5-1-2014

Scalable, Efficient and Optimal Discrete-Time Rebalancing Algorithms for Log-Optimal Investment Portfolio

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SCALABLE, EFFICIENT AND OPTIMAL DISCRETE-TIME REBALANCING ALGORITHMS FOR LOG-OPTIMAL INVESTMENT PORTFOLIO

by

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A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in Engineering

at
The University of Wisconsin-Milwaukee

May, 2014

ABSTRACT

SCALABLE, EFFICIENT AND OPTIMAL DISCRETE-TIME REBALANCING ALGORITHMS FOR LOG-OPTIMAL INVESTMENT PORTFOLIO

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The University of Wisconsin-Milwaukee, 2014
Under the supervision of Prof. Mukul Goyal

Portfolio rebalancing decisions are crucial to today's portfolio managers especially in high frequency algorithmic trading environment. These decisions must be made fast in dynamic market conditions. We develop computational algorithms to determine *optimal rebalance frequency (ORF)* of a class of investment portfolio for a finite investment horizon. We choose log-optimal investment portfolio which is deemed to be impractical and cost-prohibitive due to inherent need for continuous rebalancing and significant overhead of trading cost. Optimality of such portfolio is assured only when for very long term investor horizon. We study the question of *how often* a log-optimal portfolio be rebalanced for any given finite investment horizon. We develop an analytical framework to compute the *expected log of portfolio value* when a given discrete-time periodic rebalance frequency is used. For a certain class of portfolio assets, we compute the *optimal rebalance frequency*. We show that it is possible to improve investor log utility using this quasi-passive or *hybrid* rebalancing strategy.

Under the assumptions of geometric Brownian motion for assets and log-normality for sum of log-normal random variables, we find that the ORF is a piecewise function of investment horizon. One can construct this rebalance strategy function, called *ORF function*, up to a specified investment horizon given a limited trajectory of expected log of portfolio value (ELPV) when the initial portfolio is never rebalanced. We develop the analytical framework to compute the optimal rebalance strategy in linear time, a significant improvement from the previously proposed search-based quadratic time algorithm. Simulation studies show that an investor can gain significantly by adopting a discrete-time rebalancing periodically using ORF in lieu of continuous rebalancing. Finally we investigate the computational efficiency of the proposed algorithms to develop optimized versions which are scalable to portfolios comprising of large number of assets.

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List of Abbreviations

ELPV expected log of portfolio value

RIP rebalance inflection point

LURC log utility rebalance contour

ORF optimal rebalance frequency

EIPG expected instantaneous portfolio growth

OLUF optimal log utility frontier

Acknowledgements

Not long ago it seemed an impossibility to pursue the doctoral degree amidst other professional and family responsibilities. So many people contributed to make this thesis possible.

First and foremost, I express my deepest gratitude to my parents who have been a constant source of inspiration and confidence through out my life. I am also deeply indebted to my wife Pinky for her support and the sacrifices during these intense eight years of study. Sacrifices of my two boys, Suna and Luna who grew from infancy to boyhood are worth special mention.

I am heartily thankful to my supervisor, Dr. Mukul Goyal, for his encouragement, support and trust on my ability to conduct research independently. Special thanks to Dr. Dmitri Kaznachey, Executive Director, Securitized Products Trading Technology at JPMorgan Chase for his immensely useful critique and feedback. Lastly I owe my gratitude to Dr. Larry Holloway (Professor and Chair of the Department of Electrical and Computer Engineering at University of Kentucky) who as my MS thesis advisor twenty years back, taught me the way to conduct meaningful research and knowledge discovery. Lastly, I offer my appreciation to great many number of present and past colleagues at Rockwell Automation, Eaton Corporation and Fannie Mae who in many ways added to my modest knowledge bank.

Chapter 1

Introduction

The fifty year old Markowitz's Nobel Prize winning proposition of mean-variance principle has set in motion a large body of mathematical work which forms a solid foundation of the present day investment science. Investment decisions are increasingly made by sophisticated computationally heavy financial algorithms. These computational systems have become the corner stone of newest branch of engineering dubbed *financial engineering*. Financial engineering principles are only useful when modern computer science principles are utilized to build real-world financial computation systems. Indeed such financial computational systems are the foundation of almost all modern day wall-street like financial firms. Today's typical investor shuns the old day "art of investment" and takes recourse to financial engineering principles to set up and manage the investment portfolio.

Algorithmic trading systems are the foundations for many of today's commercial proprietary financial applications[1]. With increasing computational power, advanced trading systems are built to deploy complex algorithms to make intelligent buy and sell decisions of tradeable assets. Since the viable window for trading opportunity exists only for very short time period, all trading decisions need to be made extremely fast in the order of seconds. One important decision that many such trading systems have to reliably make is when and how to rebalance an investment portfolio. Too frequent rebalancing can be cost prohibitive due to transaction costs involved. Lethargic and procrastinated rebalancing decisions may also prove to be costly in terms of lost opportunity to respond to market signals.

1.1 Portfolio Rebalancing

Once the portfolio is set up after determining the proper asset mix, the investor needs to address the issue of rebalancing the portfolio. Calvet et. al. [2] have studied the behavioral aspects of portfolio rebalancing for Swedish household investors. The changes in the household risky share is decomposed into two components, viz., *passive change* driven by returns of risky assets in the absence of any trading and *active changes* resulting from household rebalancing decisions. Their regression based analysis shows strong household propensity to rebalance. Specifically, they show that wealthy and educated investors with better diversified portfolios rebalance more actively. There is strong evidence that households rebalance by holding on to losing stocks whose prices have fallen and selling winning stocks whose prices have increased. This is known as *disposition effect* which has been examined in many papers including [3] and [4].

Conventional rebalancing strategies have been studied extensively by both researchers and practioners[5][6][7]. Authors Collins et. al. [8] provide an excellent survey of modern principles and practice of rebalancing. There are three types of conventional rebalancing techniques discussed in literature[9]. In *calendar rebalancing* the portfolio mix is returned to initial asset mix in regular periodic intervals. In *rebalancing to allowed range*, the portfolio is always brought back within the allowed range of drift. In the third technique called *threshold rebalancing*, the portfolio is always rebalanced to the initial mix whenever it drifts beyond a predefined range.

A sound investment philosophy needs to consider several aspects of rebalancing. Prominent among them are the questions of *when* one should rebalance and *what* the portfolio should be rebalanced to. In this context the authors in [9] have studied the historical stock and bond return and standard deviation data between 1968 to 1991 to conclude that disciplined rebalancing can indeed boost returns. They conclude that it is beneficial to rebalance frequently so as to avoid large deviations from the original mix. The study also indicates that a monthly periodic rebalance frequency wins over other frequencies when both portfolio risk and return are considered. In a somewhat similar study Thompson [10] uses historical data between 1997 to 2002 to show that annual rebalancing frequency outperforms others such as monthly, quarterly and passive or no rebalancing. In yet another significant study, the authors in [4] find that investors actually lose as they indulge in excessive trade.

This apparent inconsistent empirical conclusions based on historical data mining add to the confusion in making sound rebalance decisions. Most of these exercises are at best qualitative in nature. Practitioners have attempted formulating ad-hoc intuitive rules to determine when to rebalance [11][12]. Although these rules do provide useful guidance to portfolio managers, they are not based on solid mathematical or analytical foundation.

More than forty years ago, Merton[13] was the first to compute an optimal dynamic portfolio strategy in continuous time framework. Since then, several researchers have enriched the field either by adding mathematical sophistication to the underlying asset dynamics[14] or optimizing for different investor utility[15]. A nagging practical issue with these continuous-time framework is the need for constant or continuous rebalancing to the optimal portfolio.

Many researchers have studied the efficacy of discrete-time rebalancing ([16], [17], [18]). It is found that the investor loss when the investor abandons active continuous trading for discrete-time rebalancing may not be substantial. Using Monte Carlo simulation, Branger et. al. in [17] conclude that in an incomplete market where derivatives are not used to construct portfolios, the utility loss is very small due to discrete-time trading. For a 10 year horizon, a passive buy-and-hold strategy will yield the same expected investor utility as continuous trading needing merely about 10 basis points[1] higher implied initial capital. For example, for such a portfolio an initial investment of $100 and $101 will produce the same terminal expected utility using continuous rebalancing and no rebalancing respectively.

Similarly, Sun et. al. in [18] developed a dynamic programming algorithm to compute

[1] One basis point equals one hundredth of a percentage point.

the optimal rebalancing schedule. They show that using the schedule, the suboptimality cost for not using continuous rebalancing is very small limited to only 5 basis points. However, the approach is computationally burdensome and suffers from the curse of dimensionality as the portfolio size grows. The runtime for a portfolio of five risky assets can be up to 75 minutes on a single PC. Subsequently Kriztman et. al. in [16] alleviated the scalability issue by using a quadratic heuristic (originally proposed in [19]) without significantly raising suboptimality cost.

In [20], Tokat explores the factors that influence a rebalancing decision when threshold rebalancing is adopted. Important among them are the asset characteristics, viz. correlation, volatility and expected return. For highly correlated assets, the prices move in the same direction preventing rapid deviation from the initial mix and obviating the need for frequent rebalancing. Higher volatility increases the risk of significant deviation from the initial mix requiring frequent rebalancing. The portfolio also drifts towards assets with higher expected returns as time progresses. Hence the need to rebalance frequently when there is significant differences among the expected returns of the assets.

Length of investment horizon also plays a role in rebalancing. Longer horizon increases the chance for portfolio deviation from the initial target mix requiring frequent rebalancing. Tokat also explores the influence of rebalance frequency for three types of return patterns of assets over time. In upward *trending markets*, less frequent rebalancing is preferred to avoid selling strongly performing assets in order to buy poorly performing assets. In *mean-reverting markets*, where the prices tend to reverse after following an upward or downward trend, faster rebalancing at opportunistic time can produce higher portfolio return. An asset must be bought after the price has fallen and must be sold when price has appreciated. When asset prices follow *random walks* ([21]) without following any pattern, less frequent rebalancing is better in producing higher expected portfolio return.

1.2 Research Outline

In this research, we set out to gain insight on the question of how often an initial portfolio be adjusted. For an investor, frequent rebalancing incurs cost in both time and money. The investor may not want to miss the opportunity to rebalance if there is a higher chance to get a better return. On the other hand, the investor will benefit by knowing when to be passive. Informed passivity brings worry-free investment and saves paying undue trading fees. Hence we explore two questions: when and how often the investor needs to rebalance and, when it is worthwhile to be passive after initial investment decision.

We assume the investor has a log utility function and chooses the *log-optimal strategy* to maximize *expected log of portfolio value (ELPV)*. Luenberger in [22] provides exhaustive analytical treatment to compute the optimal weights that the assets need to be divided in a continuous time framework. The investor has to continuously rebalance the portfolio to the initial estimation of the weights in order to achieve maximal ELPV in the long run. This form of *active* investment strategy is cost-prohibitive and even impractical due to significant overhead of rebalance and trading cost. Both researchers and practitioners generally acknowledge the severe practical limitation of this strategy due to the continuous rebalancing condition.

Log-optimal investment strategy, also known as Kelly's criterion, has long been of interest to researchers in ivestment community. [23] provides an extensive treatment on the topic. The strategy has also several limitations. The strategy is very risky in short term. The strategy can also fare poorly with potential huge losses as a result of a sequence of bad scenarios no matter how long the finite investment horizon is. The asset means need to be carefully and conservatively estimated since portfolio log growth is very sensitive to these values. Despite log-optimal strategy's established superiority over other similar investment strategies in the long run, it can take a very long time to compute.

We start by developing an analytical foundation for *passive* investment strategy wherein the investors do not rebalance at all. A natural question to ask if the investor can benefit by remaining passive and delaying the rebalance decision. In other words, instead of rebalancing the portfolio continuously to initial set of weights, can she rebalance back to the initial portfolio weights less frequently? By doing so we must not, at any time during the investment horizon, sacrifice the investor goal of maximizing the ELPV as achieved under active strategy. If such a rebalancing frequency exists, then the practical limitation set by the continuous rebalancing condition can be overcome. We show that, for certain class of portfolio assets, such a rebalance frequency[2] indeed exists. In fact the investor can choose from a range of rebalance frequencies to rebalance her portfolio to the optimal weights. We can compute the single rebalance frequency in this range that will maximize the ELPV for a given investment horizon. We first compute the duration called the *rebalance time* during which passive strategy offers higher ELPV. Even better, we then use *instantaneous portfolio growth* as the basis for determining the passive investment duration. Both of these alternative rebalancing times are used to design a *hybrid* strategy where the initial log-optimal portfolio can be rebalanced, not continuously but at predetermined rebalanced frequency without ever degrading investor's log utility criteria. We prove that the use of the improved rebalance time will potentially offer higher overall ELPV over any other higher rebalance time.

After establishing the analytical relationship between passive and hybrid strategies, we present a numerical algorithm to compute the *optimal rebalance frequency (ORF)* that maximizes the ELPV for a given investment horizon. Simulation studies show that the passive strategy analytical framework accurately estimates the ELPV. While the analytical framework for hybrid strategy performs well, it demonstrates better fidelity for higher rebalance time with shorter investment horizon.

[2]In this thesis the term *rebalancing frequency* is generously used to describe the *time interval* in between two rebalancing events.

It is necessary to contrast our approach to that followed by Kuhn and Luenberger in [24]. The authors formulate and solve the problem of maximizing the log-optimal portfolio's expected log growth rate when a periodic discrete-time rebalancing is used. The resulting portfolio weights may differ from those in optimal continuous-time rebalancing. They demonstrate that for long-term investors continuous rebalancing only slightly outperforms discrete-rebalancing if the investor chooses a rebalancing interval slightly shorter than a year. In our proposed approach, the investor maximizes the expected log growth for a more realistic short term horizon while rebalancing periodically to the same weights used in optimal continuous-time rebalancing. These asset weights are not guaranteed to be optimal when they are used with discrete-time rebalancing or when the investment horizon is finite. We find analytical solution for finding best possible periodic discrete-time rebalancing frequency to use when a finite-term investor opts to use the initial optimal asset weights optimal for continuous-time rebalancing. In this sense, our proposed approach adheres to calendar rebalancing to rebalance to the initial portfolio mix periodically.

Unfortunately Kuhn and Luenberger do not analyze the finite-horizon investment outcome. They provide analytical expressions for optimal weights only for simplistic portfolio consisting of one risky and one risk-free asset for any given periodic discrete-time rebalanced interval.

We merely want to know if the investor can afford to wait a certain finite time $\tau \neq 0$ to rebalance. This proposition obviates the need to continuously rebalance, yet achieves the same or higher level of ELPV. In order to answer this question, we first analyze the portfolio dynamics in a purely passive approach when the investor does not rebalance at all. This is an alternative extreme approach that follows a diametrically opposite investment philosophy about rebalancing compared to the purely active continuous rebalancing log-optimal approach.

Outline of various chapters in this thesis is as follows. In chapter 2 we establish the basic notations used in thesis and review the basics of log-optimal portfolios where the investor actively rebalances the portfolio continuously. In chapter 3 we develop the mathematical framework for the evolution of the portfolio when the investor stops adjusting the portfolio after initial setup. We develop the analytical framework necessary to estimate the moments of log of portfolio growth under such passive strategy. This analysis helps us to define and propose our initial candidates for rebalance frequency. In chapter 4 we study the use of the rebalance frequency to periodically rebalance the portfolio to obtain higher investor utility for log-optimal investors. We estimate the portfolio growth when a discrete-time periodic rebalancing is adopted in this so-called hybrid strategy. After establishing the mathematical relationship between the evolution of ELPV under passive and hybrid strategies, we compute the *optimal rebalance frequency (ORF)* maximizing terminal ELPV. In the chapter 5, we use software optimization techniques to make the search based ORF algorithm, which is by design quadratic in time, efficient and scalable to large number of assets. In chapter 6 we establish the mathematical foundations based on which we significantly simplify the computational steps required for ORF. In chapter 7, using simulation we examine the accuracy of the ORF algorithms developed in the previous chapter. Finally in chapter 8 we conclude the findings of this dissertation outlining relevant future research topics.

Chapter 2

Log-Optimal Portfolio And Active Rebalancing

In continuous time multi-period portfolio optimization, log-optimal portfolio appeal to many investors. In this framework the investor seeks to maximize *expected log of portfolio value (ELPV)*. In this chapter we review the existing mathematical foundation behind log-optimal strategy. After listing the basic notations in section 2.1, we review the basics of log-optimal portfolio in section 2.2. Lastly, we present a generic algorithm to execute a rebalancing investment strategy.

2.1 Notations

Suppose the investor has the choice of setting up an investment portfolio from a set of N *risky* financial assets and a *risk-free* asset. Typical risky assets are stocks and funds, and often are correlated with other risky financial assets. These risky assets $i = 1, \ldots, N$ are provided with *a priori* expected returns and standard deviations. We assume that returns are stationary random variables and hence the expected return and standard deviations don't change over time. We consider risk-free asset $i = N + 1$ such as T-bills offering constant fixed rate of return. We will use the following symbols in our mathematical derivations and analysis for $\forall i, j = 1$ to $N + 1$.

T = investment horizon in years (periods)

μ_i = expected rate of return for asset i

σ_i = standard deviation for asset i

ρ_{ij} = correlation between returns of asset i and j

σ_{ij} = covariance of asset i and $j = \rho_{ij}\sigma_i\sigma_j$

w_i = proportion of investment in asset i in portfolio for log-optimal allocation

$\mu_p(t)$ = expected rate of return of portfolio of assets at time t

$\sigma_p(t)$ = standard deviation of portfolio of assets at time t

$V(t)$ = value (in dollars) of portfolio at time t

Without loss of generality, throughout our analysis we will assume an initial value of $V(0) = 1\$$. For asset $N+1$ which is risk-free, we will use $r_f = \mu_{N+1}$ alternatively. Since the asset is risk free, we also have $\sigma_{N+1} = 0$ and

$$\rho_{(N+1)j} = \rho_{j(N+1)} = 0 \ \forall j = 1 \text{ to } N \tag{2.1}$$

2.2 Active Portfolio

2.2.1 Asset Price Dynamics

In this section we briefly review the well known dynamics of asset prices. For more details and discussion of asset price modeling the reader may refer to [21] and [22]. We assume that asset price dynamics follows Geometric Brownian motion. Geometric Brownian motion assumption is widely used in financial assets and derivative valuations ([25]).

$$dS(t) = \mu S(t)dt + \sigma S(t)dz \tag{2.2}$$

where

$S(t)$ = Asset price at time t.

μ = expected rate of return of the asset expressed in decimal form.

σ = volatility of the asset price.

Variable $dz = \epsilon\sqrt{dt}$ follows *Wiener process*, where $\epsilon \sim \phi(0, 1)$ is the standard normal variable. Rearranging equation 2.2, *instantaneous rate of return* of the asset will be,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz \tag{2.3}$$

In this paper we assume both the rate of return and volatility are constants for a given asset.

In this setting, asset price $S(t)$ has a lognormal distribution.

$$ln\ S(t) \sim \phi[ln\ S(0) + (\mu - \frac{\sigma^2}{2})t, \sigma^2 t] \tag{2.4}$$

The first and second terms in the above equation represent the mean and variance of the distribution respectively. Lognormality assumption precludes any negative price for assets. Expected value and variance of asset prices are given by the following relationships:

$$E[S(t)] = S(0)e^{\mu t} \tag{2.5}$$

$$Var[S(t)] = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1) \tag{2.6}$$

Lognormality of asset prices also lead to the following relationships of expected and variance of log of growth of asset price:

$$E[ln\{\frac{S(t)}{S(0)}\}] = \nu t \tag{2.7}$$

$$Var[ln\{\frac{S(t)}{S(0)}\}] = \sigma^2 t \tag{2.8}$$

where, asset *growth rate* ν is given by:

$$\nu = \mu - \frac{\sigma^2}{2} \tag{2.9}$$

Let the *continuously compounded rate of return* per annum realized between time 0 and t be denoted by x. The asset price in terms of x is given by the following expression ([21]):

$$S(t) = S(0)e^{xt} \tag{2.10}$$

From equations 2.4 and 2.10, x can be characterized by the following normal distribution:

$$x \sim \phi\Big[\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{t}\Big] \tag{2.11}$$

Note that x is a stationary random normal variable whose variance is a function of the time duration for which the rate of return is compounded. However, for simplicity of notation, we will denote x without specifying the duration as a parameter. In all future analysis, duration is always explicitly specified as a multiple of x in the context.

Risk-free asset dynamics is a special case of the risky asset dynamics described above. From equation 2.2:

$$dS_{N+1}(t) = r_f S_{N+1}(t)dt \tag{2.12}$$

The future price of risk-free asset will be deterministic and follows from equation 2.10:

$$S_{N+1}(t) = S_{N+1}(0)e^{r_f t} \tag{2.13}$$

2.2.2 Log-optimal Portfolio

In this investment strategy, portfolio weights are *continuously rebalanced* to maximize the long term growth rate of log of portfolio return. The reader can find a good treatment of this strategy in [22]. Log-optimal and semi-log optimal portfolios are also analyzed in [26].

Since the portfolio is constructed using assets $i = 1$ through $N+1$ with each asset taking up w_i proportion of the total investment outlay, we have:

$$\sum_{i=1}^{N+1} w_i = 1 \tag{2.14}$$

Note again that portfolio consists of $N + 1$ assets, one risk-free and N risky assets. If $V(t)$ is the value of the portfolio, then the *instantaneous rate of return of the portfolio* is equal to the weighted sum of the instantaneous rates of returns of the individual assets, i.e.

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{N+1} w_i \frac{dS_i(t)}{S_i(t)} \tag{2.15}$$

Substituting equation 2.3 in equation 2.15 we get,

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{N+1} (w_i \mu_i dt + w_i \sigma_i dz) \tag{2.16}$$

In the above equation, the first term is a fixed term with variance 0. The second term is a stochastic term with mean 0 and variance given by:

$$\begin{aligned} Var[\sum_{i=1}^{N+1} w_i \sigma_i dz] &= E(\sum_{i=1}^{N+1} w_i \sigma_i dz)^2 - (E(\sum_{i=1}^{N+1} w_i \sigma_i dz))^2 = E(\sum_{i=1}^{N+1} w_i \sigma_i dz)^2 \\ &= E(\sum_{i=1}^{N+1} w_i \sigma_i dz)(\sum_{j=1}^{N} w_j \sigma_j dz) = \sum_{i,j=1}^{N+1} w_i \sigma_{ij} w_j dt \end{aligned} \tag{2.17}$$

Note that in the above simplification, the second term goes away as it is the square of sum of expected values of multiples of standard normal variables. The expected value of a multiple of standard normal variable is zero ([27]). Now, we can write equation 2.16 in the following geometric Brownian motion form analogous to the dynamics of asset price in equation 2.2:

$$dV(t) = \mu_p V(t) dt + \sigma_p V(t) dz \tag{2.18}$$

where the mean and variance of the portfolio are given by:

$$\mu_p = \sum_{i=1}^{N+1} w_i \mu_i \tag{2.19}$$

$${\sigma_p}^2 = \sum_{i,j=1}^{N+1} w_i \sigma_{ij} w_j \tag{2.20}$$

Analogous to asset price dynamics, applying *Itô's lemma* ([25]) portfolio value $V(t)$ has a lognormal distribution.

$$ln\ V(t) \sim \phi[ln\ V(0) + (\mu_p - \frac{\sigma_p^2}{2})t, \sigma_p^2 t] \tag{2.21}$$

From above lognormality relationships, we can derive the expected value and variance for the growth of portfolio and log of portfolio in the following equations:

$$E[V(t)] = e^{\mu_p t} \tag{2.22}$$

$$Var[V(t)] = e^{2\mu_p t}(e^{\sigma_p^2 t} - 1) \tag{2.23}$$

$$E[ln\{V(t)\}] = \nu_p t \tag{2.24}$$

$$Var[ln\{V(t)\}] = \sigma_p^2 t \tag{2.25}$$

where, portfolio *growth rate* ν_p is given by:

$$\nu_p = \mu_p - \frac{\sigma_p^2}{2} \tag{2.26}$$

For notational simplicity we will use $\chi(t)$ to denote the ELPV at time t. Since $V(0) = 1$, we can rewrite equations 2.24 as,

$$\chi(t) = \nu_p t \tag{2.27}$$

In the log-optimal portfolio, the growth rate ν_p is maximized by solving the following optimization problem:

$$\begin{aligned} &\underset{\mathbf{w}}{\text{maximize}} \quad \nu_p \\ &\text{subject to} \quad \sum_{i=1}^{N+1} w_i = 1 \end{aligned}$$

$\mathbf{w}$ defines the vector of asset weights. The solution to the above optimization problem is to select the weight of each risky asset i satisfying the following relationship ([22]):

$$\sum_{j=1}^{N} \sigma_{ij} w_j = \mu_i - r_f \tag{2.28}$$

There will be N linear equations corresponding to each risky asset with same number of unknown weight variables. We can then solve for the values of the portfolio weights for risky assets. Finally we can find out the portfolio weight w_{N+1} of the risk free asset using equation 2.14. We will extend the example used in [22] for demonstrating different investment strategies studied in this paper. In this example, there are three risky assets, $i = 1, 2$ and 3. A portfolio manager or an investor needs to specify the asset mean, variance and correlation coefficients. She also specifies the risk free rate and investment horizon. The following is the set of input parameters specified for this example:

1. Initial portfolio value: $V(0) = \$1$

2. Mean vector:

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix} = \begin{bmatrix} 0.24 & 0.20 & 0.15 \end{bmatrix}$$

3. Asset standard deviation vector:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 0.3000 & 0.2646 & 0.1732 \end{bmatrix}$$

4. Asset correlation coefficients:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.2520 & 0.1925 \\ 0.2520 & 1.0000 & -0.2182 \\ 0.1925 & -0.2182 & 1.0000 \end{bmatrix}$$

5. Risk-free rate: $r_f = 0.1$

6. Investment horizon: $T = 30$ years.

We can derive the covariance matrix from the given asset variances and correlation coefficients using Matlab like syntax for matrix operations:

$$\mathbf{S} = \boldsymbol{\rho}. * (\boldsymbol{\Sigma}' * \boldsymbol{\Sigma}) \tag{2.29}$$

In the above syntax, $\boldsymbol{\Sigma}'$ is the compliment of $\boldsymbol{\Sigma}$ and $.*$ is element-wise multiplication of two matrices.

Using equation 2.29 we obtain:

$$\mathbf{S} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0.09 & 0.02 & 0.01 \\ 0.02 & 0.07 & -0.01 \\ 0.01 & -0.01 & 0.03 \end{bmatrix}$$

Using the above matrix notations, system of linear equations in 2.28 can be written as:

$$\mathbf{S}\mathbf{w} = \boldsymbol{\mu} - r_f \tag{2.30}$$

We can solve the above set of linear equations easily by using a linear equation solver package. In Matlab, we can solve for $\mathbf{w}$ by using the backslash or matrix left division operator:

$$\mathbf{w} = \mathbf{S}/(\boldsymbol{\mu} - r_f) \tag{2.31}$$

For the example investment problem, we obtain:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1.0509 \\ 1.3818 \\ 1.7770 \end{bmatrix}$$

Using equation 2.14, we can derive the portfolio weight for risk free asset $w_4 = -3.2098$. The negative sign indicates that the risk free asset needs to be borrowed. A portfolio set up using the above weights will maximize the ELPV in the long run if the weights are always maintained to the original value by continuously rebalancing.

The mean μ_{opt} and variance σ^2_{opt} of the portfolio corresponding to the set of optimum weights can be computed using equation 2.19 and 2.20 respectively:

$$\mu_{opt} = 0.4742, \sigma^2_{opt} = 0.3742$$

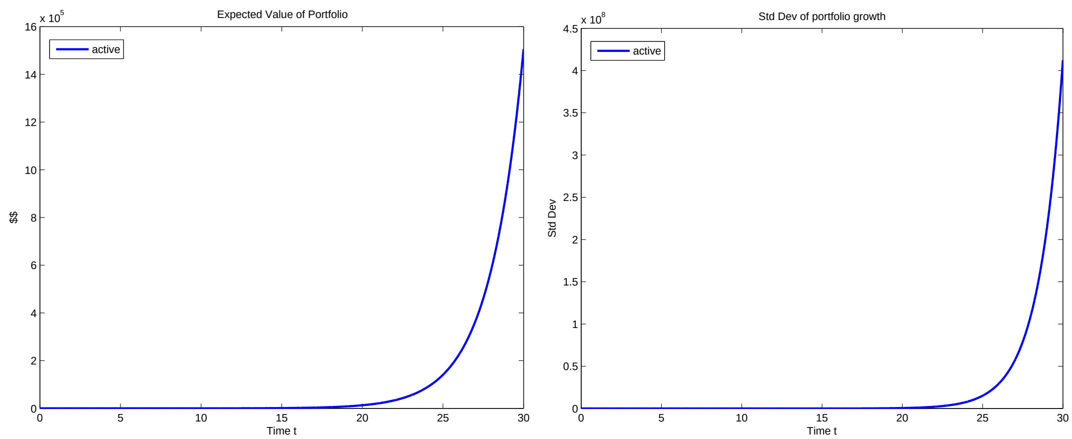
Using equation 2.26 constant growth rate $\nu_p = 0.2871$.

We now summarize the above steps in the form of an computational algorithm. Algorithm 1 computes the optimal weight vector and the corresponding growth rate for the active investment strategy. The algorithm takes in the mean, variance and correlation vectors along with the constant risk free rate. It returns the portfolio growth rate, weight vector and mean vector to the calling procedure. Note that the output mean vector contains the risk free rate, i.e. the mean of the risk-free asset as well.

Algorithm 1 ComputeLogOptimalParams

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,N

$\mathbf{w} \leftarrow \mathbf{S}/\left(\boldsymbol{\mu} - r_f\right)$ # equation 2.31
$\mu[N+1] \leftarrow r_f$
$wSum \leftarrow 0$
for $i = 1$ to N **do**
  $wSum \leftarrow wSum + w[i]$
end for
$w[N+1] \leftarrow 1 - wSum$ # equation 2.14
$\mathbf{S} \leftarrow \mathbf{S}$ # augmented with risk-free asset covariances
$\mu_p \leftarrow 0, v_p \leftarrow 0$
for $i = 1$ to N+1 **do**
  $\mu_p \leftarrow \mu_p + w[i]\mu[i]$ # equation 2.19
  for $j = 1$ to N+1 **do**
    $v_p \leftarrow v_p + w[i]\sigma[i,j]w[j]$ # equation 2.20
  end for
end for
$\nu_p \leftarrow \mu_p - \frac{1}{2}v_p$ # equation 2.26
return $\nu_p, \mathbf{w}, \boldsymbol{\mu}$,$\mathbf{S}$



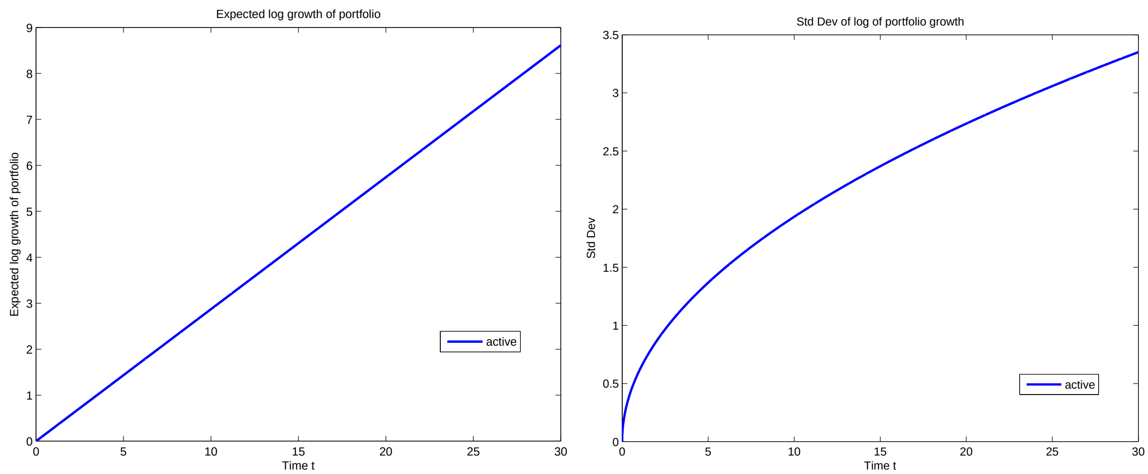
(a) (b)

Figure 2.1 Moments of active portfolio value

Figure 2.1(a) plots the expected value of the portfolio in active strategy using equations 2.19 and 2.22. This seems to have rich growth potential to an investor. However, an investor also needs to look at the risk in this strategy. One measure of the risk is the variability or standard deviation of this portfolio value given by equation 2.23 and traced in figure 2.1(b). We can see that the upside potential of the portfolio growth comes at the expense of exponential increase in variability of standard deviation of the portfolio value.

The reader is reminded that the active log-optimal strategy maximizes the *log* of portfolio growth. For such log investor utility, we need to look at the ELPV of portfolio over the investment horizon as given by equation 2.27 and plotted in figure 2.2(a). The uncertainty or risk in this estimation is given by equation 2.23 and plotted in figure 2.2(b). Notice that unlike exponential growth of standard deviation for the portfolio growth, the standard deviation of log of portfolio growth shows only quadratic growth.

Before we discuss alternative investment strategies, we will outline two important well-known properties of the log-optimal active strategy as stated in [22]. Suppose Z is an alternative investment strategy other than log-optimal active strategy.



(a) (b)

Figure 2.2 Moments of active log of portfolio value

1. Log-optimal strategy maximizes the expected portfolio ν_p growth rate in the long run, i.e. $\nu_p \geq \lim_{t\to\infty} \nu_p^Z(t)$.

2. Suppose $V^Z(t)$ is the value of portfolio at t under any investment strategy other than log-optimal strategy. Then, $E[\frac{V^Z(t)}{V(t)}] \leq 1$.

An obvious, yet important characteristic of active strategy is that it satisfies *reinvestment principle*. In other words, it produces identical portfolio value when the assets are liquidated in the middle and reinvested back in the same assets in the same proportion as before. From equation 2.22 it is easy to see how this is satisfied in active strategy. If $V(0)$, $V(t')$ and $V(t)$ are the portfolio values at time 0, t' and t such that $0 < t' < t$ then,

$$E[V(t)] = E[V(0)]e^{\mu_p t} = E[V(0)]e^{\mu_p t'}e^{\mu_p(t-t')} = E[V(t')]e^{\mu_p(t-t')} \tag{2.32}$$

Algorithm 2 outlines the generic steps for executing an investment strategy that uses a given rebalancing frequency τ. At every rebalancing time, it uses the market price for the assets to compute the total portfolio value (step 4 through 10). In steps 11 through 13, asset count is recomputed after rebalancing the portfolio to the initial optimal weights. A trader must buy and sell assets appropriately to arrive at the new asset counts. The algorithm returns the terminal ELPV χ.

Algorithm 2 ExecuteRebalanceStrategy

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,N,T,τ

$V = 1$ # Initial investment of \$1
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}, \mathbf{S}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\rho}, r_f, N)$
for $t = 0$ to T by τ **do**
  if $t \geq \tau$ **then**
    $V \leftarrow 0$
    for $j = 1$ to $N + 1$ **do**
      Obtain $P_t[j]$
      $V \leftarrow V + P_t[j] * acnt[j]$ # total portfolio value
    end for
  end if
  for $j = 1$ to $N + 1$ **do**
    $acnt[j] \leftarrow \frac{w[j]V}{P_t[j]}$ # rebalance portfolio to $\mathbf{w}$
  end for
end for
$V \leftarrow 0$
for $j = 1$ to $N + 1$ **do**
  $V \leftarrow V + P_T[j] * acnt[j]$ # liquidate the portfolio at horizon T
end for
$\chi = log(V)$
return χ

For a practical implementation, active strategy can employ daily rebalancing to emulate closely the effect of continuous rebalancing. Since in a typical year, there are 252 trading days, one can set $\tau = \frac{1}{252} = 0.004$ year. Thus, one will invoke the following command to execute active strategy for 30 year horizon:

$$\chi = ExecuteRebalanceStrategy(30, 0.004)$$

In the above statement, we have assumed that all other input parameters specific to the given set of portfolio assets have already been provided.

Chapter 3

Passive Strategy And Stable Rebalancing

In the prior chapter we elaborated the log-optimal strategy for portfolio growth where the portfolio is continuously rebalanced with a periodicity of $\tau = 0$. Upon close scrutiny of the growth rate of the portfolio ν_p specified by equation 2.26, one finds that by not rebalancing, the portfolio effective mean μ_p deteriorates simultaneously decreasing the variance σ_p^2. For a short time if the second effect dominates the first, it will result in a net increase in growth rate. During this time the investor will benefit by avoiding continuous rebalancing. In this chapter, we will develop the framework to assess the nature of portfolio growth when the investor sets up the portfolio with the optimal weight vector $\mathbf{w}$ and never rebalances throughout the investment horizon T. Consequently, we assume the rebalance frequency under such *passive strategy* to be $\tau = \infty$.

Throughout our analysis, we use the pertinent rebalance frequency as the superscript with parameters. All parameters for passive strategy will have a superscript of ∞. In the absence of any such superscript, the parameter pertains to active strategy. Note that the initial investment parameters enumerated under section 2.1 will be applicable to all strategies discussed in this thesis.

