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Statistical Hyperbolicity of Relatively Hyperbolic Groups

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STATISTICAL HYPERBOLICITY
of Relatively Hyperbolic Groups

by

Jeremy Osborne

A Dissertation Submitted in
Partial Fulfillment of the
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DOCTOR OF PHILOSOPHY
in
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at

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ABSTRACT

STATISTICAL HYPERBOLICITY
of Relatively Hyperbolic Groups

by

Jeremy Osborne

The University of Wisconsin-Milwaukee, 2014
Under the Supervision of G. Christopher Hruska

In this work, we begin by defining what it means for a group to be statistically hyperbolic. We then give several examples of groups, including non-elementary δ -hyperbolic groups, which either are statistically hyperbolic or are not. Following that, we define what it means for a group to be relatively hyperbolic. Finally, in the main portion of this work, we show that groups which are relatively hyperbolic, with a few additional conditions in place, must also be statistically hyperbolic.

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Chapter 1

Introduction

The idea of statistical hyperbolicity was first introduced by Moon Duchin, Samuel Lelièvre, and Christopher Mooney in [DLM12]. The intuitive meaning of statistical hyperbolicity of a group can be summed up as follows: On average, random pairs of points x, y on a sphere of the Cayley graph of the group almost always have the property that $d(x, y)$ is nearly equal to $d(x, e) + d(e, y)$ where e is the origin. More precisely, for a group G with finite generating set \mathcal{S} , let the function

$$E(G, \mathcal{S}) := \lim_{n \rightarrow \infty} \frac{1}{|\mathcal{S}_n|^2} \sum_{x, y \in \mathcal{S}_n} \frac{1}{n} d(x, y)$$

Then a finitely generated group G with finite generating set \mathcal{S} is *statistically hyperbolic* when

$$E(G, \mathcal{S}) = 2,$$

as in Definitions 2.1 and 2.2.

In some cases, an analogous definition to that above can be used for spaces which do not necessarily represent the Cayley graph of a group. Many results were found to this end in [DLM12]. Specifically, it was found that the following are statistically hyperbolic:

- Non-elementary hyperbolic groups with any finite generating set.
- $(H \times K, \mathcal{S})$ when H is non-elementary hyperbolic, K is finitely generated, \mathcal{S} is a split finite generating set for $H \times K$, and with generators projected to the

factors from \mathcal{S} that $\lambda_H > \lambda_K$. (We define the growth rate, λ_G , for a group G in the beginning of Chapter 5.)

- For any $m, p \geq 2$, the Diestel-Leader graph $\text{DL}(m, p)$.
- The lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$ where $m \geq 2$ is finite for certain generating sets.
- Teichmüller spaces with the Teichmüller metric. This result was due to Dowdall, Duchin, and Masur in [DDM14].

In [DLM12], Duchin, Lelièvre, and Mooney found that \mathbb{Z}^d is not statistically hyperbolic for all $d \geq 1$ and any finite generating set \mathcal{S} . Duchin and Mooney also discovered in [DM11] that the integer Heisenberg group $H(\mathbb{Z})$ with any finite generating set is not statistically hyperbolic. They further showed that $E(H(\mathbb{Z}), \mathcal{S}) = \frac{19}{31}$ using the standard generating set \mathcal{S} .

Of particular interest is that groups which are non-elementary hyperbolic are also statistically hyperbolic. However, it was not known whether relatively hyperbolic groups were also statistically hyperbolic. So, in the Main Theorem of this work we see that non-elementary relatively hyperbolic groups, with one additional condition, are indeed also statistically hyperbolic. Thus, the Main Theorem generalizes the result for non-elementary hyperbolic groups from [DLM12].

Main Theorem. *If G is a non-elementary relatively hyperbolic group with finite generating set \mathcal{S} and a finite set of peripheral subgroups \mathbb{P} having the parabolic gap property, then G is statistically hyperbolic.*

For a definition of the parabolic gap property, see Definition 5.1.

Chapter 2

Foundational Definitions and Examples

We will begin by defining the formula for sprawl, which allows us to define statistical hyperbolicity in groups.

Definition 2.1. [DLM12] Let G be a group with finite generating set \mathcal{S} . The *sprawl* of (G, \mathcal{S}) is

$$E(G, \mathcal{S}) := \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y)$$

when the limit exists, where S_n is a sphere centered at the origin $e \in G$ of radius n in the Cayley graph of G . When the limit does not exist, we say that the sprawl does not exist.

It is clear that $0 \leq E(G, \mathcal{S}) \leq 2$ for any finitely generated group (G, \mathcal{S}) since $0 \leq d(x, y) \leq 2n$ for all $x, y \in S_n$. The value of E is not quasi-isometry invariant and could vary depending on the choice of generating set, as shown in [DLM12].

Definition 2.2. [DLM12] We say a finitely generated group (G, \mathcal{S}) is *statistically hyperbolic* if the sprawl is maximal, or in other words, $E(G, \mathcal{S}) = 2$.

To help give a sense of statistical hyperbolicity, $E(G, \mathcal{S}) = 2$ means that, for almost every random pair of points x, y on a sphere of the Cayley graph of the group, $d(x, y)$ is nearly equal to $d(x, e) + d(e, y)$ where e is the origin.

We now present a few examples of calculating the sprawl of a group. To begin, we intentionally chose groups for which the calculation of the sprawl is straightforward, and then progress to the proof for non-elementary hyperbolic groups being statistically hyperbolic.

Example 2.3. The sprawl of a free group on one generator is 1. In other words, $E(\mathbb{Z}, \{1\}) = 1$.

Proof. Notice that for $n \geq 1$, there are only two elements in the sphere S_n . Thus, it is clear that

$$\begin{aligned} E(\mathbb{Z}, \{1\}) &= \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4} \cdot \frac{1}{n} (4n) \\ &= 1 \end{aligned} \quad \square$$

Example 2.4. The free group on two generators with the standard generating set is statistically hyperbolic, i.e. $E(\mathbb{F}_2, \{a, b\}) = 2$.

Proof. Let $n \geq 1$ and fix $x \in S_n$. Also, observe that there are four main “branches” emanating from the origin. With $x \in S_n$ being fixed, we see that for $y \in S_n$, $d(x, y) = 2n$ if y is on one of the three main branches that does not contain x . If $y \in S_n$ is on the main branch which also contains x , we see that $d(x, y) = 2n - 2$ on two out of the three “sub-branches.” Similarly, if $y \in S_n$ is on the sub-branch which also contains x , we see that $d(x, y) = 2n - 4$ on two out of three of “sub-sub-branches.” Continuing this pattern and keeping x fixed, we find that

$$\begin{aligned} \sum_{y \in S_n} d(x, y) &= |S_n| \left(\frac{3}{4}(2n) + \frac{1}{4} \cdot \frac{2}{3}(2n - 2) + \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{2}{3}(2n - 4) + \cdots + 0 \right) \\ &= |S_n| \left(\frac{3}{2}n + \frac{1}{6}(2n - 2) + \frac{1}{18}(2n - 4) + \cdots + 0 \right) \\ &= |S_n| \left(\frac{3}{2}n + \sum_{k=1}^n \left(\frac{1}{3} \right)^k (n - k) \right) \end{aligned}$$

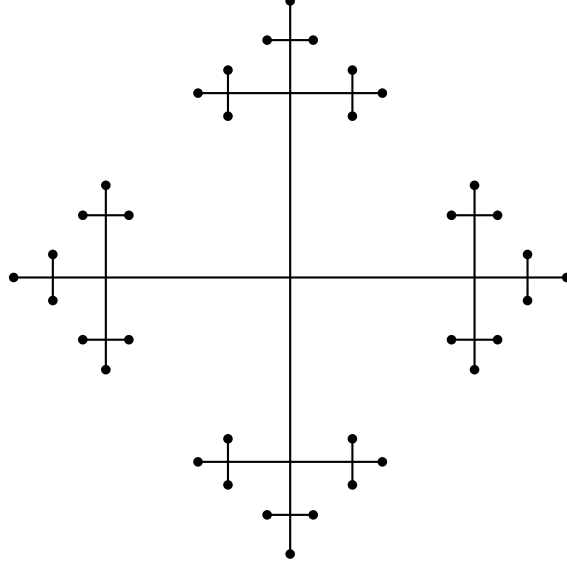


Figure 2.1: A sphere of radius 3 in the Cayley graph of \mathbb{F}_2 .

