CONTENT.HSG.SRT.C.8

Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

The EngageNY excels within this standard by providing a plethora of application problems in which learners can use the Pythagorean Theorem. We present an example in the figure below. The EngageNY material earns a grade of 3 per our rubric for SRT.C.8.

1.

a. The bus drops you off at the corner of H Street and 1st Street, approximately 300 ft. from school. You plan to walk to your friend Janneth's house after school to work on a project. Approximately how many feet will you have to walk from school to Janneth's house? Round your answer to the nearest foot. (Hint: Use the ratios you developed in Lesson 25.)

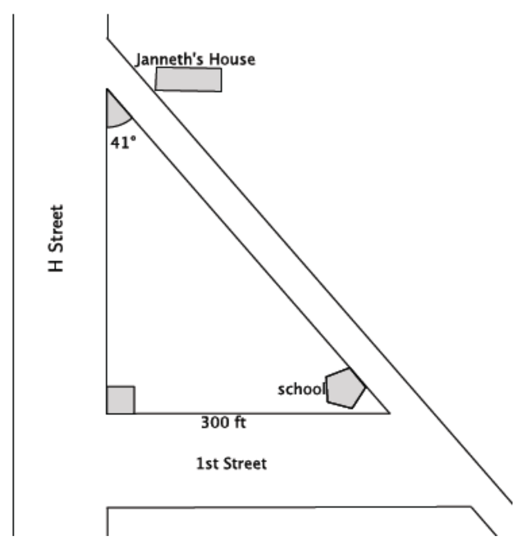


Figure 34- SRT.C.8 Activity (ENYS, M2, TE, L28, p. 1)

Apply trigonometry to general triangles

CCSS.MATH.CONTENT.HSG.SRT.D.9. Derive the formula A = (1/2) ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

We note that SRT.D.9 is a "plused" standard, but that EngageNY does cover the content within it (ENYS, M2, TE, L31). The learner is walked through a series of 11 observations, ultimately leading to the derivation of the formula A = (½) ab sin(C). We therefore assign the EngageNY a grade of 3 per our rubric for this content standard.

CCSS.MATH.CONTENT.HSG.SRT.D.10. Prove the Laws of Sines and Cosines and use them to solve problems.

Very little is done in the EngageNY student material to develop the Laws of Sines and Cosines. The burden is almost completely on the teacher to explain these laws (ENYT, M2, TE, L32). The teaching material does provide detailed guidance to the instructor in convincing students that these laws are true. For example, the instructor should demonstrate the Law of Sines by giving a specific example (a so-called 1-2-sqrt(3) right triangle). The student version relevant to this standard is a set of exercises to which the learners should apply the laws (ENYS, M2, TE, L32) (e.g. "find the unknowns of the following triangle"). This methodology does not provide much confidence that the learner would have a sufficient understanding of the proofs of these laws. The material, for example, would do well to provide more guiding exercises for the learner. We assign the EngageNY a grade of 2 per our rubric for SRT.D.10 due to a salient lack of exercises from which learners could gain the "main idea" of proofs of these laws.

CCSS.MATH.CONTENT.HSG.SRT.D.11. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

EngageNY's treatment of this standard has been addressed in the discussion of SRT.D.10. That is to say, the material does provide exercises to which learners can apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (ENYS, M2, TE, L32-33). Specifically, the material gives both application and non-application exercises relevant under this standard. It is a murky endeavor to gauge whether the learner "understands" since it depends completely on the instructor as our grading of EngageNY under SRT.D.10 implies. Nevertheless, we must assume the instructor will provide the learner with requisite explanations and proofs, and therefore we assign EnageNY a grade of 3 per our rubric for SRT.D.11.

Summary of EngageNY Alignment with Common Core Strand Similarity/Trigonometry

In every standard except two within the Similarity/Trigonometry content strand (i.e. SRT 2 & 10), EngageNY scored a grade of 3 per our rubric. In both SRT2 and SRT10, EngageNY earned a grade of 2. Hence, EngageNY averaged a grade of approximately 2.82 for this strand. This corresponds to a good to excellent alignment with the Common Core State Standards.

Within this strand, EngageNY's strengths lie in providing most relevant definitions and statements of theorems. Its major weakness lies in putting nearly all of the burden of explanation of the Laws of Sines and Cosines on the instructor. While the proofs of those laws are relatively simple in the context of the course, it would certainly do the material well to at least provide a few activities aimed at developing the structure of a proof in the student version.