Lemma 1. *Consider an initial portfolio with value* \$1 *constructed using* N *risky assets with weights* w_i*,* $i = 1, \ldots, N$ *and a risk-free asset with weight* w_0*. When left unadjusted, the portfolio will grow such that the value* $V^\infty(t)$ *at any subsequent time* $t > 0$ *will be given by:*

$$V^\infty(t) = \sum_{i=1}^{N+1} w_i e^{x_i t} \tag{3.1}$$

where x_i *is a random normal variable specified by equation 2.11.*

Proof. At $t = 0$ the value of the portfolio invested in asset i is $V(0)w_i$. This translates to the number of shares n_i to be purchased and held for asset i at time $t = 0$:

$$n_i = \frac{w_i}{S_i(0)}$$

Since the portfolio remains unadjusted, the value of n_i shares of asset i at time $t > 0$ will be:

$$V_i^{\infty}(t) = \frac{w_i}{S_i(0)} S_i(t) = \frac{w_i}{S_i(0)} S_i(0) e^{x_i t} = w_i e^{x_i t} \tag{3.2}$$

We have used equation 2.10 in simplifying the above. Now the result in equation 3.1 follows since the portfolio value is the sum of values of constituent assets.

□

Hence the value of the passive portfolio is characterized by a sum of correlated random variables as per equation 3.1. We now review some of the statistical properties of log normal random variable. The reader can find a very good overview in [28]. A comprehensive treatment of log-normal distribution will be found in [29].

Let Y be a normal random variable with mean m and standard deviation s.

Let X be another random variables such that:

$$X = e^Y$$

X is said to be a *log-normal* random variable since logarithm of the variable follows normal distribution. The first two moments of X are given as below:

$$E[X] = e^{m + \frac{s^2}{2}} \tag{3.3}$$

$$Var[X] = (e^{s^2} - 1)E[X]^2 = (e^{s^2} - 1)e^{2m + s^2} \tag{3.4}$$

When there are two correlated random normal variables Y_i with mean m_i, standard deviation s_i and correlation coefficient ρ_{12}, the covariance between the corresponding log-normal variables $X_i = e^{Y_i}$ for $i = 1, 2$ are given by:

$$Cov[X_1, X_2] = (e^{\rho_{12}s_1s_2} - 1)E[X_1]E[X_2] = (e^{\rho_{12}s_1s_2} - 1)e^{m_1+\frac{s_1^2}{2}}e^{m_2+\frac{s_2^2}{2}} \tag{3.5}$$

Given log-normal X, one can compute the variance s^2 and the mean m of the underlying normal variable Y by using the following relationships:

$$s^2 = ln(1 + \frac{Var[X]}{E[X]^2}) \tag{3.6}$$

$$m = ln(E[X]) - \frac{1}{2}ln(1 + \frac{Var[X]}{E[X]^2}) = ln(E[X]) - \frac{1}{2}s^2 \tag{3.7}$$

Now we can proceed to compute the statistics for the passive portfolio evolution.

Lemma 2. *Under passive investment strategy, the expected value of portfolio at any time* $t > 0$ *is the weighted sum of the individual expected asset growths, i.e.,*

$$E\Big[V^\infty(t)\Big] = \sum_{i=1}^{N+1} w_i e^{\mu_i t} \tag{3.8}$$

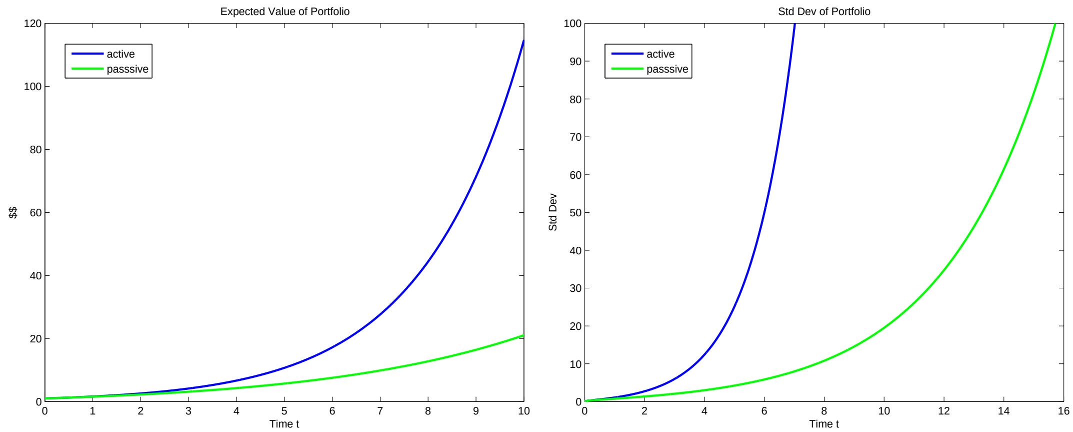
Proof. From equation 3.1, we can compute the passive portfolio growth as:

$$\begin{aligned} V^\infty(t) &= \sum_{i=1}^{N+1} w_i e^{x_i t} = \sum_{i=1}^{N+1} e^{ln(w_i)+x_i t} \\ \Rightarrow E\Big[V^\infty(t)\Big] &= E\Big[\sum_{i=1}^{N+1} e^{ln(w_i)+x_i t}\Big] = \sum_{i=1}^{N+1} E[e^{ln(w_i)+x_i t}] \end{aligned} \tag{3.9}$$

We have made use of the fact that the expected value of a sum of random variables is same as the sum of expected values of the individual random variables ([27]). Now, given that x_i's are normal random variables as specified in equation 2.11, $ln(w_i) + x_i t$ will also be normal with the following moments:

$$ln(w_i) + x_i t \sim \phi[ln(w_i) + (\mu_i - \frac{\sigma_i^2}{2})t, \sigma_i^2 t] \tag{3.10}$$

Note that $Var(aX + b) = a^2 Var(X)$ for any random variable X and constants a and b.



(a) (b)

Figure 3.1 Expected Value and Std Dev of Portfolio Growth.

We can now find out the first moment of $e^{ln(w_i)+x_i t}$ using log-normal properties of equation 3.3,

$$E[e^{ln(w_i)+x_i t}] = e^{ln(w_i)+(\mu_i - \frac{\sigma_i^2}{2})t + \frac{\sigma_i^2 t}{2}} = e^{ln(w_i)+\mu_i t} = w_i e^{\mu_i t} \tag{3.11}$$

Substituting the above in equation 3.9 we get the desired result. □

Figure 3.1(a) shows the evolution of ELPV for our example investment scenario. In this case the passive strategy produces lower ELPV than the active strategy.

Lemma 3. *Under passive investment strategy, the variance of portfolio growth at any time $t > 0$ is given by:*

$$Var[V^{\infty}(t)] = \sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1) \tag{3.12}$$

Proof. Similar to lemma 2, variance of passive portfolio growth is:

$$Var[V^{\infty}(t)] = \sum_{i,j=1}^{N+1} Cov[e^{ln(w_i)+x_i t}, e^{ln(w_j)+x_j t}] \tag{3.13}$$

The reader may refer [30] for the rule to obtain the sum of correlated random variables.

We use equation 3.5 and 3.11 to simplify equation 3.13:

$$
\begin{aligned}
Cov[e^{ln(w_i)+x_it}, e^{ln(w_j)+x_jt}] &= (w_ie^{\mu_it})(w_je^{\mu_jt})(e^{\rho_{ij}\sigma_i\sqrt{t}\sigma_j\sqrt{t}} - 1) \\
&= w_iw_je^{(\mu_i+\mu_j)t}(e^{\rho_{ij}\sigma_i\sigma_jt} - 1) \\
&= w_iw_je^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1)
\end{aligned} \tag{3.14}
$$

Substituting the results in equation 3.14 in equation 3.13, we obtain the desired passive portfolio variance expression of equation 3.12. □

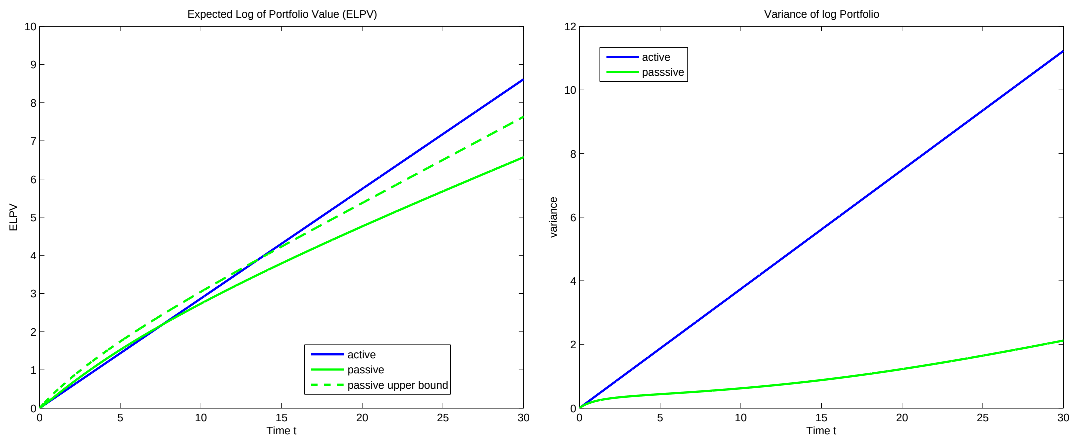
Figure 3.1(b) shows the evolution of variance of portfolio growth for our example investment scenario. In this case the passive strategy has less variance compared to the active strategy. This alone indicates that passive strategy will be less risky which is good for risk-averse investors.

The reader is reminded that the active strategy is optimal only when the ELPV given in equation 2.24 is maximized for the investor. In order to have a fair portfolio performance comparison between active and passive strategy we need to analyze the ELPV under passive strategy.

The problem here is to compute the first and if possible, the second moment of the log of the portfolio growth under passive strategy. Using equation 3.1:

$$
ln(V^{\infty}(t)) = ln(\sum_{i=1}^{N+1} w_ie^{x_it}) \tag{3.15}
$$

The need to characterize the sum of lognormal variables arises in many domains. There have been many approximations to characterize the probability density function for sum of log normal. Two analytical methods to determine the moments of sum of correlated random variables widely used by researchers in many engineering disciplines. The first one proposed by Fenton and Wilkinson in 1960 is still being used because of its simplicity and analytical tractability ([31]). More recently, the second method was proposed in [32]. Both of these methods assume that the sum of lognormal is also lognormal. [33] compare the two approaches to formulate the outage probability in a mobile radio systems. Fenton's approach allows the use of closed form analytical expression for the moments of log of sum



(a) (b)

Figure 3.2 Expected Value and Variance of Log of Portfolio Growth.

of lognormal random variables. Schwartz and Yeh method employs a recursive algorithm to obtain the moments. In this paper, we will adapt Fenton's method because of its analytical tractability.

Lemma 4. *The variance of the log of portfolio growth under passive strategy is given by:*

$$\Upsilon^{\infty}(t) = Var[ln(V^{\infty}(t))] = ln\Big(1 + \frac{\sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}(e^{\sigma_{ij}t} - 1)}{(\sum_{i=1}^{N+1} w_i e^{\mu_i t})^2}\Big) \tag{3.16}$$

Proof. We assume that sum of lognormal random variables is also lognormal as is assumed in Fenton-Wilkinson approach. Thus as per equation 3.1 the passive portfolio growth $V^{\infty}(t)$ is lognormal. This implies that log of passive portfolio growth $ln(V^{\infty}(t))$ is normal.

Using lognormal property given by equation 3.6, we obtain,

$$\Upsilon^{\infty} = Var\Big[ln\Big(V^{\infty}(t)\Big)\Big] = ln\Big(1 + \frac{Var[V^{\infty}(t)]}{E[V^{\infty}(t)]^2}\Big) \tag{3.17}$$

Substituting the values of expected value and variance of portfolio growth from equations 3.8 and 3.12 in the above equation we obtain the desired result in equation of 3.16. □

Now we derive the ELPV which is the investor utility in log-optimal investment strategy.

Lemma 5. *The expected log of portfolio value (ELPV) under passive strategy is given by:*

$$\chi^{\infty}(t) = E[ln(V^{\infty}(t))] = ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) - \frac{1}{2}\Upsilon^{\infty}(t) \tag{3.18}$$

Proof. The derivation is straightforward when we follow the lognormal assumption in lemma 4 and using lognormal property given by equation 3.7 and expected value equation 3.8. □

The expected value thus obtained is an approximation due to the inherent log-normality assumption in Fenton-Wilkinson's approach. Using Jenson's inequality ([34]) we can derive a true upper bound.

Lemma 6. *The ELPV under passive strategy will always be bounded, i.e.,*

$$\chi^{\infty}(t) \leq ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) \tag{3.19}$$

Proof. Knowing that logarithm is a concave function and using Jensen's inequality:

$$\chi^{\infty}(t) = E[ln(V^{\infty}(t))] \leq ln(E[V^{\infty}(t)]) \tag{3.20}$$

Substituting expected value expression from equation 3.8 we obtain equation 3.19. □

Notice that the estimation in equations 3.18 obtained using Fenton-Wilkinson approach will always meet the upper bound condition of equation 3.19. This is easy to see as the first term in equation 3.18 is the upper bound. The estimation is always going to be less than this bound as the variance term in the equation will always be positive.

Figure 3.2(a) shows the comparison of ELPV for our example investment scenario. We see that the passive strategy provides better performance for the initial few years. Since the investor wants to maximize the ELPV, he will choose passive strategy over active strategy for this initial period since the passive strategy offers higher ELPV. Passive strategy will be seen as more favorable if we had considered the transaction cost incurred in continuous rebalancing used in active strategy.

As a result of the log-normality assumption inherent in Fenton-Wilkinson approach, we notice the analogous nature of the passive portfolio growth in equation 3.18 and the corresponding equation under active strategy in equation 2.27. Comparing both these equations portfolio growth rate under passive strategy ν_p^∞ will be given by,

$$\nu_p^\infty(t) = \frac{\chi^\infty(t)}{t} = \frac{1}{t}ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) - \frac{1}{2}(\frac{1}{t}Var[ln(V^\infty(t))]) = \mu_p^\infty - \frac{\sigma_p^{\infty 2}}{2} \tag{3.21}$$

where, mean μ_p^∞ and standard deviation σ_p^∞ of passive portfolio are given respectively by,

$$\mu_p^\infty(t) = \frac{1}{t}ln(\sum_{i=1}^{N+1} w_i e^{\mu_i t}) \tag{3.22}$$

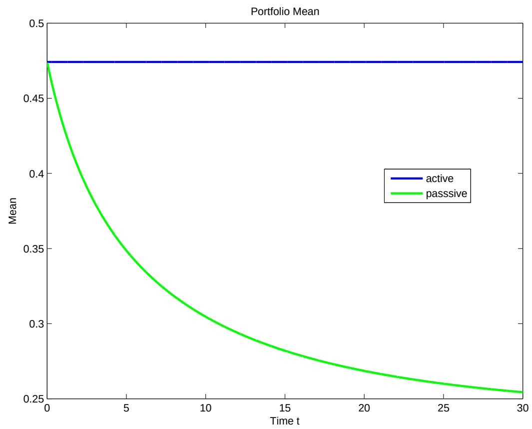
$$\sigma_p^{\infty 2}(t) = \frac{1}{t}Var[ln(V^\infty(t))] \tag{3.23}$$

One can compare equations 3.21, 3.22 and 3.23 with their counterpart equations 2.26, 2.19 and 2.20 for active strategy. Notice that the passive portfolio mean, standard deviation and growth rate are all time varying unlike the corresponding active strategy parameters. This is observed in the figures 3.3(a) and 3.3(b) for our example investment portfolio. Notice that the portfolio mean is lower than the corresponding mean in active strategy. However due to reduced portfolio standard deviation, we can still obtain higher portfolio growth rate under passive strategy for the initial period.

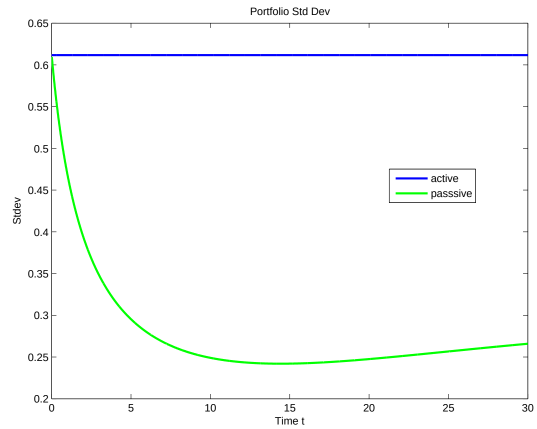
Similar to the ELPV, the growth rate in figure 3.4(a) demonstrates why the investor should choose to remain passive and not exercise her continuous rebalancing option to maximize his investment potential. It is prudent to only rebalance when the growth rate starts to fall below the active strategy.

Analogous to equations 2.22 and 2.23, we can now alternatively express the mean and variance of passive portfolio growth as following:

$$E[V^\infty(t)] = e^{\mu_p^\infty(t)t} \tag{3.24}$$



(a)



(b)

Figure 3.3 Portfolio mean and standard deviation evolution.

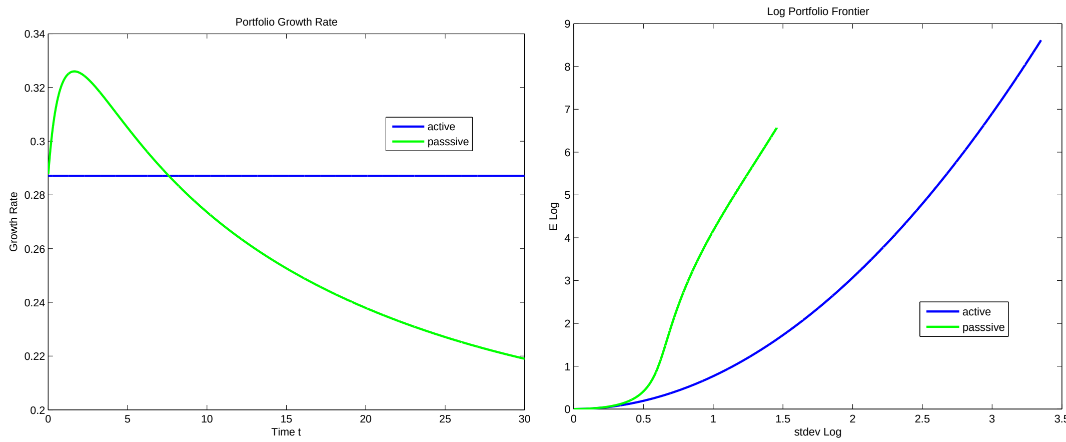
$$Var[V^{\infty}(t)] = e^{2\mu_p^{\infty}(t)t}(e^{\sigma_p^{\infty 2}(t)t} - 1) \tag{3.25}$$

It is easy to show the equivalence of equations 3.8 and 3.24. Similarly, equations 3.12 and 3.25 are also equivalent.

We now end the analysis of passive strategy by looking at the mean-variance plot comparison of log portfolio as shown in figure 3.4(b). Notice that for our example portfolio, the plot for passive strategy lies above the plot for active strategy for the entire investment period. In other words, for a given standard deviation, passive portfolio will have higher ELPV. In this sense, the investor will find the passive strategy more favorable if she is willing to take a given level of risk quantified by the standard deviation of log of portfolio growth.

3.1 Simple Rebalancing

As discussed previously, to attain the investor's log-optimality goal, the investor may not need to continuously rebalance. Figures 3.2(a) and 3.4(a) illustrate existence of opportunity to stay passive during the period when the ELPV and corresponding growth rate are higher than those in active strategy. We define this period $\tau_c > 0$ to be the *simple* rebalance time.



(a) (b)

Figure 3.4 Growth rate and mean variance of log of portfolio.

During $(0\ \tau_c)$, passive strategy offers higher investor log utility as captured in the following condition:

$$\exists \tau_c \text{ s.t. } \chi^{\infty}(t) > \nu_p t, \forall t \in (0\ \tau_c) \tag{3.26}$$

The investor continues to use passive strategy until passive log utility drops and equals that of active strategy. In the absence of transaction cost, this first rebalance time τ_c will be determined by the point of intersection of equations 2.27 and 3.18 as in figure 3.2(a). We can express this mathematically as follows:

$$\chi^{\infty}(\tau_c) = \chi(\tau_c) = \nu_p \tau_c,\ \ \tau_c > 0 \tag{3.27}$$

When τ_c exists, it is hard to obtain a closed loop solution for τ_c by solving equation 3.27 because of the non-linear nature of equation 3.18. However we can numerically solve the equation to obtain τ_c. Algorithm 3 outlines the computational steps required to compute the simple rebalance time τ_c for a given set of investment parameters. It records the time when the passive ELPV exceeds active ELPV. This is determined in lines 20 through 22.

Algorithm 3 ComputeSimpleRebalanceFrequency

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,T,δT,N

$\tau_c \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \mathbf{S}, r_f, N)$
if $!IsPassiveStrategyPossible(\mathbf{w}, \boldsymbol{\mu}, \mathbf{S})$ **then**
    return τ_c
end if
$\chi^{\infty} \leftarrow 0, \chi \leftarrow 0$
for $t = 0$ to T by δT **do**
    $X \leftarrow 0, Y \leftarrow 0$
    for $i = 1$ to N+1 **do**
        $X \leftarrow X + w[i]e^{\mu[i]t}$ # equation 3.8
        for $j = 1$ to N+1 **do** # equation 3.12
            $Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma[i,j]t} - 1)$
        end for
    end for
    $\chi^{\infty} \leftarrow ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})$ # equation 3.18
    $\chi \leftarrow \nu_p t$ # equation 2.27
    if $\chi^{\infty} < \chi$ **then** # If passive ELPV falls below active ELPV
        return $\tau_c \leftarrow t - \delta T$
    end if
end for
return τ_c

For our illustrative example we find $\tau_c = 7.61$ years. So, the investor's ELPV will be higher if the investment is unadjusted for 7.61 years than if it is to be rebalanced continuously to the optimum weights $\mathbf{w}$.

3.2 Stable Rebalancing

Is this rebalance time τ_c optimal? Can we do even better in maximizing ELPV? In order to answer these questions we must investigate the robustness of τ_c. Note that the estimation of τ_c is based on the information available at time $t = 0$. Will our decision to rebalance change before the expected scheduled rebalance time τ_c expire?

We define a rebalance strategy to be *stable* if passive ELPV exceeds active ELPV throughout the passive investment period of τ_c. More formally, a stable rebalancing strategy satisfies the following condition:

$$E[ln(\frac{V^{\infty}(t,t+dt)}{V^{\infty}(t,t)})] \geq E[ln(\frac{V(t,t+dt)}{V^{\infty}(t,t)})], \forall t \in (0\ \tau_c) \text{ and } dt \to 0 \tag{3.28}$$

Here we have expanded our notation to denote the time when the ELPV is measured. For example, $V(t,t')$ denotes the value of portfolio at time t' estimated at time t. The stability principle states that as the portfolio grows passively, at each time point before the rebalancing time, the investor should always expect to get higher or equal ELPV using passive strategy. Should her expectation of log of portfolio value using active strategy at any time be higher during passivity, she will opt to immediately switch to active strategy by rebalancing the portfolio to the set of initial optimal weights $\mathbf{w}$. A rebalancing interval τ_s is stable if the investor does not see the opportunity to switch to active strategy throughout the open interval $(0\ \tau_s)$.

Note that the denominator in the right hand side of equation 3.28 is $V^{\infty}(t,t)$, not $V(t,t)$. This is because up until time t portfolio follows passive strategy to attain the value of portfolio $V^{\infty}(t,t)$. At this time t, the investor examines the possibility to rebalance and switch to active strategy if needed.

The right hand side of inequality in equation 3.28 is $\nu_p dt$ as growth rate ν_p is always constant under active strategy. We now compute the left hand side of inequality in equation 3.28 in the following lemma.

Lemma 7. *The time t estimation of expected log growth under passive strategy for time $t+dt$ will be the difference of the initial growth estimation for the investment duration $t+dt$ and t, i.e.*

$$E[ln(\frac{V^{\infty}(t,t+dt)}{V^{\infty}(t,t)})] = E[ln(V^{\infty}(0,t+dt)] - E[ln(V^{\infty}(0,t)] \tag{3.29}$$

Proof. Consider an initial investment amount of $V(0)$. From equation 3.1, under passive strategy

$$V^{\infty}(0,t) = \sum_{i=1}^{N+1} w_i e^{x_i t} \tag{3.30}$$

Similarly for investment duration $t+dt$,

$$V^{\infty}(0,t+dt) = \sum_{i=1}^{N+1} w_i e^{x_i(t+dt)} \tag{3.31}$$

We also know that passive strategy will adhere to reinvestment principle. In other words, the portfolio value archived for duration $t+dt$ will be same as the net portfolio value achieved by first investing \$1 for duration t and then reinvesting $V^{\infty}(0,t)$ for duration dt. Mathematically,

$$V^{\infty}(0,t+dt) = \sum_{i=1}^{N+1} w_i e^{x_i(t)} \sum_{i=1}^{N+1} w_i(t) e^{x_i(dt)} \tag{3.32}$$

Note that in equation 3.32, we have to use the weights at time t that have evolved and changed from their initial optimal values since no rebalancing to these original weights are done in passive strategy.

Equating equations 3.31 and 3.32, we get

$$\sum_{i=1}^{N+1} w_i e^{x_i(t+dt)} = \sum_{i=1}^{N+1} w_i e^{x_i(t)} \sum_{i=1}^{N+1} w_i(t) e^{x_i(dt)} \tag{3.33}$$

Readjusting the terms,

$$\sum_{i=1}^{N+1} w_i(t)e^{x_i(dt)} = \frac{\sum_{i=1}^{N+1} w_i e^{x_i(t+dt)}}{\sum_{i=1}^{N+1} w_i e^{x_i(t)}} \tag{3.34}$$

From lemma 1and using equation 3.1 we can write the portfolio value estimated at time t (instead of time 0) for the next dt as:

$$V^\infty(t, t+dt) = V^\infty(t,t) \sum_{i=1}^{N+1} w_i(t)e^{x_i dt} \tag{3.35}$$

First taking logarithm and then taking expectation on both sides, we obtain:

$$E[ln(\frac{V^\infty(t,t+dt)}{V^\infty(t,t)})] = E[ln(\sum_{i=1}^{N+1} w_i(t)e^{x_i dt})] \tag{3.36}$$

Substituting equation 3.34 in equation 3.36,

$$\begin{aligned} E[ln(\frac{V^\infty(t,t+dt)}{V^\infty(t,t)})] &= E[ln(\frac{\sum_{i=1}^{N+1} w_i e^{x_i(t+dt)}}{\sum_{i=1}^{N+1} w_i e^{x_i(t)}})] \\ &= E[ln(\sum_{i=1}^{N+1} w_i e^{x_i(t+dt)})] - E[ln(\sum_{i=1}^{N+1} w_i e^{x_i(t)})] \end{aligned} \tag{3.37}$$

Using equations 3.30 and 3.31, we arrive at the desired result of equation 3.29. □

We can rewrite equation 3.29 in our familiar $\chi(.)$ notation as follows:

$$\chi^\infty(t, t+dt) = \chi^\infty(0, t+dt) - \chi^\infty(0,t) \tag{3.38}$$

Lemma 7 proves that for the same horizon, as time passes, our estimation of passive ELPV undergoes parallel downward shift as depicted in the *ELPV contours* in figure 3.5. The active ELPV also parallel shifts downward by $\nu_p t$ at successive estimation time point t. As one moves along the estimation time line, the difference between passive and active ELPV declines. After sometime passive ELPV is no more advantageous over its active counterpart as depicted in figure 3.5. At that time, it is no more prudent to continue the portfolio passively and it needs to be rebalanced to the optimal set of weights. A fresh rebalance to these weights will reset the passive growth rate to its original value.

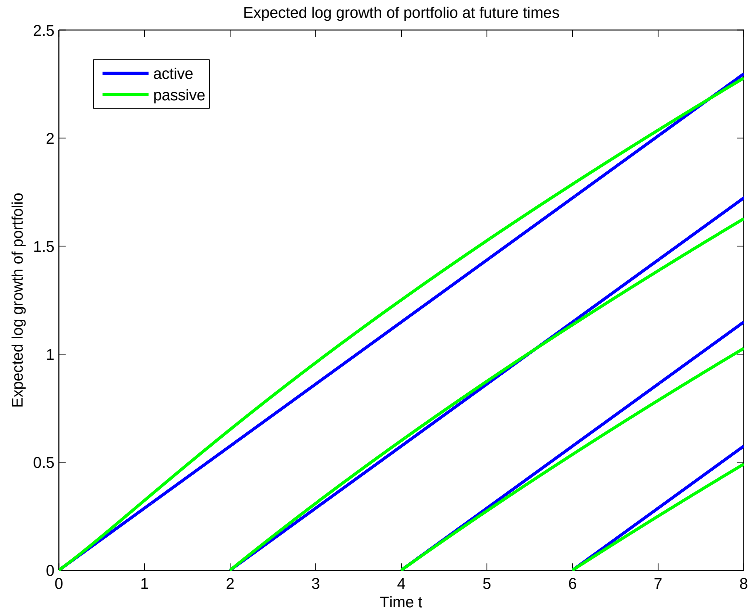


Figure 3.5 ELPV at future times.

Lemma 8. *Portfolio rebalance time τ_s is stable if the initial estimation of rate of change of passive ELPV is higher than optimal log growth rate ν_p during the passive investment period $(0\ \tau_s)$, i.e.*

$$\frac{d\chi^{\infty}(0,t)}{dt} \geq \nu_p, \forall t \in (0\ \tau_s) \tag{3.39}$$

Proof. As noted earlier, the right hand side of equation 3.28 is $\nu_p dt$. Substituting equation 3.38 in the left hand side of equation 3.28, we get $\forall t \in (0\ \tau_s)$:

$$\begin{aligned} &\chi^{\infty}(0,t+dt) - \chi^{\infty}(0,t) \geq \nu_p dt \\ \Rightarrow &\frac{\chi^{\infty}(0,t+dt) - \chi^{\infty}(0,t)}{dt} \geq \nu_p \end{aligned} \tag{3.40}$$

Letting $dt \to 0$, we get the desired equation 3.39. □

Above lemma 8 states a very important result. It says that one can compute the rebalance time at time $t = 0$, by merely taking the derivative of passive ELPV with respect to time t and equating it to optimal log growth rate of ν_p. This rebalance time τ_s shall be stable in the sense that at any time t' before τ_s, the passive investor's ELPV shall be higher than ν_p in the immediate future. Thus the investor has no incentive to shift to the continuous rebalance active strategy at any time before τ_s.

Intuitively the derivative of ELPV with respect to time t is the *expected instantaneous portfolio growth (EIPG)* in the log domain. We will use ξ to denote expected EIPG. In this notation, we can write,

$$\xi = \frac{d\chi(t)}{dt} = \nu_p \tag{3.41}$$

Using equation 2.27, we see that under active strategy the EIPG $\xi = \nu_p$, an invariant of time. Using equivalent notation for passive strategy, we can write,

$$\xi^{\infty}(t) = \frac{d\chi^{\infty}(t)}{dt} \tag{3.42}$$

One needs to distinguish between EIPG and portfolio growth rate. Portfolio growth rate is the average portfolio growth for a specified duration of time. Instantaneous portfolio growth at any time is the incremental growth that is achieved for a infinitely small time interval. In the context of this paper, both are defined for log of portfolio value. From equation 3.41, under active strategy, these two measures are always equal and invariant of time.

The lemma 8 merely states that one needs to continue using passive strategy as long as the EIPG offered by passive strategy is higher than or equal to that under active strategy. It also states that the stable rebalancing is possible only when the following condition is satisfied:

$$\exists \tau_s \text{ s.t. } \xi^{\infty}(t) > \nu_p, \forall t \in (0\ \tau_s) \tag{3.43}$$

Assuming that above condition is satisfied, the investor benefits by adopting passive strategy until τ_s, when the need to rebalance arises. At τ_s, the EIPG for passive strategy becomes equal to that for active strategy, i.e. ν_p.

$$\xi^{\infty}(\tau_s) = \nu_p \tag{3.44}$$

We are now set to compute τ_s in terms of the given initial investment parameters.

Lemma 9. *The portfolio rebalance time τ_s is the solution of the following equation:*

$$\frac{1}{X(t)}\Big[X'(t) - \frac{1}{2}\frac{X(t)Y'(t) - 2X'(t)Y(t)}{X(t)^2 + Y(t)}\Big] = \nu_p \tag{3.45}$$

where,
$X(t)$ *= expected portfolio value at time t, given by equation 3.8*
$Y(t)$ *= variance of portfolio value at time t, given by equation 3.12*

$$X'(t) = \frac{dX}{dt} = \sum_{i=1}^{N+1} w_i \mu_i e^{\mu_i t} \tag{3.46}$$

$$Y'(t) = \frac{dY}{dt} = \sum_{i,j=1}^{N+1} w_i w_j e^{(\mu_i+\mu_j)t}[(\mu_i + \mu_j)(e^{\sigma_{ij}t} - 1) + \sigma_{ij}e^{\sigma_{ij}t}] \tag{3.47}$$

Proof. We start with the resulting equation 3.39 of lemma 8. The stable rebalance time τ_s is given by the solution of the following equation when passive EIPG equals the EIPG under active strategy which is ν_p:

$$\frac{d\chi^{\infty}(t)}{dt} = \nu_p \tag{3.48}$$

Note, for simplicity we have removed the first time index from above equation and assume initial time for these estimation.

Using our notations, we can rewrite equation 3.16:

$$Var\Big[ln\Big(V^{\infty}(t)\Big)\Big] = ln\Big(1 + \frac{Y(t)}{X^2(t)}\Big) \tag{3.49}$$

Moreover, using our notations and equation 3.49, we can rewrite equation 3.18:

$$\begin{aligned}\chi^{\infty}(t) = E\Big[ln(V^{\infty}(t))\Big] &= ln(X(t)) - \frac{1}{2}Var\Big[ln(V^{\infty}(t))\Big] \\ &= ln(X(t)) - \frac{1}{2}ln\Big(1 + \frac{Y(t)}{X^2(t)}\Big)\end{aligned} \tag{3.50}$$

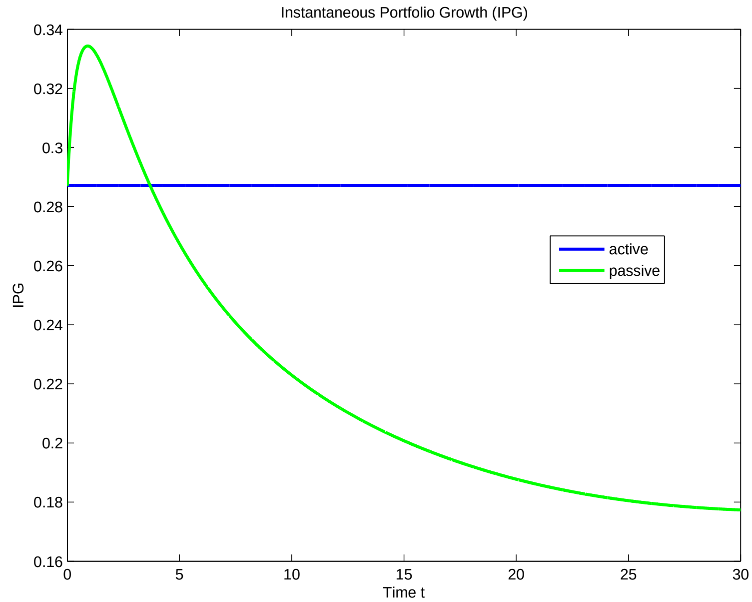


Figure 3.6 EIPG comparison with passive strategy.

Taking the first derivative of equation 3.50, we get:

$$\begin{aligned}
\frac{d\chi^{\infty}(t)}{dt} &= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}\left(1+\frac{Y(t)}{X^2(t)}\right)' \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}\left(\frac{Y(t)}{X^2(t)}\right)' \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{X^2(t)}{X^2(t)+Y(t)}\frac{X^2(t)Y'(t)-2X(t)X'(t)Y(t)}{X^4(t)} \\
&= \frac{X'(t)}{X(t)} - \frac{1}{2}\frac{1}{X^2(t)+Y(t)}\frac{X(t)Y'(t)-2X'(t)Y(t)}{X(t)} \\
&= \frac{1}{X(t)}\left[X'(t) - \frac{1}{2}\frac{X(t)Y'(t)-2X'(t)Y(t)}{X(t)^2+Y(t)}\right] \qquad (3.51)
\end{aligned}$$

□

Figure 3.6 plots the EIPG for passive strategy following equation 3.51 for our example investment scenario. As per lemma 9 non-zero intersection of the passive and the active EIPG curves give the stable rebalance time $\tau_s = 3.7$ for the portfolio.

Notice that the simple rebalance time $\tau_c = 7.61$ determined by algorithm 3 is much longer. Even though the investor can attain the same ELPV as active strategy by remaining passive for $\tau_c = 7.61$ years, after $\tau_s = 3.7$ years, her incremental ELPV shall be smaller compared to that offered by active strategy. We will soon see that by rebalancing earlier after $\tau_s = 3.7$ years, she can increase the potential gain measured in terms of ELPV.

We now define $\psi^\infty(t) = \chi^\infty(t) - \chi(t)$ which is the *excess growth* relative to active strategy. We show that the excess passive growth $\psi^\infty(t)$ is a monotonously increasing function for $0 < t < \tau_s$.