Thus, we have $\sum_{x,y \in S_n} d(x,y) = |S_n|^2 \left(\frac{3}{2}n + \sum_{k=1}^n \left(\frac{1}{3} \right)^k (n-k) \right)$. Hence,

$$\begin{aligned}
 E(\mathbb{F}_2, \{a, b\}) &= \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \sum_{x,y \in S_n} \frac{1}{n} d(x,y) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \cdot |S_n|^2 \cdot \frac{1}{n} \cdot \left(\frac{3}{2}n + \sum_{k=1}^n \left(\frac{1}{3} \right)^k (n-k) \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{2} + \frac{1}{n} \cdot \sum_{k=1}^n \left(\frac{1}{3} \right)^k (n-k) \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{2} + \frac{1}{n} \cdot \sum_{k=1}^n \frac{n}{3^k} - \frac{1}{n} \cdot \sum_{k=1}^n \frac{k}{3^k} \right)
 \end{aligned}$$

Since $\sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{3}{4}$, we see that

$$E(\mathbb{F}_2, \{a, b\}) \geq \frac{3}{2} + \sum_{k=1}^{\infty} \left(\frac{1}{3} \right)^k = 2$$

Therefore, since $E(\mathbb{F}_2, \{a, b\}) \geq 2$ and $0 \leq E(G, \mathcal{S}) \leq 2$ for all groups G , we

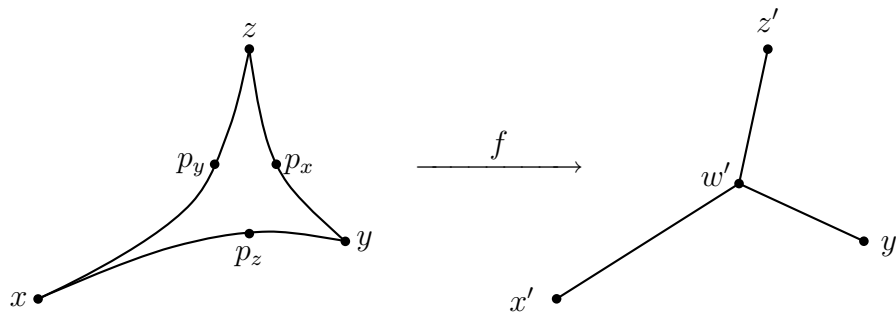


Figure 2.2: A δ -thin triangle, $\Delta = xyz$, mapped to a tripod, T .

know $E(\mathbb{F}_2, \{a, b\}) = 2$. □

We now wish to define hyperbolicity for groups and observe that non-elementary δ -hyperbolic groups are, in fact, statistically hyperbolic. The following is the definition of δ -thin, which leads directly to the definition of δ -hyperbolic.

Definition 2.5. Given a geodesic triangle $\Delta = xyz$ in a geodesic space X , by Definition 2.16 of [GdlH90], there exists a unique tripod $T = x'y'z'$ with points labeled as in Figure 2.2 where the following equations hold:

$$d(x, p_y) = d(x, p_z) = d(x', w')$$

$$d(y, p_x) = d(y, p_z) = d(y', w')$$

$$d(z, p_x) = d(z, p_y) = d(z', w')$$

Let $f : \Delta \rightarrow T$ be the natural map as in Definition 2.16 of [GdlH90]. We say that Δ is δ -thin if, for all $p' \in T$, $\text{diam}(f^{-1}(p')) \leq \delta$.

Now, using the definition of δ -thin, we can define a hyperbolic space, and consequently, a hyperbolic group.

Definition 2.6. A geodesic space X is δ -hyperbolic for $\delta \geq 0$ if all geodesic triangles in X are δ -thin. A finitely generated group G with generators \mathcal{S} is (word) hyperbolic if its Cayley graph, $\Gamma = \text{Cayley}(G, \mathcal{S})$ is a hyperbolic metric space.

Definition 2.7. A hyperbolic group G is said to be *non-elementary* if it is not virtually cyclic. (A group is *virtually cyclic* if it contains a finite index cyclic subgroup.)

Notice that as a result of the next theorem, if a group is a non-elementary hyperbolic group, then it is statistically hyperbolic. We provide a more detailed proof here than that which can be found in [DLM12]. The proof also gives a flavor of what the proof of the Main Theorem will entail.

Theorem 2.8. [DLM12] *Let G be a non-elementary δ -hyperbolic group. Then $E(G, \mathcal{S}) = 2$ for any finite generating set \mathcal{S} .*

Proof. A point z is (metrically) *between* two points x and y if $d(x, z) + d(z, y) = d(x, y)$. A set Z is *between* two points x and y if there exists a point $z \in Z$ such that z is between x and y .

Choose r such that $0 < r < 1$ and choose $x \in S_n$. Let $x' \in S_{\lfloor rn \rfloor}$ be between e and x . We wish to bound the number of $w \in S_n$ which are “close” to x . We will specifically consider any point $w \in S_n$ to be close to x when the ball $B_{2\delta}(x')$ is between e and w . In other words, w is close to x if there exists $w' \in B_{2\delta}(x') \cap S_{\lfloor rn \rfloor}$ which is between e and w . Clearly, the number of such w' is at most $|B_{2\delta}(x')|$. Also, since such a $w' \in S_{\lfloor rn \rfloor}$ is between e and w , we know $d(w', w) = n - \lfloor rn \rfloor$. For a fixed w' of this type, we then know that the number of $w \in S_n$ where the geodesic from e to w passes through w' can be no more than $|S_{n-\lfloor rn \rfloor}|$. Hence, the total number of $w \in S_n$ which are close to x is at most $|B_{2\delta}| \cdot |S_{n-\lfloor rn \rfloor}|$.

Assume $v \in S_n$ is not close to x . So, the geodesics $[e, x]$ and $[e, v]$ are separated by a distance of at least 2δ at time $\lfloor rn \rfloor$. By the definition of δ -thin on triangle exv , this implies that there exist points t_x, t_v on the geodesic $[x, v]$ such that $d(v', t_v) \leq \delta$ and $d(x', t_x) \leq \delta$. Also, since we know that $d(x', v') > 2\delta$, the point t_x must be between x and t_v on $[x, v]$ as in Figure 2.3. Additionally, since $d(v, v') > n - \lfloor rn \rfloor$ and $d(v', t_v) \leq \delta$, we know that $d(v, t_v) \geq n - \lfloor rn \rfloor - \delta$. Similarly, $d(x, t_x) \geq n - \lfloor rn \rfloor - \delta$. Thus, $d(x, v) \geq d(x, t_x) + d(v, t_v) \geq 2(n - \lfloor rn \rfloor - \delta) \geq 2(n - rn - \delta)$. Hence,

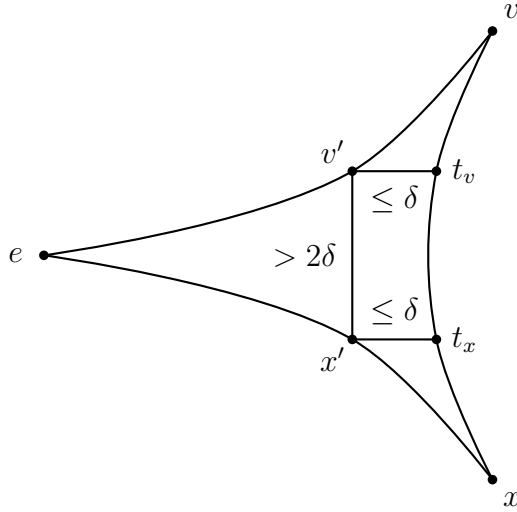
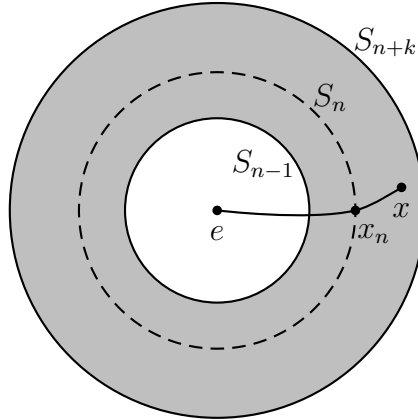