Circles

Understand and apply theorems about circles

CCSS.MATH.CONTENT.HSG.C.A.1. Prove that all circles are similar.

Exercises exist in the EngageNY material that aim to convince the learner that all circles are similar. For example, circles of unequal circumference are presented to the learner, and the learner is asked what similarity transformation would be required to move one circle onto the other (ENYS, M5, L7). The proof, however, is provided by the instructor. However, since exercises exist tailored to structuring a proof, and the proof is given by the instructor, the EngageNY material meets the demand of this content standard. It earns a grade of 3 per our rubric for C.A.1.

CCSS.MATH.CONTENT.HSG.C.A.2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

EngageNY excels in providing all content under this standard. For example, the material provides activities tailored toward noticing that: "The measure of an inscribed angle is half the angle measure of its intercepted arc. The measure of a central angle is equal to the angle measure of its intercepted arc" (ENYS, M5, L7, p. 82). We assign EngageNY a grade of 3 per our rubric for C.A.2 since all content is covered at the level required by the Common Core.

CCSS.MATH.CONTENT.HSG.C.A.3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

The demands of this standard are quite straightforward: the EngageNY material has the learner construct inscribed and circumscribed circles of a triangle (ENYS, M5, L1&L5). Furthermore, properties of quadrilaterals inscribed in circles are established in several instances, and proofs of those properties are developed in tandem. The figure below is illustrative. Since the content is covered thoroughly with numerous examples, we assign a grade of 3 per our rubric for EngageNY in standard C.A.3.

Let's look at relationships between inscribed angles.

a. **Examine the inscribed polygon below. Express x in terms of y and y in terms of x. Are the opposite angles in any quadrilateral inscribed in a circle supplementary? Explain.**

$x = 180° - y$; $y = 180° - x$. The angles are supplementary.

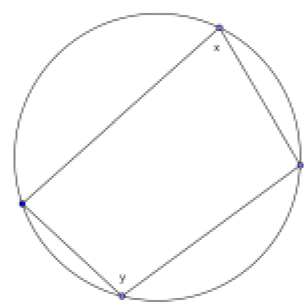


Figure 35- C.A.3. Activity (ENYS, M5, L5, p. 59)

(+)CCSS.MATH.CONTENT.HSG.C.A.4. Construct a tangent line from a point outside a given circle to the circle.

This is a "plused" standard, and EngageNY makes no explicit effort to make the relevant construction. Therefore it earns a grade of 0 per our rubric, but since it is "plused," the grade will not factor into EngageNY's strand alignment average.

Find arc lengths and areas of sectors of circles

> CCSS.MATH.CONTENT.HSG.C.B.5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Again, the EngageNY material completely covers the content under this standard. To illustrate the manner in which EngageNY delivers on this content, let us consider its derivation for the area of a sector of a circle. It begins by providing specific examples of finding areas of sectors like the area of a quarter circle with a specific area. The exercises (and discussion by the instructor) evolve to considering how to determine the area of a sector defined by an arc measuring a specific number of degrees (ENYS&T, M5, L9, p. 59-60). The material goes on (via instructor) to deliver a standard derivation of the general formula for the area of a sector. The other content under this standard is handled similarly, and therefore we assign the EngageNY material a grade of 3 per our rubric for C.B.5.

Summary of EngageNY Alignment with Common Core Strand Circles

For the content strand "Circles," the EngageNY material averages a "perfect" 3 (of course excluding the "plused" standard). Again, the EngageNY material excels in providing effective setups to the properties and theorems listed per content standard. However, it is important emphasize that the material often puts a significant burden on the instructor to really follow through on proving important theorems. While there is not anything ostensibly troublesome about that setup, as we have pointed out, the material really should provide some development material either to motivate or make sense out of proofs. In this strand EngageNY succeeded in that respect.

Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

CCSS.MATH.CONTENT.HSG.GPE.A.1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

The EngageNY material has the instructor provide the equation of a circle of a given center and radius after asking if certain equations could represent circles (ENYT&S, M5, L18). There is minimal to no effort in the student material to derive the equation of a circle of a given center and radius using the Pythagorean Theorem. There are, however, standard activities such as using the method of completing the square to find the center and radius of a circle given by an equation (ENYS, M5, L18, p. 131). Since the EngageNY material only delivers on the second demand in this standard, we assign the EngageNY material a grade of 2 per our rubric for GPE.A.1.