Lemma 10. *$\psi^\infty(t)$, the excess growth produced by passive strategy is increasing in the range $t \in (0\ \tau_s)$.*

Proof. We need to prove that $\psi'^\infty(t) > 0$, $\forall t \in (0\ \tau_s)$. Let's start with the derivative of $\psi^\infty(t)$.

$$\psi'^\infty(t) = \frac{d(\chi^\infty(t) - \nu_p t)}{dt} = \frac{d(\chi^\infty(t))}{dt} - \nu_p = \xi^\infty(t) - \nu_p \tag{3.52}$$

By definition of passive strategy, one needs to continue without rebalancing till $\xi^\infty(t) > \nu_p$. Using equation 3.43, $\xi^\infty(t) > \nu_p$, $\forall t \in (0\ \tau_s)$ implying $\psi'^\infty(t) > 0$. □

Lemma 11. *$\psi^\infty(t)$, the excess growth produced by passive strategy is maximized at τ_s.*

Proof. In order to prove that τ_s is a relative maxima, we need to prove the following two:

$$\psi'^\infty(\tau_s) = 0 \tag{3.53}$$

$$\psi''^\infty(\tau_s) < 0 \tag{3.54}$$

Proof for equation 3.53 is straightforward:

$$\psi'^\infty(\tau_s) = \xi^\infty(\tau_s) - \xi(\tau_s) = \xi^\infty(\tau_s) - \nu_p = 0 \tag{3.55}$$

We have used the results of lemma 10 above. Hence we proved equation 3.53.

To prove equation 3.54, we will use fundamental definition of differentiation.

$$\begin{aligned} \psi''^\infty(\tau_s) &= \lim_{d\tau \to 0} \frac{\psi'^\infty(\tau_s + d\tau) - \psi'^\infty(\tau_s)}{d\tau} = \lim_{d\tau \to 0} \frac{(\chi'^\infty(\tau_s + d\tau) - \nu_p) - (\chi'^\infty(\tau_s) - \nu_p)}{d\tau} \\ &= \lim_{d\tau \to 0} \frac{\chi'^\infty(\tau_s + d\tau) - \chi'^\infty(\tau_s)}{d\tau} = \lim_{d\tau \to 0} \frac{\xi^\infty(\tau_s + d\tau) - \xi^\infty(\tau_s)}{d\tau} \end{aligned} \tag{3.56}$$

By definition of stable strategy $\xi^\infty(\tau_s) = \nu_p$ and $\xi^\infty(\tau_s + d\tau) < \nu_p$. Therefore $\psi''^\infty(\tau_s) < 0$, proving equation 3.54. □

We summarize the computational steps in the form of algorithm 4 required to compute τ_s.

Algorithm 4 ComputeStableRebalanceFrequency

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,N,T,δT

$\tau_s \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}, \mathbf{S}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \mathbf{S}, r_f, N)$
if $!IsPassiveStrategyPossible(\mathbf{w}, \boldsymbol{\mu}, \mathbf{S})$ **then**
  return τ_s
end if
$\psi \leftarrow 0$
for $t = 0$ to T by δT **do**
  $X \leftarrow 0, Y \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X \leftarrow X + w[i]e^{\mu[i]t}$ # equation 3.8
    for $j = 1$ to N+1 **do**
      # equation 3.12
      $Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma[i,j]t} - 1)$
    end for
  end for
  $\chi^{\infty} \leftarrow ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})$ # equations 3.17 and 3.18
  $\psi_{prev} \leftarrow \psi, \psi \leftarrow \chi^{\infty} - \nu_p t$
  if $\psi \leq \psi_{prev}$ **then** # lemma 11
    # reached stable rebalance time
    return $\tau_s \leftarrow t$
  end if
end for
return τ_s

3.3 Eligibility For Discrete-Time Rebalancing

Thus far we have not shed any light on conditions for the existence of rebalance time allowing an investor to take advantage of passive investment strategy. First, without going through formal mathematical proof, we will discuss the existence of initial rebalance time τ_c. When τ_c does not exist, then the opportunity to remain passive during certain duration of investment horizon will not be possible. In this case, the investor has to continuously rebalance in order to maximize her log utility for the given horizon. For rebalance time τ_c to exist the following two conditions must hold:

$$\chi^{\infty}(\tau_c - d\tau_c) > \chi(\tau_c - d\tau_c) \text{ where, } 0 < d\tau_c < \tau_c \text{ and } d\tau_c \to 0 \tag{3.57a}$$

$$\chi^{\infty}(\tau_c + d\tau_c) < \chi(\tau_c + d\tau_c) \text{ where, } 0 < d\tau_c < \tau_c \text{ and } d\tau_c \to 0 \tag{3.57b}$$

From equation 2.27, $\chi(t)$ is a monotonically increasing function for $t \geq 0$ since its first derivative, the portfolio growth rate ν_p is a positive constant. This assumes the investor is profit seeking and chooses the assets for positive growth rate only. We also know that at $t = 0$, $\chi(t) = \chi^{\infty}(t) = ln[V(0)] = 0$. Hence, if equation 3.18 is a monotonically decreasing function for the given set of input investment parameters, then condition specified in equation 3.57a will never be satisfied for any $t > 0$. If, however, equation 3.18 is monotonically increasing with its first derivative or slope higher than ν_p, then opportunity to remain passive exists. In other words, opportunity for passive strategy exists if the time zero EIPG is higher under passive strategy. However, this is not sufficient to establish the existence condition as it turns out that at $t = 0$, the EIPG is same for both passive and active strategy.

Lemma 12. *Time zero EIPG are equal under passive and active strategies, i.e.*

$$\xi^{\infty}(0) = \xi = \nu_p \tag{3.58}$$

Proof. Using equation 3.51,

$$\xi^{\infty}(0) = \frac{d\chi^{\infty}(t)}{dt}|_{t=0} = \frac{1}{X(0)}[X'(0) - \frac{1}{2}\frac{X(0)Y'(0) - 2X'(0)Y(0)}{X(0)^2 + Y(0)}] \tag{3.59}$$

Substituting $t = 0$ in equations 3.8, 3.12, 3.46 and 3.47 respectively, we obtain:

$$X(0) = \sum_{i=1}^{N+1} w_i = 1 \tag{3.60a}$$

$$Y(0) = 0 \tag{3.60b}$$

$$X'(0) = \sum_{i=1}^{N+1} w_i \mu_i \tag{3.60c}$$

$$Y'(0) = \sum_{i,j=1}^{N+1} w_i w_j \sigma_{ij} \tag{3.60d}$$

Substituting the above set of values in equation 3.59, we obtain:

$$\xi^{\infty}(0) = \sum_{i=1}^{N+1} w_i \mu_i - \frac{1}{2} \sum_{i,j=1}^{N+1} w_i w_j \sigma_{ij} = \mu_p - \frac{\sigma_p^2}{2} = \nu_p = \xi \tag{3.61}$$

We have used the relationships of equations 2.19, 2.20 and 2.26 in the above derivation. □

We now know two properties of EIPG. First, active strategy has constant EIPG, ν_p. Secondly, as per lemma 12, both active and passive strategy start out with the same EIPG at time zero. Consequently, to obtain higher passive ELPV for a non-zero initial time interval, the passive portfolio must have an increasing EIPG at time zero. Founded on this premise, we establish the condition for existence of opportunity to stay passive and rebalance in the following lemma.

Lemma 13. *Passive strategy is feasible only when the EIPG is an increasing function at time $t = 0$ satisfying the following relationship:*

$$[X''(0) - X'(0)^2] - \frac{1}{2}[Y''(0) - Y'(0)^2] + 2X'(0)Y'(0) \geq 0 \tag{3.62}$$

where $X'(0)$ and $Y'(0)$ are given by equations 3.60c and 3.60d respectively. $X''(0)$ and $Y''(0)$ are the time zero values of the second derivatives of $X(t)$ and $Y(t)$ respectively and are given as follows:

$$X''(0) = \sum_{i=1}^{N+1} w_i {\mu_i}^2 \tag{3.63a}$$

$$Y''(0) = \sum_{i,j=1}^{N+1} w_i w_j \sigma_{ij} [2(\mu_i + \mu_j) + \sigma_{ij}] \tag{3.63b}$$

Proof. $\xi^\infty(t)$ is an increasing function at $t = 0$ when its first derivative is positive. Hence differentiating equation 3.51, we obtain:

$$\begin{aligned}\frac{d\xi^\infty(t)}{dt} &= \frac{d}{dt}\frac{1}{X}\Big(X' - \frac{1}{2}\frac{XY' - 2X'Y}{X^2+Y}\Big)\\ &= \frac{d}{dt}\Big(\frac{X'}{X} - \frac{1}{2}\frac{Y'}{X^2+Y} + \frac{X'Y}{X(X^2+Y)}\Big)\\ &= \frac{X''X - X'X'}{X^2} - \frac{1}{2}\frac{Y''(X^2+Y) - Y'(2XX'+Y')}{(X^2+Y)^2}\\ &\quad + \frac{(X''Y + X'Y')X(X^2+Y) - X'Y(3X^2X' + XY' + X'Y)}{X^2(X^2+Y)^2}\end{aligned} \tag{3.64}$$

For clarity, we have omitted time t from the variable notations above. For example, X above denotes $X(t)$. Using equations 3.60a and 3.60b, the value of the derivative of equation 3.64 at $t = 0$ will be given by:

$$\begin{aligned}\frac{d\xi^\infty(t)}{dt}|_{t=0} &= X''(0) - X'(0)^2 - \frac{1}{2}Y''(0) + X'(0)Y'(0) + \frac{1}{2}Y'(0)^2 + X'(0)Y'(0)\\ &= [X''(0) - X'(0)^2] - \frac{1}{2}[Y''(0) - Y'(0)^2] + 2X'(0)Y'(0)\end{aligned} \tag{3.65}$$

$X''(0)$ in equation 3.63a is obtained by differentiating equation 3.46 and substituting $t = 0$:

$$X''(t) = \frac{d^2X}{dt^2} = \sum_{i=1}^{N+1} w_i{\mu_i}^2 e^{\mu_i t} \tag{3.66}$$

Similarly, $Y''(0)$ in equation 3.63b is obtained by differentiating equation 3.47 and substituting $t = 0$:

$$\begin{aligned}Y''(t) = \frac{d^2Y}{dt^2} = \sum_{i,j=1}^{N+1} & w_i w_j(\mu_i+\mu_j)e^{(\mu_i+\mu_j)t}[(\mu_i+\mu_j)(e^{\sigma_{ij}t}-1) + \sigma_{ij}e^{\sigma_{ij}t}]\\ &+ w_i w_j e^{(\mu_i+\mu_j)t}[\sigma_{ij}(\mu_i+\mu_j)e^{\sigma_{ij}t} + \sigma_{ij}^2 e^{\sigma_{ij}t}]\end{aligned} \tag{3.67}$$

□

Hence, according lemma 13, the opportunity to take advantage of intermittent rebalancing and adherence to passive strategy is entirely determined by the set of asset mean and covariance characteristics. We now present this result in the form of the following algorithm 5.

Algorithm 5 IsPassiveStrategyPossible

Require: $\mathbf{w}$,$\boldsymbol{\mu}$,$\mathbf{S}$,N

$X' \leftarrow 0, X'' \leftarrow 0, Y' \leftarrow 0, Y'' \leftarrow 0$
for $i = 1$ to N+1 **do**
  $X' \leftarrow X' + w[i]\mu[i]$ # equation 3.60c
  $X'' \leftarrow X'' + w[i]\mu[i]^2$ # equation 3.63a
  for $j = 1$ to N+1 **do**
    $Y' \leftarrow Y' + w[i]w[j]\sigma[i,j]$ # equation 3.60d # equation 3.63b
    $Y'' \leftarrow Y'' + w[i]w[j]\sigma[i,j](2(\mu[i] + \mu[j]) + \sigma[i,j])$
  end for
end for
# equation 3.62
if $(X'' - X'^2) - \frac{1}{2}(Y'' - Y'^2) + 2X'Y' \geq 0$ **then**
  return true
else
  return false
end if

Chapter 4

Hybrid Strategy And Optimal Rebalancing

There is little incentive for the investor to resort to continuous rebalancing if passive strategy yields equal or higher portfolio growth for a given finite horizon. As depicted in figure 3.2(a), for the example portfolio, passive strategy outperforms active strategy for the initial investment period of 7.61 years determined by the point of intersection of equations 2.24 and 3.18. This initial passive investment period will be longer if transaction costs are to be considered.

Thus far, we have analyzed the nature of log of portfolio growth when the investor opts to remain passive without performing any rebalancing. We have shown that for certain assets characteristics the investor can get higher ELPV during the initial investment period. During this investment period, which we term as rebalance time, the investor should not opt to rebalance the portfolio. First, we derived the initial rebalance time τ_c when the ELPV is higher under passive strategy. Then, we designed a stable rebalance time τ_s when the EIPG of portfolio is higher under passive strategy. One can show that for any given set of assets $\tau_s \leq \tau_c$.

We now explore the nature of the portfolio growth after the first rebalance to determine the subsequent rebalance times. While determining the set of rebalance points or times, we must ensure that the investor utility, i.e. the ELPV, must not fall below the baseline value obtained using active rebalancing strategy. We define such investment strategy, the investor uses intermittent non-continuous rebalancing to maintain equal or higher ELPV throughout the investment horizon. In the simple approach, she will wait to rebalance as long as the ELPV remains higher than the corresponding active strategy value. In the stable rebalancing approach, she will only rebalance when the EIPG dips below the corresponding value of ν_p under active strategy. Our goal is to find the periodic frequency $\tau = \tau_o$ at which the investor can rebalance the portfolio to the initial optimal weights to maximize portfolio growth for the intended investment horizon. The frequency τ is the time interval measured in years. Under such a *hybrid strategy* the portfolio is rebalanced periodically every τ years till the end of investment horizon. We use superscript $\tau \neq \infty$ to denote a hybrid strategy that uses τ as the rebalance frequency.

We use superscript $\tau \neq \infty$ to denote a hybrid strategy that uses τ as the rebalance frequency. Note that during the initial simple rebalance time period $(0\ \tau_c]$, hybrid strategy growth is identical to that of passive strategy. Thus we must satisfy equations 3.26 and 3.27. Using the expanded notation, the time 0 estimation of the ELPV satisfies the following two conditions:

$$\chi^{\tau_c}(0, \delta t) > \nu_p \delta t, \text{ where } 0 < \delta t < \tau_c \tag{4.1}$$

$$\chi^{\tau_c}(0, \tau_c) = \nu_p \tau_c \tag{4.2}$$

From fundamental definition,

$$\begin{aligned} \chi^{\tau_c}(0, \delta t) &= E[ln(V^{\infty}(0, \delta t))] \\ &= E[\sum_{i=1}^{N+1} w_i e^{x_i \delta t})] \\ &= E[ln(\sum_{i=1}^{N+1} w_i e^{x_i \delta t})] \\ &= E[ln(\sum_{i=1}^{N+1} w_i e^{x_i \delta t})] \end{aligned} \tag{4.3}$$

Hence, equation 4.1 implies,

$$E[ln(\sum_{i=1}^{N+1} w_i e^{x_i \delta t})] > \nu_p \delta t \tag{4.4}$$

Similarly, from fundamental definition we can derive:

$$\chi^{\tau_c}(0, \tau_c) = E[ln(\sum_{i=1}^{N+1} w_i e^{x_i \tau_c})] \tag{4.5}$$

Hence, combining equations 4.2 and 4.5 we obtain:

$$E[ln(\sum_{i=1}^{N+1} w_i e^{x_i \tau_c})] = \nu_p \tau_c \tag{4.6}$$

4.1 Simple Hybrid Strategy

Theorem 1. *Let τ_c, the initial simple rebalance time satisfying equation 3.26 and 3.27 exist (and which can be computed using algorithm 3). Then $i\tau_c$ will also be a rebalance time for simple hybrid strategy, where $i \in \mathbb{N}$.*

Proof. We need to prove that for all $i \in \mathbb{N}$, i.e. for initial and all subsequent rebalance periods, $(i\tau_c\ (i+1)\tau_c]$, the following two conditions analogous to equations 4.1 and 4.2 must also hold.

$$\chi^{\tau_c}(i\tau_c, i\tau_c + \delta t) > \nu_p(i\tau_c + \delta t),\ \forall i \in \mathbb{N}, \delta t < \tau_c \tag{4.7}$$

$$\chi^{\tau_c}(i\tau_c, (i+1)\tau_c) = \nu_p((i+1)\tau_c),\ \forall i \in \mathbb{N} \tag{4.8}$$

We will prove both of these equations by the method of induction. For initial step when $i = 0$, both equations 4.7 and 4.8 becomes equations 4.1 and 4.2 respectively. By definition of τ_c these will be true. For the inductive step, assume equation 4.7 and 4.8 hold for $i = k$ and hence $k\tau_c$ is also a rebalance time. That is,

$$\chi^{\tau_c}(k\tau_c, k\tau_c + \delta t) > \nu_p(k\tau_c + \delta t) \tag{4.9}$$

$$\chi^{\tau_c}(k\tau_c, (k+1)\tau_c) = \nu_p((k+1)\tau_c) \tag{4.10}$$

Equation 4.10 indicates that $(k+1)\tau_c$ is a rebalance point. At rebalance points the ELPV will always be equal to the corresponding value under active strategy. This value will not be driven by the time of estimation. Hence equation 4.10 can be written as:

$$\chi^{\tau_c}(., (k+1)\tau_c) = E[ln(V^{\infty}(., (k+1)\tau_c))] = \nu_p((k+1)\tau_c) \tag{4.11}$$

We must show that equations 4.7 and 4.8 also hold for $i = k+1$, i.e.

$$\chi^{\tau_c}((k+1)\tau_c, (k+1)\tau_c + \delta t) > \nu_p((k+1)\tau_c + \delta t) \tag{4.12}$$

$$\chi^{\tau_c}((k+1)\tau_c,(k+2)\tau_c) = \nu_p((k+2)\tau_c) \tag{4.13}$$

Following similar steps as of the derivation of equation 4.3,

$$\begin{aligned}
&\chi^{\tau_c}((k+1)\tau_c,(k+1)\tau_c+\delta t) \\
&= E[ln(V^{\infty}((k+1)\tau_c,(k+1)\tau_c+\delta t))] \\
&= E[ln(V^{\infty}((k+1)\tau_c,(k+1)\tau_c)\sum_{i=1}^{N+1} w_i e^{x_i\delta t})] \\
&= E[ln(V^{\infty}((k+1)\tau_c,(k+1)\tau_c))] + E[ln(\sum_{i=1}^{N+1} w_i e^{x_i\delta t})]
\end{aligned} \tag{4.14}$$

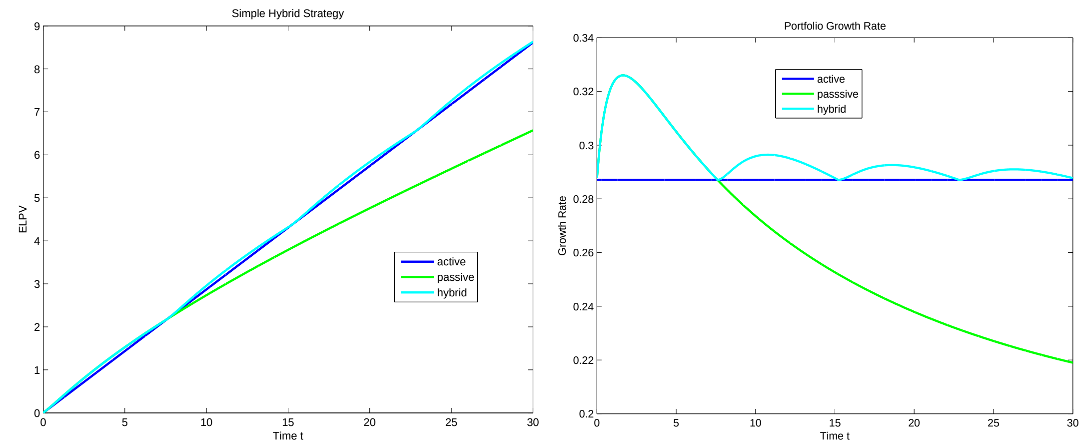
We have made use of the fact that $(k+1)\tau_c$ is a rebalance time and hence the initial asset weights are used. Now let's look at the two terms in the above equation. The first term is given by equation 4.11. The value of the second term is given by equation 4.4. Thus we establish the required relationship given by equation 4.12. To prove equation 4.13, we start with the LHS:

$$\begin{aligned}
&\chi^{\tau_c}((k+1)\tau_c,(k+2)\tau_c) \\
=&E[ln(V^{\infty}((k+1)\tau_c,(k+2)\tau_c))] \\
=&E[ln(V^{\infty}((k+1)\tau_c,(k+1)\tau_c)\sum_{i=1}^{N+1} w_i e^{x_i\tau_c})] \\
=&E[ln(V^{\infty}((k+1)\tau_c,(k+1)\tau_c))] + E[ln(\sum_{i=1}^{N+1} w_i e^{x_i\tau_c})]
\end{aligned} \tag{4.15}$$

Once again, we have made use of the fact that $(k+1)\tau_c$ is a rebalance time and hence the initial asset weights are used. As before, the first term is given by equation 4.11. The value of the second term is given by equation 4.6. Thus we establish the required relationship given by equation 4.13 and hence the equation 4.8.

This completes the proof of the theorem establishing the need to rebalance the assets to the initial optimal weights $\mathbf{w}$ at a periodic interval of τ_c. □

As per theorem 1 under simple hybrid strategy the portfolio needs to be rebalanced at $\tau_c, 2\tau_c, 3\tau_c, \ldots$ regular time intervals in order to attain or exceed investor log utility for a given finite investment horizon. This is illustrated in figure 4.1(a) for our example investment portfolio. The portfolio only needs to be rebalanced successively at $7.61, 15.22$ and 22.83 years during the 30 year investment period.



(a) (b)

Figure 4.1 Expected value and growth rate of log portfolio.

We now determine the ELPV under hybrid strategy when the portfolio is rebalanced periodically. Using the next theorem, we show that one can compute the ELPV for hybrid strategy using the ELPV from passive strategy. Hence, we name this theorem as the passive to hybrid *growth map theorem*. The theorem is applicable for all rebalancing scenarios including simple and stable rebalancing. Before we state and prove the theorem, we will state and prove two hypothesis concerning periodic rebalancing. The first one is called the *law of additive growth* whereas the second one is termed as *law of multiplicative growth*. First we state and prove the law of additive growth.

Lemma 14. *Passive portfolio growth is additive, i.e.*

$$\chi^\tau(k\tau + t') = \chi^\tau(k\tau) + \chi^\infty(t'),\ \forall k \in \mathbb{N}^+,\ \tau \in \mathbb{R}^+,\ \text{and } 0 < t' < \tau \tag{4.16}$$

where $k\tau$ is the most recent time when the portfolio is rebalanced and t' is the time for which the portfolio grows passively after $k\tau$.

Proof. Since $k\tau$ is the most recent rebalance time, the portfolio growth at $k\tau + t'$ is given by:

$$V^\tau(k\tau + t') = V^\tau(k\tau) \sum_{i=1}^{N+1} w_i e^{x_i(t')} \tag{4.17}$$

Taking first log and then expected value on both sides, we obtain,

$$\chi^{\tau}(k\tau + t') = \chi^{\tau}(k\tau) + E[ln(\sum_{i=1}^{N+1} w_i e^{x_i(t')})] = \chi^{\tau}(k\tau) + \chi^{\infty}(t') \tag{4.18}$$

□

Lemma 15. *Portfolio growth multiplies with the number of times periodic rebalancing is executed, i.e*

$$\chi^{\tau}(k\tau) = k\chi^{\infty}(\tau), \ \forall k \in \mathbb{N}^{+}, \, \tau \in \mathbb{R}^{+} \tag{4.19}$$

where τ *is the periodic rebalance frequency.*

Proof. We prove this lemma by method of induction. For the base case $k = 1$, equation 4.19 is trivially true. We then assume equation 4.19 holds for k and prove below that it also holds for $k + 1$. For $k + 1$, we need to prove,

$$\chi^{\tau}(\overline{k+1}\tau) = (k+1)\chi^{\infty}(\tau) \tag{4.20}$$

We start with RHS of above equation 4.20:

$$\begin{aligned}(k+1)\chi^{\infty}(\tau) &= k\chi^{\infty}(\tau) + \chi^{\infty}(\tau) = \chi^{\tau}(k\tau) + \chi^{\infty}(\tau), \text{ as equation 4.19 holds for } k. \\ &= \chi^{\tau}(k\tau + \tau), \text{ applying law of additive growth, lemma 14} \\ &= \chi^{\tau}(\overline{k+1}\tau) = LHS \end{aligned} \tag{4.21}$$

That completes the proof of equation 4.19 by induction. □

Theorem 2. *Assume that* $\chi^{\tau}(t) = \chi^{\infty}(t), \ \forall \ t \in (0 \ \tau]$ *is known following equation 3.18. Then* $\forall \ t > \tau > 0$,

$$\chi^{\tau}(t) = \begin{cases} \nu_p t & \text{if } \tau = 0 \\ k\chi^{\infty}(\tau) + \chi^{\infty}(t') & \text{otherwise} \end{cases} \tag{4.22}$$

where $t = k\tau + t'$, $k = \lfloor t/\tau \rfloor$ *and* $t' = t \bmod \tau$.

Proof. At the very outset, note that we consciously treat $\tau = 0$ case to be same as the active strategy for consistency of results between different strategies. Additionally while computing k and t', we avoid divide-by-zero scenarios. We only need to prove:

$$\chi^{\tau}(k\tau + t') = k\chi^{\infty}(\tau) + \chi^{\infty}(t') \tag{4.23}$$

We start with LHS of above equation 4.23.

$$\begin{aligned}\chi^{\tau}(k\tau + t') &= \chi^{\tau}(k\tau) + \chi^{\infty}(t'), \text{ applying law of additive growth, lemma 14} \\ &= k\chi^{\infty}(\tau) + \chi^{\infty}(t'), \text{ applying law of multiplicative growth, lemma 15} \quad = LHS \end{aligned} \tag{4.24}$$

□

The growth map theorem 2 establishes the relationship between passive and hybrid strategy ELPVs. It states that under any hybrid strategy where rebalancing is done with periodicity of τ, the expected log of portfolio growth at subsequent rebalancing points can be obtained by multiplying the ELPV at the first rebalance point by the number of times the portfolio has been rebalanced to the initial optimal weights. Once we obtain the ELPV at the last rebalance point, growth for any additional time $t' < \tau$ will occur following the passive trajectory identical to the initial rebalance period. An important aspect of this lemma is that the proposition is true for any positive finite value of period rebalance frequency τ, not just simple or stable rebalance frequencies. For any hybrid strategy with periodic rebalance frequency τ, once the passive χ^{∞} trajectory is calculated for the initial duration up to the first rebalance time, i.e. $[0\ \tau]$, we can completely construct the χ^{τ} trajectory for any future investment horizon.

For our example portfolio, after 30 years, the ELPV will be 8.6125 and 8.632 under active and simple hybrid strategy respectively. In terms of the ELPV, throughout the investment period, we expect to outperform the continuous rebalance active strategy. We will now prove this assertion using the next lemma. In real life investment, the performance of hybrid strategy will even be better once we factor in the cost of rebalancing.

Lemma 16. *Simple hybrid strategy will always outperform active strategy, i.e.* $\chi^{\tau_c}(t) \geq \chi(t)$.

Proof. Using the results of growth map theorem 2:

$$\chi^{\tau_c}(t) = k\chi^{\infty}(\tau_c) + \chi^{\infty}(t') \tag{4.25}$$

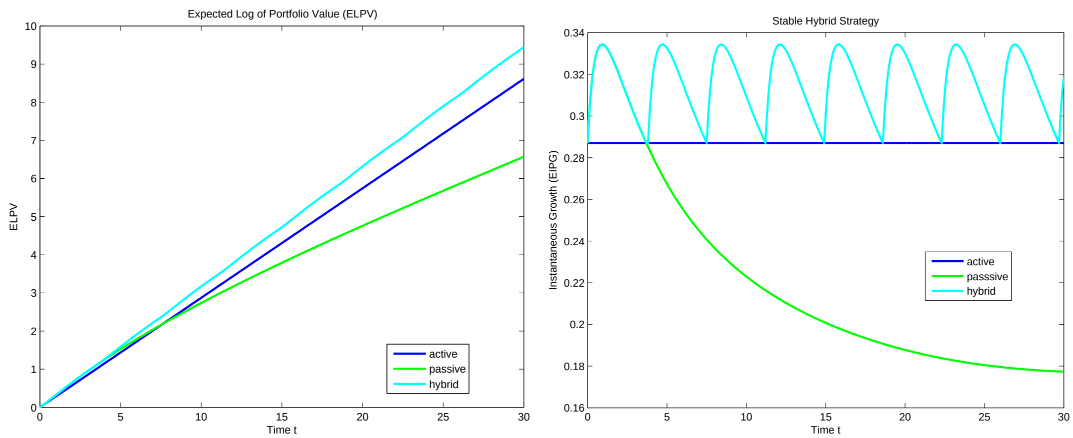
where $t = k\tau_c + t'$, $k = \lfloor t/\tau_c \rfloor$ and $t' = t \bmod \tau_c$.

Using the same notations,

$$\chi(t) = \nu_p t = \nu_p(k\tau_c + t') = k\nu_p\tau_c + \nu_p t' = k\chi(\tau_c) + \chi(t') \tag{4.26}$$

By definition, during the initial rebalance period $[0\ \tau_c)$, passive strategy outperforms active strategy, i.e.

$$\chi^{\infty}(t') \geq \chi(t'), \forall t' \in [0\ \tau_c) \tag{4.27}$$



(a) (b)

Figure 4.2 ELPV and EIPG in stable hybrid strategy

We know from equation 3.27,

$$\chi^{\infty}(\tau_c) = \chi(\tau_c) \tag{4.28}$$

Using the relations of equations 4.25 through 4.28, we obtain $\chi^{\tau_c}(t) \geq \chi(t)$. $\square$

4.2 Stable Hybrid Strategy

Similar to theorem 1, we will now derive subsequent stable rebalance times for stable hybrid strategy.

Theorem 3. *Let τ_s, the initial stable rebalance time satisfying equation 3.43 exists (and which can be computed using algorithm 4). Then $i\tau_s$ will also be a rebalance time for a stable hybrid strategy, where i is the set of natural numbers including 0, i.e. $i \in \mathbb{N}$.*

The proof is similar to the proof of theorem 1. For brevity we provide the proof in appendix A.1. Figure 4.2(b) shows the evolution of EIPG under stable hybrid strategy. The growth is never allowed to slip below the corresponding value ν_p under active strategy. Under such a strategy the EIPG during the entire investment horizon always remains higher or equal to that under active strategy. Using lemma 2, we can obtain the ELPV under stable hybrid strategy as follows:

$$\chi^{\tau_s}(t) = k_s \chi^{\infty}(\tau_s) + \chi^{\infty}(t'_s) \tag{4.29}$$

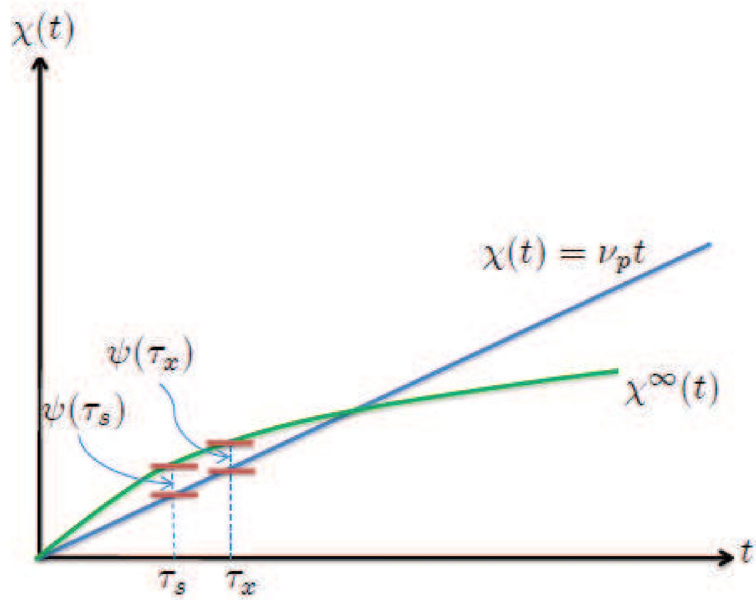


Figure 4.3 Illustration of excess growth at rebalance frequency τ_s and τ_x.

where $t = k\tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ and $t'_s = t \bmod \tau_s$. As shown in figure 4.2(a), stable hybrid strategy yields higher expected log of portfolio growth. We will formalize this property in the form of theorem 4 below.

Theorem 4. *Stable hybrid strategy will always outperform a hybrid strategy with higher rebalancing frequency, i.e. for any investment horizon* $t > \tau_x$, $\chi^{\tau_s}(t) > \chi^{\tau_x}(t)$, $\forall \tau_x > \tau_s$.

Proof. Using the results of theorem 2:

$$\chi^{\tau_s}(t) = k_s \chi^\infty(\tau_s) + \chi^\infty(t'_s) \tag{4.30}$$

where $t = k_s\tau_s + t'_s$, $k_s = \lfloor \frac{t}{\tau_s} \rfloor$ and $t'_s = t \bmod \tau_s$. Similarly,

$$\chi^{\tau_x}(t) = k_x \chi^\infty(\tau_x) + \chi^\infty(t'_x) \tag{4.31}$$

where $t = k_x\tau_x + t'_x$, $k_x = \lfloor \frac{t}{\tau_x} \rfloor$ and $t'_x = t \bmod \tau_x$. Figure 4.3 depicts the two different rebalance frequencies under consideration relative to the simple rebalance point τ_c. We need to prove the following inequality:

$$\begin{aligned}
&\chi^{\tau_s}(t) > \chi^{\tau_x}(t)\\
\Rightarrow &k_s\chi^{\infty}(\tau_s) + \chi^{\infty}(t'_s) > k_x\chi^{\infty}(\tau_x) + \chi^{\infty}(t'_x)\\
\Rightarrow &k_s[\chi(\tau_s) + \psi^{\infty}(\tau_s)] + [\chi(t'_s) + \psi^{\infty}(t'_s)] > k_x[\chi(\tau_x) + \psi^{\infty}(\tau_x)] + [\chi(t'_x) + \psi^{\infty}(t'_x)]\\
\Rightarrow &k_s[\nu_p\tau_s + \psi^{\infty}(\tau_s)] + [\nu_p t'_s + \psi^{\infty}(t'_s)] > k_x[\nu_p\tau_x + \psi^{\infty}(\tau_x)] + [\nu_p t'_x + \psi^{\infty}(t'_x)]\\
\Rightarrow &\nu_p[k_s\tau_s - k_x\tau_x] + [k_s\psi^{\infty}(\tau_s) - k_x\psi^{\infty}(\tau_x)] > \nu_p[t'_x - t'_s] + [\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)]\\
\Rightarrow &\nu_p[t - t'_s - t + t'_x] + [k_s\psi^{\infty}(\tau_s) - k_x\psi^{\infty}(\tau_x)] > \nu_p[t'_x - t'_s] + [\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)]\\
\Rightarrow &k_s\psi^{\infty}(\tau_s) - k_x\psi^{\infty}(\tau_x) > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) \qquad (4.32)
\end{aligned}$$

Since, $\tau_s < \tau_x$, we know that $k_s \geq k_x$. Let's define $\triangle k = k_s - k_x$ and substitute in the above inequality.

$$\begin{aligned}
&(k_x + \triangle k)\psi^{\infty}(\tau_s) - k_x\psi^{\infty}(\tau_x) > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)\\
\Rightarrow &k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k\psi^{\infty}(\tau_s) > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) \qquad (4.33)
\end{aligned}$$

We will now separately consider two possible cases for the value of $\triangle k$.

Case 1 - $\triangle k \geq 1$: The worst case scenario for equation 4.33 is when we consider the maximum possible value for the RHS expression. This will occur when $\psi^{\infty}(t'_s) \to 0$ and $\psi^{\infty}(t'_x) \to \psi^{\infty}(\tau_s)$(using lemma 11). Hence it is sufficient to prove:

$$\begin{aligned}
&k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k\psi^{\infty}(\tau_s) > max[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)]\\
\Rightarrow &k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + \triangle k\psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_s)\\
\Rightarrow &k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] + [\triangle k - 1]\psi^{\infty}(\tau_s) > 0 \qquad (4.34)
\end{aligned}$$

Again using lemma 11, we know $\psi^{\infty}(\tau_s) > \psi^{\infty}(\tau_x)$. We are considering investment horizons $t > \tau_x$. Hence $k_x \geq 1$. For this case, $[\triangle k - 1] \geq 0$. Lastly for valid passive strategy we need to have positive excess growth, i.e. $\psi^{\infty}(\tau_s) > 0$. With these conditions, inequality 4.34 will always hold.

Case 2 - $\triangle k = 0$: Under this scenario, inequality 4.33 is simplified to:

$$k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] > \psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) \qquad (4.35)$$

We now show that the above inequality 4.35 always holds since $k_x[\psi^{\infty}(\tau_s) - \psi^{\infty}(\tau_x)] > 0$ and $[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] < 0$. Since $\triangle k = 0, k_s = k_x$. To prove that $[\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s)] < 0$, we will start from the definition of horizon t:

$$\begin{aligned}
&t = k_s\tau_s + t'_s = k_x\tau_x + t'_x\\
\Rightarrow &k_x\tau_s + t'_s = k_x\tau_x + t'_x, \text{ since } k_s = k_x\\
\Rightarrow &k_x(\tau_x - \tau_s) = t'_s - t'_x\\
\Rightarrow &t'_s - t'_x > 0, \text{ since } \tau_x > \tau_s, k_x \geq 1\\
\Rightarrow &t'_s > t'_x\\
\Rightarrow &\tau_s > t'_s > t'_x, \text{ since } \tau_s > t'_s\\
\Rightarrow &\psi(\tau_s) > \psi(t'_s) > \psi(t'_x), \text{ using lemma 10}\\
\Rightarrow &\psi^{\infty}(t'_x) - \psi^{\infty}(t'_s) < 0 \qquad (4.36)
\end{aligned}$$

It is easy to prove that $k_x[\psi^\infty(\tau_s) - \psi^\infty(\tau_x)] > 0$. We know that $k_x \geq 1$ since we are concerned with investment horizon $t \geq \tau_x$ here. We also know by means of lemma 11 that $\psi^\infty(\tau_s) > \psi^\infty(\tau_x)$. Hence we showed that inequality 4.35 is always true since LHS is positive whereas RHS is negative. □

Corollary 1. *Stable hybrid strategy will outperform simple hybrid strategy for any investment horizon exceeding* τ_c*, i.e.* $\chi^{\tau_s}(t) > \chi^{\tau_c}(t), \forall t > \tau_c$.