Figure 2.3: A δ -thin triangle where $v \in S_n$ is not close to $x \in S_n$.

$$\sum_{x,y \in S_n} d(x,y) \geq 2(n - rn - \delta) (|S_n| - |B_{2\delta}| \cdot |S_{n-\lfloor rn \rfloor}|) \cdot |S_n|$$

Coornaert proved in [Coo93] that for every non-elementary hyperbolic group with a fixed, finite generating set, there are bounded coefficients of exponential growth. Specifically, there exist $c_1, c_2 > 0$ and $\lambda > 1$ such that $c_1 \lambda^n \leq \beta(n) \leq c_2 \lambda^n$ for all $n \in \mathbb{N}$ where $\beta(n) = |B(e, n)|$ is the growth function of the group in question (growth function is defined in the beginning of Chapter 5). A group meeting the conditions above is said to have *definite exponential growth*. Clearly, we can choose $k \in \mathbb{N}$ such that $c_1 \lambda^k > c_2$. Then that gives us $\beta(n+k) \geq c_1 \lambda^{n+k} > c_2 \lambda^n$. By Coornaert's result, we have $\beta(n-1) \leq c_2 \lambda^{n-1}$. Let B be an annulus such that $B = B(e, n+k) - B(e, n-1)$ as in Figure 2.4. Then we can see that $|B| \geq c_2 \lambda^n - c_2 \lambda^{n-1}$. For each $x \in B$, choose a geodesic $[e, x]$. Clearly, this geodesic $[e, x]$ and S_n have a single point of intersection which we will call x_n . This process defines a surjective map $\varphi : B \rightarrow S_n$ by $\varphi(x) = x_n$. Let $N = |B(e, k)|$. Then φ is at most N -to-1. So, $|S_n| \geq \frac{1}{N} |B| \geq \frac{1}{N} (c_2 \lambda^n - c_2 \lambda^{n-1})$. Thus we see that

Figure 2.4: The annulus, B .

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{|S_{n-\lfloor rn \rfloor}|}{|S_n|} &= \lim_{n \rightarrow \infty} \frac{\beta(n - \lfloor rn \rfloor) - \beta(n - \lfloor rn \rfloor - 1)}{|S_n|} \\
&\leq \lim_{n \rightarrow \infty} \frac{c_2 \lambda^{n-\lfloor rn \rfloor} - c_1 \lambda^{n-\lfloor rn \rfloor - 1}}{\frac{1}{N} (c_2 \lambda^n - c_2 \lambda^{n-1})} \\
&= \lim_{n \rightarrow \infty} \frac{\lambda^{n-\lfloor rn \rfloor - 1} (c_2 \lambda - c_1)}{\lambda^{n-1} \left(\frac{c_2}{N} (\lambda - 1) \right)} \\
&= \lim_{n \rightarrow \infty} \frac{1}{\lambda^{\lfloor rn \rfloor}} \cdot \frac{c_2 \lambda - c_1}{\frac{c_2}{N} (\lambda - 1)} \\
&= 0
\end{aligned}$$

The previous two inequalities now give us

$$\begin{aligned}
E(G, \mathcal{S}) &= \lim_{n \rightarrow \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y) \\
&\geq \lim_{n \rightarrow \infty} \frac{2(n - rn - \delta) (|S_n| - |B_{2\delta}| \cdot |S_{n-\lfloor rn \rfloor}|) \cdot |S_n|}{n |S_n|^2} \\
&= \lim_{n \rightarrow \infty} \frac{2(1 - r - \delta/n) (|S_n| - |B_{2\delta}| \cdot |S_{n-\lfloor rn \rfloor}|)}{|S_n|} \\
&= \lim_{n \rightarrow \infty} 2(1 - r - \delta/n) - \lim_{n \rightarrow \infty} \frac{2(1 - r - \delta/n) \cdot |B_{2\delta}| \cdot |S_{n-\lfloor rn \rfloor}|}{|S_n|} \\
&= 2(1 - r)
\end{aligned}$$

Since $r \in (0, 1)$ was arbitrary, let $r \rightarrow 0$. Therefore, $E(G, \mathcal{S}) = 2$. □

Chapter 3

Relatively hyperbolic groups.

Now that we have dealt with the concepts of sprawl and statistical hyperbolicity, we turn our attention to relatively hyperbolic groups. It should be noted that there are several equivalent definitions of relative hyperbolicity for groups. In this work, we will outline the definition we believe is the simplest to present.

Definition 3.1. [Far98] Let G be a finitely generated group with generating set \mathcal{S} where \mathbb{P} is a finite set of subgroups of G . For each $g \in G$ and $P \in \mathbb{P}$, we have a left coset gP . For each such left coset, we will add a vertex $v(gP)$ to $\Gamma = \text{Cayley}(G, \mathcal{S})$. We then add edges connecting each element of gP to $v(gP)$. The resulting graph, $\hat{\Gamma}$, is known as the *coned-off Cayley graph*.

Definition 3.2. A graph K is *fine* if each edge of K is contained in only finitely many circuits of length n for each value of n . Here, a *circuit* is a closed path where there are exactly two edges ending at each vertex. In other words, it is a closed path where each vertex is crossed over only once. The length of a circuit is the total number of edges in the circuit.

Definition 3.3. [Bow12] A finitely generated group G with generating set \mathcal{S} is *relatively hyperbolic* with respect to a finite set of subgroups \mathbb{P} if the coned-off Cayley graph $\hat{\Gamma}$ is δ -hyperbolic and fine. The subgroups $P \in \mathbb{P}$ are the *peripheral subgroups* of G and the left cosets gP where $g \in G$ and $P \in \mathbb{P}$ are *peripheral cosets*.

Definition 3.4. A relatively hyperbolic group G with generating set \mathcal{S} and finite set of peripheral subgroups \mathbb{P} is said to be *non-elementary* if it is not virtually cyclic (see Definition 2.7) and $G \notin \mathbb{P}$.

Now that we have defined what a relatively hyperbolic group is, we provide a concrete example so the reader may more clearly visualize what such a group is like.

Example 3.5. $\mathbb{Z}^2 * \mathbb{Z} \cong \pi_1(T^2 \wedge S^1)$, where T^2 is a torus and S^1 a circle, is hyperbolic relative to \mathbb{Z}^2 . It is one of the simplest examples of a group that is relatively hyperbolic and tree-like, but not hyperbolic.

Chapter 4

Showing $\mathbb{Z}^2 * \mathbb{Z}$ to be Statistically Hyperbolic

In this chapter, we will show that $\mathbb{Z}^2 * \mathbb{Z}$ is statistically hyperbolic using its standard generating set. This result is a special case of the Main Theorem in which we can give a more direct, elementary argument that does not rely on definite exponential growth, as does the proof of Theorem 2.8 for the δ -hyperbolic case, nor does it rely on Yang's result in Theorem 5.2, as does the proof of the Main Theorem.

Consider the universal cover of $T^2 \wedge S^1$ and suppose we have $B(e, n)$, a ball of radius n centered at the origin as shown in Figure 4.1. Define S_n to be the sphere which is the boundary of $B(e, n)$. We wish to define types of “planes” found in this ball and then count the number of each type of “plane” we find. Let a *plane of type k* be a bounded copy of \mathbb{Z}^2 within the universal cover of $T^2 \wedge S^1$ where the number of points of the plane which intersect S_n is exactly $4k$. In this section, we will refer to a plane of type k as P_k as in Figure 4.2. We also wish to make one exception, in the case of P_0 . We define P_0 as above where the number of points of the plane intersecting S_n is exactly 1. We use the notation $|P_k|$ to mean the number of planes of type k in $B(e, n)$. In order to somewhat simplify the notation in the following proofs, let us define $a_k = |P_{n-k}|$ when $n \in \mathbb{N}$ is fixed.