CCSS.MATH.CONTENT.HSG.GPE.A.2. Derive the equation of a parabola given a focus and directrix.

This material was covered in EngageNY's "Algebra II" (ENY-AlgebraII, M1, TC, L33, p. 163). We present the manner in which it is presented there in the figure below.

Definition: A *parabola with directrix* L *and focus point* F is the set of all points in the plane that are equidistant from the point F and line L.

Figure 2 to the right illustrates this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ showing that for any point P on the parabola, the distance between P and F is equal to the distance between P and the line L.

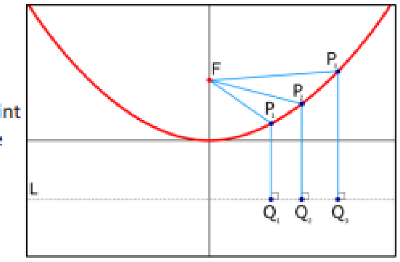


Figure 2

All parabolas have the reflective property illustrated in Figure 3 Rays parallel to the axis will reflect off the parabola and through the focus point, F.

Thus, a mirror shaped like a rotated parabola would satisfy Newton's requirements for his telescope design.

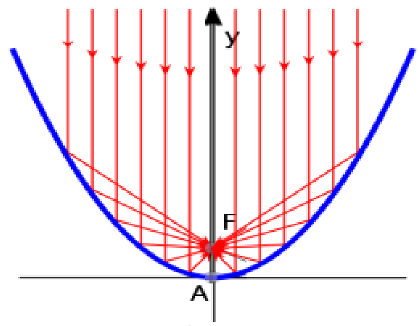
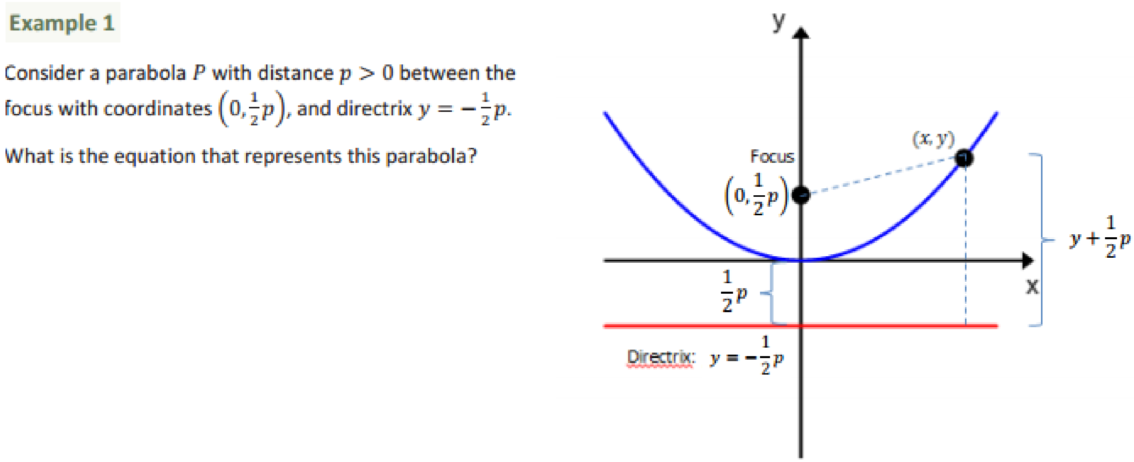


Figure 3

Figure 36- Definition of "parabola with directrix L and focus point F (ENYS-Algebra II, M1, TC, L33, p. 163-4)

So, the EngageNY material provides the definition of a parabola first, and later it establishes some of the intuition behind "how parabolas work." In the subsequent section in "Algebra II," the EngageNY material asks the learner (with assistance by her/his instructor) to derive the vertex form of a parabola. The following figure is illustrative. We assign the EngageNY material a rubric score of 3 because it had developed the content relevant under this standard prior to the "typical" time in which a learner would take geometry.


Example 1

Consider a parabola P with distance $p > 0$ between the focus with coordinates $\left(0, \frac{1}{2}p\right)$, and directrix $y = -\frac{1}{2}p$.