Proposition in corollary 1 is directly evident from theorem 4. For our running investment example, the investor will obtain ELPV of 9.447 under stable hybrid strategy as compared to the 8.632 and 8.6125 obtained under simple hybrid and active strategy respectively. Stable hybrid strategy yields about 9.7% higher ELPV compared to baseline active strategy whereas simple hybrid strategy merely yields 0.23% higher ELPV.

4.3 Optimal Hybrid Strategy

The obvious question now is if there exists a rebalance frequency at which the ELPV is maximum for a given investment horizon. The key to find the answer is to study the results of growth map theorem 2. The theorem provides the ELPV attained for a given horizon T when a particular rebalance frequency τ is used. We will rewrite equation 4.22 as a function of $T > 0$ and τ:

$$\chi^\tau(T) = \lfloor \frac{T}{\tau} \rfloor \chi^\infty(\tau) + \chi^\infty(T \bmod \tau) \tag{4.37}$$

As mentioned before, in order to indicate passive strategy we can merely set the rebalance frequency to ∞. A hybrid strategy will have a rebalance frequency τ, such that $0 < \tau < T$.

We need to obtain the partial derivative of equation 4.37 with respect to τ to search for a maxima. In its current form equation 4.37 is expressed in terms of floor and mod functions which are non-continuous piecewise linear functions. It turns out it is difficult to differentiate this equation. Thus, our first attempt is to follow a numerical approach to search for the maxima of the equation.

For a given investment horizon, one can use equation 4.37 to compute the ELPV for any value of rebalance frequency τ. Figure 4.4(a) plots $\chi^\tau(30)$ for various values of τ. We notice that using a frequency of $\tau_o = 1.76$ year the ELPV is maximized to 9.75 in $T = 30$ years. Notice further that, the investor may use any rebalance frequency in $(0\ \tau_c]$ to obtain higher ELPV than if continuous rebalancing had been used. However, a rebalance frequency $\tau \in (0\ \tau_o)$ is not *efficient* since there is always a corresponding rebalance frequency $\tau' \in [\tau_o\ \tau_c)$ which will produce equal ELPV. More formally,

$$\forall \tau \in (0\ \tau_o), \exists \tau' \in [\tau_o\ \tau_c) \text{ s.t. } \chi^{\tau}(T) = \chi^{\tau'}(T) \tag{4.38}$$

The investor will pay higher transaction cost for using τ instead of τ'. Therefore the investor will have no incentive to use τ when she can afford to remain passive longer without degrading her terminal ELPV. In fact, she will improve her terminal ELPV when non-zero transaction costs are considered. Thus, investor will consider using a rebalance frequency τ only if it is on the *efficient rebalance frontier*, i.e. $\tau \in (\tau_o\ \tau_c)$. Figure 4.4(a) depicts the frontier as the shaded portion of the plot.

The algorithm 6 outlines the computational steps to search for the optimal rebalance frequency (ORF) τ_o which maximizes the left hand side of the equation 4.37 for any given investment horizon T. The computational burden is greatly reduced as the search needs to be performed only in the range of $\tau \in (0\ \tau_s]$ as per theorem 4.

We now state the following corollary which is quite obvious from the exposition thus far.

Corollary 2. *If for any given portfolio, τ_c, τ_s and τ_o are the simple, stable and ORFs respectively, then the following must hold true:*

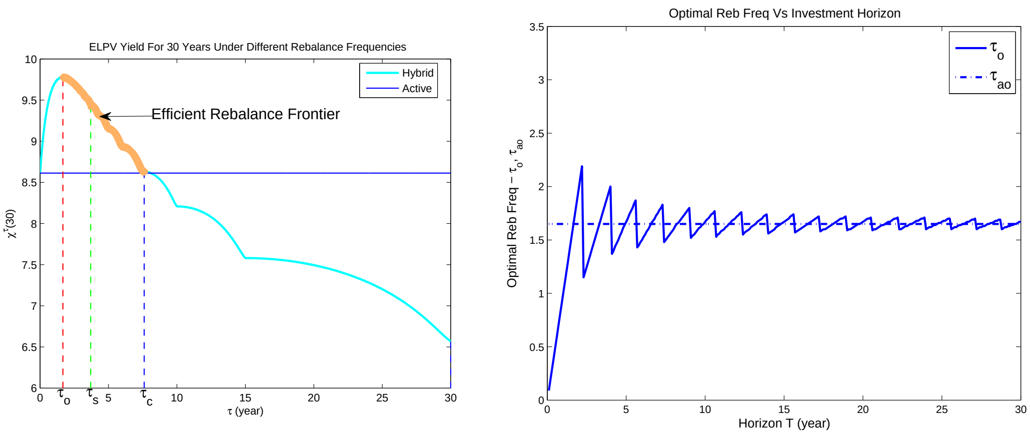
$$\tau_o \leq \tau_s \leq \tau_c \tag{4.39}$$

The above relationship is also demonstrated in figure 4.4(a). Using algorithm 6 we can determine the optimal frequency for various values of investment horizon. Figure 4.4(b) illustrates the variation of τ_o against horizon T for our example portfolio. It is interesting to observe the fluctuation pattern of τ for different values of T. The fluctuation is vigorous for smaller values of T. As T increases, the amplitude of the fluctuation decreases. One would

Algorithm 6 ComputeOptimalRebFreq

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,T,δT,N

$[\nu_p, \mathbf{w}, \boldsymbol{\mu}, \mathbf{S}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \mathbf{S}, r_f, N)$
$\tau_o \leftarrow 0,\ \chi^{\tau_o} \leftarrow \nu_p T$ # default continuous rebalancing
if $!IsPassiveStrategyPossible(\mathbf{w}, \boldsymbol{\mu}, \mathbf{S})$ **then**
  return (τ_o, χ^{τ_o})
end if
$m \leftarrow 0, \psi \leftarrow 0$
for $t = 0$ to T by δT **do**
  $m \leftarrow m + 1, X \leftarrow 0, Y \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X \leftarrow X + w[i]e^{\mu[i]t}$ # equation 3.8
    for $j = 1$ to N+1 **do**
      # equation 3.12
      $Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma[i,j]t} - 1)$
    end for
  end for
  $\chi^{\infty}[m] \leftarrow ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})$ # equations 3.17 and 3.18
  if $t = 0$ **then**
    $\tau_o = 0,\ \chi^{\tau_o} = \nu_p T$
  else
    # theorem 2
    $k \leftarrow \lfloor \frac{T}{t} \rfloor,\ t' \leftarrow T \bmod t,\ m' \leftarrow \chi^{\infty}(\lfloor \frac{t'}{\delta t} + 0.5 \rfloor)$
    $\chi^t \leftarrow k\chi^{\infty}[m] + \chi^{\infty}[m']$ # theorem 2
    if $\chi^t > \chi^{\tau_o}$ **then** # look for maximum χ^{τ}
      $\tau_o \leftarrow t,\ \chi^{\tau_o} \leftarrow \chi^t$
    end if
  end if
  $\psi_{prev} \leftarrow \psi, \psi \leftarrow \chi^{\infty}[m] - \nu_p t$
  if $\psi \leq \psi_{prev}$ **then** # lemma 10 and 11
    # reached stable rebalance time
    return (τ_o, χ^{τ_o})
  end if
end for
return (τ_o, χ^{τ_o}) # stop searching, theorem 4



(a) (b)

Figure 4.4 Optimal rebalancing frequency and its fluctuation with investment horizon

expect that for very large horizon, the optimal frequency will converge to a single value. We now prove that is indeed the case. Henceforth we will call this converged frequency as *asymptotic optimal* rebalance frequency and denote it by τ_{ao}.

Theorem 5. *For sufficiently large values of investment horizon T, the optimal frequency will asymptotically converge to τ_{ao}, the time at which instantaneous growth becomes equal to passive portfolio growth ν_p^∞, i.e.*

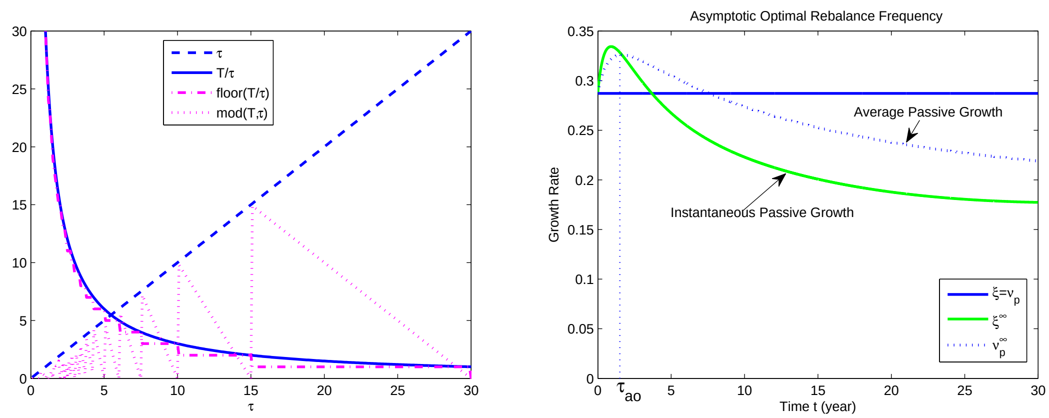
$$\xi^\infty(\tau_{ao}) = \nu_p^\infty(\tau_{ao}), \text{ where } \nu_p^\infty(t) = \frac{\chi^\infty(t)}{t} \tag{4.40}$$

Proof. Using the growth map theorem 2 we can write:

$$\chi^\tau(T) = \lfloor T/\tau \rfloor \chi^\infty(\tau) + \chi^\infty(T \bmod \tau) \tag{4.41}$$

From theorem 4 we know that $\tau_o \leq \tau_s$. Note that for optimality of τ, our interest is only in $\forall \tau \leq \tau_s$. We assume that horizon T is sufficiently large, such that $T \gg \tau_s > \tau_o$. Thus, $\lfloor \frac{T}{\tau} \rfloor \gg 1$. We also know that $\tau > (T \bmod \tau)$ implying that $\chi^\infty(\tau) > \chi^\infty(T \bmod \tau)$ since the passive portfolio growth will always be an increasing function of time for $t < \tau_s$ courtesy lemma 10. Combining these two, we get $\lfloor \frac{T}{\tau} \rfloor \chi^\infty(\tau) \gg \chi^\infty(T \bmod \tau)$. In other words, the first term involving floor function shall dominate the second term. Hence, as a first order simplification we can ignore the second term:

$$\chi^\tau(T) \approx \lfloor \frac{T}{\tau} \rfloor \chi^\infty(\tau) \tag{4.42}$$



(a) (b)

Figure 4.5 Illustration of derivation of asymptotic ORF τ_{ao} for $T = 30$ years.

Furthermore, using the illustration in figure 4.5(a), for $T \gg \tau$, $\lfloor \frac{T}{\tau} \rfloor \approx \frac{T}{\tau}$. Applying this second order of simplification, we obtain:

$$\chi^{\tau}(T) \approx \frac{T}{\tau}\chi^{\infty}(\tau) \tag{4.43}$$

In order to determine the value of τ at which the LHS of equation 4.43 is maximized, we take the partial derivative:

$$\frac{\partial \chi^{\tau}(T)}{\partial \tau} \approx \frac{\partial\left(\frac{T}{\tau}\chi^{\infty}(\tau)\right)}{\partial \tau} \approx -\frac{T}{\tau^2}\chi^{\infty}(\tau) + \frac{T}{\tau}\frac{\partial \chi^{\infty}(\tau)}{\partial \tau} \approx \frac{T}{\tau}\left(\frac{\partial \chi^{\infty}(\tau)}{\partial \tau} - \frac{1}{\tau}\chi^{\infty}(\tau)\right) \tag{4.44}$$

Setting 4.44 to zero, we obtain the value of τ_{ao} at which the hybrid portfolio growth value is maximized.

$$\begin{aligned}
&\frac{T}{\tau_{ao}}\left(\left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao})\right) = 0 \\
\Rightarrow &\left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} - \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao}) = 0, \text{ since } T \neq 0 \text{ and } \tau_{ao} \neq \infty \\
\Rightarrow &\left|\frac{\partial \chi^{\infty}(\tau)}{\partial \tau}\right|_{\tau=\tau_{ao}} = \frac{1}{\tau_{ao}}\chi^{\infty}(\tau_{ao}) \Rightarrow \xi^{\infty}(\tau_{ao}) = \nu_p^{\infty}(\tau_{ao})
\end{aligned} \tag{4.45}$$

□

Figure 4.5(b) illustrates the application of the above theorem to compute τ_{ao} for our example portfolio. In this case τ_{ao} is found to be 1.65 years. Observe from figure 4.4(b) how τ_o fluctuates around τ_{ao} as the investment horizon changes. We now present our last algorithm 7 to compute τ_{ao}.

Algorithm 7 ComputeAsymptoticOptimalRebalanceFrequency

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,N,T,δT

$\tau_{ao} \leftarrow 0$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}, \mathbf{S}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \mathbf{S}, r_f, N)$
if $!IsPassiveStrategyPossible(\mathbf{w}, \boldsymbol{\mu}, \mathbf{S})$ **then**
  return τ_{ao}
end if
$\xi^\infty \leftarrow 0, \nu_p^\infty \leftarrow 0$
for $t = \delta T$ to T by δT **do**
  $\xi_{prev}^\infty \leftarrow \xi^\infty, \nu_{p_{prev}}^\infty \leftarrow \nu_p^\infty$
  $X \leftarrow 0, X' \leftarrow 0, Y \leftarrow 0, Y' \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X \leftarrow X + w[i]e^{\mu[i]t}$ # equation 3.8
    $X' \leftarrow X' + w[i]\mu[i]e^{\mu[i]t}$ # equation 3.46
    for $j = 1$ to N+1 **do**
      # equation 3.12
      $Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma(i,j)t} - 1)$
      # equation 3.47
      $Y' \leftarrow Y' + w[i]w[j]e^{(\mu[i]+\mu[j])t}[(\mu[i] + \mu[j])(e^{\sigma[i,j]t} - 1) + \sigma[i,j]e^{\sigma[i,j]t}]$
    end for
  end for
  $\xi^\infty \leftarrow \frac{1}{X}\left[X' - \frac{1}{2}\frac{XY' - 2X'Y}{X^2 + Y}\right]$ # equation 3.51
  $\nu_p^\infty \leftarrow \frac{1}{t}\left(ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})\right)$ # equation 3.21
  if $\xi^\infty \leq \nu_p^\infty$ and $\xi_{prev}^\infty \geq \nu_{p_{prev}}^\infty$ **then**
    return $\tau_{ao} = t$
  end if
end for
return τ_{ao}

Chapter 5

Scalability and Efficiency

In the previous chapter, we investigated the possibility of discrete time trading for log-optimal portfolios. We outlined an analytical approach and an algorithm 6 to compute the optimal rebalance frequency (ORF) for a log-optimal investor with a finite investment horizon. The investor will rebalance periodically to the optimal portfolio weights to maximize the expected log of portfolio value (ELPV) for a given investment horizon. This periodic rebalancing approach obviates the need to continuously rebalancing the portfolio which is impractical to implement in real life investment.

A natural question to ask is if the investor can utilize Monte-Carlo simulation to compute the true underlying ORF instead of using a rebalancing algorithm 6. The core issue in simulation has always been the tradeoff between speed and accuracy[35]. The accuracy of simulation largely depends on the number of paths and size of discrete time step. Unfortunately, the speed of simulation to determine the ORF is not suitable for most modern-day investment scenarios. It is more so in today's era of algorithmic, micro-second and high-frequency trading environment that demands extremely fast determination of ORF in a dynamic changing market[36].

Later in chapter 7, we will present the Monte-Carlo simulation results for a portfolio with three risky and one risk-free assets. Using $20,000$ Monte-Carlo paths and 0.01 year time steps using a dual core 2.20 GHz, 4 GB Intel Pentium computer, the simulation takes a few days to complete. This large latency is unsuitable for most dynamic investment systems. The duration grows exponentially if we increase the asset count in the portfolio.

In this chapter, we investigate the scalability of the optimal algorithm 6. We analyze and improve the performance of the ORF algorithm 6. As we will see, inherently it is an $O(N^2)$ algorithm. Yet the speed of the algorithm can greatly improve by performing software optimizations. With such efficiency improvement the algorithm can be applied to large scale financial applications with hundreds of assets in a portfolio.

The rest of this chapter is organized as follows. Section 5.1 analyzes its run time. Section 5.2 discusses various optimization steps to be applied to improve efficiency of the algorithm. Section 5.3 presents the algorithm run time measurements and computes the gain in algorithm speed attributed to the optimizations in the previous section.

5.1 Computational Analysis

For our computational analysis, we only consider the core of algorithm 6 ignoring the computational load of one time invocation of algorithm 1 and 5 to compute the log optimal parameters and check for existence of ORF respectively. Algorithm 6 uses only scalars for all but one of the computed variables inside the asset loop (line 10). χ^∞ in line 20 is the only vector that needs to cache the previously computed values. The necessity of this caching arises due to the use of growth map theorem to compute the expected value of log of portfolio growth under hybrid strategy. Eliminating the use of unnecessary vectors makes the algorithm more efficient since storage and access of elements become very expensive as the size of the vectors (same as the number of assets used) grow. For example, in the covariance matrix $\mathbf{\Sigma}$, the first row and column have the covariances of risky assets with the risk-free asset.

We now analyze the following three loops in the algorithm which are the key drivers of runtime performance.

1. Outer time loop: Starts at line 7. The loop count is determined by precision of time discretization δT. The worst case loop count occurs when stable rebalance frequency τ_s does not exist for the asset class under consideration. Otherwise, the loop count is determined by the value of stable rebalance frequency τ_s.

$$L_t = \begin{cases} \frac{\tau_s}{\delta T} + 1, & \text{if } \exists\ \tau_s > 0; \\ \frac{T}{\delta T} + 1, & \text{otherwise;} \end{cases} \tag{5.1}$$

2. Outer asset loop: Starts at line 10. The loop count is proportional to $N + 1$, the number of risky and risk-free assets.

$$L_{ao} = L_t(N + 1) \tag{5.2}$$

3. Inner asset loop: Starts at line 13. The loop count depends on the outer asset loop count. The total number of times this loop is executed given by:

$$\begin{aligned} L_{ai} &= L_t(N + (N-1) + (N-2) + \cdots + 1) \\ &= L_t \frac{N(N+1)}{2} \end{aligned} \tag{5.3}$$

Equation 5.3 is translated into run time upper bound of $O(N^2)$ as well as lower bound of $\Omega(N^2)$. The reader can refer to [37] for the derivation of upper and lower bounds for summations.

It is more insightful to investigate the number of instructions executed as part of the algorithm. Let's denote η_t, η_{ao} and η_{ai} as the number instructions executed in the respective loops. For this exercise we count each instruction only once against the innermost loop it is in. The total number of instructions executed during the program is given by the following function:

$$\begin{aligned} \phi(N) &= \eta_t L_t + \eta_{ao} L_{ao} + \eta_{ai} L_{ai} \\ &= \eta_t L_t + \eta_{ao} L_t(N+1) + \eta_{ai} L_t \frac{N(N+1)}{2} \\ &= L_t(\eta_t + \eta_{ao}(N+1) + \eta_{ai} \frac{N(N+1)}{2}) \\ &= L_t(\eta_t + (N+1)(\eta_{ao} + 0.5\eta_{ai} N)) \end{aligned} \tag{5.4}$$

Assuming that the cost of executing each instruction is fixed, any optimization should reduce the total number of executed instructions $\phi(N)$ given by equation 5.4. Note that $\eta_t L_t$ is a fixed cost and does not increase as we increase the size of the portfolio to include more assets. Scanning the algorithms, we also observe that η_t consists of only a few number of mostly simple mathematical assignments essential to the algorithm. There will not be any substantial benefit to the overall algorithm performance by reducing η_t. On the contrary, any reduction in η_{ao} or η_{ai}, especially the later will increase the performance significantly as the size of N increases. Our optimization steps in the following section are primarily geared towards reducing η_{ao} and η_{ai}.

5.2 Optimization Steps

We apply following series of incremental optimization steps to algorithm 6.

1. Use covariance matrix: This simple step to replace all $\rho_{ij}\sigma_i\sigma_j$ terms by σ_{ij} terms of the covariance matrix $\mathbf{S}$ eliminates large number of multiplication instructions. Each such step involves at least 2 multiplications of double values. There are three instances of use of these expressions. Eliminating all of them to use σ_{ij} will reduce η_{ai} by 6. We will also replace two inputs $\mathbf{\Sigma}$ and $\boldsymbol{\rho}$ with only one input, the covariance matrix $\mathbf{S}$.

2. Loop splitting, precomputing and caching: The algorithm computes and uses the values for $w[i]e^{\mu[i]t}$, $\forall 1 \leq i \leq N$ in multiple lines (11,12,15 and 17). Note that some of these values are recomputed during the execution. The performance will improve if these values are precomputed and cached before the algorithm uses them. Without this caching mechanism these are computed $\frac{(N+1)(N+2)}{2}$ times corresponding to the one diagonal half of the i-j matrix space including the diagonal elements. With caching, however, these terms need only be computed $N+1$ times. This will save a net $\frac{N(N+1)}{2}$ times computing these terms.

The caching can be achieved by splitting the outer asset loop into two: separating the first iteration (i.e. $i = 1$) from the rest. In the first iteration all $N+1$ terms for $w[i]e^{\mu[i]t}$ can be pre-computed and cached. In the remaining part of the loop, these cached values will be reused without recomputing them.

3. Eliminate variance and covariance for risk-free asset: Note that the first iteration of the inner and outer asset loops involve computations with risk-free assets. The variance and co-variance terms will be zero in this iteration. Thus we eliminate the need to compute all the variance and covariance terms, viz. Y and Y' inside the inner loop. This further reduces the instruction counts η_{ao} and η_{ai}.

4. Common term refactoring: This is the final optimization step where common terms in expressions are identified and factored out to compute once. For example in the inner asset loop, the term $e^{\sigma[i,j]t}$ is a common term in both computing Y and Y' expressions.

Algorithm 8 outlines the resulting the optimized version to compute the ORF for log-optimal portfolios.

5.3 Performance Measurement

We implemented both versions of the algorithms presented here in Matlab environment for performance measurement. In our implementation we used only fundamental computational operations avoiding any usage of Matlab specific operators, such as matrix multiplication. We also implemented another version using Matlab matrix and summation operations avoiding any explicit for-loop constructs. This matrix-based implementation takes advantage of Matlab's underlying *Basic Linear Algebra Subprograms (BLAS)* library, a set of external linear algebra routines optimized for fast computation of low-level matrix operations. We have presented this Matlab algorithm version in the appendix.

To obtain a fair comparison, we carefully chose the algorithm input parameters $\boldsymbol{\mu}$, $\mathbf{S}$ and $\mathbf{w}$ so that the stable rebalance frequency $\boldsymbol{\tau_s}$ does not exist. This ensures that each run of the algorithm goes through till the end of the horizon $\boldsymbol{T}$ without breaking in the middle. Thus for the given asset parameters and horizon $\boldsymbol{T}$, we capture the worst case execution time. We considered the horizon value of $\boldsymbol{T = 10}$ years and $\boldsymbol{\delta T = 0.01}$ for our experiments. We ran and measured the execution time using Matlab cpu-time measurement instructions,

Algorithm 8 RebFreqCoreOptimized

Require: $\boldsymbol{\mu}$,$\mathbf{S}$,r_f,T,δT,N

$\tau_o \leftarrow 0$, $\chi^{\tau_o} \leftarrow \nu_p T$ # default continuous rebalancing
$[\nu_p, \mathbf{w}, \boldsymbol{\mu}, \mathbf{S}] \leftarrow ComputeLogOptimalParams(\boldsymbol{\mu}, \mathbf{S}, r_f, N)$
if $!IsPassiveStrategyPossible(\mathbf{w}, \boldsymbol{\mu}, \mathbf{S})$ **then**
  return (τ_o, χ^{τ_o})
end if
$m \leftarrow 0$
for $t = 0$ to T by δT **do**
  $m \leftarrow m + 1$
  $X \leftarrow 0, X' \leftarrow 0, Y \leftarrow 0, Y' \leftarrow 0$
  $A[1] \leftarrow w[1]e^{\mu[1]t}$
  $X \leftarrow X + A[1]$, $X' \leftarrow X' + \mu[1]A[1]$
  for $j = 2$ to N+1 **do**
    $A[j] \leftarrow w[j]e^{\mu[j]t}$
  end for
  for $i = 2$ to N+1 **do**
    $X \leftarrow X + A[i]$, $X' \leftarrow X' + \mu[i]A[i]$
    for $j = 2$ to N+1 **do**
      $c \leftarrow A[i]A[j]$, $b \leftarrow e^{\sigma[i,j]t}$
      $Y \leftarrow Y + c(b-1)$
      $Y' \leftarrow Y' + c((\mu[i] + \mu[j])(b-1) + \sigma[i,j]b)$
    end for
  end for
  $\chi^\infty[m] \leftarrow ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})$ # equations 3.17 and 3.18
  $\xi^\infty \leftarrow \frac{1}{X}[X' - \frac{1}{2}\frac{XY' - 2X'Y}{X^2 + Y}]$ # equation 3.51
  if $t = 0$ **then**
    $\chi^{\tau_o} = \nu_p T$, $\tau_o = 0$
  else
    $k \leftarrow \lfloor \frac{T}{t} \rfloor$, $t' \leftarrow T \bmod t$, $m' \leftarrow 1 + \lfloor \frac{t'}{\delta t} \rfloor$ # theorem 2
    # theorem 2
    $\chi^t \leftarrow k\chi^\infty[m] + \chi^\infty[m']$
    if $\chi^t > \chi^{\tau_o}$ **then** # look for maximum χ^H
      $\chi^{\tau_o} \leftarrow \chi^t$, $\tau_o \leftarrow t$
    end if
    if $\xi^\infty < \nu_p$ **then**
      # found stable rebalance time?
      return (τ_o, χ^{τ_o})
    end if
  end if
end for
return (τ_o, χ^{τ_o}) # stop searching, theorem 4

tic and ***toc***. We ran each of the three versions of algorithms with increasing asset size $\boldsymbol{N}$ and captured average execution time over 200 identical trials. The data is presented in the following table 5.1. All the measurements are taken in a standard commercially available dual core 2.20 GHz, 4 GB Intel Pentium personal computer.

Table 5.1 Algorithm Execution Time (Seconds) Comparison

N	original	matrix based	optimized
2	0.0256	0.0203	0.0093
3	0.0316	0.02	0.0094
5	0.046	0.0215	0.0111
8	0.073	0.0242	0.0144
16	0.1597	0.0382	0.0286
32	0.4075	0.0861	0.0744
64	1.2634	0.2826	0.2513
128	4.2851	1.285	0.9116
256	15.9081	6.7322	3.6558
400	37.8103	16.2942	9.1331
600	83.4313	37.8169	21.8522

Figure 5.1 plots the performance data of table 5.1. Observe that even though the matrix-based implementation offers significant speed improvement, the optimized algorithm 8 is the most efficient one. Speed improvement is specially significant for large values of $\boldsymbol{N}$. In order to compute magnitude of speed improvement, we fitted a polynomial curve of degree 2 (i.e. $\boldsymbol{aN^2 + bN + c}$) corresponding to runtime of $\boldsymbol{O(N^2)}$ to each of the curves in figure 5.1. The fitting for each curve is very tight with negligible error estimates. Table 5.2 presents the polynomial coefficients for each algorithm. To quantify the speed gain we neglected the lower order coefficients $\boldsymbol{b}$ and $\boldsymbol{c}$ to define the *speed gain* over the original algorithm as the coefficient ratio $\boldsymbol{a}$ of original algorithm to optimized algorithm. We observe that the matrix-based algorithm offers twice the speed of the original algorithm. Algorithm 8, however wins by offering a speed gain of 3.44 over the original one. This speed improvement is more visible for large size of $\boldsymbol{N}$. For instance, when the portfolio consists of 600 risky assets, ORF can be computed within 21 seconds compared to 83 seconds taken by the original algorithm.

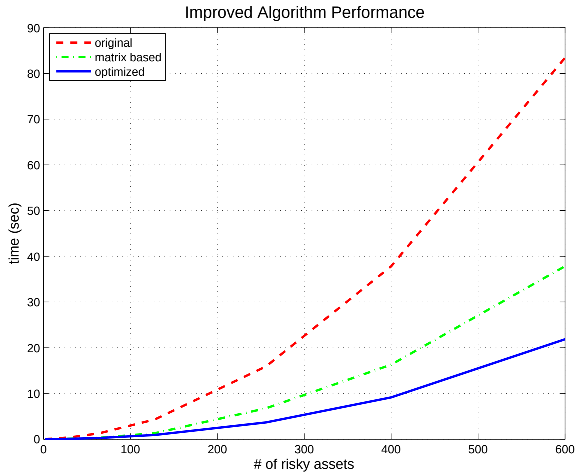


Figure 5.1 Algorithm performance in seconds when asset size $\boldsymbol{N}$ increases.

Table 5.2 Polynomial Curve Coefficients and Algorithm Speed Gain

	a	**b**	**c**	**Speed Gain**
optimized	**0.000065**	**-0.002834**	**0.067825**	**3.44**
matrix based	**0.000110**	**-0.002846**	**0.031052**	**2.04**
original	**0.000223**	**0.005010**	**0.014020**	**1.00**

It is important to note that the speed measured and presented here is the worst case speed. A typical investment scenario will run much faster since stable rebalance frequency occurs much before the horizon of 10 years considered here. For our optimization and analysis we did not consider exploiting parallelism inherent in the algorithm. Grid based computing is the foundation of many financial application used for commercial purpose. By exploiting the parallelism, the algorithm can be run in sub-second time interval in a multiprocessor or grid architecture.

Chapter 6

Computing Optimal Rebalance Frequency Function

It turns out that periodic *optimal rebalance frequency (ORF)* is a function of the investment horizon. Often investment portfolio managers have to make rebalancing decisions for funds invested by multitude of diverse investors. These investors have different preference for the length of investment horizon. For example, a pension fund manager needs to worry about investors of all ages. Hence the fund manager has to make rebalance decision for a continuum of investment horizons. In such scenarios, there is a need to compute the value of ORF function for a range of investment horizon. One can construct this rebalance strategy function, called *ORF function*, up to a specified investment horizon given a limited trajectory of expected log of portfolio value (ELPV) when the initial portfolio is never rebalanced. The search based algorithm 8 even after software optimization is inherently quadratic in time. The computing time rapidly explodes as the range of investment horizon expands.

Another limitation of our approach is that it assumes static return and risk characteristics for constituent assets. The limitation can however be obviated by re-computing the ORF if the market dynamics is changed with new values of risk and returns. For this approach to be viable, our algorithm needs to be computationally more efficient so that ORF can be adjusted dynamically with the change in market dynamics. Our goal is to compute ORF taking no more than a few milliseconds for very large portfolio size. With this efficiency goal, even smaller investment boutiques can take advantage of our proposed approach. These smaller investment firms typically can not afford high end grid or parallel computing platforms.

By using mathematical analysis, we reduce the complexity of the algorithm to linear time in two steps. First we show that the ORF can only be chosen from a finite set of numbers determined only by the intended investment horizon. Thus we obtain substantial improvement in performance for finding ORF by limiting the search space to a discrete and countable set instead of a continuous range of numbers. Then we show that the entire investment horizon range can be divided into non-overlapping piecewise segments. The ORF within each horizon segment is the ratio of the investment horizon to a fixed positive integer called the *rebalance divisor*. Therefore we reduce the task of computing the ORF function to merely finding the horizon segmentation points called *rebalance inflection point (RIP)* and the corresponding rebalance divisors.

The rest of the chapter is organized in the following manner. In section 6.2 we investigate approaches to reduce the continuous time search space to a finite countable discrete space for finding ORF. In section 6.3 we analytically compute the ORF eliminating the need for any search-based algorithm. Using this approach we present a linear time algorithm to compute the ORF function. We then measure and compare the computational complexity of three increasingly sophisticated algorithms in section 6.5.

6.1 Optimal Rebalance Frequency (ORF) Function

The obvious question now is if there exists a rebalance frequency at which the ELPV is maximum for a given investment horizon. The key to find the answer is to study the results of growth map theorem 2. The theorem provides the ELPV attained for a given horizon $\boldsymbol{T}$ when a particular rebalance frequency $\boldsymbol{\tau}$ is used.

We need to obtain the partial derivative of equation 4.22 with respect to $\boldsymbol{\tau}$ to search for a maxima. In its current form equation 4.22 is expressed in terms of floor and mod functions which are non-continuous piecewise linear functions. It turns out it is difficult to differentiate this equation. Thus, our first attempt is to follow a numerical approach to search for the maxima of the equation for a given $\boldsymbol{T}$ when $\boldsymbol{\tau}$ is varied.

$$\tau_o(T) = \tau, \text{ s.t. } \max_{0 \leq \tau \leq min(\tau_s, T)} \chi^{\tau}(T) \tag{6.1}$$

Figure 4.4(b) illustrates the variation of $\tau_o(T)$ when horizon T is varied from 0 to 30 years for our example portfolio. We call $\tau_o(T)$ as the *ORF function* for the portfolio. Observe that ORF function is piecewise linear and shows a sawtooth like pattern of fluctuation. The amplitude of fluctuation is larger for smaller values of T. As T increases, the amplitude decreases. For very large horizon, the ORF converges to a single asymptotic value τ_{ao}. Theorem 5 establishes the convergence condition. For the example portfolio τ_{ao} is found to be $\mathbf{1.65}$ years. We now outline the steps needed to compute the ORF function as per the specification in equation 6.1 in the form of algorithm 10.

Since the τ_o has an upper bound in τ_s, in the worst case the algorithm needs only the passive trajectory χ^{∞} from 0 to τ_s. Algorithm 9 summarizes the computational steps involved in computing the ELPV for passive portfolio. The algorithm computes the evolution of χ^{∞} until stable rebalance frequency τ_s is found. The algorithm tracks the value of excess growth ψ^{∞} to determine when τ_s is reached. Since we need to know the value of χ^{∞} only till τ_s in order to compute the ORF of the portfolio. For every discrete time horizon t, algorithm 10 searches for the ORF between 0 and $min(\tau_s, t)$ that yields the maximum ELPV. There are two discrete time for-loops starting at line 3 and 5. A similar algorithm 13 to compute τ_m from the unimodal trajectory of ξ^{∞} is outlined in the appendix.

For any given horizon T, the algorithm has to examine a set of candidate rebalance frequencies before selecting the optimal choice of τ_o. We define this set as *rebalance frequency domain* $\Im_s(T)$. For this algorithm $\Im_s(T)$ contains the following elements:

$$\Im_s(T) = \{m\delta t : \forall m \in \mathbb{N}^+ \text{ s.t. } m \leq \lfloor \frac{min(\tau_s, T)}{\delta t} \rfloor\} \tag{6.2}$$

For our example portfolio we have seen the value of $\tau_s = \mathbf{3.7}$. For a reasonable value of $\delta t = \mathbf{0.001}$, the cardinality of $\Im_s(\mathbf{30})$ shall be $\lfloor \frac{min(\mathbf{3.7}, \mathbf{30})}{\mathbf{0.001}} \rfloor = \mathbf{3,700}$. Thus one has to

Algorithm 9 ComputePassivePortfolio

Require: $\boldsymbol{\mu}$,S,w,$\boldsymbol{T}$,$\boldsymbol{\delta t}$,$\boldsymbol{N}$,$\boldsymbol{\nu_p}$
$m \leftarrow 0, \psi \leftarrow 0, \tau_s \leftarrow T$
for $t = 0$ to T by δt **do**
  $m \leftarrow m + 1, X \leftarrow 0, Y \leftarrow 0$
  for $i = 1$ to N+1 **do**
    $X \leftarrow X + w[i]e^{\mu[i]t}$ # equation 3.8
    for $j = 1$ to N+1 **do**
      # equation 3.12
      $Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma[i,j]t} - 1)$
    end for
  end for
  $\chi^\infty[m] \leftarrow ln(X) - \frac{1}{2}ln(1 + \frac{Y}{X^2})$ # equations 3.16 and 3.18
  $\psi_{prev} \leftarrow \psi, \psi \leftarrow \chi^\infty[m] - \nu_p t$
  if $\psi \leq \psi_{prev}$ **then**
    $\tau_s \leftarrow t$
    return (χ^∞, τ_s)
  end if
end for
return (χ^∞, τ_s)

Algorithm 10 ComputeORFfcn_Search_$\boldsymbol{\tau_o}$

Require: $\boldsymbol{\mu}$,S,w,$\boldsymbol{T}$,$\boldsymbol{\delta t}$,$\boldsymbol{N}$,$\boldsymbol{\nu_p}$
$[\chi^\infty, \tau_s] \leftarrow ComputePassivePortfolio(\mu, \mathrm{S}, \mathrm{w}, T, \delta t, N, \nu_p)$
$m \leftarrow 0$
for $t = 0$ to T by δt **do**
  $m \leftarrow m + 1, \mathrm{T}[m] \leftarrow t, \tau_o[m] = 0, \chi^{\tau_o}[m] = \nu_p t$ # continuous rebalancing
  for $\tau = \delta t$ to $min(\tau_s, t)$ by δt **do**
    $k \leftarrow \lfloor \frac{t}{\tau} \rfloor$, $t' \leftarrow t \bmod \tau$, $m' \leftarrow \chi^\infty(\lfloor \frac{t'}{\delta t} + 0.5 \rfloor)$ # theorem 2
    $\chi^\tau \leftarrow k\chi^\infty[m] + \chi^\infty[m']$ # theorem 2
    if $\chi^\tau > \chi^{\tau_o}[m]$ **then**
      $\tau_o[m] \leftarrow t$, $\chi^{\tau_o}[m] \leftarrow \chi^\tau$
    end if
  end for
end for
return $(\mathrm{T}, \tau_o, \chi^{\tau_o})$

search for $\mathbf{3,700}$ possible candidates to find the ORF $\boldsymbol{\tau_o(30)}$. The search space increases as the horizon value is increased from 0 to $\boldsymbol{\tau_s = 3.7}$. It remains the same for any investment horizon longer than $\boldsymbol{\tau_s = 3.7}$.