Lemma 4.1. *For the group $\mathbb{Z}^2 * \mathbb{Z}$ with the standard generating set \mathcal{S} where $n \in \mathbb{N}$*

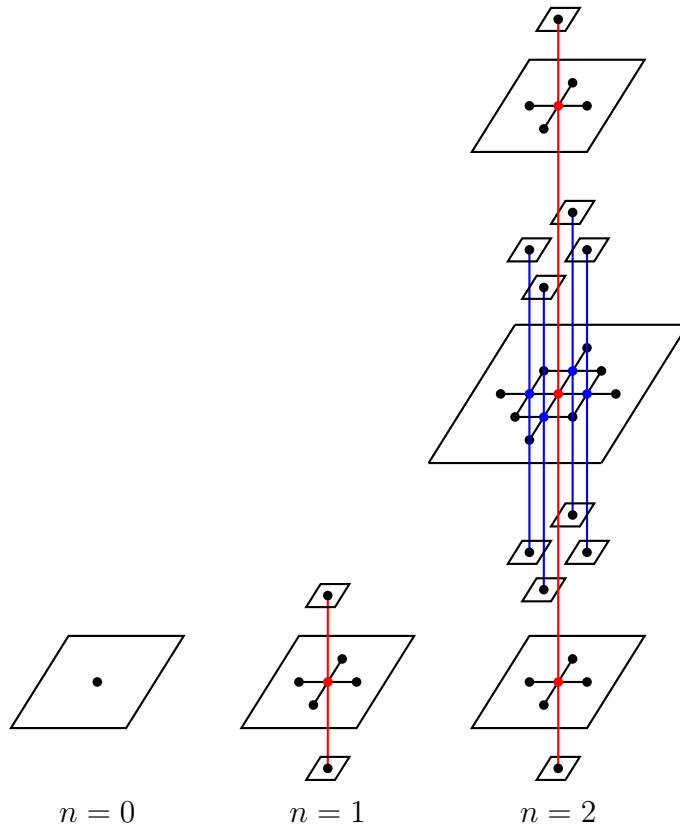


Figure 4.1: Balls of radius n in universal cover of $T^2 \wedge S^1$ for $n = 0, 1, 2$

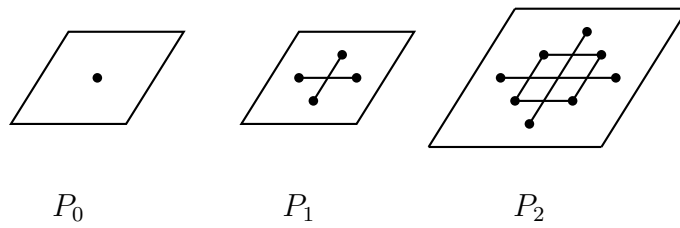


Figure 4.2: Planes of type k for $k = 0, 1, 2$

is fixed, we can find the recursive formula

$$a_k = a_{k-1} + 8a_{k-2} + 16a_{k-3} + \cdots + 8(k-2)a_1 + 8(k-1)a_0$$

for $k \geq 4$ for the number of planes of each type in the universal cover of $T^2 \wedge S^1$ where $a_k = |P_{n-k}|$.

Proof. Let $n \in \mathbb{N}$ be fixed. The first observation we make is that $a_0 = 1$, and that P_n has the identity at its center. We will refer to this plane as the “main plane.” Additionally, we also see that $a_1 = 2$, since there are two planes of type $n-1$ emanating from the main plane. We will refer to the set of all planes emanating from one of the P_{n-1} , including the plane P_{n-1} in question, as a “stem.” There are eight planes of type $n-2$ which emanate from the main plane and only one plane of type $n-2$ emanating from each stem. Thus, $a_2 = 10$. All of the previous observations can be made by simply looking at a ball of radius 2. From this point forward, a_k where $k > 2$ can be found by realizing that each $P_{n-(k-1)}$ will have one P_{n-k} emanating from it, each $P_{n-(k-2)}$ will have 8 of the P_{n-k} emanating from it, each $P_{n-(k-3)}$ will have 16 of the P_{n-k} emanating from it, and so on, so that each $P_{n-(k-j)}$ where $j \leq k$ will have $8(j-1)$ of the P_{n-k} emanating from it.

Using the steps outlined above, we can find that for $k \geq 4$ we have

$$a_k = a_{k-1} + 8a_{k-2} + 16a_{k-3} + \cdots + 8(k-2)a_1 + 8(k-1)a_0 \quad \square$$

Corollary 4.2. For the group $\mathbb{Z}^2 * \mathbb{Z}$ with the standard generating set \mathcal{S} , $a_k = 3a_{k-1} + 5a_{k-2} + a_{k-3}$ for $k \geq 4$.

Proof. (Induction on k .) Fix $n \in \mathbb{N}$. As our base case, let $k = 4$. Then,

$$\begin{aligned} a_4 &= 3a_3 + 5a_2 + a_1 \\ &= 3 \cdot 42 + 5 \cdot 10 + 2 \\ &= 178 \end{aligned}$$

Now assume the claim is true for all values of $k \leq m-1$. So, we know by Lemma 4.1

$$a_m = a_{m-1} + 8a_{m-2} + 16a_{m-3} + \cdots + 8(m-2)a_1 + 8(m-1)a_0$$

Thus, we see that

$$\begin{aligned} a_m - a_{m-1} &= a_{m-1} + 8a_{m-2} + 16a_{m-3} + \cdots + 8(m-2)a_1 + 8(m-1)a_0 \\ &\quad - a_{m-2} - 8a_{m-3} - \cdots - 8(m-3)a_1 - 8(m-2)a_0 \\ &= a_{m-1} + 7a_{m-2} + 8a_{m-3} + \cdots + 8a_1 + 8a_0 \end{aligned}$$

Hence,

$$a_m = 2a_{m-1} + 7a_{m-2} + 8a_{m-3} + \cdots + 8a_1 + 8a_0$$

Now we see that by using our induction hypothesis and the equation above, we have

$$\begin{aligned} a_m &= 2a_{m-1} + 7a_{m-2} + 8a_{m-3} + 8a_{m-4} + 8a_{m-5} + \cdots + 8a_1 + 8a_0 \\ &= 3a_{m-1} + 4a_{m-2} + 3a_{m-3} + 7a_{m-4} + 8(a_{m-5} + \cdots + a_1 + a_0) \\ &= 3a_{m-1} + 5a_{m-2} + 2a_{m-4} + 7a_{m-5} + 8(a_{m-6} + \cdots + a_1 + a_0) \\ &= 3a_{m-1} + 5a_{m-2} + a_{m-3} \end{aligned} \quad \square$$

Corollary 4.3. *For the group $\mathbb{Z}^2 * \mathbb{Z}$ with the standard generating set \mathcal{S} , there exist $\lambda > 1$ and $c > 0$ such that $c\lambda^n \leq |S_n| \leq (8n^2 + 2)\lambda^n$. (Specifically, $\lambda = 2 + \sqrt{5}$.)*

Proof. From Lemma 4.1 and Corollary 4.2, we have $|P_{n-k}| = a_k = 3a_{k-1} + 5a_{k-2} + a_{k-3}$ for $k \geq 4$ where $a_1 = 2$, $a_2 = 10$, and $a_3 = 42$. Thus, the characteristic equation of a_k is $x^3 = 3x^2 + 5x + 1$. Hence, the roots of the equation are $x = -1, 2 + \sqrt{5}, 2 - \sqrt{5}$. Thus, we know that

$$|P_{n-k}| = a_k = c_1(-1)^k + c_2(2 + \sqrt{5})^k + c_3(2 - \sqrt{5})^k$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Using the initial conditions, we find that

$$|P_{n-k}| = a_k = \frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^k + \frac{5 + \sqrt{5}}{5}(2 - \sqrt{5})^k$$