What is the equation that represents this parabola?

Figure 37- Capstone activity to derive the formula for a parabola with line of symmetry x=0 (ENYS- Algebra II, M1, TC, L34, p. 171)

> CCSS.MATH.CONTENT.HSG.GPE.A.3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

This is a “plused” standard, and the EngageNY material makes no attempt to deliver on its demands. Therefore it earns a grade of 0 per our rubric. However, that grade will not factor into its average alignment in this strand because it is “plused.”

Use coordinates to prove simple geometric theorems algebraically

> CCSS.MATH.CONTENT.HSG.GPE.B.4. Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).

The EngageNY material delivers within this standard by covering the exact examples provided in the standard. For example, given four points, learners are asked to prove that the quadrilateral with those points as vertices is a rectangle using what they know together with the relevant algebra (ENYS, M4, TD, L14, p.66). Even further, the learners are asked whether points lie on the diagonals of that same quadrilateral. Similar examples abound throughout the geometry material getting students to apply algebraic techniques in coordinate-based proofs. We therefore assign the EngageNY material a grade of 3 per our rubric for GPE.B.4.

CCSS.MATH.CONTENT.HSG.GPE.B.5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

The EngageNY material provides basic examples for learners to speculate about regarding slope criteria for parallel and perpendicular lines. Most of the burden is on the instructor to convince the learner that the criteria are true (ENYT, M4, TB, L8). The figure below demonstrates the argument the instructor is directed to give under this standard. Since the material provides discovery activities relevant under this standard, and the formulation of the proof to be offered by the instructor is highly explanatory, we assign the EngageNY material a grade of 3 per our rubric for GPE.B.5.

Students will begin the lesson with the following activity using geometry software to reinforce the theorem studied in Lesson 6, which states that given points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, and $D(d_1, d_2)$, $\overline{AB} \perp \overline{CD}$ if and only if $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$.

- Construct two perpendicular segments and measure the abscissa (the x-coordinate) and ordinate (the y-coordinate) of each of the endpoints of the segments. (You may extend this activity by asking students to determine whether the points used must be the endpoints. This is easily investigated using the dynamic geometry software by creating free moving points on the segment and watching the sum of the products of the differences as the points slide along the segments.)
- Calculate $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2)$.
- Note the value of the sum and observe what happens to the sum as they manipulate the endpoints of the perpendicular segments.

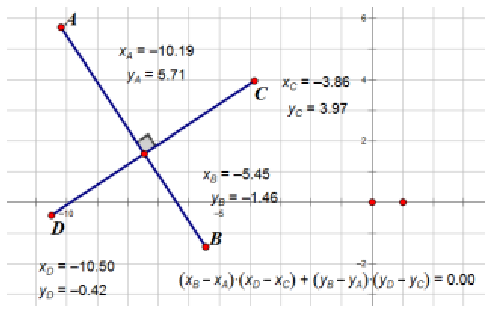


Figure 38- Explanatory material to prove the slope criterion for perpendicular lines (ENYT, TB, L8, p.81)

CCSS.MATH.CONTENT.HSG.GPE.B.6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

The EngageNY material has the learner use coordinates to find points on directed line segments between two given points that partition the segment in a given ratio (ENYS&T, M4, TD, L13). The treatment includes finding midpoints of line segments between two given points. We assign the EngageNY material a grade of 3 for GPE.B.6 per our rubric since the material completely covers the demands of this standard.

CCSS.MATH.CONTENT.HSG.GPE.B.7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

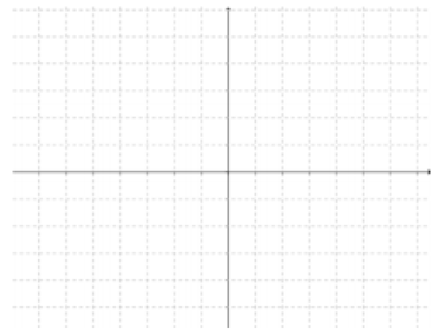
This standard is very straightforward in its demands, and we note that EngageNY provides numerous activities and explanations (via instructor) relevant to this standard.

We provide a relevant example below. EngageNY earns a grade of 3 per our rubric for GPE.B.7.