6.2 Reducing the Search Space

For large value of horizon $\boldsymbol{T}$ and small $\boldsymbol{\delta t}$ the search space $\Im_{\boldsymbol{s}}(\boldsymbol{T})$, $\lfloor \frac{\boldsymbol{min(\tau_s,T)}}{\boldsymbol{\delta t}} \rfloor$ can become very large. The algorithm needs to use a small $\boldsymbol{\delta t}$ for an acceptable accurate optimal solution. A very small value of $\boldsymbol{\delta t}$ will provide a more precise solution at the cost of increasing the search space and causing increasing computational burden. Our goal is to find alternative algorithms with reduced cardinality of search space to obtain better run-time performance. With subsequent mathematical analysis we will precisely achieve this goal. We show that one needs to search only a much smaller set of possible candidates.

6.2.1 Discrete Rebalance Divisor

With the help of the following lemma we will see that not all time values between 0 and $\boldsymbol{\tau_s}$ are candidates for rebalance frequency domain.

Lemma 17. *The rebalance frequency domain of any log-optimal portfolio is restricted to only the factors of horizon* $\boldsymbol{T}$ *with positive integer divisors, i.e.*

$$\Im_{\boldsymbol{k}}(\boldsymbol{T}) = \{\frac{\boldsymbol{T}}{\boldsymbol{k}} : \forall \boldsymbol{k} \in \mathbb{N}^+\} \tag{6.3}$$

Proof. Using the results of growth map theorem 2:

$$\boldsymbol{\chi^{\tau}(T) = k'\chi^{\infty}(\tau) + \chi^{\infty}(T - k'\tau)} \tag{6.4}$$

where $\boldsymbol{\tau}$ is a rebalance frequency and $\boldsymbol{k'} = \lfloor \boldsymbol{T/\tau} \rfloor \in \mathbb{N}$ is the set of positive natural numbers including 0. Taking the partial derivative with respect to $\boldsymbol{\tau}$, we obtain:

$$\begin{aligned} \frac{\partial \boldsymbol{\chi^{\tau}(T)}}{\partial \boldsymbol{\tau}} &= \boldsymbol{k'}\frac{\partial \boldsymbol{\chi^{\infty}(\tau)}}{\partial \boldsymbol{\tau}} + \frac{\partial \boldsymbol{\chi^{\infty}(T - k'\tau)}}{\partial \boldsymbol{\tau}} \\ &= \boldsymbol{k'\xi^{\infty}(\tau) - k'\xi^{\infty}(T - k'\tau)} \end{aligned} \tag{6.5}$$

To find the rebalance frequency $\boldsymbol{\tau = \tau_o}$ at which $\boldsymbol{\chi^{\tau}(T)}$ is maximized, we set the partial derivative in equation 6.5 to zero and solve for $\boldsymbol{\tau_o}$:

$$\begin{aligned}
&\boldsymbol{k'\xi^{\infty}(\tau_o) - k'\xi^{\infty}(T - k'\tau_o) = 0} \\
\Rightarrow &\boldsymbol{k'\xi^{\infty}(\tau_o) = k'\xi^{\infty}(T - k'\tau_o)} \\
\Rightarrow &\boldsymbol{\xi^{\infty}(\tau_o) = \xi^{\infty}(T - k'\tau_o)} \\
\Rightarrow &\boldsymbol{\tau_o = T - k'\tau_o} \\
\Rightarrow &\boldsymbol{\tau_o = \frac{T}{k' + 1}}
\end{aligned} \tag{6.6}$$

Note that $\boldsymbol{k' = 0}$ is true only when horizon $\boldsymbol{\tau_o \geq T}$ implying adherence to passive strategy. Equation 6.6 provides generic solutions for $\boldsymbol{\tau_o}$. Substituting $\boldsymbol{k = k' + 1}$ such that $\boldsymbol{k} \in \mathbb{N}^+$ in equation 6.6 we arrive at the following relationship:

$$\boldsymbol{\tau_o = \frac{T}{k}} \tag{6.7}$$

Hence, $\Im_{\boldsymbol{k}}\boldsymbol{(T)}$ can only have the factors for $\boldsymbol{T}$ as specified in equation 6.3. □

Corollary 3. *The ELPV at* $\boldsymbol{\tau_o = \left(\frac{T}{k}\right)}$ *is given by:*

$$\boldsymbol{\chi^{\tau_o}(T) = k\chi^{\infty}\left(\frac{T}{k}\right)} \tag{6.8}$$

Proof. The derivation is straight forward. Substituting equation 6.6 in equation 6.4, the maximum ELPV when $\boldsymbol{\tau_o}$ is used as the rebalance frequency:

$$\begin{aligned}
\boldsymbol{\chi^{\tau_o}(T)} &= \boldsymbol{k\chi^{\infty}\left(\frac{T}{k}\right) + \chi^{\infty}(T - \frac{kT}{k})} \\
&= \boldsymbol{k\chi^{\infty}\left(\frac{T}{k}\right) + \chi^{\infty}(0) = k\chi^{\infty}\left(\frac{T}{k}\right)}
\end{aligned} \tag{6.9}$$

□

Lemma 17 restricts the rebalance frequency domain to only a infinite set of rational numbers. Henceforth we describe the positive integer $\boldsymbol{k}$ as the *rebalance divisor* of the portfolio. A rebalance divisor divides the prescribed horizon into $\boldsymbol{k}$ equal segments. The portfolio has to be rebalanced after each segment to attain the terminal ELPV. In this way the wealth grows following passive dynamics for $\boldsymbol{k}$ equal time periods. The ELPV at the end of the passive period $\boldsymbol{\frac{T}{k}}$ multiplies $\boldsymbol{k}$ fold at the end of the horizon $\boldsymbol{T}$.

Notice that the rebalance frequency domain is entirely determined only by the length of investment $\boldsymbol{T}$ independent of any other portfolio characteristics. For example, any given portfolio that is eligible for rebalancing and has $\boldsymbol{T = 30}$ year investment horizon, ORF $\boldsymbol{\tau_o}$ for a log-optimal investor must belong to $\boldsymbol{\Im_k(30) = \{30, 15, 10, 7.5, 6, 5, 4.3, 3.8, 3.3, 3, \ldots\}}$. If the intended horizon is $\boldsymbol{T = 12}$ years, then the investor must choose $\boldsymbol{\tau_o}$ from $\boldsymbol{\Im_k(12) = \{12, 6, 4, 3, 2.4, 2, 1.7, 1.5, \ldots\}}$. Portfolio parameters only help determine the ORF from the domain.

There are infinite choices of rebalance divisors. Any search algorithm has to search for infinite possible alternative divisors to find the optimal $\boldsymbol{\tau_o}$. It turns out we can do even better by finding upper and lower bounds for the rebalance divisor. Consequently we restrict the domain $\boldsymbol{\Im_k(T)}$ to only a finite and countable set. From corollary 2 we know that $\boldsymbol{\tau_o}$ has portfolio specific upper bound $\boldsymbol{\tau_s}$ such that $\boldsymbol{\tau_o \leq \tau_s}$. This leads to a lower bound $\boldsymbol{k_{mn}}$ given by:

$$\boldsymbol{k_{mn} = max(1, \lceil \frac{T}{\tau_s} \rceil)} \tag{6.10}$$

The rebalance frequency domain still remains an infinite set as follows:

$$\boldsymbol{\Im_k(T) = \{\frac{T}{k} : \forall k \in \mathbb{N}^+ \text{ and } k \geq k_{mn}\}} \tag{6.11}$$

For our example portfolio using $\boldsymbol{\tau_s = 3.7}$ the domain is now reduced to $\boldsymbol{\{3.3, 3, 2.7, 2.5, \ldots\}}$ and $\boldsymbol{\{3, 2.4, 2, 1.7, 1.5, \ldots\}}$ for $\boldsymbol{\tau_o}$ for $\boldsymbol{30}$ and $\boldsymbol{12}$ year horizons respectively. We now prove that the ORF $\boldsymbol{\tau_o}$ will also have a lower bound which further restricts the domain to a countable finite set.

Lemma 18. *For any given portfolio with horizon* $\boldsymbol{T}$, *assume* $\boldsymbol{\xi^\infty(t)}$ *is* unimodal *in* $\boldsymbol{0 \leq t \leq T}$ *with the unique maxima at* $\boldsymbol{\tau_m}$. *The ELPV* $\boldsymbol{\chi^{\tau_o}(T)}$ *is maximized for ORF* $\boldsymbol{\tau_o \in \Im_k(T)}$ *such that* $\boldsymbol{\tau_o \geq \tau_m}$.

Proof. Using the results of lemma 17, we know that the ORF $\boldsymbol{\tau_o \in \Im_k(T)}$. Let's choose two rebalance divisors $\boldsymbol{k+1}$ and $\boldsymbol{k}$. The corresponding rebalance frequencies are $\boldsymbol{\tau_1 = \frac{T}{k+1}}$ and $\boldsymbol{\tau_2 = \frac{T}{k}}$ belong to $\boldsymbol{\Im_k(T)}$. By definition $\boldsymbol{\tau_1 < \tau_2}$. We need to prove that if $\boldsymbol{\tau_1 < \tau_m}$,

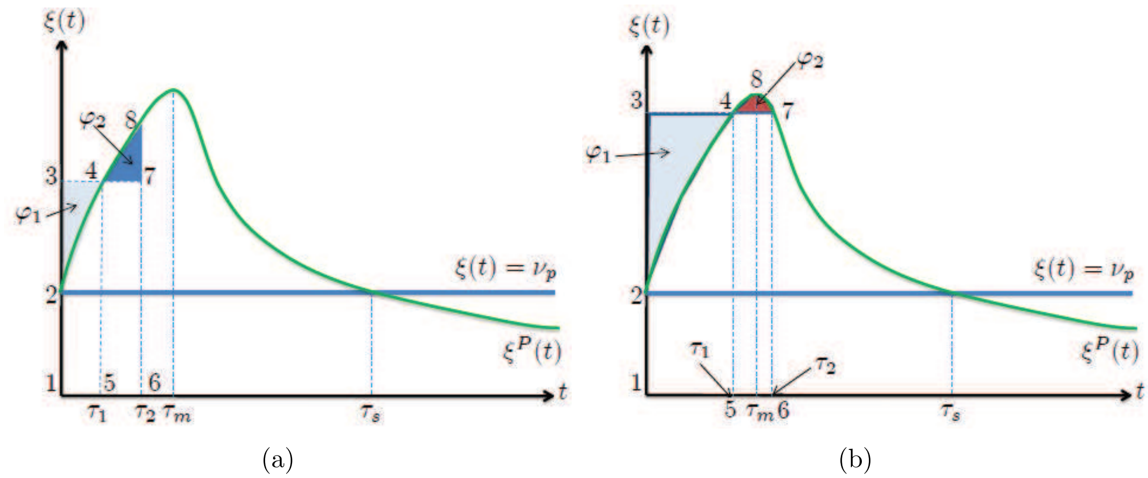


Figure 6.1 Two possible scenarios for deriving the lower bound for rebalance frequency.

$\boldsymbol{\tau_2}$ will always outperform $\boldsymbol{\tau_1}$ in generating higher ELPV for horizon $\boldsymbol{T}$. $\boldsymbol{\tau_2}$ may take any value on either side of $\boldsymbol{\tau_m}$. Mathematically, it will suffice to prove the following:

$$\begin{aligned}
&\chi^{\tau_1}(T) < \chi^{\tau_2}(T) \\
\Rightarrow&(k+1)\chi^{\infty}(\tau_1) < k\chi^{\infty}(\tau_2), \text{ using lemma 17} \\
\Rightarrow&\frac{k+1}{k}\chi^{\infty}(\tau_1) < \chi^{\infty}(\tau_2) \\
\Rightarrow&\frac{1}{k}\chi^{\infty}(\tau_1) < \chi^{\infty}(\tau_2) - \chi^{\infty}(\tau_1) \\
\Rightarrow&\frac{1}{k}\int_0^{\tau_1} \xi^{\infty}(t)\,\mathrm{d}t < \int_0^{\tau_2} \xi^{\infty}(t)\,\mathrm{d}t - \int_0^{\tau_1} \xi^{\infty}(t)\,\mathrm{d}t
\end{aligned} \tag{6.12}$$

We consider two possible scenarios for inequality 6.12 as illustrated in figure 6.1. The first scenario in figure 6.1(a) is applicable when $\boldsymbol{\tau_1 < \tau_2 \leq \tau_m}$. The LHS of inequality 6.12 is given by:

$$\begin{aligned}
&\frac{1}{k}\int_0^{\tau_1} \xi^{\infty}(t)\,\mathrm{d}t \\
=&\frac{1}{k}(\text{Area of region 1-2-4-5}) \\
=&\frac{1}{k}(\text{Area of region 1-3-4-5}) - \frac{1}{k}(\text{Area of region 2-3-4}) \\
=&\left(\frac{1}{k}\right)(\tau_1)\xi^{\infty}(\tau_1) - \frac{\varphi_1}{k} \\
=&\frac{T}{k(k+1)}\xi^{\infty}(\tau_1) - \frac{\varphi_1}{k}
\end{aligned} \tag{6.13}$$

The RHS of inequality 6.12 is given by:

$$\begin{aligned}
&\int_0^{\tau_2} \xi^\infty(t)\,dt - \int_0^{\tau_1} \xi^\infty(t)\,dt\\
=&\text{(Area of region 4-5-6-7-8)}\\
=&\text{(Area of region 4-5-6-7)} + \text{(Area of region 4-7-8)}\\
=&(\tau_2 - \tau_1)\xi^\infty(\tau_1) + \varphi_2 = \Big(\frac{T}{k} - \frac{T}{k+1}\Big)\xi^\infty(\tau_1) + \varphi_2\\
=&\frac{T}{k(k+1)}\xi^\infty(\tau_1) + \varphi_2 \qquad (6.14)
\end{aligned}$$

Since $\varphi_1, \varphi_2 > 0$, comparing equations 6.13 and 6.14 we prove that inequality 6.12 holds true.

With the help of scenario 1, we proved that the investor should prefer to use the *largest* value of rebalance frequency out of all possible rebalance frequencies in the interval of 0 to τ_m. This largest frequency shall have a corresponding rebalance divisor of $\lfloor \frac{T}{\tau_m} \rfloor$. We now show that the next higher rebalance frequency shall always be a better choice for the investor to attain higher expected utility. All rebalance frequency candidates in the interval of 0 to τ_m shall be suboptimal for the investor. Therefore the ORF τ_o shall always be higher than τ_m.

As illustrated in figure 6.1(b), τ_1 is the largest rebalance frequency candidate less than τ_m with a rebalance divisor of $k + 1$. τ_2 is the next higher rebalance frequency candidate with a rebalance divisor of k. We need to prove that the investor shall attain higher ELPV when rebalance frequency of τ_2 instead of τ_1 is used.

First we reckon that for sufficiently large value of horizon $T \gg \tau_m$, $k = \lfloor \frac{T}{\tau_m} \rfloor \gg 1$. For scenario 2, we can derive the values of τ_1 and τ_2 as follows:

$$\begin{aligned}
\tau_1 &= \frac{T}{k+1} = \frac{k\tau_m + t'}{k+1}, \text{ where } 0 \le t' < \tau_m\\
&= \tau_m - \frac{\tau_m - t'}{k+1} \qquad (6.15a)\\
\tau_2 &= \frac{T}{k} = \frac{k\tau_m + t'}{k} = \tau_m + \frac{\tau_m}{k} \qquad (6.15b)
\end{aligned}$$

For $k \gg 1$, both $\frac{\tau_m - t'}{k+1}$ and $\frac{\tau_m}{k}$ shall be small compared to τ_m. Hence τ_1 and τ_2 will be very close to τ_m. This is illustrated in figure 6.1(b) where it is assumed $\xi^\infty(\tau_1) = \xi^\infty(\tau_2)$. As in scenario 1, the LHS of inequality 6.12 is given by equation 6.13. We now derive the

RHS of inequality 6.12 as follows:

$$\begin{aligned}
&\int_0^{\tau_2} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\tau_1} \xi^\infty(t)\,\mathrm{d}t \\
=&\text{(Area of region 4-5-6-7-8)} \\
=&\text{(Area of region 4-5-6-7)} + \text{(Area of region 4-7-8)} \\
=&(\tau_2 - \tau_1)\xi^\infty(\tau_1) + \varphi_2 = \Big(\frac{T}{k} - \frac{T}{k+1}\Big)\xi^\infty(\tau_1) + \varphi_2 \\
=&\frac{T}{k(k+1)}\xi^\infty(\tau_1) + \varphi_2
\end{aligned} \tag{6.16}$$

Once again, since $\varphi_1, \varphi_2 > 0$, comparing equations 6.13 and 6.16 we prove that inequality 6.12 holds true for scenario 2. Therefore an investor will never choose any rebalancing frequency $\tau < \tau_m$. □

Thus we establish upper and lower bounds for the ORF as τ_s and τ_m respectively. While τ_s determines the lower bound of rebalance divisor as per equation 6.10, τ_m determines the upper bound k_{mx} as per equation 6.17 below:

$$k_{mx} = max(1, \lfloor \frac{T}{\tau_m} \rfloor) \tag{6.17}$$

Figure 6.2 illustrates the values of upper and lower bounds of rebalance divisors for various investment horizons. Notice that, for low values of horizon, $k_o = 1$ outperforms all other rebalance divisors. Hence the investor will follow passive strategy for such low value of investment horizon. As the horizon increases, we observe that k_o increases in steps of 1 resulting in faster ORF τ_o for longer horizon. As per theorem 5, for sufficiently large horizon τ_o converges to τ_{ao}. We refine the rebalance frequency domain further to a finite countable set as follows:

$$\Im_k(T) = \{\frac{T}{k} : \forall k \in \mathbb{N}^+ \text{ and } k_{mn} \le k \le k_{mx}\} \tag{6.18}$$

Contrast the above rebalance frequency domain $\Im_k(T)$ to the prior domain $\Im_s(T)$ defined in equation 6.2. For our example portfolio, the values for $\tau_m = 0.91$ and $\tau_s = 3.7$ lead to $k_{mn} = 9$ and $k_{mx} = 32$ for horizon $T = 30$ years. This restricts the rebalance

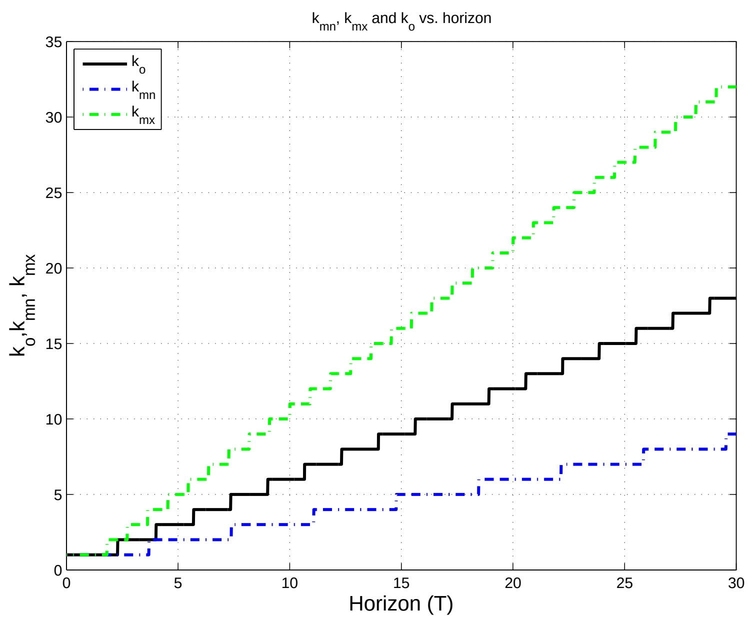


Figure 6.2 Bounding the rebalance divisors for various investment horizons.

frequency domain to $\Im_k(30) = \{3.3, 3, 2.7, 2.5, \ldots, 1.0, 0.97, 0.94\}$. Thus one has to search for only 24 possible candidates to find the optimal frequency $\tau_o(30)$. Similarly an investment horizon of 12 years leads to $k_{mn} = 4$ and $k_{mx} = 13$. One has to search the domain $\Im_k(12) = \{3, 2.4, 2, 1.7, 1.5, 1.33, 1.2, 1.09, 1, 0.92\}$ consisting of only 10 discrete values to find $\tau_o(12)$. There is significant reduction in search space compared to earlier search based rebalance domain. For any horizon value $\forall T \geq 3.7$, $\Im_s(T)$ has a cardinality of 370 and 3,700 for δt values of 0.01 and 0.001 respectively. Given $\Im_k(T)$, one can define the *ORF* similar to the version in equation 6.1 as:

$$\tau_o(T) = \tau, \text{ s.t. } \max_{k_{mn} \leq k \leq k_{mx}} k\chi^{\infty}(\frac{T}{k}) \tag{6.19}$$

Algorithm 11 outlines the steps to compute the trajectory of ORF function up to the specified horizon T. It takes the portfolio parameters and returns three vectors, viz. $\mathbf{T}$, the horizon time point vector, τ_o, the ORF function vector for each discrete horizon time point and χ^{τ_o}, the vector containing the ELPV if the corresponding ORF were to be used. For each horizon time point the algorithm only searches for the optimal value of rebalance divisor k_o between

Algorithm 11 ComputeORFfcn_Search_$\boldsymbol{k_o}$

Require: $\boldsymbol{\mu}$,S,w,$\boldsymbol{T}$,$\boldsymbol{\delta t}$,$\boldsymbol{N}$,$\boldsymbol{\nu_p}$

$[\chi^{\infty}, \tau_s] \leftarrow \boldsymbol{ComputePassivePortfolio}(\mu, \mathrm{S}, \mathrm{w}, T, \delta t, N, \nu_p)$
$\tau_m \leftarrow \boldsymbol{ComputeTauMax}(\mu, \mathrm{S}, \mathrm{w}, T, \delta t, N)$
$m \leftarrow 1, \mathrm{T}[m] \leftarrow 0, \tau_o[m] \leftarrow 0, \chi^{\tau_o}[m] \leftarrow 0$
for $t = \delta t$ to T by δt **do**
  $m \leftarrow m + 1, \mathrm{T}[m] \leftarrow t, \tau_o[m] \leftarrow 0, \chi^{\tau_o}[m] \leftarrow \nu_p t$
  $k_{mn} \leftarrow max(1, \lceil \frac{t}{\tau_s} \rceil), k_{mx} \leftarrow max(1, \lfloor \frac{t}{\tau_m} \rfloor)$
  for $k = k_{mn}$ to k_{mx} by 1 **do**
    $\tau = \frac{t}{k}$
    if $k\chi^{\infty}(\lfloor \frac{\tau}{\delta t} + 0.5 \rfloor) > \chi^{\tau_o}[m]$ **then**
      $\tau_o[m] \leftarrow \tau, \chi^{\tau_o}[m] \leftarrow k\chi^{\infty}(\lfloor \frac{\tau}{\delta t} + 0.5 \rfloor)$
    end if
  end for
end for
return $(\mathrm{T}, \tau_o, \chi^{\tau_o})$

k_{mn} and k_{mx} that maximizes the ELPV under hybrid strategy.

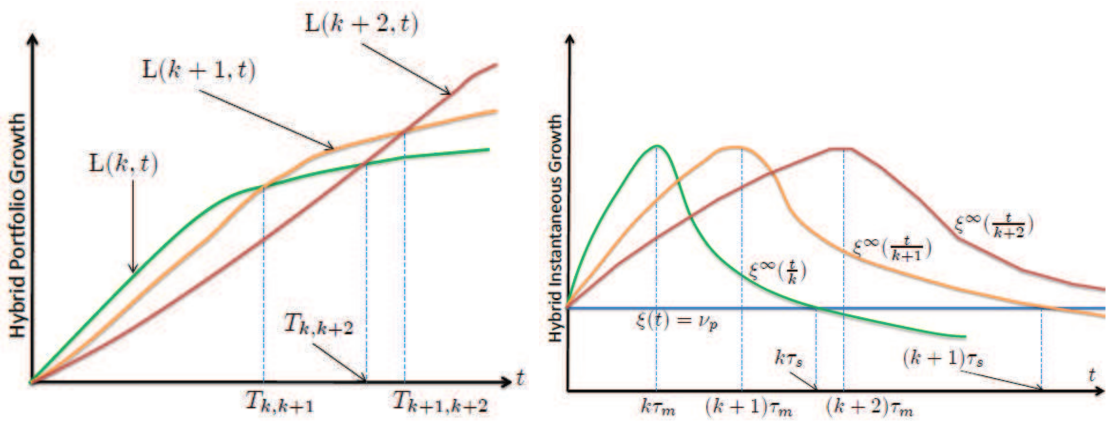
6.3 Rebalance Divisor Optimality

Comparing equation 6.1 and 6.19, we see that we have significantly reduced the search space. In this section we will explore the possibility to completely avoid searching for the ORF. The first step in this direction is to analyze the nature of the equation 6.19.

6.3.1 Log Utility Rebalance Contour (LURC)

The hybrid portfolio evolution is governed by the function $k\chi^{\infty}(\frac{t}{k})$. The investor has a finite choice of such evolution paths or contours, one for each possible value of k between k_{mn} and k_{mx}. For any horizon T, τ_o is determined by the rebalance divisor k defining the contour that yields the maximum ELPV for $t = T$[1]. We describe each such contour as *log utility rebalance contour (LURC)*. The function $\text{Ł}: (\mathbb{N}^+, \mathbb{R}^+) \mapsto \mathbb{R}^+$ defines time evolution

[1]In subsequent discussions we use both t and T interchangeably. We prefer to use t when the horizon value is used in the context of a variable of a function.



(a) (b)

Figure 6.3 Illustrations of LURC, inflection point and EIPG of LURC.

of the $\boldsymbol{k^{th}}$ LURC as follows:

$$\text{Ł}(\boldsymbol{k}, \boldsymbol{t}) = \boldsymbol{k}\boldsymbol{\chi}^{\infty}\Big(\frac{\boldsymbol{t}}{\boldsymbol{k}}\Big), \forall \boldsymbol{k} \in \mathbb{N}^{+}, \forall \boldsymbol{t} \in \mathbb{R}^{+} \tag{6.20}$$

The LURC function $\text{Ł}(\boldsymbol{k}, \boldsymbol{t})$ defines evolution of ELPV if the investor adopts a hybrid strategy rebalancing after every $\frac{\boldsymbol{t}}{\boldsymbol{k}}$ time interval. Due to the results of the lemmas 17 and 18, for a given horizon, the investor needs to adopt a hybrid strategy that corresponds to one of the finite possible LURCs to optimize his log utility.

The functional notation in general describes all the three strategies we have discussed so far. $\text{Ł}(\boldsymbol{0}, \boldsymbol{t})$ and $\text{Ł}(\boldsymbol{1}, \boldsymbol{t})$ describe the ELPV under active and passive strategies respectively. The following are some generic equivalent notations:

$$\text{Ł}(\boldsymbol{0}, \boldsymbol{t}) = \boldsymbol{\chi}^{\boldsymbol{0}}(\boldsymbol{t}) = \boldsymbol{\chi}(\boldsymbol{t}) \tag{6.21a}$$

$$\text{Ł}(\boldsymbol{1}, \boldsymbol{t}) = \boldsymbol{\chi}^{\boldsymbol{t}}(\boldsymbol{t}) = \boldsymbol{\chi}^{\infty}(\boldsymbol{t}) \tag{6.21b}$$

$$\text{Ł}(\boldsymbol{k}, \boldsymbol{t}) = \boldsymbol{\chi}^{\frac{\boldsymbol{t}}{\boldsymbol{k}}}(\boldsymbol{t}) = \boldsymbol{k}\boldsymbol{\chi}^{\infty}\Big(\frac{\boldsymbol{t}}{\boldsymbol{k}}\Big) \tag{6.21c}$$

Given a horizon $\boldsymbol{t}$ there are only finite such LURCs we need to consider corresponding to all possible rebalance divisors $\boldsymbol{k}$ used in defining $\Im_{\boldsymbol{k}}(\boldsymbol{t})$. Figure 6.3(a) illustrates conceptual

LURCs for three values of rebalance divisors $\boldsymbol{k}$, $\boldsymbol{k+1}$ and $\boldsymbol{k+2}$. Before we proceed we will derive a few important properties of LURC.

Lemma 19. *The EIPG of the $\boldsymbol{k^{th}}$ LURC is given by:*

$$\frac{\partial L(\boldsymbol{k}, \boldsymbol{t})}{\partial \boldsymbol{t}} = \boldsymbol{\xi}^{\infty}\left(\frac{\boldsymbol{t}}{\boldsymbol{k}}\right) \tag{6.22}$$

Proof. The proof is straight forward. Differentiating equation 6.20 with respect $\boldsymbol{t}$ gives us the following:

$$\frac{\partial \mathrm{L}(\boldsymbol{k}, \boldsymbol{t})}{\partial \boldsymbol{t}} = \boldsymbol{k} \frac{\partial \boldsymbol{\chi}^{\infty}(\frac{t}{k})}{\partial (\frac{t}{k})} \frac{\partial (\frac{t}{k})}{\partial \boldsymbol{t}} = \boldsymbol{\xi}^{\infty}\left(\frac{\boldsymbol{t}}{\boldsymbol{k}}\right) \tag{6.23}$$

□

Let's look at the nature of EIPG evolution for various LURCs. The first LURC with $\boldsymbol{k = 1}$ always evolves following the passive portfolio growth pattern. As we increase the value of $\boldsymbol{k}$, EIPG of LURC becomes increasingly flatter as illustrated in figure 6.3(b). EIPG of all LURCs have the same maximum value of $\boldsymbol{\xi}^{\infty}(\boldsymbol{\tau_m})$ although occurring at different time points. The EIPG of $\boldsymbol{k^{th}}$ LURC maximizes at $\boldsymbol{k\tau_m}$. Note that for a given time interval, the area under the EIPG of $\boldsymbol{k^{th}}$ LURC calculates the change in ELPV during the interval. Because of the EIPG asymmetry, two different LURCs will have different levels of performance in generating ELPV for different lengths of horizon. As an example, referring to the figure 6.3(a), after horizon $\boldsymbol{T_{k,k+1}}$, $\boldsymbol{(k+1)^{th}}$ LURC surpasses $\boldsymbol{k^{th}}$ LURC in performance to generate higher ELPV.

6.3.2 Inflection Point

We describe the horizon at which two LURCs intersect as a *inflection point*. At any inflection point one LURC's performance surpasses the performance of another LURC. We will denote the inflection point of two different LURCs for $\boldsymbol{k^{th}}$ and $\boldsymbol{k'^{th}}$ as $\boldsymbol{T_{k,k'}}$. As an illustration figure 6.3(a) shows three different inflection points generated by $\boldsymbol{k^{th}}$, $\boldsymbol{(k+1)^{th}}$ and $\boldsymbol{(k+2)^{th}}$ LURCs. We describe $\boldsymbol{T_{k,k+a}}, \boldsymbol{a} \in \mathbb{N}^{+}$ as the $\boldsymbol{a^{th}}$ inflection point for the $\boldsymbol{k^{th}}$ LURC.

For example $\boldsymbol{T_{k,k+1}}$, $\boldsymbol{T_{k,k+2}}$ and $\boldsymbol{T_{k,k+3}}$ are the first, second and third inflection points respectively for the $\boldsymbol{k^{th}}$ LURC. Note that by definition $\boldsymbol{k'^{th}}$ inflection point of $\boldsymbol{k^{th}}$ LURC is same as $\boldsymbol{k^{th}}$ inflection point of $\boldsymbol{k'^{th}}$ LURC. Using these notations, $\boldsymbol{T_{k,k'} = T_{k',k}}$, $\forall \boldsymbol{k \neq k', \{k,k\}'} \in \mathbb{N}^{+}$. As a convention, we prefer to use $\boldsymbol{T_{k,k'}}$ where $\boldsymbol{k < k'}$ to denote the inflection point of $\boldsymbol{k^{th}}$ and $\boldsymbol{k'^{th}}$ LURCs.

Lemma 20. *An inflection point shall have a lower bound given by:*

$$\boldsymbol{T_{k,k+a} > (k+a)\tau_m}, \ \forall \boldsymbol{a} > \in \mathbb{N}^{+} \tag{6.24}$$

Proof. For simplicity of notation, we will use $\boldsymbol{T_{ka} = T_{k,k+a}}$. We start with the definition of $\boldsymbol{T_{k,k+a}}$.

$$\begin{aligned} & \text{Ł}(\boldsymbol{k, T_{ka}}) = \text{Ł}(\boldsymbol{k+a, T_{ka}}) \\ \Rightarrow & \boldsymbol{k\chi^{\infty}\Big(\frac{T_{ka}}{k}\Big) = (k+a)\chi^{\infty}\Big(\frac{T_{ka}}{k+a}\Big)} \end{aligned} \tag{6.25}$$

Using lemma 19 we know that EIPG for $\text{Ł}(\boldsymbol{k,t})$ is increasing and peaks at $\boldsymbol{k\tau_m}$. Thus $\text{Ł}(\boldsymbol{k,t})$ has higher EIPG than that of $\text{Ł}(\boldsymbol{k+a,t})$ when $\boldsymbol{t \leq k\tau_m}$. Hence $\boldsymbol{T_{ka} > k\tau_m}$ since $\text{Ł}(\boldsymbol{k,t})$ will not intersect $\text{Ł}(\boldsymbol{k+a,t})$ otherwise. Therefore,

$$\boldsymbol{T_{ka} > k\tau_m \Rightarrow \frac{T_{ka}}{k} > \tau_m}$$

$$\Rightarrow \boldsymbol{\frac{T_{ka}}{k} = \tau_m + \Delta t_1}, \text{ for some } \boldsymbol{\Delta t_1 > 0} \tag{6.26a}$$

$$\Rightarrow \boldsymbol{T_{ka} = k(\tau_m + \Delta t_1)} \tag{6.26b}$$

We will prove the lemma's proposition by contradiction. Suppose the proposition is not true, i.e.

$$\boldsymbol{T_{ka} \leq (k+a)\tau_m \Rightarrow \frac{T_{ka}}{k+a} \leq \tau_m}$$

$$\Rightarrow \boldsymbol{\frac{T_{ka}}{k+a} = \tau_m - \Delta t_2}, \text{ for some } \boldsymbol{\Delta t_2 \geq 0} \tag{6.27a}$$

$$\Rightarrow \boldsymbol{T_{ka} = (k+a)(\tau_m - \Delta t_2)} \tag{6.27b}$$

Substituting equations 6.26a and 6.27a in equation 6.25, we obtain:

$$\boldsymbol{k\chi^{\infty}(\tau_m + \Delta t_1) = (k+a)\chi^{\infty}(\tau_m - \Delta t_2)} \tag{6.28}$$

Furthermore, from equations 6.26b and 6.27b, we obtain:

$$\boldsymbol{k(\tau_m + \Delta t_1) = (k+a)(\tau_m - \Delta t_2)} \tag{6.29}$$

Both equations 6.28 and 6.29 can only be true when passive portfolio evolution function, $\chi^\infty(t)$ is linear and takes the form of $\chi^\infty(t) = ct$ for any constant c. However, we know from equation 3.18 that $\chi^\infty(t)$ is not linear. This contradiction shows that the supposition is false and so the given proposition of this lemma is true. □

Lemma 21. *Let $T_{k,k+a}$ be the a^{th} inflection point for the k^{th} LURC where $\{k,a\} \in \mathbb{N}^+$. The relative performance of the k^{th} and $(k+a)^{th}$ LURCs shall meet the following constraints:*

$$Ł(k,t) > Ł(k+a,t), \ \forall 0 < t < T_{k,k+a} \tag{6.30a}$$
$$Ł(k,t) = Ł(k+a,t), \ \forall t = T_{k,k+a} \tag{6.30b}$$
$$Ł(k,t) < Ł(k+a,t), \ \forall t > T_{k,k+a} \tag{6.30c}$$

Proof. We know that for initial investment of one dollar, all LURCs start with zero ELPV. From lemma 19, for any horizon t, the EIPG for k^{th} and $(k+a)^{th}$ LURCs shall be given by $\xi^\infty(\frac{t}{k})$ and $\xi^\infty(\frac{t}{k+a})$ respectively. From the unimodality assumption of EIPG, $\xi^\infty(\frac{t}{k})$ will be increasing till $t = k\tau_m$. Observe that during the interval of $(0\ k\tau_m]$, $\xi^\infty(\frac{t}{k+a})$ is also increasing, albeit at a slower rate. During $(0\ k\tau_m]$, due to higher EIPG, k^{th} LURC will have higher ELPV than $(k+a)^{th}$ LURC. Since a LURC is monotonically increasing k^{th} LURC will eventually catch up with $(k+a)^{th}$ LURC at $T_{k,k+a}$. This proves that equation 6.30a holds.