Putting it all together, we now see that for a sphere of radius n ,

$$\begin{aligned} |S_n| &= 4n|P_n| + 4(n-1)|P_{n-1}| + \cdots + 4(n-k)|P_{n-k}| \\ &\quad + \cdots + 4|P_1| + |P_0| \\ &= |P_0| + \sum_{k=0}^{n-1} 4(n-k)|P_{n-k}| \\ &= \frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^n + \frac{5 + \sqrt{5}}{5}(2 - \sqrt{5})^n \\ &\quad + \sum_{k=0}^{n-1} 4(n-k) \left(\frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^k + \frac{5 + \sqrt{5}}{5}(2 - \sqrt{5})^k \right) \\ &\leq \frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^n + \frac{5 + \sqrt{5}}{5}(2 + \sqrt{5})^n \\ &\quad + \sum_{k=0}^{n-1} 4(n-k) \left(\frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^k + \frac{5 + \sqrt{5}}{5}(2 + \sqrt{5})^k \right) \\ &= 2(2 + \sqrt{5})^n + \sum_{k=0}^{n-1} 4(n-k) \left(2(2 + \sqrt{5})^k \right) \\ &\leq 2(2 + \sqrt{5})^n + 8n^2(2 + \sqrt{5})^n \\ &= (8n^2 + 2)(2 + \sqrt{5})^n \end{aligned}$$

Clearly, we also see that

$$|S_n| \geq \frac{5 - \sqrt{5}}{5}(2 + \sqrt{5})^n \geq \frac{1}{2}(2 + \sqrt{5})^n$$

□

Theorem 4.4. *The group $\mathbb{Z}^2 * \mathbb{Z}$ with standard generating set \mathcal{S} is statistically hyperbolic.*

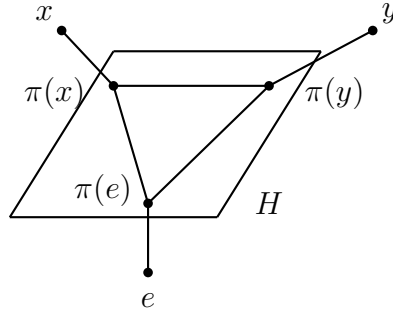


Figure 4.3: Triangle Δexy in $\text{Cayley}(\mathbb{Z}^2 * \mathbb{Z}, \mathcal{S})$

Proof. Choose $r \in (0, 1)$ and $x \in S_n$. For each $g \in S_n$, we define the set $A_g = \{g' \in S_{\lfloor rn \rfloor} \mid d(e, g') + d(g', g) = d(e, g)\}$. Note that $\Gamma(G, \mathcal{S})$ has a tree-like structure. So, every embedded loop must lie in a single copy of \mathbb{Z}^2 . This gives us that A_g must lie in a single copy of \mathbb{Z}^2 for all $g \in S_n$.

Now choose $y \in S_n$ such that A_x and A_y are contained in different peripheral cosets and consider the triangle Δexy . Let \mathbb{H} be the set of all peripheral cosets which intersect both $[e, x]$ and $[e, y]$. Define $H \in \mathbb{H}$ to be the peripheral coset where $d(x, H) \leq d(x, H')$ and $d(y, H) \leq d(y, H')$ for all $H' \in \mathbb{H}$.

Let $\Gamma(G, \mathcal{S}) = \text{Cayley}(G, \mathcal{S})$ be the Cayley graph of G with standard generating set \mathcal{S} and let $\pi : \Gamma \rightarrow H$ be the nearest point projection map. We know this map exists due to the tree-like structure of $\text{Cayley}(G, \mathcal{S})$. So, $\pi(x)$, $\pi(y)$, and $\pi(e)$ are as labeled in Figure 4.3.

By way of contradiction, suppose $\pi(e) \notin B(e, \lfloor rn \rfloor)$. Then $A_x = A_y = A_{\pi(e)}$. This contradicts the assumption that A_x and A_y are contained in different peripheral cosets. Also by way of contradiction, suppose $\pi(e) \in B(e, \lfloor rn \rfloor)$ and that $\pi(x), \pi(y) \notin B(e, \lfloor rn \rfloor)$. Thus, $A_x \subseteq H$ and $A_y \subseteq H$. This also contradicts our assumption of the choice of $y \in S_n$. Hence, assuming that A_x and A_y are in different peripheral cosets implies that $\pi(e) \in B(e, \lfloor rn \rfloor)$ and that at least one of $\{\pi(x), \pi(y)\}$ is in $B(e, \lfloor rn \rfloor)$.

Without loss of generality, assume $\pi(x) \in B(e, \lfloor rn \rfloor)$. Thus, any geodesic between x and y must pass through $x' \in S_{\lfloor rn \rfloor}$ for some $x' \in A_x$. So, assuming that

A_x and A_y are in different peripheral cosets gives us

$$\begin{aligned} d(x, y) &\geq d(x, x') + d(x', y) \\ &\geq n - \lfloor rn \rfloor + n - \lfloor rn \rfloor \\ &\geq 2(n - rn) \end{aligned}$$

Assuming that $x \in S_n$ is fixed, we wish to set a lower bound on the number of choices for $y \in S_n$ where A_x and A_y are in different peripheral cosets. We will accomplish this by bounding the number of $z \in S_n$ for which A_x and A_z are in the same peripheral coset. Let $B(e, n)$ denote a ball of radius n with its center at e . Let $\beta_G(n) = |B(e, n)|$ and $\beta_P(n) = |B(e, n)|$ when we restrict our ball to be contained in just one peripheral coset of our group. We see that $|A_x| \leq \beta_P(\lfloor rn \rfloor)$ where equality is only attained if $e \in H$. Thus, assuming A_x and A_z are in the same peripheral coset, we see that $|A_z| \leq \beta_P(\lfloor rn \rfloor)$. For each $z' \in A_z$, the number of elements $z \in S_n$ for which e, z' , and z lie on a geodesic is bounded by $|S_n - \lfloor rn \rfloor|$. Thus, the total number of possible $z \in S_n$ for which A_x and A_z lie on the same peripheral coset is bounded above by $\beta_P(\lfloor rn \rfloor)|S_n - \lfloor rn \rfloor|$. Therefore, the number of choices for $y \in S_n$ such that A_x and A_y are in different peripheral cosets is bounded below by $|S_n| - \beta_P(\lfloor rn \rfloor)|S_n - \lfloor rn \rfloor|$.

From Corollary 4.3, $c\lambda^n \leq |S_n| \leq (8n^2 + 2)\lambda^n$ for all n for some $\lambda > 1$ and $c > 0$. Hence, we see that

$$\begin{aligned}
E(G, \mathcal{S}) &= \lim_{n \rightarrow \infty} \frac{\sum_{x, y \in \mathcal{S}_n} d(x, y)}{n \cdot |\mathcal{S}_n|^2} \\
&\geq \lim_{n \rightarrow \infty} \frac{2(n - rn) \cdot |\mathcal{S}_n| \cdot (|\mathcal{S}_n| - \beta_P(\lfloor rn \rfloor) \cdot |\mathcal{S}_{n - \lfloor rn \rfloor}|)}{n \cdot |\mathcal{S}_n|^2} \\
&= \lim_{n \rightarrow \infty} 2(1 - r) - \lim_{n \rightarrow \infty} \frac{2(1 - r)\beta_P(\lfloor rn \rfloor)|\mathcal{S}_{n - \lfloor rn \rfloor}|}{|\mathcal{S}_n|} \\
&\geq 2(1 - r) - 2(1 - r) \lim_{n \rightarrow \infty} \frac{(4rn)^2(8n^2 + 2)\lambda^{n - \lfloor rn \rfloor}}{c\lambda^n} \\
&= 2(1 - r) - \frac{64r^2(1 - r)}{c} \lim_{n \rightarrow \infty} \frac{(4n^4 + n^2)\lambda^{n - \lfloor rn \rfloor}}{\lambda^n} \\
&= 2(1 - r) - \frac{64r^2(1 - r)}{c} \lim_{n \rightarrow \infty} \frac{(4n^4 + n^2)}{\lambda^{\lfloor rn \rfloor}} \\
&= 2(1 - r)
\end{aligned}$$

Since $r \in (0, 1)$ was arbitrary, letting $r \rightarrow 0$ gives $E(G, \mathcal{S}) = 2$. □

Chapter 5

Proof of the Main Theorem

We now wish to have a fixed setup for an arbitrary relatively hyperbolic group which we can utilize for the remainder of this work. Let $(G, \mathcal{S}, \mathbb{P})$ denote a non-elementary relatively hyperbolic group G with finite generating set \mathcal{S} and a finite set of peripheral subgroups \mathbb{P} . Let $\beta_G(n)$ be the *growth function* for the entire group G defined as the number of elements $g \in G$ such that the word length of g , using the generating set \mathcal{S} , is no more than n . In other words, the growth function is the size of a ball of radius n , or equivalently $\beta_G(n) = |B(e, n)|$. Now consider the induced growth function of each $P \in \mathbb{P}$, denoted as $\beta_P(n)$, where we are continuing to use the word metric based on the generating set \mathcal{S} . We define $\beta_{\mathbb{P}}(n) = \max_{P \in \mathbb{P}} \beta_P(n)$. The *growth rate* of $\beta_G(n)$ is defined to be $\lambda_G = \lim_{n \rightarrow \infty} \sqrt[n]{\beta_G(n)}$ and the *growth rate* of $\beta_{\mathbb{P}}(n)$ is $\lambda_{\mathbb{P}} = \lim_{n \rightarrow \infty} \sqrt[n]{\beta_{\mathbb{P}}(n)}$. Note that the preceding limits exist due to the submultiplicativity of the growth functions, as can be seen in Chapter VI, Section C of [dlH00].