A quadrilateral region is defined by the system of inequalities below:

$y \le x + 5$ $\quad$ $y \ge x - 4$ $\quad$ $y \le 4$ $\quad$ $y \ge -\frac{5}{4}x - 4$

a. Sketch the region.



b. Determine the vertices of the quadrilateral.

c. Which quadrilateral is defined by these inequalities? How can you prove your conclusion?

d. Find the perimeter of the quadrilateral region.

Figure 39- Find the perimeter of a quadrilateral region (ENYS, M4, TC, L11, p. 51)

Summary of EngageNY Alignment with Common Core Strand Expressing Geometric Properties with Equations

The EngageNY material earned an average of approximately 2.86 under this content strand (excluding plused standards). This corresponds to a good to excellent alignment with the Common Core Standards. We note that this strand is the weakest showing of EngageNY's alignment thus far. Its strengths within this standard lie in providing definitions, statements of theorems, and proofs (most frequently by the instructor). However, significant weaknesses are obvious when EngageNY does not follow through on all the content demanded under specific standards (e.g. GPE.A.1).

Geometric Measurement & Dimension

Explain volume formulas and use them to solve problems

> CCSS.MATH.CONTENT.HSG.GMD.A.1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

EngageNY delivers on all of the content under this standard, including a discussion of Cavalieri's principle as discussed under the following ("plused") standard. To illustrate EngageNY's typical coverage of the first half of this standard, learners are presented with an informal argument for the formula for the volume of a cylinder (simply by formulating the area of a cross section multiplied by the height of the cylinder). Since the EngageNY material covers all content within this standard, we assign it a grade of 3 per our rubric for GMD.A.1.

> CCSS.MATH.CONTENT.HSG.GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Interestingly, the EngageNY material utilizes the example of 3d printing to explain the intuition behind Cavalieri's principle (ENYT, M3, TB, L13). That is to say, 3-d printing prints "layer by layer" according to a schematic, but the 3-d printer has physical, planar limitations (both planes being parallel, though the technique used varies by type of 3d-printer). The material goes on to give an informal argument regarding the use of Cavalieri's principle for the formulas for the volume of a sphere in particular (ENYS, M3, TB, L12-13). Though this standard is "plused," we assign the EngageNY material a grade of 3 per our rubric for this standard as it covers the action item "give an informal proof" in a very explanatory manner.

CCSS.MATH.CONTENT.HSG.GMD.A.3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

This standard is extremely straightforward. We note that the EngageNY material provides a plethora of examples relevant to this standard. We provide an example in the figure below. The EngageNY material earns a grade of 3 per our rubric for this content standard.

A cone fits inside a cylinder so that their bases are the same and their heights are the same, as shown in the diagram below. Calculate the volume that is inside the cylinder but outside of the cone. Give an exact answer.

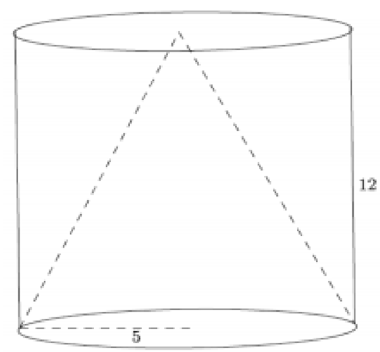


Figure 40- Volume of a cylinder without the volume of the cone inside (ENYS, M3, TB, L11, p. 68)

Visualize relationships between two-dimensional and three-dimensional objects

CCSS.MATH.CONTENT.HSG.GMD.B.4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

The EngageNY material's coverage under this standard has largely already been exemplified in the discussion of its treatment of Cavalieri's Principle. There are many examples throughout the curriculum where learners are asked to identify 2-d cross-sections of 3-d objects (e.g. ENYS, M3, TB, L10). Furthermore, there are indeed examples asking learners to visualize rotations of 2-d objects about, e.g., axes. We therefore assign the EngageNY material a grade of 3 per our rubric for this content standard due to its complete coverage of the content.

Summary of EngageNY Alignment with Common Core Strand Geometric Measurement & Dimension

The EngageNY material excelled within this content strand. It scored a "perfect" 3 among all standards, including the "plused" standard related to Cavlieri's Principle. We note that the exercises the EngageNY material has the learners do are not simple (for example, in Figure 40, the learners are asked to find the volume of a difference of volumes.