Equation 6.30b holds from the definition of inflection point $T_{k,k+a}$.

Using the results of lemma 20, the following relationship shall always be satisfied:

$$\begin{aligned}&\frac{T_{k,k+a}}{k} > \frac{T_{k,k+a}}{k+a} > \tau_m\\ \Rightarrow &\xi^\infty\Big(\frac{T_{k,k+a}}{k}\Big) < \xi^\infty\Big(\frac{T_{k,k+a}}{k+a}\Big)\end{aligned} \tag{6.31}$$

Equation 6.31 holds since both $\frac{T_{k,k+a}}{k}$ and $\frac{T_{k,k+a}}{k+a}$ fall on the decreasing part of passive EIPG curve ξ^∞. Hence at $T_{k,k+a}$, $(k+a)^{th}$ LURC will have higher EIPG than k^{th} LURC. Note that at $T_{k,k+a}$ both the LURCs yield equal ELPV. But due to higher EIPG at $T_{k,k+a}$, for $t > T_{k,k+a}$, $(k+a)^{th}$ LURC will remain higher than k^{th} LURC satisfying equation 6.30c. □

Lemma 22. *Let $T_{k,k+b}$ be the b^{th} inflection point for the k^{th} LURC. At $T_{k,k+b}$, k^{th} LURC shall have higher ELPV than higher order LURCs. Mathematically,*

$$Ł(k, T_{k,k+b}) > Ł(k+a, T_{k,k+b}), \ \forall a > b, \ \{k,a,b\} \in \mathbb{N}^+ \tag{6.32}$$

Proof. For notational simplicity we will use T_{kb} to denote $T_{k,k+b}$. By the definition of inflection point we can write:

$$\begin{aligned}&k\chi^\infty\Big(\frac{T_{kb}}{k}\Big) = (k+b)\chi^\infty\Big(\frac{T_{kb}}{k+b}\Big)\\ \Rightarrow &\chi^\infty\Big(\frac{T_{kb}}{k}\Big) = \frac{(k+b)}{b}\Big[\chi^\infty\Big(\frac{T_{kb}}{k}\Big) - \chi^\infty\Big(\frac{T_{kb}}{k+b}\Big)\Big] = \frac{(k+b)}{b}\Delta\chi_k^{k+b}\end{aligned} \tag{6.33}$$

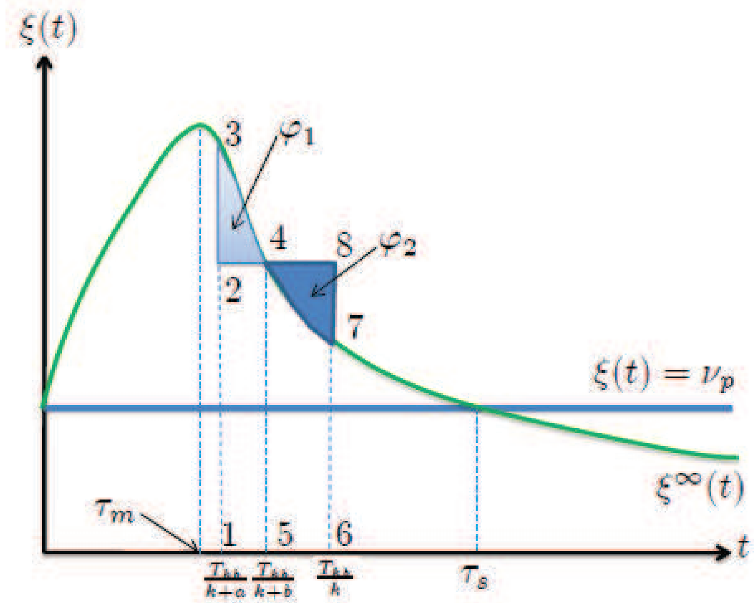


Figure 6.4 Illustration for proof of lemma 22.

Where, using the illustration in figure 6.4, $\boldsymbol{\Delta\chi_k^{k+b}}$ is given by:

$$\boldsymbol{\Delta\chi_k^{k+b}} = \boldsymbol{\chi^\infty\Big(\frac{T_{kb}}{k}\Big) - \chi^\infty\Big(\frac{T_{kb}}{k+b}\Big)}$$
$$= \boldsymbol{\int_0^{\frac{T_{kb}}{k}} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+b}} \xi^\infty(t)\,\mathrm{d}t} \tag{6.34}$$

Figure 6.4 illustrates the positions of different horizon points, viz. $\boldsymbol{\frac{T_{kb}}{k}}$, $\boldsymbol{\frac{T_{kb}}{k+b}}$ and $\boldsymbol{\frac{T_{kb}}{k+a}}$. Note that due to the lower and upper bounds set by lemmas 20 and 26, the following relationship between the horizon points holds:

$$\boldsymbol{\tau_s > \frac{T_{kb}}{k} > \frac{T_{kb}}{k+b} > \tau_m} \tag{6.35}$$

The horizon point $\boldsymbol{\frac{T_{kb}}{k+a}}$ may lie anywhere within $\boldsymbol{(0\ \frac{T_{kb}}{k+b}]}$. This figure places these time points on the right hand side of $\boldsymbol{\tau_m}$ representing the worst case scenario that one needs to prove.

Expanding equation 6.32, we need to prove the following:

$$\boldsymbol{k\chi^\infty\Big(\frac{T_{kb}}{k}\Big) > (k+a)\chi^\infty\Big(\frac{T_{kb}}{k+a}\Big)}$$
$$\boldsymbol{\Rightarrow k\Big[\chi^\infty\Big(\frac{T_{kb}}{k}\Big) - \chi^\infty\Big(\frac{T_{kb}}{k+a}\Big)\Big] > a\chi^\infty\Big(\frac{T_{kb}}{k+a}\Big)}$$
$$\boldsymbol{\Rightarrow k\Delta\chi_k^{k+a} > a\Big[\chi^\infty\Big(\frac{T_{kb}}{k}\Big) - \Delta\chi_k^{k+a}\Big]}$$
$$\boldsymbol{\Rightarrow (k+a)\Delta\chi_k^{k+a} > a\chi^\infty\Big(\frac{T_{kb}}{k}\Big)} \tag{6.36}$$

Where, using the illustration in figure 6.4, $\Delta\chi_k^{k+a}$ is given by:

$$\begin{aligned}\Delta\chi_k^{k+a} &= \chi^\infty\Big(\frac{T_{kb}}{k}\Big) - \chi^\infty\Big(\frac{T_{kb}}{k+a}\Big)\\ &= \int_0^{\frac{T_{kb}}{k}} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^\infty(t)\,\mathrm{d}t \end{aligned} \tag{6.37}$$

Substituting the value of $\chi^\infty\Big(\frac{T_{kb}}{k}\Big)$ from equation 6.33 in equation 6.36, we need to prove the following:

$$\begin{aligned}
&(k+a)\Delta\chi_k^{k+a} > \frac{a(k+b)}{b}\Delta\chi_k^{k+b}\\
\Rightarrow&(k+a)[\Delta\chi_k^{k+a} - \Delta\chi_k^{k+b}] > \frac{k(a-b)}{b}\Delta\chi_k^{k+b}\\
\Rightarrow&(k+a)\Big[\int_0^{\frac{T_{kb}}{k}} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^\infty(t)\,\mathrm{d}t\\
&-\int_0^{\frac{T_{kb}}{k}} \xi^\infty(t)\,\mathrm{d}t + \int_0^{\frac{T_{kb}}{k+b}} \xi^\infty(t)\,\mathrm{d}t\Big] > \frac{k(a-b)}{b}\Delta\chi_k^{k+b}, \text{ using equations 6.34 and 6.37}\\
\Rightarrow&(k+a)\Big[\int_0^{\frac{T_{kb}}{k+b}} \xi^\infty(t)\,\mathrm{d}t - \int_0^{\frac{T_{kb}}{k+a}} \xi^\infty(t)\,\mathrm{d}t\Big] > \frac{k(a-b)}{b}\Delta\chi_k^{k+b}\\
\Rightarrow&(k+a)\Delta\chi_{k+b}^{k+a} > \frac{k(a-b)}{b}\Delta\chi_k^{k+b}
\end{aligned} \tag{6.38}$$

We can simplify LHS of equation 6.38 below:

$$\begin{aligned}
&(k+a)\Delta\chi_{k+b}^{k+a}\\
=&(k+a)(\text{Area of region 1-2-3-4-5})\\
=&(k+a)[(\text{Area of region 1-2-4-5}) + (\text{Area of region 2-3-4})]\\
=&(k+a)\Big[\Big(\frac{T_{kb}}{k+b} - \frac{T_{kb}}{k+a}\Big)\xi^\infty\Big(\frac{T_{kb}}{k+b}\Big) + \varphi_1\Big]\\
=&(k+a)\Big[\frac{(a-b)T_{kb}}{(k+b)(k+a)}\xi^\infty\Big(\frac{T_{kb}}{k+b}\Big) + \varphi_1\Big]\\
=&\frac{(a-b)T_{kb}}{(k+b)}\xi^\infty\Big(\frac{T_{kb}}{k+b}\Big) + (k+a)\varphi_1
\end{aligned} \tag{6.39}$$

We can simplify RHS of equation 6.38 below:

$$\begin{aligned}
&\frac{k(a-b)}{b}\Delta\chi_k^{k+b}\\
=&\frac{k(a-b)}{b}(\text{Area of region 4-5-6-7})\\
=&\frac{k(a-b)}{b}[(\text{Area of region 4-5-6-7-8}) - (\text{Area of region 4-7-8})]\\
=&\frac{k(a-b)}{b}\Big[\Big(\frac{T_{kb}}{k}-\frac{T_{kb}}{k+b}\Big)\xi^{\infty}\Big(\frac{T_{kb}}{k+b}\Big)-\varphi_2\Big]\\
=&\frac{k(a-b)}{b}\Big[\frac{bT_{kb}}{k(k+b)}\xi^{\infty}\Big(\frac{T_{kb}}{k+b}\Big)-\varphi_2\Big]\\
=&\frac{(a-b)T_{kb}}{(k+b)}\xi^{\infty}\Big(\frac{T_{kb}}{k+b}\Big)-\frac{k(a-b)}{b}\varphi_2
\end{aligned} \tag{6.40}$$

Since $\varphi_1, \varphi_2 > 0$, $a > b$ and $k \geq 1$, we see that the terms $(k+a)\varphi_1$ and $\frac{k(a-b)}{b}\varphi_2$ are both positive. Comparing equations 6.39 and 6.40 we conclude that equation 6.38 holds true. □

Lemma 23. *Higher order inflection points of a LURC are always longer. Equivalently if $T_{k,k+a}$ and $T_{k,k+b}$ be the a^{th} and b^{th} reflection points respectively for the k^{th} LURC, then the following must be true:*

$$T_{k,k+a} > T_{k,k+b},\ \forall a > b,\ \{k,a,b\} \in \mathbb{N}^+ \tag{6.41}$$

Proof. We will prove this proposition by contradiction. Suppose the proposition of this lemma is not true. Then either $T_{k,k+a} < T_{k,k+b}$ or $T_{k,k+a} = T_{k,k+b}$. Let's first suppose $T_{k,k+a} < T_{k,k+b}$.

Since $a > b$, from lemma 22 we can write:

$$Ł(k, T_{k,k+b}) > Ł(k+a, T_{k,k+b}) \tag{6.42}$$

Using the results of lemma 21, from equation 6.30c we obtain:

$$Ł(k, t) < Ł(k+a, t),\ \forall t > T_{k,k+a} \tag{6.43}$$

Then under the assumption that $T_{k,k+b} > T_{k,k+a}$, the following must be true:

$$Ł(k, T_{k,k+b}) < Ł(k+a, T_{k,k+b}) \tag{6.44}$$

This contradiction in equations 6.42 and 6.44 shows that the supposition $T_{k,k+a} < T_{k,k+b}$ is false.

Let's suppose $T_{k,k+a} = T_{k,k+b}$. Then by fundamental definition of inflection point, at $T_{k,k+b}$ all three LURCs, viz. k^{th}, $(k+a)^{th}$ and $(k+b)^{th}$ shall have identical ELPV. Mathematically,

$$Ł(k, T_{k,k+b}) = Ł(k+a, T_{k,k+b}) = Ł(k+b, T_{k,k+b}) \tag{6.45}$$

Once again we arrive at contradictory results between equation 6.42 and 6.45. Therefore the supposition $\boldsymbol{T_{k,k+a} = T_{k,k+b}}$ is also false. Thus the given proposition of this lemma is true. □

Lemma 24. *LURCs with higher rebalance divisor shall have longer inflection point of the same order. Equivalently if* $\boldsymbol{T_{k-a,k}}$ *and* $\boldsymbol{T_{k-b,k}}$ *be the* $\boldsymbol{k^{th}}$ *order reflection points for* $\boldsymbol{(k-a)^{th}}$ *and* $\boldsymbol{(k-b)^{th}}$ *LURCs respectively, then the following must be true:*

$$\boldsymbol{T_{k-a,k} < T_{k-b,k}},\ \forall \boldsymbol{k > a > b},\ \{k,a,b\} \in \mathbb{N}^{+} \tag{6.46}$$

Proof. Using the results of lemma 23 we know that the following must be true:

$$\boldsymbol{T_{k-a,k-b} < T_{k-a,k}} \tag{6.47}$$

Using the results of lemma 21 and equation 6.30c we also know that the following must be true:

$$Ł(\boldsymbol{k-a,t}) < Ł(\boldsymbol{k-b,t}),\ \forall \boldsymbol{t > T_{k-a,k-b}} \tag{6.48}$$

From equations 6.47 and 6.48 we obtain:

$$Ł(\boldsymbol{k-a,T_{k-a,k}}) < Ł(\boldsymbol{k-b,T_{k-a,k}}) \tag{6.49}$$

By definition $Ł(\boldsymbol{k-a,T_{k-a,k}}) = Ł(\boldsymbol{k,T_{k-a,k}})$, i.e. at $\boldsymbol{T_{k-a,k}}$ the ELPV for the $\boldsymbol{k^{th}}$ and $\boldsymbol{(k-a)^{th}}$ LURCs are identical. Therefore,

$$Ł(\boldsymbol{k,T_{k-a,k}}) < Ł(\boldsymbol{k-b,T_{k-a,k}}) \tag{6.50}$$

Again using the results of lemma 21 and equation 6.30a we know that the following must be true:

$$Ł(\boldsymbol{k-b,t}) > Ł(\boldsymbol{k,t}),\ \forall \boldsymbol{t < T_{k-b,k}} \tag{6.51}$$

Comparing equations 6.50 and 6.51, we conclude that the following relationship must hold true:

$$\boldsymbol{T_{k-a,k} < T_{k-b,k}} \tag{6.52}$$

□

We now state and prove *inflection points seriality* theorem. Theorem 6 states that a given LURC's inflection points increase as the intersecting LURC's rebalance divisor increases. For example the inflection points of the LURC with $\boldsymbol{k = 4}$ shall maintain an increasing sequence of $\boldsymbol{T_{1,4} < T_{2,4} < T_{3,4} < T_{4,5} < T_{4,6} < \dots}$.

Theorem 6. *A LURC has distinct and increasing inflection points, i.e.*

$$\boldsymbol{T_{k,k_1} > T_{k,k_2},\ \forall k_1 > k_2,\ k \neq k_1 \neq k_2,\ \{k, k_1, k_2\} \in \mathbb{N}^+} \tag{6.53}$$

Proof. Note that $\boldsymbol{T_{k,k}}$ is not defined since an inflection point has to involve two different LURCs. There are two cases we need to consider. First case is when $\boldsymbol{k > k_1 > k_2}$. For such cases, according to lemma 24 $\boldsymbol{T_{k_1,k} > T_{k_2,k}}$ which is equivalent to $\boldsymbol{T_{k,k_1} > T_{k,k_2}}$.

The second case is when $\boldsymbol{k_1 > k_2 > k}$. For such cases, according to lemma 23 $\boldsymbol{T_{k,k_1} > T_{k,k_2}}$. To complete the proof we need to show that the maximum inflection point for first case is less than the minimum inflection point for the second case. In other words, we need to prove the following:

$$\boldsymbol{T_{k-1,k} < T_{k,k+1}} \tag{6.54}$$

We will prove the proposition in equation 6.54 by contradiction. Suppose the proposition is not true, i.e. one of the following two equations must hold:

$$\boldsymbol{T_{k-1,k} > T_{k,k+1}} \tag{6.55a}$$

$$\boldsymbol{T_{k-1,k} = T_{k,k+1}} \tag{6.55b}$$

According to lemma 22, $\boldsymbol{(k-1)^{th}}$ and $\boldsymbol{k^{th}}$ LURCs shall have higher ELPV than $\boldsymbol{(k+1)^{th}}$ LURC for investment horizon of $\boldsymbol{T_{k-1,k}}$. Mathematically,

$$\text{Ł}(\boldsymbol{k, T_{k-1,k}}) > \text{Ł}(\boldsymbol{k+1, T_{k-1,k}}) \tag{6.56}$$

From equation 6.30c of lemma 21, the following must hold:

$$\text{Ł}(\boldsymbol{k, t}) < \text{Ł}(\boldsymbol{k+1, t}),\ \boldsymbol{\forall t > T_{k,k+1}} \tag{6.57}$$

From equations 6.55a and 6.57, the following relationship must hold:

$$\text{Ł}(\boldsymbol{k, T_{k-1,k}}) < \text{Ł}(\boldsymbol{k+1, T_{k-1,k}}) \tag{6.58}$$

We observe that equation 6.58 contradicts equation 6.56. Therefore the proposition in equation 6.55a must be false.

Suppose equation 6.55b is true. Then substituting equation 6.55b in inequality 6.56 we obtain:

$$\text{Ł}(\boldsymbol{k, T_{k,k+1}}) > \text{Ł}(\boldsymbol{k+1, T_{k,k+1}}) \tag{6.59}$$

Moreover from equation 6.30b of lemma 21, the following must hold:

$$\text{Ł}(\boldsymbol{k, T_{k,k+1}}) = \text{Ł}(\boldsymbol{k+1, T_{k,k+1}}) \tag{6.60}$$

Once again we arrive at contradiction in inequalities 6.59 and 6.60. Therefore equation 6.55b must also be false. Thus the relationship in equation 6.54 holds and hence we establish the proposition of this theorem. □

Lemma 25. *The $\boldsymbol{k^{th}}$ LURC shall have higher expected log utility than all other LURCs with lower rebalance divisor $\boldsymbol{k' < k}$ for any time horizon longer than inflection point $\boldsymbol{T_{k-1,k}}$. Equivalently, the following must hold true:*

$$Ł(\boldsymbol{k, t}) > Ł(\boldsymbol{k', t}),\ \forall \boldsymbol{t} > \boldsymbol{T_{k-1,k}},\ \boldsymbol{k > k'},\ \{\boldsymbol{k, k'}\} \in \mathbb{N}^{+} \tag{6.61}$$

Proof. We will use induction to prove this lemma. The base case is when $\boldsymbol{k = 2}$ with only permissible value of $\boldsymbol{k' = 1}$. We must prove that:

$$Ł(\boldsymbol{2, t}) > Ł(\boldsymbol{1, t}), \forall \boldsymbol{t} > \boldsymbol{T_{1,2}} \tag{6.62}$$

This is true due to the results of lemma 21 and equation 6.30c for $\boldsymbol{k = a = 1}$. Assume that the hypothesis in equation 6.61 holds for $\boldsymbol{k}$. We also know from theorem 6 that the following relationship holds for inflection points:

$$\boldsymbol{T_{k,k+1}} > \boldsymbol{T_{k-1,k}} \tag{6.63}$$

Using the above relationship in equation 6.63 we can rewrite slightly less restrictive form of equation 6.61 as below:

$$Ł(\boldsymbol{k, t}) > Ł(\boldsymbol{k', t}),\ \forall \boldsymbol{t} > \boldsymbol{T_{k,k+1}} \tag{6.64}$$

We must now prove that equation 6.61 holds for $\boldsymbol{k+1}$, i.e. the following must also be true:

$$Ł(\boldsymbol{k+1, t}) > Ł(\boldsymbol{k', t}),\ \forall \boldsymbol{t} > \boldsymbol{T_{k,k+1}},\ \boldsymbol{k+1 > k'} \tag{6.65}$$

Once again using equation 6.30c of lemma 21, for $\boldsymbol{a = 1}$ we obtain:

$$Ł(\boldsymbol{k+1, t}) > Ł(\boldsymbol{k, t}), \forall \boldsymbol{t} > \boldsymbol{T_{k,k+1}} \tag{6.66}$$

Equations 6.64 and 6.66 jointly imply that equation 6.65 is true. Thus we establish lemma 25. □

Thus far we have explored important properties of LURCs, rebalance divisors and inflection points. These set of properties will enable us to derive the maximum achievable ELPV with one of the permissible values of rebalance divisor. It turns out that the value of this optimum rebalance divisor depends on the desired investment horizon. We can divide the horizon into linear segments separated by predetermined inflection points. For each of the horizon segments an optimum rebalance divisor can be assigned that maximizes the ELPV for the horizon. Thus each non-overlapping horizon segment can be associated with a *distinct* optimum rebalance divisor.

We now state and prove the *rebalance divisor optimality* theorem.

Theorem 7. *The rebalance divisor $\boldsymbol{k}$ maximizes the ELPV for any investment horizon between $\boldsymbol{T_{k-1,k}}$ and $\boldsymbol{T_{k,k+1}}$. Mathematically,*

$$Ł(\boldsymbol{k},\boldsymbol{t}) \geq Ł(\boldsymbol{k'},\boldsymbol{t}),\ \forall \boldsymbol{t} \in (\boldsymbol{T_{k-1,k}}\ \boldsymbol{T_{k,k+1}}],\ \boldsymbol{k'} \neq \boldsymbol{k},\ \{\boldsymbol{k},\boldsymbol{k'}\} \in \mathbb{N}^+ \tag{6.67}$$

Proof. Combining equations 6.30a and 6.30b of lemma 21, we know that the following is true:

$$Ł(\boldsymbol{k},\boldsymbol{t}) \geq Ł(\boldsymbol{k_h},\boldsymbol{t}),\ \forall \boldsymbol{t} \in (\boldsymbol{0}\ \boldsymbol{T_{k,k_h}}],\ \boldsymbol{k_h} > \boldsymbol{k},\ \{\boldsymbol{k},\boldsymbol{k_h}\} \in \mathbb{N}^+ \tag{6.68}$$

Furthermore using lemma 25, we obtain:

$$Ł(\boldsymbol{k},\boldsymbol{t}) > Ł(\boldsymbol{k_l},\boldsymbol{t}),\ \forall \boldsymbol{t} > \boldsymbol{T_{k-1,k}},\ \boldsymbol{k} > \boldsymbol{k_l},\ \{\boldsymbol{k},\boldsymbol{k_l}\} \in \mathbb{N}^+ \tag{6.69}$$

However from inflection point seriality theorem 6, we know that the following order of inflection points holds:

$$\boldsymbol{T_{k,k_h}} \geq \boldsymbol{T_{k-1,k}},\ \forall \boldsymbol{k_h} > \boldsymbol{k} \tag{6.70a}$$

$$\boldsymbol{T_{k-1,k}} > \boldsymbol{T_{k,k+1}},\ \forall \boldsymbol{k_h} > \boldsymbol{k} \tag{6.70b}$$

Using the relationship of inequality 6.70a, a less restrictive form of inequality 6.68 is as follows:

$$Ł(\boldsymbol{k},\boldsymbol{t}) \geq Ł(\boldsymbol{k_h},\boldsymbol{t}),\ \forall \boldsymbol{t} \in (\boldsymbol{0}\ \boldsymbol{T_{k,k+1}}] \tag{6.71}$$

Similarly, using the relationship of inequality 6.70b, a less restrictive form of inequality 6.69 is as follows:

$$Ł(\boldsymbol{k},\boldsymbol{t}) > Ł(\boldsymbol{k_l},\boldsymbol{t}),\ \forall \boldsymbol{t} \in (\boldsymbol{T_{k-1,k}}\ \boldsymbol{T_{k,k+1}}] \tag{6.72}$$

Equations 6.71 and 6.72 jointly imply the validity of the hypothesis in equation 6.67. □

6.3.3 Rebalance Inflection Point (RIP)

Figure 6.5 illustrates a subset of possible LURCs with rebalance divisors $\boldsymbol{k-1}$ through $\boldsymbol{k+2}$. These LURCs participate in determining three distinct inflection points, viz. $\boldsymbol{T_{k-1,k}}$, $\boldsymbol{T_{k,k+1}}$ and $\boldsymbol{T_{k+1,k+2}}$. As per theorem 7, the optimal rebalance divisor for any investment horizon between $\boldsymbol{T_{k-1,k}}$ and $\boldsymbol{T_{k,k+1}}$ shall be $\boldsymbol{k}$. Therefore the maximum possible ELPV shall be determined by $\boldsymbol{k^{th}}$ LURC as depicted by the bold uppermost segment during this horizon interval. Similarly the optimal rebalance divisor for any investment horizon between $\boldsymbol{T_{k,k+1}}$

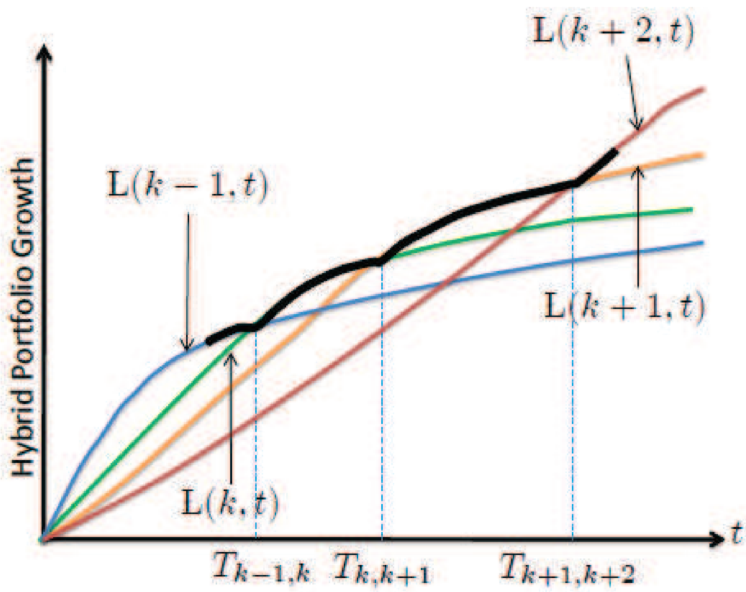


Figure 6.5 Rebalance Inflection Point (RIP) and Optimal Log Utility Frontier (OLUF) (in dark bold line).

and $\boldsymbol{T_{k+1,k+2}}$ shall be $\boldsymbol{k+1}$. In this case $\boldsymbol{(k+1)^{th}}$ LURC shall determine the maximum possible ELPV traced in bold.

The inflection points of interest here are the ones which are generated by two adjacent LURCs. We term such a special inflection point $\boldsymbol{T_{k,k+1}}$, $\forall \boldsymbol{k} \in \mathbb{N}^{+}$ as *rebalance inflection point (RIP)*. For completeness we assume $\boldsymbol{T_{0,1}}$ as the zeroth RIP with a value of $\mathbf{0}$. For brevity of notation, henceforth we will drop the second subscript for specifying a RIP. Thus $\boldsymbol{T_k}$ denotes the $\boldsymbol{k^{th}}$ RIP equivalent to the expanded notation of $\boldsymbol{T_{k,k+1}}$. In this parlance, $\mathbf{0}$ is the zeroth RIP, $\boldsymbol{T_1}$ is the first RIP and so on.

By virtue of theorem 7 the entire investment horizon axis can be divided into piecewise intervals by the series of RIPs, $\{\boldsymbol{T_1, T_2, T_3, T_4, T_5 \ldots}\}$ with associated optimum rebalance divisors as $\{\mathbf{1, 2, 3, 4, 5, \ldots}\}$.

From the results of lemma 20, we have already seen that the $\boldsymbol{k^{th}}$ RIP has a lower bound as follows:

$$\boldsymbol{(k+1)\tau_m < T_k}, \ \forall \boldsymbol{k} \in \mathbb{N}^{+} \tag{6.73}$$

We now show that the RIPs also have an upper bound derived in lemma 26.

Lemma 26. *The* $\boldsymbol{k^{th}}$ *RIP shall have an upper bound of* $\boldsymbol{k\tau_s}$, *i.e.*

$$\boldsymbol{T_k < k\tau_s,\ \forall k \in \mathbb{N}^+} \tag{6.74}$$

Proof. From the results of lemma 21 we know that for all values of $\boldsymbol{t > T_k}$, the value of $\text{Ł}(\boldsymbol{k+1, t})$ exceeds $\text{Ł}(\boldsymbol{k, t})$ and vice versa. Then to prove that $\boldsymbol{T_k < k\tau_s}$ it is suffice to show the following:

$$\begin{aligned}
&\text{Ł}(\boldsymbol{k+1, k\tau_s}) > \text{Ł}(\boldsymbol{k, k\tau_s}) \\
\Rightarrow&(\boldsymbol{k+1})\boldsymbol{\chi^\infty}(\frac{\boldsymbol{k\tau_s}}{\boldsymbol{k+1}}) > \boldsymbol{k\chi^\infty}(\frac{\boldsymbol{k\tau_s}}{\boldsymbol{k}}) \\
\Rightarrow&(\boldsymbol{k+1})\boldsymbol{\chi^\infty}(\boldsymbol{\tau_s'}) > \boldsymbol{k\chi^\infty}(\boldsymbol{\tau_s})
\end{aligned} \tag{6.75}$$

We have substituted $\boldsymbol{\tau_s' = \frac{k}{k+1}}$ above. Note that $\boldsymbol{\tau_s' < \tau_s}$. We know from lemma 11 that $\boldsymbol{\psi^\infty(t)}$ $(= \boldsymbol{\chi^\infty(t) - \chi(t)})$, the excess growth produced by passive strategy is maximized at $\boldsymbol{\tau_s}$. Therefore, the following must hold true:

$$\begin{aligned}
&\boldsymbol{\psi^\infty(\tau_s) > \psi^\infty(\tau_s')} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) - \chi(\tau_s) > \chi^\infty(\tau_s') - \chi(\tau_s')} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) > \chi^\infty(\tau_s') + \chi(\tau_s) - \chi(\tau_s')} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) > \chi^\infty(\tau_s') + \nu_p\tau_s - \nu_p\tau_s'}, \text{ using 3.42} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) > \chi^\infty(\tau_s') + \nu_p(\tau_s - \tau_s')} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) > \chi^\infty(\tau_s') + \nu_p(\tau_s - \frac{k}{k+1}\tau_s)} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s) > \chi^\infty(\tau_s') + \nu_p\frac{\tau_s}{k+1}}
\end{aligned} \tag{6.76}$$

Substituting equation 6.76 in equation 6.75, it is suffice to show that:

$$\begin{aligned}
&\boldsymbol{(k+1)\chi^\infty(\tau_s') > k\Big(\chi^\infty(\tau_s') + \nu_p\frac{\tau_s}{k+1}\Big)} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s') > \nu_p\frac{k}{k+1}\tau_s} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s') > \nu_p\tau_s'} \\
\Rightarrow&\boldsymbol{\chi^\infty(\tau_s') > \chi(\tau_s')}
\end{aligned} \tag{6.77}$$

By definition of stable rebalancing for all $\boldsymbol{\tau_s' < \tau_s}$, ELPV will always be higher under passive strategy compared to active strategy. Hence the above equation 6.77 will always hold true. ☐

In prior section 6.2 we showed that the optimal rebalance divisor for a given horizon must be bounded by $\boldsymbol{k_{mn}}$ and $\boldsymbol{k_{mx}}$ given by equations 6.10 and 6.17 respectively. We must now examine that the choice of $\boldsymbol{k}$ as specified in rebalance divisor optimality theorem 7 conforms to these upper and lower bounds as well.

Lemma 27. *For any horizon* $\boldsymbol{T}$ *between RIPs* $\boldsymbol{T_{k-1}}$ *and* $\boldsymbol{T_k}$ *following must hold true:*

$$\boldsymbol{k} \geq \left\lceil \frac{\boldsymbol{T}}{\boldsymbol{\tau_s}} \right\rceil \tag{6.78a}$$

$$\boldsymbol{k} \leq \left\lfloor \frac{\boldsymbol{T}}{\boldsymbol{\tau_m}} \right\rfloor \tag{6.78b}$$

Proof. By definition of horizon $\boldsymbol{T}$ the following is true:

$$\boldsymbol{T} \leq \boldsymbol{T_k} \Rightarrow \frac{\boldsymbol{T}}{\boldsymbol{\tau_s}} \leq \frac{\boldsymbol{T_k}}{\boldsymbol{\tau_s}} \tag{6.79}$$

We can rewrite the inequality 6.74 as follows:

$$\boldsymbol{k} > \frac{\boldsymbol{T_k}}{\boldsymbol{\tau_s}} \tag{6.80}$$

Inequalities 6.79 and 6.80 together imply the following:

$$\boldsymbol{k} > \frac{\boldsymbol{T}}{\boldsymbol{\tau_s}} \tag{6.81}$$

Since $\boldsymbol{k}$ takes only positive integer values, inequality 6.81 implies inequality 6.78a. Again by definition of horizon $\boldsymbol{T}$ the following is true:

$$\boldsymbol{T} > \boldsymbol{T_{k-1}} \Rightarrow \frac{\boldsymbol{T}}{\boldsymbol{\tau_m}} > \frac{\boldsymbol{T_{k-1}}}{\boldsymbol{\tau_m}} \tag{6.82}$$

We can rewrite the inequality 6.73 as follows:

$$\boldsymbol{k} < \frac{\boldsymbol{T_{k-1}}}{\boldsymbol{\tau_m}} \tag{6.83}$$

Inequalities 6.82 and 6.83 together imply the following:

$$\boldsymbol{k} < \frac{\boldsymbol{T}}{\boldsymbol{\tau_m}} \tag{6.84}$$

Since $\boldsymbol{k}$ takes only positive integer values, inequality 6.84 implies inequality 6.78b. □

6.3.4 Optimal Rebalance Frequency (ORF) Function

The uppermost LURC between the two adjacent RIPs determines the maximum achievable ELPV following hybrid strategy. By combining these optimum contours for all the non-overlapping horizon segments we obtain the *optimal log utility frontier (OLUF)* traced in bold in figure 6.5. We can completely specify the OLUF, $Ł_{\boldsymbol{o}}(\boldsymbol{t})$ representing the maximum possible ELPV for all investment horizon $\boldsymbol{t} \in \mathbb{R}$ as follows:

$$Ł_{\boldsymbol{o}}(\boldsymbol{t}) = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{t} = \mathbf{0} \\ Ł(\boldsymbol{k}, \boldsymbol{t}) & \text{if } \boldsymbol{t} \in (\boldsymbol{T_{k-1,k}}\ \boldsymbol{T_{k,k+1}}],\ \forall \boldsymbol{k} \in \mathbb{N}^+ \end{cases} \tag{6.85}$$

Given the OLUF specification we can compute the ORF function to determine the ORF for any given horizon $\boldsymbol{t}$ as follows:

$$\boldsymbol{\tau_o}(\boldsymbol{t}) = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{t} = \mathbf{0} \\ \frac{\boldsymbol{t}}{\boldsymbol{k}} & \text{if } \boldsymbol{t} \in (\boldsymbol{T_{k-1,k}}\ \boldsymbol{T_{k,k+1}}],\ \forall \boldsymbol{k} \in \mathbb{N}^+ \end{cases} \tag{6.86}$$

Algorithm 12 is the final algorithm that outlines the steps to compute the ORF function. It starts with the value of 1 as the optimal rebalance divisor. As the horizon is increased, it checks to see if the next RIP is reached. If so, it increments the optimal rebalance divisor by 1. It also records the asymptotic ORF when the values of two consecutive RIPs are within a specified error tolerance. It stops to look for subsequent RIPs thereafter.

6.3.5 An Example

We use our familiar four-asset portfolio example to illustrate the concepts discussed so far. Table 6.1 presents the values of rebalance divisor, RIP, ORF at RIP and the error, i.e the deviation of rebalance frequency from the previous iteration. Note how the error diminish as we increase $\boldsymbol{k}$. This is due to the rebalance frequency convergence theorem 5. Given an error tolerance we can stop computing the RIP further since frequency $\frac{\boldsymbol{T_k}}{\boldsymbol{k}}$ can be approximated to the last computed value when the tolerance is reached.