Definition 5.1. $(G, \mathcal{S}, \mathbb{P})$ has the *parabolic gap property* if $\lambda_G > \lambda_{\mathbb{P}}$.

Theorem 5.2. [Yan13] *Given $(G, \mathcal{S}, \mathbb{P})$, there exists $C > 0$ such that*

$$\frac{1}{C} \lambda_G^n < \beta_G(n) < C \lambda_G^n \text{ for all } n \geq 0$$

Lemma 5.3. [DS05b] *Given $(G, \mathcal{S}, \mathbb{P})$, for each $L < \infty$ there is a constant $K =$*

$K(L) < \infty$ such that for any two peripheral cosets $gP \neq g'P'$ we have

$$\text{diam}(\mathcal{N}_L(gP) \cap \mathcal{N}_L(g'P')) < K$$

with respect to the metric $d_{\mathcal{S}}$.

Definition 5.4. Let (G, \mathcal{S}) be a group with generating set \mathcal{S} . For a fixed $x \in G$ and $N \in \mathbb{N}$, we call the set $\{x' \in S_N \mid d(e, x') + d(x', x) = d(e, x)\}$ the N^{th} geodesic layer of x , or simply the *geodesic layer* when N and x are understood. In other words, each x' in the geodesic layer lies on a geodesic between e and x at a distance N from e .

After choosing $r \in \mathbb{R}$ such that $0 < r < 1$, we define A_x to be the $[rn]^{\text{th}}$ geodesic layer of x for a given $n \in \mathbb{N}$.

Lemma 5.5. *Given $(G, \mathcal{S}, \mathbb{P})$ and $x \in G$, the geodesic layer A_x either is a bounded set, or A_x is within an L -neighborhood of a single peripheral coset gP for a fixed constant L .*

Proof. Let $x \in G$ and choose a geodesic c from e to x . Fix a constant $L > 0$. Let K be a constant for the given L , so that for any pair of peripheral cosets $gP \neq g'P'$ in G , we have $\text{diam}(\mathcal{N}_L(gP) \cap \mathcal{N}_L(g'P')) < K$ as in Lemma 5.3. Clearly, if $\text{diam}(A_x) < K$ then A_x is a bounded set. However, if $\text{diam}(A_x) \geq K$ then A_x must be within the L -neighborhood of a unique peripheral coset gP , by our choice of K . \square

Definition 5.6. [DS05a] Let G be a group and let \mathbb{P} be a finite set of subgroups in G . Then G is *(*)-relatively hyperbolic with respect to \mathbb{P}* if there exist a finite generating set \mathcal{S} of G , and two constants σ and δ such that the following property holds.

(*) For every geodesic triangle ABC in the Cayley graph of G with respect to \mathcal{S} , there exists a coset gP such that the closed σ -neighborhood $\overline{\mathcal{N}}_{\sigma}(gP)$ intersects each of the sides of the triangle, and the entrance (and exit) points A_1, B_1, C_1 (and B_2, C_2, A_2) of the sides $[A, B], [B, C], [C, A]$ in $\overline{\mathcal{N}}_{\sigma}(gP)$ satisfy

$$d(A_1, A_2) < \delta, \quad d(B_1, B_2) < \delta, \quad d(C_1, C_2) < \delta$$

as in Figure 5.1.

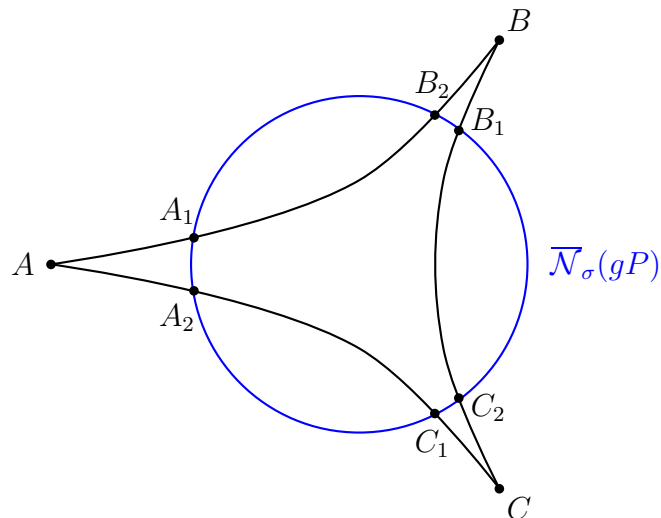


Figure 5.1: A triangle which meets the conditions for (*)-relative hyperbolicity.

Lemma 5.7. [DS05a] Every group G that is relatively hyperbolic with respect to a finite set of finitely generated subgroups \mathbb{P} is also (*)-relatively hyperbolic with respect to \mathbb{P} .

Theorem 5.8 (Main Theorem). *If G is a non-elementary relatively hyperbolic group with finite generating set \mathcal{S} and a finite set of peripheral subgroups \mathbb{P} having the parabolic gap property, then G is statistically hyperbolic.*

Proof. Using the assumption that G has the parabolic gap property and Theorem 5.2, we will choose constants $\lambda > 1$, $C > 0$ and $C' > 0$ where $\frac{1}{C}\lambda_G^n < \beta_G(n) < C\lambda_G^n$ for all $n \geq 0$ and $\lambda_{\mathbb{P}} < \lambda < \lambda_G$ such that $\beta_{\mathbb{P}}(n) \leq C'\lambda^n$ holds for sufficiently large n as described at the beginning of this chapter. Fix a value of $n \in \mathbb{N}$ so that the above inequalities hold. Choose $x \in S_n$ and $r \in \mathbb{R}$ such that $0 < r < 1$. Let $A_x = \{x' \in S_{[rn]} | d(e, x') + d(x', x) = d(e, x)\}$ be the $[rn]^{th}$ geodesic layer of x . Fix a geodesic from e to x and let $x' \in [e, x] \cap A_x$. Consider any triangle Δexy having the chosen geodesic $[e, x]$ as one of its sides. We now use the setup from Definition 5.6 to have a peripheral coset gP which lies in the “center” of Δexy and proceed to simplify our work by “thinning down” Δexy until it is nearly a tripod, as in Figure 5.2. In order to accomplish this, choose $u, v, w \in gP$ such that

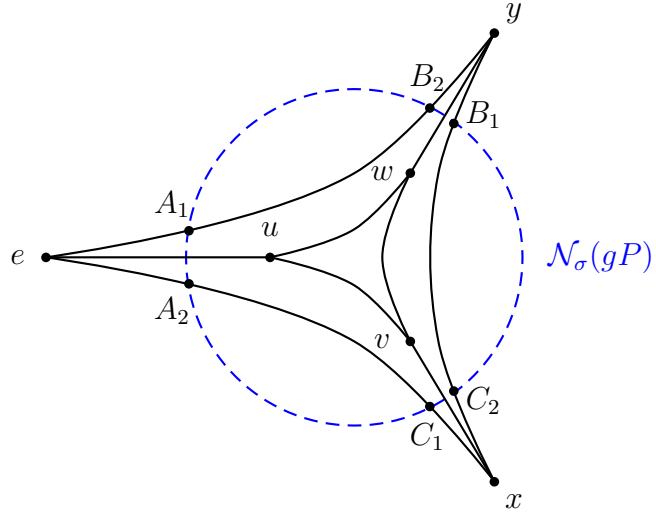


Figure 5.2: A triangle “thinned down” almost to a tripod.