Modeling with Geometry

Apply geometric concepts in modeling situations

CCSS.MATH.CONTENT.HSG.MG.A.1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

This standard is extremely straightforward, and the EngageNY material exhibits numerous examples of modeling objects by using shapes, measures, and the properties of the shapes. For example, learners are asked to estimate the circumference of the Earth based on what they know (ENYS, M2, TC, L19, p. 129). Since the EngageNY material provides numerous examples relevant under this standard that involve the learner's previously established knowledge, we assign it a grade of 3 per our rubric for MG.A.1.

CCSS.MATH.CONTENT.HSG.MG.A.2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

Again, this standard is straightforward to the extent that it simply demands applications related to the application of concepts of density based on area and volume. For example the learner is asked the following: “A metal cup full of water has a mass of 1,000 g. The cup itself has a mass of 214.6 g. If the cup has both a diameter and a height of 10 cm, what is the approximate density of water?” (NYSE, M3, TB, L8, p. 49). Clearly this activity falls under this standard, and it aims to get the learner to utilize area/volume to understand the question of density. Since the EngageNY material provides a numerous amount of activities based on developed knowledge, we assign it a grade of 3 per our rubric for MG.A.2.

Summary of EngageNY Alignment with Common Core Strand Modeling with Geometry

The EngageNY material aligns at a “perfect 3” with the Common Core Strand Modeling with Geometry. The material excels at providing a diverse group of application and modeling problems which learners can use to solidify their understanding of the relevant mathematics. No significant drawback is noticeable of the EngageNY treatment under this strand.

General Conclusions Regarding the EngageNY Material Alignment with the Common Core Geometry Domain

As detailed above, the EngageNY material aligns at a good to excellent or excellent level with the Common Core across all geometry domain strands. The strengths of the EngageNY material lie in its strong efforts to provide precise definitions, statements of theorems, and exercises to which learners can apply those definitions and theorems. However, there was an instance where the EngageNY completely missed covering some content. Our speculation is that the content was omitted due to either the EngageNY material being unfinished or a concern for time. Since the EngageNY material is meant to be covered in the traditionally-structured secondary school geometry class, a concern for time is certainly relevant. We reserve further analysis for the following section.

CONCLUSION AND RECOMMENDATIONS

Geometry Standard Strand	Utah MVP Mean	New York EngageNY Mean
Congruence	2.54	3
Similarity & Trigonometry	2.55	2.82
Circles	3	3
Expressing Geometric Properties with Equations	3	2.86
Geometric Measurement & Dimension	2.67	3
Modeling with Geometry	3	3

Table 5. Overall Summary of Average Grades per Strand

Both of the curricula we analyzed aligned at a good to excellent level, according to our rubric, across all Common Core geometry content strands. In all but the "Expressing Geometric Properties with Equations" strand, the EngageNY material aligned higher under each strand compared to the MVP material. Each curriculum had strengths and weaknesses. The EngageNY material excelled at providing precise definitions, statements of relevant theorems, and examples of proofs for the learner. The MVP material, on the other hand, excelled at providing highly intuitive discovery exercises very consistently throughout the material.

Each curriculum had weaknesses as well. The EngageNY material omitted some content, perhaps due to a concern for time. The MVP material, on the other hand, tended to gloss over providing precise definitions, statements of relevant theorems, and examples of proofs for the learner.

We find that the structuring of the course material (i.e. "integrated" vs "traditional") played no significant role in whether or not the material aligned with the Common Core Standards. Each curriculum was able to work students up from what we might call low Van Hiele levels to more advanced levels of geometric understanding over the timeframe assumed. More than that, however, each curriculum— for the most part— did that while aligning well with the Common Core.

Our analysis points toward the recommendation to authors of geometry curricula that they cover all material listed per Common Core Standard, clearly develop performance related items, and provide examples of worked exercises and proofs— no matter the pathway chosen. We have seen that geometry curricula can align strongly with the Common Core while still being highly explanatory. This is not to say that very formal definitions, statements of theorems, and examples of proof technique should not be provided in text material. To the contrary, these ingredients are extremely helpful to curricula in aligning with the Common Core and, indeed, in preparing a learner for higher mathematics. It is simply the case that continuity and developmental-discovery oriented material can help the average learner succeed by helping her/him develop geometric reasoning abilities. In this time where U.S. students are struggling to compete with much of the rest of the developed world in mathematics performance, certainly we ought to provide our students with as much help as we can.

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