Algorithm 12 ComputeORFfcn_Search_T_k

Require: μ,S,w,T,δt,N,ν_p,ϵ

$[\chi^{\infty}, \tau_s] \leftarrow ComputePassivePortfolio(\mu, \mathrm{S}, \mathrm{w}, T, \delta t, N, \nu_p)$
$m \leftarrow 1, \mathrm{T}[m] \leftarrow 0, \tau_o[m] \leftarrow 0, \chi^{\tau_o}[m] \leftarrow 0, k \leftarrow 1, \tau_{ao} \leftarrow -1$
for $t = \delta t$ to T by δt **do**
  $m \leftarrow m + 1, \mathrm{T}[m] \leftarrow t$
  if $\tau_{ao} = -1$ **then**
    if $k\chi^{\infty}(\lfloor \frac{t}{k\delta t} + 0.5 \rfloor) < (k+1)\chi^{\infty}(\lfloor \frac{t}{(k+1)\delta t} + 0.5 \rfloor)$ **then**
      $\mathrm{R}[k] \leftarrow t$ # found the next RIP
      if $k > 1$ **then**
        if $\| \frac{\mathrm{R}[k]}{k} - \frac{\mathrm{R}[k-1]}{k-1} \| \leq \epsilon$ **then**
          $\tau_{ao} \leftarrow \frac{t}{k}$ # asymptotic ORF reached
        end if
      end if
      $k \leftarrow k + 1$ # current optimal rebalance divisor
    end if
    $\tau_o[m] \leftarrow \frac{t}{k}, \chi^{\tau_o}[m] \leftarrow k\chi^{\infty}(\lfloor \frac{\tau_o[m]}{\delta t} + 0.5 \rfloor)$
  else
    $\tau_o[m] \leftarrow \tau_{ao}, \chi^{\tau_o}[m] \leftarrow \lfloor \frac{t}{\tau_{ao}} \rfloor \chi^{\infty}(\lfloor \frac{\tau_{ao}}{\delta t} + 0.5 \rfloor)$
  end if
end for
return $(\mathrm{T}, \tau_o, \chi^{\tau_o})$

Table 6.1 Rebalance Inflection Points (RIPs)

k	T_k	$\frac{T_k}{k}$	$Error$	k	T_k	$\frac{T_k}{k}$	$Error$
1	2.2916	2.2916	-	10	17.2725	1.7273	0.0086
2	4.0127	2.0063	0.2853	11	18.9221	1.7202	0.0071
3	5.6906	1.8969	0.1095	12	20.5713	1.7143	0.0059
4	7.3548	1.8387	0.0582	13	22.2202	1.7092	0.0050
5	9.0129	1.8026	0.0361	14	23.8690	1.7049	0.0043
6	10.6676	1.7779	0.0246	15	25.5175	1.7012	0.0038
7	12.3204	1.7601	0.0179	16	27.1659	1.6979	0.0033
8	13.9719	1.7465	0.0136	17	28.8142	1.6950	0.0029
9	15.6225	1.7358	0.0107	18	30.4624	1.6924	0.0026

Assume that the RIPs have been computed as in table 6.1. We will now illustrate how one determines the ORF for a specified investment horizon $\boldsymbol{T}$. As an example consider the specified horizon values in table 6.2. From table 6.1, we notice that the optimal rebalance divisor for any investment horizon from 0 to 2.2916 is 1. Therefore for $\boldsymbol{T = 1}$, the ORF $\boldsymbol{\tau_o = \frac{T}{k} = 1}$. Hence if the log-optimal investor desires to invest for 1 year, she should adhere to passive strategy without any rebalancing. With the passive strategy the investor shall have an ELPV of $\mathbf{0.3229}$. If instead the investor uses a lower rebalance frequency of $\mathbf{0.8}$, then the ELPV shall be lowered to $\mathbf{0.3161}$. Suppose the investor has a desire to invest till $\boldsymbol{T = 6}$ years. From table 6.1, the optimum rebalance divisor shall be 4 since $\boldsymbol{5.6906 < T < 7.3548}$ implying an ORF of $\mathbf{1.5}$. This will generate $\mathbf{2.0862}$ as the ELPV. A lower ($\boldsymbol{\tau_l}$) or a higher ($\boldsymbol{\tau_h}$) rebalancing frequency shall generate lower ELPV for this horizon. Similarly for $\mathbf{30}$ year horizon the optimum rebalance divisor and frequency are $\mathbf{18}$ and $\mathbf{1.67}$ respectively resulting in a maximum ELPV of $\mathbf{9.7991}$. This is the same value we have found earlier and is depicted in figure 4.4(a).

The last example investment horizon we consider is $\boldsymbol{T = 40}$. Let's assume that we will accept an ORF error threshold of $\boldsymbol{\epsilon = 0.0026}$. Using table 6.1, we will use the data for the highest RIP in the very last row. We have assumed that for this RIP the ORF $\boldsymbol{\frac{T_k}{k}}$ is very close to the asymptotic rebalance frequency $\boldsymbol{\tau_{ao}}$, i.e. $\boldsymbol{\tau_{ao} \approx \frac{T_k}{k}}$. Instead of computing more higher order RIPs, we merely impute the optimum rebalance divisor by using $\lfloor \boldsymbol{\frac{T}{\tau_{ao}}} \rfloor$. For $\boldsymbol{\epsilon = 0.0026}$, we can apply this imputation for all $\boldsymbol{T > 30.4624}$. For $\boldsymbol{T = 40}$, the imputation results in an optimum rebalance divisor of $\mathbf{23}$. Note that we would have obtained the same optimum divisor had we continued computing higher order RIP equal to or higher than $\boldsymbol{T = 40}$. We would have to compute six additional RIPs, viz. $\mathbf{32.1105, 33.7586, 35.4065, 37.0545, 38.7023}$ and $\mathbf{40.3502}$ corresponding to rebalance divisors of $\mathbf{19}$ through $\mathbf{24}$. This would have resulted an optimum rebalance divisor of $\mathbf{24}$ instead of the imputed value of $\mathbf{23}$.

Table 6.2 Investment Horizon and ORF

T	k_o	τ_o	χ^{τ_o}	τ_l	χ^{τ_l}	τ_h	χ^{τ_h}
1	1	1.00	0.3229	0.80	0.3161	-	-
6	4	1.50	2.0862	1.20	1.9479	1.75	1.8447
30	18	1.67	9.7991	1.58	9.7854	1.76	9.7516
40	24	1.67	13.0654	1.60	13.0390	1.74	13.0442

Appendix D.3 presents a series of Matlab functions to compute ORF function. Figure 6.6 presents the generated ORF values for various investment horizons. As expected, for sufficiently long horizon values ORF converges to the asymptotic value τ_{ao}. For relatively smaller horizon values ORF fluctuates around this asymptotic value. Finally, figure 6.7 presents the corresponding ELPV yields when the respective ORF is used to periodically rebalance the portfolio to the initial optimal weights. Note that for all investment horizon, hybrid strategy with discrete-time rebalancing outperforms long-optimal strategy with continuous rebalancing in generating higher ELPV.

6.4 Asymptotic Growth Rate

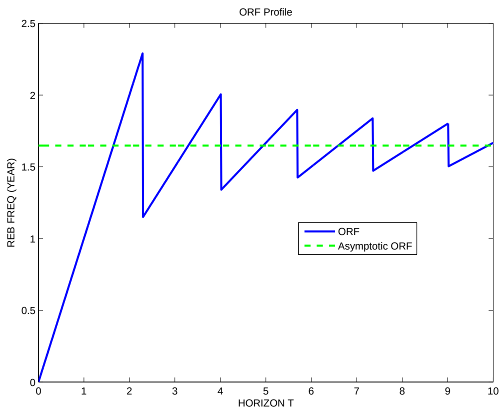
In the previous chapter we defined asymptotic optimal rebalance frequency τ_{ao} for long-term investors with infinite horizon, i.e. $T \to \infty$. Thus,

$$\lim_{T\to\infty} T(\tau_o) = \lim_{T\to\infty} \frac{T}{k} = \tau_{ao} \tag{6.87}$$

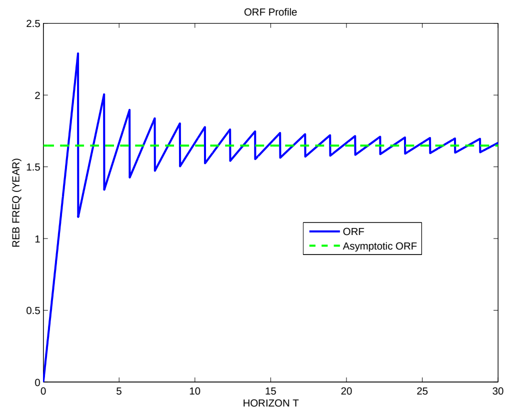
With the help of theorem 5, we established that at τ_{ao} the EIPG and portfolio growth are same under passive strategy as specified in 4.40. What is the long run log growth rate that is achieved when τ_{ao} is used as ORF? With the help of the following lemma we show that growth rate that can be achieved under such hybrid strategy shall be equal to the growth rate achieved at τ_{ao} when passive strategy is followed.

Lemma 28. *For sufficiently large value of investor horizon, the maximum growth rate achieved under hybrid strategy is the growth rate at τ_{ao} under passive strategy.*

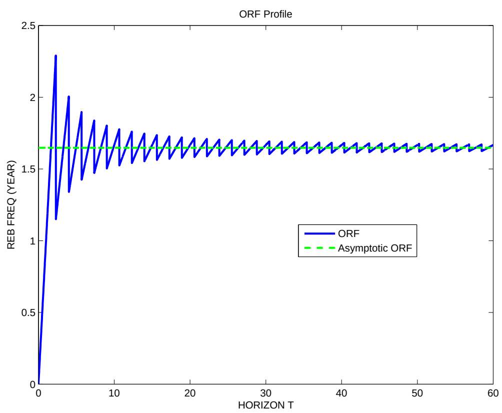
$$\nu^{\tau_{ao}}(\infty) = \nu^{\infty}(\tau_{ao}) \tag{6.88}$$



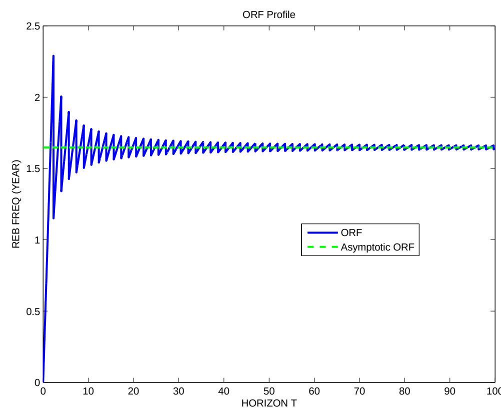
(a) Horizon = 10 Years



(b) Horizon = 30 Years



(c) Horizon = 60 Years



(d) Horizon = 100 Years

Figure 6.6 ORF profile for various lengths of horizon.

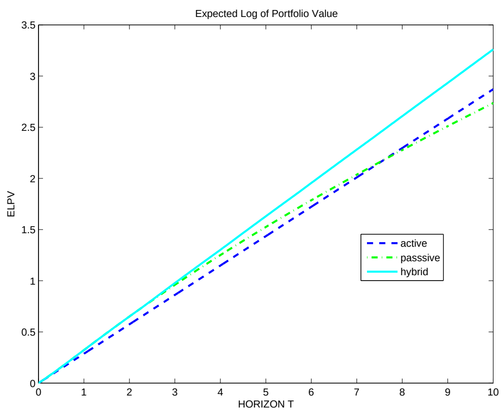
Proof. Using equation 6.8 for passive ELPV, we can compute the growth rate when $\boldsymbol{\tau_{ao}}$ is used as the ORF for very long horizon:

$$\begin{aligned}\boldsymbol{\nu^{\tau_{ao}}(\infty)} &= \lim_{\boldsymbol{T}\to\infty} \frac{\boldsymbol{k}}{\boldsymbol{T}}\boldsymbol{\chi^{\infty}}\left(\frac{\boldsymbol{T}}{\boldsymbol{k}}\right) = \frac{\mathbf{1}}{\boldsymbol{\tau_{ao}}}\boldsymbol{\chi^{\infty}(\tau_{ao})}, \text{ using equation 6.87} \\ &= \boldsymbol{\nu^{\infty}(\tau_{ao})} \end{aligned} \tag{6.89}$$

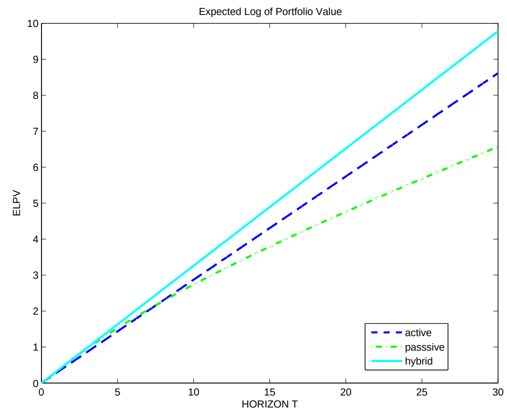
□

Readers are reminded of the growth map theorem 2 that establishes the relationship between passive and hybrid strategy ELPVs. Analogously lemma 28 establishes the relationship between passive and hybrid portfolio growth rates for long-run investments.

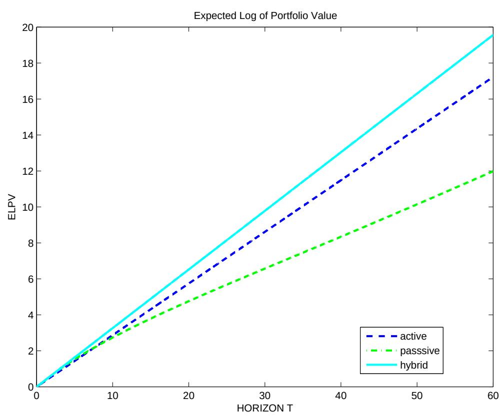
For finite horizon, we have proved that for certain class of portfolio assets, it is possible



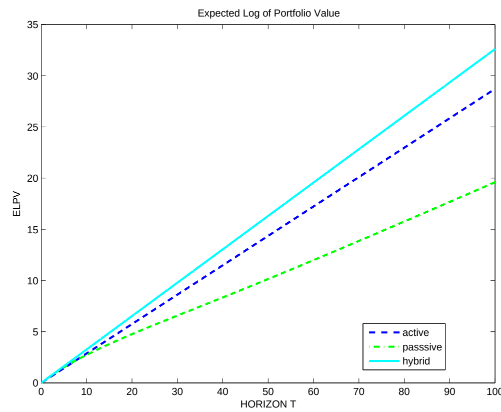
(a) Horizon = 10 Years



(b) Horizon = 30 Years



(c) Horizon = 60 Years



(d) Horizon = 100 Years

Figure 6.7 Hybrid ELPV yield for ORF values in figure 6.6.

to abandon continuous rebalancing in favor of more realistic discrete rebalancing to obtain higher ELPV and growth rates. Intuitively one can imagine that by repeating the same finite horizon and discretely rebalanced investment strategy for many times, we can obtain higher ELPV and growth rates for long-run investments as well. We now formally prove this.

Lemma 29. *For a given portfolio for which asymptotic ORF* $\boldsymbol{\tau_{ao} > 0}$, *hybrid strategy shall generate higher growth rate than active continuously rebalanced log-optimal strategy in the long-run. Mathematically,*

$$\boldsymbol{\nu^{\tau_{ao}}(\infty) \geq \nu_p} \tag{6.90}$$

Proof. Using the results of lemma 28, it is sufficient to prove that:

$$\boldsymbol{\nu^{\infty}(\tau_{ao}) \geq \nu_p} \tag{6.91}$$

Using the results of theorem 5, it is sufficient to prove that:

$$\boldsymbol{\xi}^{\infty}(\boldsymbol{\tau_{ao}}) \geq \boldsymbol{\nu_p} \tag{6.92}$$

Since, under active strategy the EIPG is same as the constant growth rate, using equation 3.41 it is sufficient to prove that:

$$\boldsymbol{\xi}^{\infty}(\boldsymbol{\tau_{ao}}) \geq \boldsymbol{\xi} \tag{6.93}$$

From the definition of $\boldsymbol{\tau_s}$, we know:

$$\boldsymbol{\xi}^{\infty}(\boldsymbol{\tau}) \geq \boldsymbol{\xi}, \forall \boldsymbol{\tau} \in [\mathbf{0}\ \boldsymbol{\tau_s}] \tag{6.94}$$

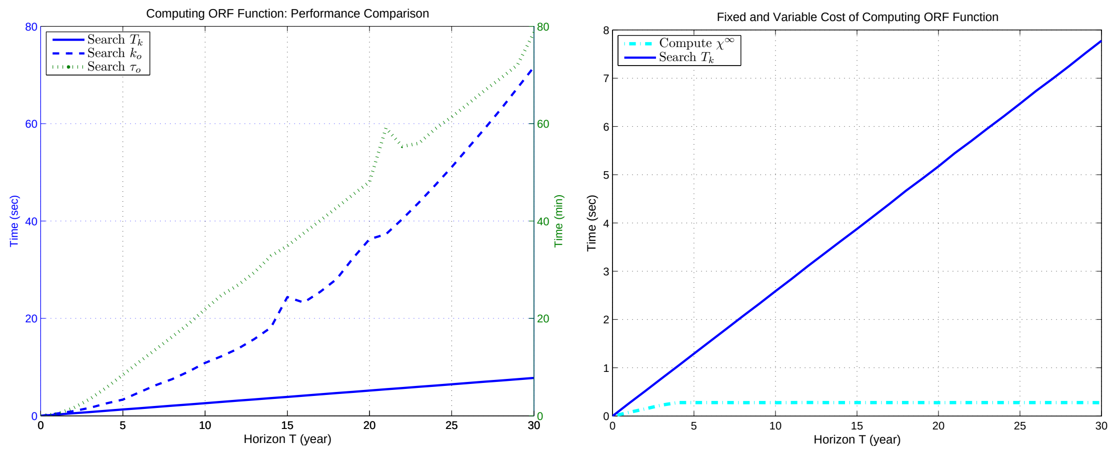
Since, the upper bound for ORF is $\boldsymbol{\tau_s}$, we know that $\mathbf{0} \leq \boldsymbol{\tau_{ao}} \leq \boldsymbol{\tau_s}$. Hence equation 6.93 follows from equation 6.94.

□

6.5 Computational Efficiency ORF Function

All three algorithms to compute ORF function depend on algorithm 9 to compute the ELPV during the horizon range of $[\mathbf{0}\ \boldsymbol{\tau_s}]$ when passive strategy is followed. Algorithm 9 has $\boldsymbol{O(N^2)}$ complexity and the computational cost rises with more assets in the investment portfolio. This is the *variable cost* component of the ORF function algorithms. From the hitherto computational analysis we know that the algorithm 9 is scalable to large number of assets.

The rest of the ORF function algorithms excluding the one time call to algorithm 9 has a *fixed cost* component, invariant of the number of assets. The complexity of the fixed cost is driven by the length of horizon $\boldsymbol{T}$ and the time discretization value $\boldsymbol{\delta t}$. We use $\mathbb{T} = \frac{\boldsymbol{T}}{\boldsymbol{\delta t}}$ to denote the number of discrete horizon points used by the algorithm. Algorithm 10 has a complexity of $\mathbf{O}\left(\mathbb{T}^{\mathbf{2}}\right)$ which is quadratic in time. Let $\mathbb{K}$ be the domain of rebalance divisors that algorithm 11 searches to find optimum $\boldsymbol{k_o}$. Note that $\mathbb{K}$ is only limited to positive integer values between $\boldsymbol{k_{mn}}$ and $\boldsymbol{k_{mx}}$. Hence the valid rebalance divisor domain $\mathbb{K}$ is much smaller compared to the time domain $\mathbb{T}$. Consequently algorithm 11 has a significantly improved complexity of $\mathbf{O}\left(\mathbb{K}\mathbb{T}\right)$. Finally we designed a linear algorithm 12 which has $\mathbf{O}\left(\mathbb{T}\right)$ complexity.



(a) (b)

Figure 6.8 Algorithm performance comparison.

Figure 6.8(a) shows the execution timings of Matlab implementation of the three algorithms. The timings are generated for our example portfolio with four assets. Small values of $\boldsymbol{\delta t} = \mathbf{0.0001}$ and $\boldsymbol{\epsilon} = \mathbf{0.0001}$ are used in order to achieve high accuracy of rebalance strategy. The measurements are taken in a 64-bit Intel 3 GHz computer with 32 GB of RAM.

To study the order of magnitude of performance improvements, we compare the times taken by the algorithms to compute ORF function for 30 years of investment horizon. Algorithm 10, a pure search algorithm, takes slightly more than an hour[2] to compute the strategy. In comparison, algorithm 11 that only searches optimum rebalance divisor $\boldsymbol{k_o}$, reduces the computation time significantly to under one minute. The final algorithm 12 that searches only the RIPs brings down the time to **6.5** seconds. Notice that for higher values of horizon, the performance difference between the algorithms widens rapidly.

As depicted in figure 6.8(b) the variable cost of computing the ELPV is negligible in comparison to the fixed cost of computing the ORF function. This is true even when there is a large number of assets in the portfolio. In chapter 5, an optimized version of algorithm 9

[2]The right hand side time axis is in minutes and is applicable to algorithm 10.

implementation takes less than one second for a portfolio consisting of 128 assets.

Chapter 7

Simulation And Error Estimation

7.1 Methodology

As part of this study, we used Monte Carlo simulation to examine the accuracy of the analytical results presented in this paper. The simulation is run for the familiar portfolio example with four assets. The asset price equation used in generating Monte Carlo paths is given as follows ([21]):

$$S(t+dt) = S(t)e^{(\mu - \frac{\sigma^2}{2})dt + \sigma\epsilon\sqrt{dt}} \tag{7.1}$$

Alternatively one can also use equation 2.2 to generate Monte Carlo paths for correlated asset prices. As explained in [21], equation 7.1 is valid for any value of dt whereas equation 2.2 is accurate only when dt is very small.

To reduce variance in simulation, an antithetic variable is used ([21]). For every asset price path generated using a set of random correlated standard normal variables ϵ, another path using $-\epsilon$ is generated. A total of $20,000$ such Monte Carlo paths for correlated prices are generated using a discrete time step of 0.01 year for both T and τ.

An initial $\$1$ investment fund is distributed among the four assets as per the optimized proportion determined by $\mathbf{w}$. For each price path, the allocated funds are periodically rebalanced to the initial optimal weights $\mathbf{w}$ at the specified optimal frequency. Portfolio growth is computed as the average of the terminal portfolio values over all the price paths. Thus as an example, for horizon 30 years, the terminal value $\widehat{\chi}^{\tau}(30)$ is computed for each value of τ between dt to 30 years at an increment of dt. This process is repeated for each investment horizon value T up to 30 years at an increment of dt. The true ORF, for a given horizon $\boldsymbol{T}$ is the frequency at which simulation produced highest portfolio value. Appendix E lists the Matlab programs used for producing the simulation data.

We have two primary goals here. First, we want to measure the error or equivalently the accuracy of the ORF we compute in this paper. Second, a larger goal, is to assess potential *loss to investor* if she would use this ORF recommendation to execute the optimal hybrid strategy. This loss has to be estimated by the differential wealth creation using the true ORF $\widehat{\boldsymbol{\tau}}_{\boldsymbol{o}}$ and the analytical ORF $\boldsymbol{\tau_o}$. For our purpose, we will use the following two functions to estimate the loss and the percentage loss to investor respectively if $\boldsymbol{\tau_o}$ is used as ORF:

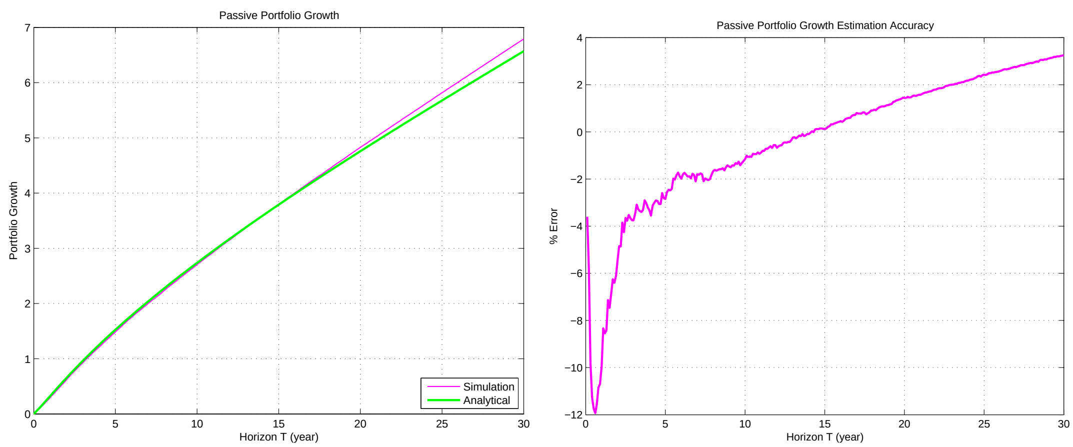
$$\boldsymbol{L(t, \tau_o) = \widehat{\chi}^{\widehat{\tau}_o(t)}(t) - \widehat{\chi}^{\tau_o(t)}(t)} \tag{7.2}$$

$$\%\boldsymbol{L(t, \tau_o)} = \frac{\boldsymbol{\widehat{\chi}^{\widehat{\tau}_o(t)}(t) - \widehat{\chi}^{\tau_o(t)}(t)}}{\boldsymbol{\widehat{\chi}^{\widehat{\tau}_o(t)}(t)}} \text{ x } \mathbf{100} \tag{7.3}$$

It is important to understand the significance of equation 7.2. First and foremost, we are measuring the logarithmic loss. The first term in the numerator is the true ELPV value when the true ORF $\widehat{\boldsymbol{\tau}}_{\boldsymbol{o}}$ is used. This is the best case expected log portfolio growth that is possible if the investor had known and used the true ORF $\widehat{\boldsymbol{\tau}}_{\boldsymbol{o}}$. The second term is the true ELPV had the investor used the recommendation $\boldsymbol{\tau_o}$ computed using the analytical framework. In some sense, this is the *realized* ELPV for the investor. Note that we consider true $\widehat{\boldsymbol{\chi}}$ instead of analytical $\boldsymbol{\chi}$ in the second term. Investor has only control over weather to use the ORF predicted by our analytical framework. Once used, she will obtain only the true underlying ELPV. We assume both $\widehat{\boldsymbol{\tau}}_{\boldsymbol{o}}$ and $\boldsymbol{\tau_o}$ change with horizon $\boldsymbol{t}$.

7.2 Active Strategy Accuracy

We validated the correctness of active strategy by setting the rebalance frequency to a near zero value of **0.001** year for the example 4-asset portfolio. We recorded **8.5675429** as the average log of terminal portfolio value over all the paths for an investment horizon of **30** years. This is compared against the theoretical value of $\boldsymbol{\nu_p T = 0.2871}$ x $\mathbf{30 = 8.613}$. The simulated value is close to the theoretical value within **0.53%** error.



(a) (b)

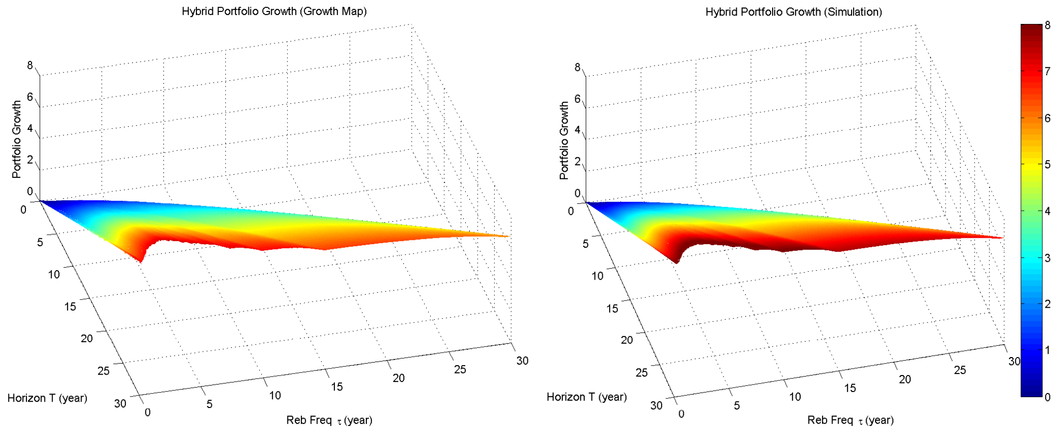
Figure 7.1 Accuracy of passive portfolio growth estimation.

7.3 Passive Strategy Accuracy

As plotted in figure 7.1(a), passive portfolio growth values from simulation closely track the values computed analytically using equation 3.18. Analytical approach slightly overestimates the portfolio growth values at the short-end of investment horizon while underestimating for longer horizons. For the entire horizon, except for the first two years, the analytical passive portfolio growth values are within +/-**5%** (figure 7.1(b)) of the true values obtained in simulation. Higher error percentages observed during the initial two year period is mostly because of the division by very small numbers.

7.4 Growth Map Theorem Accuracy

To examine the accuracy of growth map theorem 2, hybrid portfolio growth is computed according to the theorem for every possible combination of $\boldsymbol{T}$ and $\boldsymbol{\tau}$ using the passive trajectory of portfolio growth obtained in simulation as shown in figure 7.2(a). Compare this with the corresponding values obtained in simulation. As plotted in figures 7.3, except for small values of $\boldsymbol{\tau}$, there is very little deviation of the computed hybrid portfolio growth from the values obtained in simulation. Small but visible error for low values of $\boldsymbol{\tau}$ is attributed to the



(a) (b)

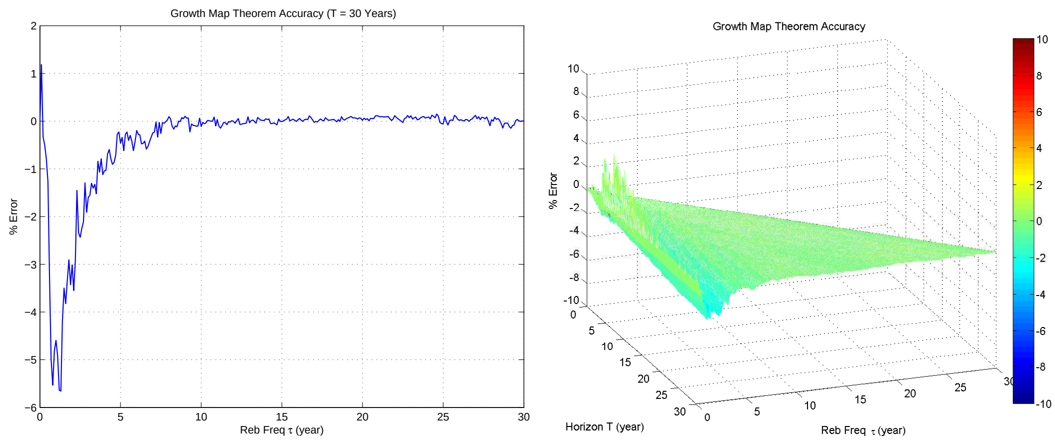
Figure 7.2 Comparison of ELPV using growth map theorem with realized ELPV.

inherent estimation error in simulation data. For horizon values higher than $\mathbf{4}$ years, there is very little deviation of the computed hybrid ELPV from the values obtained in simulation. The positive spikes in error values can be attributed to small absolute values of ELPV.

7.5 Optimal Hybrid Strategy Accuracy

Similar to the analytically computed optimal frequency, the true values obtained in simulation also exhibit saw-tooth pattern especially for lower values of horizon (figure F.4(a)). The amplitude of fluctuation diminish for large horizons. The true optimal frequencies have a midpoint of $\mathbf{2.6}$ years compared to a more conservative analytical estimate of $\boldsymbol{\tau_{ao}} = \mathbf{1.65}$ years. The true values suggest longer passivity with longer rebalancing intervals for investors than the recommendations obtained analytically.

We can trace the under estimation of $\boldsymbol{\tau_{ao}}$ by about $\mathbf{0.95}$ years from $\widehat{\boldsymbol{\tau}}_{\boldsymbol{o}}$ to the slight over estimation of $\boldsymbol{\chi^{\infty}(t)}$ in the short end as depicted in figure 7.1(a). A wide hat $\widehat{(\cdot)}$ is used to denote a parameter predicted by simulation. For simplicity of exposition we assume that this estimation error in $\boldsymbol{\chi^{\infty}(t)}$ is constant $\boldsymbol{e}$ in this short end. Using the notations used in



(a) (b)

Figure 7.3 Percentage error in estimating hybrid ELPV using growth map theorem.

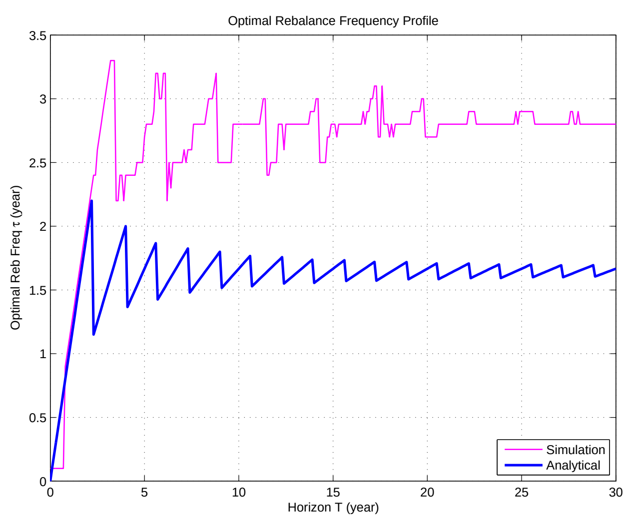


Figure 7.4 Comparison of analytical ORF with underlying true values.

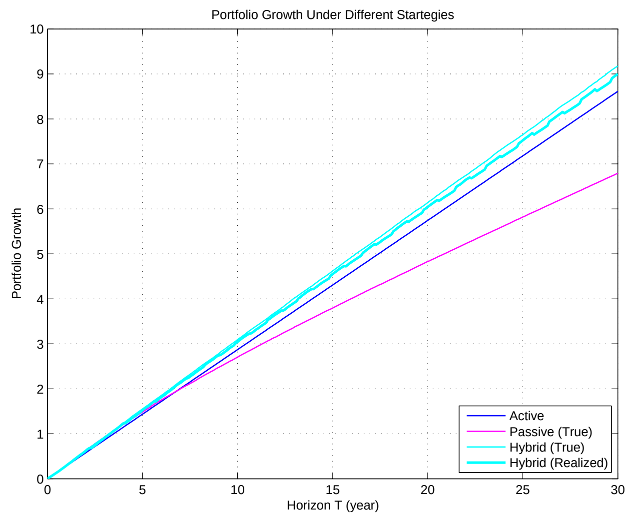


Figure 7.5 Comparison of ELPV values realized with underlying true values.

theorem 5, we can write equation 4.43 for true ELPV for hybrid strategy as:

$$\begin{aligned} \widehat{\chi}^{\tau}(T) &\approx \frac{T}{\tau}\widehat{\chi}^{\infty}(\tau) \\ &\approx \frac{T}{\tau}\Big(\chi^{\infty}(\tau) - e\Big) \end{aligned} \tag{7.4}$$

Following derivation similar to theorem 5, equation 7.4 will be maximized when the following condition holds:

$$\begin{aligned} \left|\frac{\partial\chi^{\infty}(\tau)}{\partial\tau}\right|_{\tau=\widehat{\tau}_{ao}} &= \frac{1}{\widehat{\tau}_{ao}}\Big(\chi^{\infty}(\widehat{\tau}_{ao}) - e\Big) \\ \Rightarrow \xi^{\infty}(\widehat{\tau}_{ao}) &= \nu_p^{\infty}(\widehat{\tau}_{ao}) - \frac{e}{\widehat{\tau}_{ao}} \end{aligned} \tag{7.5}$$

Hence, the value of $\widehat{\tau}_{ao}$ will be obtained by the intersection point of ξ^{∞} curve and ν_p^{∞} curve stretched downwards to adjust for the term $\frac{e}{\widehat{\tau}_{ao}}$. Referring to the illustration in figure 4.5(b), this intersection point $\widehat{\tau}_{ao}$ will occur at a higher value relative to the theoretical τ_{ao}.

Figure 7.5 plots the ELPVs for optimal hybrid strategy (using τ_o), active strategy (using $\tau = 0$) and passive strategy (using $\tau = \infty$) relative to the true underlying ELPV if $\widehat{\tau}$ were used. Corroborating our hitherto claims, hybrid optimal strategy fares better than active

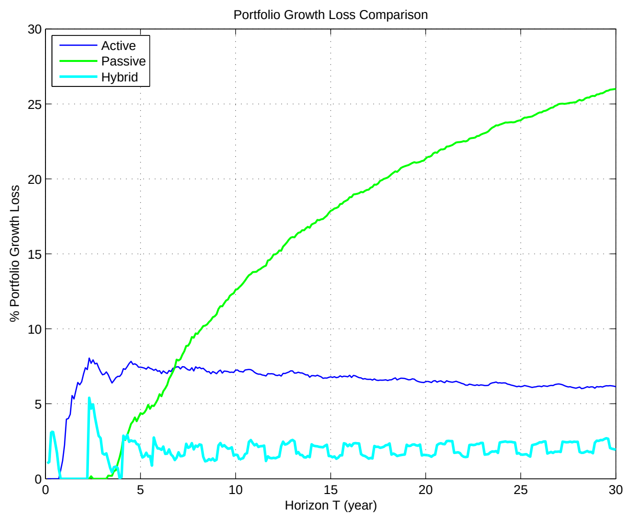


Figure 7.6 Investor loss percentage in adopting various investment strategies.

strategy for any horizon. In spite of improved performance relative to active strategy, there is some performance loss in real-term when we compare the output with that of underlying true ELPV. Potential *loss to investor* is assessed for using a rebalancing frequency τ instead of the true underlying optimal frequency $\widehat{\tau}_o$ found in simulation. The loss is estimated by the fraction of the log wealth the investor gives up by using τ instead of $\widehat{\tau}_o$:

$$L(t,\tau) = \frac{\widehat{\chi}^{\widehat{\tau}_o(t)}(t) - \widehat{\chi}^{\tau(t)}(t)}{\widehat{\chi}^{\widehat{\tau}_o(t)}(t)} \tag{7.6}$$

Corroborating our hitherto claims, as depicted in figure 7.6, hybrid optimal strategy fares better than active strategy in terms of limiting investor loss. In spite of improved performance relative to active strategy, there is small albeit observable loss in using analytically predicted τ_o. The investor incurs higher loss in active continuously rebalanced strategy even without considering the adversarial effect of transaction cost. Following hybrid optimal strategy, the investor loss in the long run is limited to **1.6%** compared to a much higher percentage of **6.2%** for active strategy. As anticipated, passive strategy is far suboptimal with higher than **25%** loss in the long run. The standard error estimate[1] of realized portfolio growth $\widehat{\chi}^{\tau_o(t)}(t)$

[1]The standard error estimate is square root of the ratio of variance of estimation to the number of

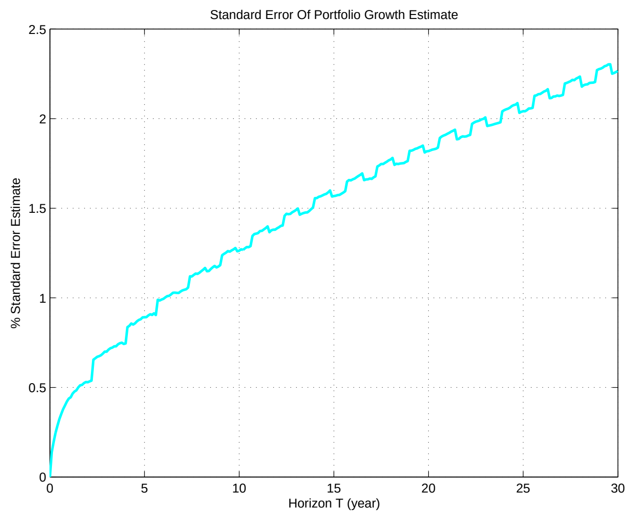


Figure 7.7 Standard error of estimate of ELPV.

used in the loss calculation is small as shown in figure 7.7.