$d(u, A_2) < \sigma$, $d(v, C_1) < \sigma$, and $d(w, B_2) < \sigma$. Then we know that $d(u, A_1) < \sigma + \delta$, $d(v, C_2) < \sigma + \delta$, and $d(w, B_1) < \sigma + \delta$. We will now show that when the geodesic $[e, x]$ is fixed in this way, then either $d(x, y)$ has a lower bound, or the number of choices for y has an upper bound where $d(x, y)$ is not sufficient for our purposes here.

Case 1: $\lfloor rn \rfloor \leq d(e, u)$.

We know by Definition 5.6 that $d(A_1, A_2) < \delta$. So, $d(A_1, e) \geq d(A_2, e) - \delta$ by the triangle inequality. Thus, $d(A_1, y) = d(e, y) - d(e, A_1) \leq n - d(A_2, e) + \delta$. We also see that $d(x', A_2) = d(A_2, e) - d(x', e) = d(A_2, e) - \lfloor rn \rfloor$. Thus, $d(x', y) \leq (d(A_2, e) - \lfloor rn \rfloor) + \delta + (n - d(A_2, e) + \delta) \leq n - \lfloor rn \rfloor + 2\delta$. Therefore, $y \in B_{n - \lfloor rn \rfloor + 2\delta}(x')$, so there is a bounded number of choices for y . Specifically, for J equal to the number of choices for such a y , we see that

$$J \leq |A_x| \cdot |S_{n - \lfloor rn \rfloor + 2\delta}|$$

Further, we know from Lemma 5.5 that A_x is either a bounded set or within an L -neighborhood of a unique peripheral coset. Choose a constant L which satisfies Lemma 5.5. If A_x is bounded, let $\text{diam}(A_x) = M$ for some constant $M \geq 0$. Then, we would have $|A_x| \leq \beta_G(M) \leq C\lambda_G^M$. Alternately, if A_x is not bounded, then

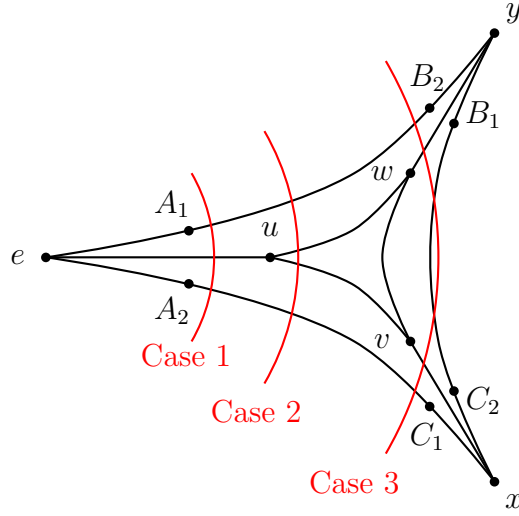


Figure 5.3: Representations of $S_{[rn]}$ for Cases 1, 2, and 3.

it must be within an L -neighborhood of a single peripheral coset. In this case, $|A_x| \leq \beta_{\mathbb{P}}(\lfloor rn \rfloor) \cdot \beta_G(L) \leq C' \lambda^{\lfloor rn \rfloor} \cdot C \lambda_G^L$. For sufficiently large n , we observe that $C' \lambda^{\lfloor rn \rfloor} \cdot C \lambda_G^L > C \lambda_G^M$. Therefore, we find that

$$\begin{aligned}
 J &\leq |A_x| \cdot |S_{n-\lfloor rn \rfloor+2\delta}| \\
 &\leq CC' \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^L \cdot (\beta_G(n - \lfloor rn \rfloor + 2\delta) - \beta_G(n - \lfloor rn \rfloor + 2\delta - 1)) \\
 &\leq CC' \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^L \cdot \left(C \lambda_G^{n-\lfloor rn \rfloor+2\delta} - \frac{1}{C} \lambda_G^{n-\lfloor rn \rfloor+2\delta-1} \right) \\
 &\leq C' \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor+L+2\delta-1} (C^2 \lambda_G - 1)
 \end{aligned}$$

Case 2: $d(e, u) < \lfloor rn \rfloor \leq d(e, u) + \min\{d(u, v), d(u, w)\}$.

Let N be the number of peripheral cosets which intersect the ball $B_{2\sigma+2\delta}(x')$. Choose one of these N cosets, gP . Notice that for each choice of y , and each choice of geodesic triangle Δexy there are corresponding points u, v, w as described above.

One might, at first, think that we could get a different point u for each choice of y . However, A_2 is the point at which our chosen geodesic $[e, x]$ first enters the closed σ -neighborhood of gP , where gP is one of the N peripheral cosets intersecting the ball $B_{2\sigma+2\delta}(x')$. And u is a point chosen arbitrarily from the ball $B_{\sigma}(A_2)$. Therefore, the number of choices for u is at most N times the size of the ball $B_{\sigma}(e)$.

We note that for $y' \in A_y$, if $y' \notin [A_1, B_2]$ but $y' \in [e, A_1] \cup [B_2, y]$, then $d(y', A_1) < \sigma + \delta$. Thus, $d(y', gP) < 2\sigma + 2\delta$. However, if $y' \in [A_1, B_2]$, then $d(y', gP) < \sigma$. So, we can say that in any case, $d(y'gP) < 2\sigma + 2\delta$. Let $y'' \in gP \cap B_{\lfloor rn \rfloor + 2\sigma + 2\delta}(e)$. Then we see that $d(u, y'') \leq d(u, A_1) + d(A_1, y') + d(y', y'') \leq \sigma + \lfloor rn \rfloor + 2\sigma + 2\delta = \lfloor rn \rfloor + 3\sigma + 3\delta$. Thus, there is a bounded number of choices for such y'' , which in turn gives us a bounded number of choices for y . Specifically, for J equal to the number of choices for such a y , we see that

$$J \leq N \cdot \beta_G(\sigma) \cdot \beta_{\mathbb{P}}(\lfloor rn \rfloor + 3\sigma + 3\delta) \cdot \beta_G(2\sigma + 2\delta) \cdot |S_{n - \lfloor rn \rfloor}|$$

The above formulation is due to first choosing u , then allowing y'' to be any point in the same peripheral coset within the requisite radius, then choosing any point y' within a radius $2\sigma + 2\delta$ of y'' , followed by any point y on the sphere centered at y' with radius $n - \lfloor rn \rfloor$. Further, we find that

$$\begin{aligned} J &\leq N \cdot \beta_G(\sigma) \cdot \beta_{\mathbb{P}}(\lfloor rn \rfloor + 3\sigma + 3\delta) \cdot \beta_G(2\sigma + 2\delta) \cdot |S_{n - \lfloor rn \rfloor}| \\ &\leq N \cdot C\lambda_G^\sigma \cdot C'\lambda^{\lfloor rn \rfloor + 3\sigma + 3\delta} \cdot C\lambda_G^{2\sigma + 2\delta} \cdot (\beta_G(n - \lfloor rn \rfloor) - \beta_G(n - \lfloor rn \rfloor - 1)) \\ &\leq N \cdot C\lambda_G^\sigma \cdot C'\lambda^{\lfloor rn \rfloor + 3\sigma + 3\delta} \cdot C\lambda_G^{2\sigma + 2\delta} \cdot \left(C\lambda_G^{n - \lfloor rn \rfloor} - \frac{1}{C}\lambda_G^{n - \lfloor rn \rfloor - 1} \right) \\ &\leq N \cdot C \cdot C'\lambda^{\lfloor rn \rfloor + 3\sigma + 3\delta} \cdot \lambda_G^{n + 3\sigma + 2\delta - \lfloor rn \rfloor - 1} \cdot (C^2\lambda_G - 1) \end{aligned}$$

Case 3: $\lfloor rn \rfloor > d(e, u) + \min\{d(u, v), d(u, w)\}$.

In this case, at least one of v, w lies in $B_{\lfloor rn \rfloor}(e)$. Without loss of generality, assume $v \in B_{\lfloor rn \rfloor}(e)$. From above, $d(v, C_2) < \sigma + \delta$. Thus, in this case,

$$\begin{aligned} d(x, y) &= d(x, C_2) + d(C_2, y) \\ &\geq d(x, v) - d(v, C_2) + d(v, y) - d(v, C_2) \\ &\geq (n - \lfloor rn \rfloor) - (\sigma + \delta) + (n - \lfloor rn \rfloor) - (\sigma + \delta) \\ &\geq 2(n - rn - \sigma - \delta) \end{aligned}$$

So now we have established that either $d(x, y)$ has a lower bound as in Case 3, or the number of choices for y has an upper bound as in Cases 1 and 2. To simplify our notation a bit, we will combine the results of Cases 1 and 2 in the following way.

Let K be a constant so that $K \geq 2N \cdot C \cdot C' \lambda^{3\sigma+3\delta} \cdot \lambda_G^{3\sigma+2\delta+L-1} \cdot (C^2 \lambda_G - 1)$. Then we observe that

$$J \leq K \cdot \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}$$

Now when we calculate the sprawl, we will leave out all the choices for y for which we did not observe a lower bound on $d(x, y)$. Thus, we have $|S_n|$ choices for $x \in S_n$ and $|S_n| - J$ choices for a corresponding $y \in S_n$ so that the distance from Case 3 above is satisfied. Hence,

$$\begin{aligned} E(G, \mathcal{S}) &= \lim_{n \rightarrow \infty} \frac{\sum_{x, y \in S_n} d(x, y)}{n \cdot |S_n|^2} \\ &\geq \lim_{n \rightarrow \infty} \frac{2(n - rn - \sigma - \delta) \cdot |S_n| \cdot (|S_n| - J)}{n \cdot |S_n|^2} \\ &\geq \lim_{n \rightarrow \infty} \frac{2(1 - r - \sigma/n - \delta/n) \cdot (|S_n| - K \cdot \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor})}{|S_n|} \\ &= \left(\lim_{n \rightarrow \infty} 2(1 - r - \sigma/n - \delta/n) \right) \left(\lim_{n \rightarrow \infty} \frac{|S_n| - K \cdot \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}}{|S_n|} \right) \\ &= 2(1 - r) \lim_{n \rightarrow \infty} \left(1 - \frac{K \cdot \lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}}{|S_n|} \right) \\ &= 2(1 - r) - 2K(1 - r) \lim_{n \rightarrow \infty} \frac{\lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}}{\beta_G(n) - \beta_G(n-1)} \\ &\geq 2(1 - r) - 2K(1 - r) \lim_{n \rightarrow \infty} \frac{\lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}}{C\lambda_G^n - \frac{1}{C}\lambda_G^{n-1}} \\ &= 2(1 - r) - 2K(1 - r) \lim_{n \rightarrow \infty} \frac{\lambda^{\lfloor rn \rfloor} \cdot \lambda_G^{n-\lfloor rn \rfloor}}{\lambda_G^n (C - \frac{1}{C\lambda_G})} \\ &= 2(1 - r) - \frac{2K(1 - r)}{C - \frac{1}{C\lambda_G}} \lim_{n \rightarrow \infty} \left(\frac{\lambda}{\lambda_G} \right)^{\lfloor rn \rfloor} \\ &= 2(1 - r) \end{aligned}$$

The final line above is due to the assumption that G has the parabolic gap property and our choice of λ , which gives us that $\lambda < \lambda_G$. Finally, since $r \in (0, 1)$

was arbitrary, letting $r \rightarrow 0$ gives $E(G, \mathcal{S}) = 2$. □

So, in conclusion, we now observe that statistical hyperbolicity is a generalization of relative hyperbolicity, assuming the group in question is non-elementary and has the parabolic gap property.

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Jeremy A. Osborne

Personal Statement

I am an excellent mathematics instructor who consistently receives outstanding teaching reviews from students, peers, and administrators. I am well-qualified for the position at your institution, both through my teaching experience and educational background. If hired, I would be an asset to the school and community in which the school is located.

Education

2008-2014 *University of Wisconsin - Milwaukee* (Milwaukee, WI)

- Earned **Doctor of Philosophy in Mathematics**
- Achieved 3.8 GPA Overall
- Recipient of GAANN Fellowship for five years (funded by U.S. Department of Education)

2012 *Oregon State University* (Corvallis, OR)

- Attended a three-day workshop on Geometric Topology

2006-2007 *Truman State University* (Kirksville, MO)

- Attended an interdisciplinary curriculum retreat (Topic: Incorporating Research into Undergraduate Curricula)
- Participated in ten-week summer research program (funded by the National Science Foundation) during which I researched Integer Sequences (Fibonacci, Lucas, and Lucas-like sequences), wrote several enrichment projects for College Algebra students, began writing a College Algebra text, developed proficiency with LaTeX software, and developed familiarity with Mathematica software

2005 *University of Central Oklahoma* (Edmond, OK)

- Attended a two-day conference on Cryptology

2001-2004 *Missouri State University* (Springfield, MO)

- Earned **Master of Science in Mathematics**
- Achieved 4.0 GPA Overall

1996-2001 *Evangel University* (Springfield, MO)

- Earned **Bachelor of Science in Mathematics**
- Completed **Minor in Physics**
- Achieved 3.5 GPA Overall (3.9 GPA in Mathematics courses)
- Member of Kappa Mu Epsilon National Mathematics Honor Society
- Member of Evangel University Math Club (President for one year)

Teaching Experience

2008-2014 **Teaching Assistant** at the *University of Wisconsin - Milwaukee*
(Milwaukee, WI)

- Courses taught:
 - Intermediate Algebra
 - Beginning/Intermediate Algebra (using ALEKS computer program)
 - Contemporary Mathematics (as a course assistant)
 - College Algebra
 - Trigonometry
 - Elementary Statistics
 - Calculus I
- Other: held office hours; tutored undergraduate mathematics
- Received “First Year Student Success Award” for dedication to the academic success of first year students

2004-2008 **Assistant Professor** at *Moberly Area Community College* (Kirksville, MO)

- Courses taught:

- Fundamentals of Algebra
 - Intermediate Algebra (including several internet-television courses)
 - College Algebra (including several internet-television courses)
 - Dual-credit College Algebra (all internet-television courses)
 - Trigonometry
 - Elementary Statistics (including one internet-television course)
 - Calculus I
- o Other: held office hours; carried out student advising; attended Department and Faculty Meetings; member of Faculty Forum; mentored a new faculty member; member of President's Faculty and Staff Advisory Council; member of Wellness Committee

2002-2004 **Adjunct Instructor** at *Ozarks Technical Community College* (Springfield, MO)

- o Courses taught:
- Intermediate Algebra
 - College Algebra
 - Trigonometry
- o Other: held office hours; assisted in writing department core final exams for Intermediate Algebra

2001-2003 **Graduate Assistant** at *Missouri State University* (Springfield, MO)

- o Taught Intermediate Algebra
- o Other: held office hours; tutored Intermediate and College Algebra

Other Work Experience

2003-2004 **Math Tutor** at *Bellwether Learning Center* (Springfield, MO)

- o Tutored Basic, Intermediate, and College Algebra as well as Geometry

- Prepared students for the Math and Science Reasoning portions of ACT
- Prepared students for the Math portion of other standardized tests (GMAT, etc.)

1998-2001 OSP Technician at *Stine Consulting* (Dillsburg, PA)

- Worked as crew leader of survey team (diagramming telecommunications)
- Developed proficiency in spreadsheets, word processors, and file management
- Developed familiarity with AutoCAD software
- Assisted in troubleshooting in-house software development

Technological Skills

Computer Software

- Microsoft Word, Excel, and PowerPoint
- Open Office
- ALEKS
- MathType
- TI-interactive
- LaTeX
- Mathematica
- AutoCAD

Computer Hardware

- Install external devices such as printers, scanners, etc.
- Install internal devices such ROM drives, hard drives, sound/video cards, RAM, etc.

Calculators

- Proficient in using all Texas Instruments calculators, including writing programs for the TI-92 and TI-89
- Proficient in using all Casio calculators

Personal Interests

- Church involvement (children's ministries, music, audio/visual technology)
- Playing piano, guitar and singing
- Reading
- Baseball
- Computers