Observation of $\widehat{\chi}^{\tau}(30)$ plot in figure 7.8 offers some interesting insights. For smaller rebalance frequency $(0 < \tau < 0.5)$, the ELPV decreases relative to continuous rebalancing case. However, as we increase the frequency beyond this range performance of rebalancing continues to improve and peaks at $\tau = 2.7$ years. For higher rebalancing frequencies the performance continues to degrade. Rebalance frequencies in the range of $1 < \tau < 6.9$ years offer higher performance over continuous rebalancing case. However an investor will always benefit to use a rebalance frequency from the rebalance efficient frontier of $2.7 \leq \tau < 6.9$ years. The reader should compare this frontier predicted by simulation with $1.67 \leq \tau < 7.61$ years which is computed by our analytical framework and illustrated in figure 4.4(a). Our analytical framework's efficient rebalance frontier includes that predicted by simulation and slightly larger on both side of the interval.

Also note the scant deviation of ELPV computed using growth map theorem to that produced by simulation. For most of the horizon they overlap except near the peak where growth map theorem appears to slightly overestimate.

simulation trials ([21])

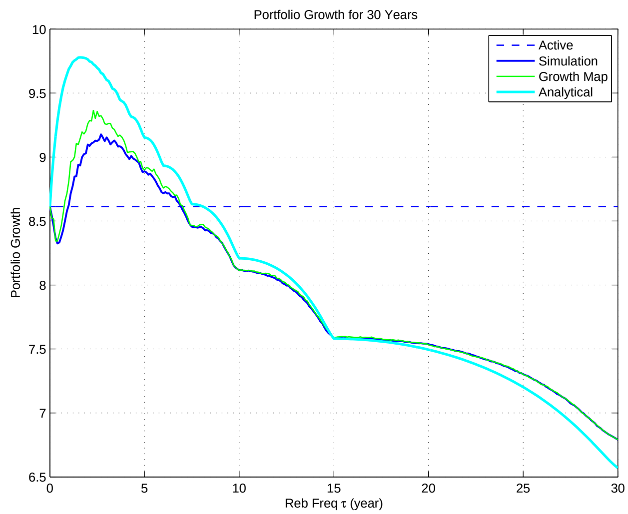


Figure 7.8 ELPV for 30 years at various rebalancing frequencies.

Please note that approximations of the true underlying values can be improved with increasingly smaller ***dt*** value. We can also improve the accuracy if we use higher number of Monte Carlo paths. Note that analytically predicted values can deviate from the true underlying values because of our central assumption of log-normality for sum of log-normal variables under Fenton-Wilkinson method.

7.6 Real Portfolio Example

The simulation is also run for a more realistic portfolio comprising of four real risky assets and the risk-free asset. The representative risky assets are chosen from S&P 100 stock index representing four different industry sectors. Exxon Mobile Corp (ticker: XOM), Amgen Inc (ticker: AMGN) and Verizon Communications Inc (ticker: VZ) stocks are picked from oil, pharmaceutical and communication industries respectively. Gold Trust exchange traded fund (ticker: GLD) is the fourth risky asset representing the commodity market. The portfolio parameters are computed using the historical daily stock prices for six years recorded between 2007 and 2013.

$$\boldsymbol{\mu} = \begin{bmatrix} \mathbf{0.0799} & \mathbf{0.1802} & \mathbf{0.1213} & \mathbf{0.1126} \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{0.2791} & \mathbf{0.2874} & \mathbf{0.2190} & \mathbf{0.2472} \end{bmatrix}$$

$$\boldsymbol{\rho} = \begin{bmatrix} \mathbf{1.0000} & \mathbf{0.5168} & \mathbf{0.1571} & \mathbf{0.5838} \\ \mathbf{0.5168} & \mathbf{1.0000} & \mathbf{-0.0249} & \mathbf{0.4126} \\ \mathbf{0.1571} & \mathbf{-0.0249} & \mathbf{1.0000} & \mathbf{-0.0097} \\ \mathbf{0.5838} & \mathbf{0.4126} & \mathbf{-0.0097} & \mathbf{1.0000} \end{bmatrix}$$

Analogous simulation results are obtained reinforcing the findings of earlier simulation with the fictitious portfolio scenario and reinforced the hitherto conclusions on the validity of analytical results. The simulation results are presented in appendix F.

Chapter 8

Conclusion And Further Research

8.1 Contradiction?

Log optimal strategy can be used in both discrete time and continuous time contexts. In the discrete time context, one determines asset weights that will lead to maximization of ELPV during a single rebalancing period and reverts to these weights at the beginning of each rebalancing period. In the continuous time context, the rebalancing period is infinitesimally small. It so happens that solving for the optimal asset weights for continuous rebalancing is quite easy. Hence often the weights calculated for continuous rebalancing are used for discrete time rebalancing as well. But one has to keep in mind that these weights are not really "log optimal" when used with discrete time rebalancing. In this thesis, we proposed a method to calculate the rebalancing period which when used with weights calculated for continuous rebalancing can lead to better performance than what can be achieved with continuous rebalancing.

The assets weights that we use are optimal only for continuous time rebalancing when the investment horizon is infinitely long. These asset weights will produce maximum log-utility neither when they are used with discrete-time rebalancing nor when the investment horizon is finite. In fact, the authors in [24] have analyzed the use of optimal asset weights which is different for different discrete-time rebalancing periodic for infinitely long horizon. They have proved that when horizon is infinite, (when using rebalancing period specific optimal weights) larger rebalancing period reduces ELPV yield.

So for a long-term log-utility investor, the best possible outcome will be when continuous-time rebalancing is used. In this sense, using discrete-time rebalancing will always be suboptimal for long-term log-utility investors. But we also claim to have discovered a discrete time rebalancing method that performs at least as well as the continuous time rebalancing and some times performs even better than the continuous time rebalancing. Is there a contradiction?

Note that our method applies to finite horizon as well. If our proposed method does better than continuous time rebalancing over a finite horizon, one can possibly apply our proposed method again and again over an infinite sequence of finite horizons and thus do better than continuous time rebalancing even over the infinite horizon!

First of all log-optimality is an optimization problem to derive the optimal portfolio composition, $\mathbf{w}$, when continuous rebalancing ($\boldsymbol{\tau} = \mathbf{0}$) is applied and when the investment horizon $\boldsymbol{T}$ is infinite. Another way to state, given $\boldsymbol{\tau} = \mathbf{0}$ and $\boldsymbol{T} = \boldsymbol{\infty}$, it finds $\mathbf{w}$ to maximize the portfolio log growth rate $\boldsymbol{\nu_p}$. It fixes $\boldsymbol{\tau} = \mathbf{0}$ and $\boldsymbol{T} = \boldsymbol{\infty}$ and solves for $\mathbf{w}$.

We assert that, for the same portfolio composition $\mathbf{w}$, it is possible to obtain higher log growth $\boldsymbol{\nu_p}$ for a shorter $\boldsymbol{T}$ using a different $\boldsymbol{\tau}$. This is evident from the MC simulation results for the example portfolio we picked for our study. For this portfolio, we do indeed obtain higher $\boldsymbol{\nu_p}$ when $\boldsymbol{T}$ is small and when no rebalancing happens ($\boldsymbol{\tau} = \boldsymbol{\infty}$). We also obtain higher $\boldsymbol{\nu_p}$ for medium horizon $\boldsymbol{T}$ (e.g. any value up to 30 years) when a non-zero rebalancing frequency is applied periodically. The optimum rebalance frequency, $\boldsymbol{\tau_o}$ seems to be function of $\boldsymbol{T}$ in the simulation (which is in line with our analytical result). Figure F.4(a) and F.4(b) represent the results entirely derived from MC simulation for each horizon point from 0 to 30 years.

Hence our optimization problem is framed differently from the log-optimality optimization problem. Given $\mathbf{w}$ (found in the above log-optimal optimization) and a specified $\boldsymbol{T}$ (not necessarily $\boldsymbol{T} = \boldsymbol{\infty}$), find the optimum rebalance frequency, $\boldsymbol{\tau_o}$, to maximize the $\boldsymbol{\nu_p}$.

Now the question is if using $\mathbf{w}$ and a non-zero discrete $\boldsymbol{\tau}$, it is possible to obtain higher $\boldsymbol{\nu_p}$ for $\boldsymbol{T} < \boldsymbol{\infty}$, is it possible to obtain higher growth when $\boldsymbol{T} = \boldsymbol{\infty}$? We proved that (refer to section 6.4, lemmas 28 and 29) it is indeed possible to obtain higher growth when $\boldsymbol{T} = \infty$ using a non-zero discrete $\boldsymbol{\tau}$. This is in line with our earlier observation that we apply the proposed method again and again over an infinite sequence of finite horizons and thus do better than continuous time rebalancing even over the infinite horizon.

So far so good till we encounter Proposition **4.1** in [24] that states that continuous-time rebalancing ($\boldsymbol{\tau} = \mathbf{0}$, $\boldsymbol{T} = \infty$, $\mathbf{w}$) outperforms discrete-time rebalancing ($\boldsymbol{\tau} \neq \mathbf{0}$, $\boldsymbol{T} = \infty$, $\mathbf{w}(\boldsymbol{\tau})$). This seems to contradict our finding that we can possibly obtain higher $\boldsymbol{\nu_p}$ when $\boldsymbol{T} = \infty$ by rebalancing to the portfolio composition $\mathbf{w}$ using a non-zero $\boldsymbol{\tau}$. We don't have an absolute refutation to this contradiction other than to point out the following plausible rationale why this proposition may not apply to our proposed method:

1. The proposition applies to portfolios when short sell is forbidden. Our approach does permit short selling and in fact, the example portfolio has negative weights for risk-free asset meaning that we can borrow money to invest in other risky assets.

2. The authors in [24] have mentioned that their results have been found to be accurate (again using MC simulation) only for small ($\approx$ 1 year) rebalancing periods. Specifically, the authors have mentioned that the analysis assumes Taylor's approximation for deriving the optimal weights when $\boldsymbol{\tau} > \mathbf{0}$. Optimal weights from the analytical results match closely with optimal weights predicted by MC simulation only for $\boldsymbol{\tau} < \mathbf{0.5}$ year. They deviate for $\boldsymbol{\tau} > \mathbf{0.5}$.

3. The authors have mentioned that rebalancing once a year is as good as applying continuous rebalancing in log-optimal solution to achieve long term growth. For our example portfolio, we found an asymptotic rebalancing periodic of $\boldsymbol{\tau_{ao}} = \mathbf{1.6}$ year (not 0) to maximize long term growth.

4. Above all, some of the results in the paper may deviate from ours since we assume log-normality for sum of log-normals by adopting Fenton's approach.

8.2 Summary

In log-optimal investment strategy, to maximize the investor's log utility in the long run, the investor continuously rebalances to the initial optimal asset weights. A more realistic investment proposition is to maximize the log-utility by rebalancing the portfolio periodically at discrete non-zero time intervals for a finite desired investment horizon. We investigated the existence of such a periodic optimal frequency by first developing an analytical framework to study the nature of the portfolio growth if it is left passive. We used Fenton-Wilkinson log-normality assumption for sum of log-normal variables to determine the first and second moments of log of portfolio growth for the passive investment. The underlying log-normal assumption in Fenton-Wilkinson approach made it possible to derive analytical expression for passive portfolio mean and variance analogous to active strategy.

We explored and proposed three different rebalancing approaches, viz. simple, stable and optimal rebalancing. Under these approaches, generally termed as hybrid strategy, the investor resorts to periodic rebalancing at a chosen frequency. In the simple rebalancing approach, the investor's criteria is to rebalance when the passive portfolio growth falls below the active portfolio growth. In stable rebalancing the investor can obtain higher terminal portfolio growth by opting for higher passive instantaneous growth during the entire investment horizon. In optimal rebalancing, the investor uses the optimum periodic rebalancing frequency that maximizes the terminal portfolio growth for the intended horizon.

We established an important relationship, called growth map theorem. For any given investment horizon and rebalance frequency, with the help of the theorem, one can compute the portfolio growth under hybrid strategy by merely knowing the evolution of the portfolio growth under passive strategy. First we identified a special rebalancing frequency $\boldsymbol{\tau_s}$ and showed that using a different rebalancing frequency $\boldsymbol{\tau} > \boldsymbol{\tau_s}$ is always suboptimal in the sense that it produces lower terminal portfolio growth. With this premise, we described an algorithm to compute optimal $\boldsymbol{\tau_o}$ by first merely computing the portfolio growth in the entire range of possible rebalancing frequency of $(0\ \tau_s]$ and then selecting the τ_o at which the portfolio growth is found to be maximized.

We analyzed the computational latency of this algorithm that searches for ORF in the continuous time range between 0 and τ_s. By applying software optimizations we were able to more than triple the speed of the rebalance frequency computation. Yet the search algorithm is quadratic by design. Therefore, the search speed is heavily dependent on the width of discrete time interval used to break this continuous range. Smaller time granularity increases the accuracy of ORF and simultaneously deteriorates the computational performance.

Further mathematical insight into log-optimal portfolio rebalancing helped us to simplify the computation when one needs to compute ORF for a given range of horizon values. We reduced the complexity of the ORF function algorithms from quadratic to linear time. First we reduced the search space by showing that there is only a discrete set of finite possible candidates for the choice of ORF. We introduced the concept of rebalance divisors which are positive integer values. A rebalance divisor divides the investment horizon into equal intervals. At the end of each interval the portfolio rebalancing is to be executed. For the first interval, the portfolio growth grows following passive strategy. The terminal value of portfolio growth is given by multiplying portfolio growth at the end of the first interval with the rebalance divisor.

We then restricted the choice of optimal frequency τ_o to only those discrete factors of horizon within the interval of $[\tau_m\ \tau_s]$. Upon further mathematical analysis we determined the unique optimal rebalance divisor for any given investment horizon without resorting to search. We introduced the concepts of LURC, RIP and OLUF. The entire horizon time axis is divided into unique non-overlapping intervals by the series of RIPs. We then serially assign an unique and increasing optimal rebalance divisor to each horizon interval. The ORF is computed by finding the unique optimal rebalance divisor assigned to the horizon interval of the specified horizon. The ORF is the ratio of the value of the given horizon to the unique optimal rebalance divisor. This enabled us to specify the ORF function as a piecewise continuous function.

Finally we derived a few key asymptotic properties of hybrid strategy with periodic discrete time rebalancing. First we showed that for sufficiently large investment horizons, optimal frequency converges to an asymptotic value τ_{ao}. At τ_{ao}, the expected portfolio growth rate is equal to the instantaneous growth when the portfolio is left to grow passively. We then proved that in the long run hybrid strategy shall produce higher portfolio growth rate compared to continuously rebalanced log-optimal strategy.

Simulation studies showed that our analytical framework predicts the passive portfolio growth very accurately. It slightly underestimates at the short end while slightly overestimating at the long end of the horizon. The growth map theorem also accurately transforms the passive portfolio values to hybrid values. The discrepancy in the passive portfolio value estimation results in a relatively smaller optimal frequency estimation. We showed that there is considerable improvement in investor log loss when the investor uses the estimated τ_o. In particular, for our portfolio example, for medium to long term investors the log loss was found to be less than 2% compared to 6% or higher if the investor had used active continuous rebalancing strategy.

8.3 Future Research

The above analytical framework is scalable to any number of risky assets to be considered for portfolio construction. Nevertheless, there are several future research topics. First, it is important to highlight the key underlying assumptions we have made to arrive at the mathematically elegant solutions for computing ORF. First, we assumed that the asset prices follow geometric Brownian motion and have static mean and standard deviations. Second, to derive mathematical expressions for passive evolution of portfolio we assumed log-normality for sum of log-normal random variables. We assumed unimodality for instantaneous growth function in order to simplify the mathematical analysis. We ignored the effect of trading cost for rebalancing as well. Our research can be extended to overcome the constraints put by one or more of these assumptions. Further research is also needed to explore mathematical framework to determine the existence of ORF for portfolios other than log-optimal.

A critical future research is to break the assumption that the asset return mean and variance are static values. It is well known that the asset returns are not invariant of time as is assumed so far [38][39]. Estimation of expected returns of the assets used to construct the portfolio has been the subject of active research [40][41][42][43]. In real tradable assets these characteristics may evolve dynamically, especially when the investment horizon is long. There are several alternative models proposed in literature to make these parameters more dynamic ([44], [45] and [39]). The authors in [46] model the variation of expected return as first-order autoregressive process. One needs to study and apply these models to modify our analytical framework suitably.

More recently alternative approaches have been proposed to model the non-stationary nature of expected returns [45] and portfolio construction using time-varying expected returns[44]. CAPM theory states that in equilibrium, the expected return of an assets has a linear relationship with the market beta of the assets as given by the following expression[47]:

$$\bar{\mu}_i = r_f + (\bar{\mu}_M - r_f)\beta_{iM} \tag{8.1}$$

where, r_f = risk-free rate

β_{iM} = market beta of asset i

$\bar{\mu}_i$ and $\bar{\mu}_M$ are expected returns of assets i and market portfolio M respectively. In this model, the asset's β_{iM} is he only parameter to be estimated to compute asset return. β_{iM} completely models asset's risk characteristics. If β_{iM} evolution in time can be modeled then one can logically model evolution of asset rate of return.

As we have noted before, we have ignored the transaction cost in our models. This simplification needs to be avoided by assuming appropriate transaction cost model suitable for the analytical framework. With reasonable transaction cost one should derive more accurate rebalance frequency than the conservative estimations presented in this paper.

We also derived the condition for existence of the rebalance possibility for a given set of input asset characteristics. The rebalance opportunity exists if the time zero instantaneous growth under passive strategy is higher than corresponding value $\boldsymbol{\nu_p}$ under active strategy. If this criterion is not satisfied, the investor will prefer to follow active strategy for some positive initial duration. Determination of an appropriate rebalancing strategy when this criteria is violated is a future research topic.

As we have noted before, we have ignored the transaction cost in our models. This simplification needs to be avoided by assuming appropriate transaction cost model suitable for the analytical framework. With reasonable transaction cost one should derive more accurate optimal frequency than the conservative estimations presented in this paper.

An alternative to obtain higher growth rate is by reducing portfolio variance by diversification. For example, if we combine several stocks with the same mean and variance, the portfolio variance will reduce and growth rate will increase. So a potential future research area is to investigate whether the diversification by itself will use up the potential for improvement that can be obtained by our proposed rebalancing method.

Lastly, there are a few limitations of the algorithm that need further research. Current algorithm, for instance, fails to compute a rebalance frequency if $\boldsymbol{X}$, the expected value of passive portfolio is negative. The algorithm is defined to find a rebalance frequency for certain class of assets where at time $\boldsymbol{t = 0}$, the instantaneous portfolio growth for passive strategy is higher than active strategy.

Appendix A

Error Amplification

Another consequence of growth map theorem 2 is that any error in the estimation of passive strategy portfolio growth projection will lead to amplified error in portfolio growth projection for hybrid strategy. We will quantify this error amplification in the following lemma.

Lemma 30. *Let the error in estimating the expected log of portfolio growth under passive and hybrid strategies are as follows:*

$$\boldsymbol{e^{\infty}(t) = \widehat{\chi}^{\infty}(t) - \chi^{\infty}(t)} \tag{A.1}$$

$$\boldsymbol{e^{\tau}(t) = \widehat{\chi}^{\tau}(t) - \chi^{\tau}(t)} \tag{A.2}$$

where $\boldsymbol{\widehat{\chi}^{\infty}(t)}$ *and* $\boldsymbol{\widehat{\chi}^{\tau}(t)}$ *are the* true *underlying expected log of portfolio growth values for a given rebalance frequency of* $\boldsymbol{\tau}$*. Then,*

$$\boldsymbol{e^{\tau}(t) = ke^{\infty}(\tau) + e^{\infty}(t')} \tag{A.3}$$

where $\boldsymbol{t = k\tau + t'}$, $\boldsymbol{k = \lfloor \frac{t}{\tau} \rfloor}$ *and* $\boldsymbol{t' = t \bmod \tau}$.

Proof. We start with LHS of equation A.3:

$$\begin{aligned}
\boldsymbol{e^{\tau}(t)} &= \boldsymbol{\widehat{\chi}^{\tau}(t) - \chi^{\tau}(t)} \text{ (using equation A.2)} \\
&= \boldsymbol{[k\widehat{\chi}^{\infty}(\tau) + \widehat{\chi}^{\infty}(t')] - [k\chi^{\infty}(\tau) + \chi^{\infty}(t')]} \\
&\quad \text{(using theorem 2)} \\
&= \boldsymbol{k[\widehat{\chi}^{\infty}(\tau) - \chi^{\infty}(\tau)] + [\widehat{\chi}^{\infty}(t') - \chi^{\infty}(t')]} \\
&= \boldsymbol{ke^{\infty}(\tau) + e^{\infty}(t')} \text{ (using equation A.1)}
\end{aligned} \tag{A.4}$$

□

As per lemma A.1, effect of any error in passive strategy for shorter investment horizon will have noticeable error amplifying effect in hybrid strategy. As the illustration in figure 4.5(a) depicts, the value of rebalance frequency τ decreases and/or the value of investment horizon t increases, the value of $k(=\lfloor\frac{T}{\tau}\rfloor)$ becomes larger. This will have an adversarial effect on the hybrid strategy estimation of portfolio growth.

We anticipate some estimation error in passive log growth estimation using equation 3.18 since there is an underlying log-normality assumption in the Fenton-Wilkinson approach to obtain the moments of a sum of log-normal random variables. In our simulation section, we will observe and study the effect of this error for the hybrid strategy.

A.1 Proof of Theorem 3

Proof. When $i = 0$, $i\tau_s = 0$ is trivially true as time 0 is the very first time when the portfolio is setup with the desired set of optimum asset weights. When $i = 1$, $i\tau_s = \tau_s$ is given as the first rebalance time after the initial setup. During this initial rebalancing period $(0\ \tau_s]$, we must satisfy equation 3.43 and 3.44. Using the expanded notation, the time 0 estimation of the passive instantaneous portfolio growth satisfies the following two conditions:

$$\xi^{\tau_s}(0,\delta t) > \nu_p, \delta t < \tau_s \tag{A.5}$$

$$\xi^{\tau_s}(0,\tau_s) = \nu_p \tag{A.6}$$

When $i \geq 1$, i.e. for all subsequent rebalance periods, $(i\tau_s\ (i+1)\tau_s]$, the following two conditions analogous to equations A.5 and A.6 must also hold.

$$\xi^{\tau_s}(i\tau_s, i\tau_s+\delta t) > \nu_p,\ \forall i\ \in\ \mathbb{N}, \delta t < \tau_s \tag{A.7}$$

$$\xi^{\tau_s}(i\tau_s,(i+1)\tau_s) = \nu_p,\ \forall i\ \in\ \mathbb{N} \tag{A.8}$$

We will prove both of these equations A.7 and A.8 by the method of induction. Let's prove first equation A.7. The base case is when $i = 0$. Then equation A.7 simply becomes equation A.5 which by definition is true. From fundamental definition,

$$\begin{aligned}
\xi^{\tau_s}(0,\delta t) &= \frac{dE[ln(V^{\infty}(0,\delta t))]}{dt} = \frac{dE[ln(V^{\infty}(0,0)\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} \\
&= \frac{dE[ln(V^{\infty}(0,0))]}{dt} + \frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} \\
&= \frac{dE[\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt}, \text{ since } V^{\infty}(0,0) = 1
\end{aligned} \tag{A.9}$$

Hence, equation A.5 implies,

$$\frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\delta t})]}{dt} > \nu_p \tag{A.10}$$

Now, assume equation A.7 holds for $i = k$ and hence $k\tau_s$ is also a rebalance time. That is,

$$\xi^{\tau_s}(k\tau_s, k\tau_s + \delta t) > \nu_p \tag{A.11}$$

To complete the proof we must show that it also holds for $i = k + 1$, i.e.

$$\xi^{\tau_s}((k+1)\tau_s, (k+1)\tau_s + \delta t) > \nu_p \tag{A.12}$$

Following similar steps as of the derivation of equation A.9,

$$\begin{aligned} & \xi^{\tau_s}((k+1)\tau_s, (k+1)\tau_s + \delta t) \\ = \quad & \frac{dE[ln(V^{\infty}((k+1)\tau_s, (k+1)\tau_s + \delta t))]}{dt} \\ = \quad & \frac{dE[ln(V^{\infty}((k+1)\tau_s, (k+1)\tau_s) \sum_{i=1}^{N+1} w_i^{x_i \delta t})]}{dt} \\ = \quad & \frac{dE[ln(V^{\infty}((k+1)\tau_s, (k+1)\tau_s))]}{dt} + \frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i \delta t})]}{dt} \end{aligned} \tag{A.13}$$

We have made use of the fact that $(k+1)\tau_s$ is a rebalance time and hence the initial asset weights are used. Now let's look at the two terms in the above equation. In the first term $V^{\infty}((k+1)\tau_s, (k+1)\tau_s)$ is a deterministic value as the estimation time is same as the time at which the portfolio value is being computed. It is same as asking for the *current* portfolio value which is known at that instant and is invariant of time. Hence the derivative of a constant (i.e. log of the constant portfolio value) will be 0. The value of the second term is given by equation A.10. Thus we establish the required relationship given by equation A.12 and hence the equation A.7.

Now let's prove equation A.8. The induction approach is similar to above with small differences. The base case is when $i = 0$. Then equation A.8 simply becomes equation A.6 which by definition is true. Similar to the derivation of equation A.9, we can show that,

$$\xi^{\tau_s}(0, \tau_s) = \frac{dE[\sum_{i=1}^{N+1} w_i^{x_i \tau_s})]}{dt} \tag{A.14}$$

Hence, equation A.6 implies,

$$\frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i \tau_s})]}{dt} = \nu_p \tag{A.15}$$

Now, assume equation A.8 holds for $i = k$ and hence $(k+1)\tau_s$ is also a rebalance time. That is,

$$\xi^{\tau_s}(k\tau_s, (k+1)\tau_s) = \nu_p \tag{A.16}$$

To complete the proof we must show that it also holds for $i = k + 1$, i.e.

$$\xi^{\tau_s}((k+1)\tau_s, (k+2)\tau_s) = \nu_p \tag{A.17}$$

Following similar steps as of the derivation of equation A.13,

$$\boldsymbol{\xi^{\tau_s}((k+1)\tau_s, (k+2)\tau_s) = \frac{dE[ln(\sum_{i=1}^{N+1} w_i^{x_i\tau_s})]}{dt} = \nu_p,} \text{ using equation A.15} \tag{A.18}$$

Thus we establish the required relationship given by equation A.17 and hence the equation A.8. This completes the proof of the theorem stating that, in order to obtain stable rebalancing, the assets need to be rebalanced to the initial optimal weights at a periodic interval of $\boldsymbol{\tau_s}$. □

Appendix B

Algorithm To Compute $\boldsymbol{\tau_m}$

Algorithm 13 ComputeTauMax

Require: $\boldsymbol{\mu}$,S,w,$\boldsymbol{T}$,$\boldsymbol{\delta t}$,$\boldsymbol{N}$

$\boldsymbol{m \leftarrow 0, \xi^{\infty} \leftarrow 0}$
for $\boldsymbol{t = 0}$ to $\boldsymbol{T}$ by $\boldsymbol{\delta t}$ **do**
    $\boldsymbol{m \leftarrow m + 1, X \leftarrow 0, Y \leftarrow 0, X' \leftarrow 0, Y' \leftarrow 0}$
    for $\boldsymbol{i = 1}$ to N+1 **do**
        # equations 3.8 and 3.46
        $\boldsymbol{X \leftarrow X + w[i]e^{\mu[i]t}, X' \leftarrow X' + w[i]\mu[i]e^{\mu[i]t}}$
        for $\boldsymbol{j = 1}$ to N+1 **do**
            # equation 3.12
            $\boldsymbol{Y \leftarrow Y + w[i]w[j]e^{(\mu[i]+\mu[j])t}(e^{\sigma[i,j]t} - 1)}$
            # equation 3.47
            $\boldsymbol{Y' \leftarrow Y' + w[i]w[j]e^{(\mu[i]+\mu[j])t}[(\mu[i] + \mu[j])(e^{\sigma[i,j]t} - 1) + \sigma[i,j]e^{\sigma[i,j]t}]}$
        end for
    end for
    # equation 3.51
    $\boldsymbol{\xi^{\infty}_{prev} \leftarrow \xi^{\infty}, \xi^{\infty} \leftarrow \frac{1}{X}[X' - \frac{1}{2}\frac{XY'-2X'Y}{X^2+Y}]}$
    if $\boldsymbol{\xi^{\infty} \leq \xi^{\infty}_{prev}}$ **then**
        return $\boldsymbol{t - \delta t}$ # max value is $\boldsymbol{\xi^{\infty}_{prev}}$
    end if
end for
return $\boldsymbol{T}$

Appendix C

Analytical Algorithms For ORF

Algorithm 14 ComputeOptimalRebFreqRebDiv

Require: μ,S,r_f,T,N,$\chi^P(t)$,$\xi^P(t)$

$[\nu_p, \mathrm{w}, \mu, \mathrm{S}] \leftarrow \boldsymbol{ComputeLogOptimalParams}(\mu, \mathrm{S}, r_f, N)$
$k_o \leftarrow 0$, $\tau_o \leftarrow 0$, $\chi^{\tau_o} \leftarrow \nu_p T$ # default continuous rebalancing
if $!\boldsymbol{IsPassiveStrategyPossible}(\mathrm{w}, \mu, \mathrm{S})$ **then**
  return $(k_o, \tau_o, \chi^{\tau_o})$
end if
$\tau_s \leftarrow \boldsymbol{solve}(\xi^P(t) = \nu_p)$ # equation 3.44 and lemma 9
$\tau_m \leftarrow \boldsymbol{ComputeTauMax}(\mu, \mathrm{S}, \mathrm{w}, T, \delta t, N)$
$k_{mn} \leftarrow \boldsymbol{max}(1, \lceil \frac{T}{\tau_s} \rceil)$, $k_{mx} \leftarrow \boldsymbol{max}(1, \lfloor \frac{T}{\tau_m} \rfloor)$
$k_{mx} = \boldsymbol{max}(1, \lfloor \frac{T}{\tau_m} \rfloor)$
# search for k_o
for $k = k_{mn}$ to k_{mx} by $\mathbf{1}$ **do**
  $\tau_o = 0$,$\chi^{\tau_o} = 0$
  $\chi^{\tau_o} = 0$
  if $k\chi^{\infty}\left(\frac{T}{k}\right) > \chi^{\tau_o}$ **then**
    $k_o \leftarrow k$, $\tau_o \leftarrow \frac{T}{k_o}$, $\chi^{\tau_o} \leftarrow k_o \chi^{\infty}(\tau_o)$
    $\chi^{\tau_o} = \frac{T}{k}$
  end if
end for
return $(k_o, \tau_o, \chi^{\tau_o})$

For investment horizons between $(k-1)^{th}$ and k^{th} pair of RIPs one needs to use $k_o = k$ as the rebalance divisor to maximize the ELPV. Between two consecutive RIPs the optimum rebalance divisor remains the same. For small investment horizon T between $\mathbf{0}$ to $T_{1,2}$ optimum rebalance divisor k_o is $\mathbf{1}$. The divisor is incremented to $\mathbf{2}$ for horizon starting at $T = T_{1,2}$. One continues to use $\mathbf{2}$ as the divisor for $T \leq T_{2,3}$. For investment horizon $T > T_{2,3}$ the rebalance divisor is incremented to $\mathbf{3}$ for optimum performance. Thus given an investment horizon $\boldsymbol{T}$, one needs to compute the lowest RIP that is equal to or higher than $\boldsymbol{T}$. The rebalance divisor of this RIP is used as the optimum rebalance divisor $\boldsymbol{k_o}$. Given this $\boldsymbol{k_o}$, $\boldsymbol{\tau_o} = \frac{\boldsymbol{T}}{\boldsymbol{k_o}}$ becomes the ORF. The reader is reminded that per theorem 5 for sufficiently long horizon, the ORF converges to $\boldsymbol{\tau_{ao}}$.

Algorithm 15 ComputeOptimalRebFreqFinal

Require: $\boldsymbol{\mu}$,S,$\boldsymbol{r_f}$,$\boldsymbol{T}$,$\boldsymbol{N}$,$\boldsymbol{\chi^P(t)}$,$\boldsymbol{\epsilon}$

1: $[\boldsymbol{\nu_p}, \mathrm{w}, \boldsymbol{\mu}, \mathrm{S}] \leftarrow \boldsymbol{ComputeLogOptimalParams(\mu, \mathrm{S}, r_f, N)}$
2: $\boldsymbol{\tau_o} \leftarrow \boldsymbol{0}$, $\boldsymbol{\chi^{\tau_o}} \leftarrow \boldsymbol{\nu_p T}$ # default continuous rebalancing
3: **if** $!\boldsymbol{IsPassiveStrategyPossible}(\mathrm{w}, \boldsymbol{\mu}, \mathrm{S})$ **then**
4:     **return** $(\boldsymbol{\tau_o}, \boldsymbol{\chi^{\tau_o}})$
5: **end if**
6: $\boldsymbol{k} \leftarrow \boldsymbol{0}$, $\boldsymbol{T_k} \leftarrow \boldsymbol{0}$
7: **while** $\boldsymbol{T > T_k}$ **do**
8:     $\boldsymbol{k \leftarrow k + 1}$,$\boldsymbol{\tau_{k-1} \leftarrow \tau_o}$
9:     $\boldsymbol{T_k \leftarrow solve\{k\chi^P\left(\frac{t}{k}\right) = (k+1)\chi^P(\frac{t}{k+1})\}}$
10:     $\boldsymbol{\tau_o \leftarrow \frac{T_k}{k}}$
11:     **if** $\boldsymbol{abs(\tau_o - \tau_{k-1}) \leq \epsilon}$ **then**
12:         $\boldsymbol{\chi^{\tau_o} \leftarrow \lfloor \frac{T}{\tau_o} \rfloor \chi^{\infty}(\tau_o)}$ # $\boldsymbol{T_k}$ converged to $\boldsymbol{\tau_{ao}}$
13:         **return** $(\boldsymbol{\tau_o}, \boldsymbol{\chi^{\tau_o}})$
14:     **end if**
15: **end while**
16: $\boldsymbol{\tau_o \leftarrow \frac{T}{k}}$, $\boldsymbol{\chi^{\tau_o} \leftarrow k\chi^{\infty}(\tau_o)}$
17: **return** $(\boldsymbol{\tau_o}, \boldsymbol{\chi^{\tau_o}})$

This logic is summarized in our final rebalance frequency computing algorithm 15. In the while loop (line number 7 through 14), we compute the $\boldsymbol{k^{th}}$ RIP $\boldsymbol{T_k}$ for increasing values of $\boldsymbol{k}$. In line 10 we compute the rebalance frequency at $\boldsymbol{T_k}$. If both the rebalance frequencies at RIPs of current and previous iteration converge within a specified error $\boldsymbol{\epsilon}$, then we use the converged value as the optimum frequency. In line 12 we compute the ELPV using this converged frequency. Note that for given horizon $\boldsymbol{T}$ we find the value of rebalance divisor ($\lfloor \frac{\boldsymbol{T}}{\boldsymbol{\tau_o}} \rfloor$) to compute the hybrid portfolio value. If the frequency does not converge then the loop continues till the lowest $\boldsymbol{k^{th}}$ RIP $\boldsymbol{T_k}$ is equal to or greater than $\boldsymbol{T}$. In line 16 we compute the optimum rebalance frequency for divisor $\boldsymbol{k}$ and the corresponding ELPV.

Appendix D

Matlab Functions To Compute ORF

D.1 Check τ Existence

```
function possible = IsPassiveStrategyPossible(MU, W, S)
possible = false;

[x N]=size(MU);%N = number of risky + risk free assets
X=1; Y=0;
dX = sum(W'.*MU);%derivative of X
%derivative of Y, same as active portfolio variance
dY = sum(sum((W*W').*S));

ddX = sum(W'.*MU.^2);%second derivative of X

MUC = repmat(MU,N,1)+repmat(MU',1,N);
ddY = sum(sum((W*W').*S.*(2*MUC + S)));%second derivative of Y

ddX_P=(ddX - dX^2) - 0.5*(ddY-dY^2) + 2*dX*dY;
if ddX_P > 0
    possible = true;
end
```

D.2 Compute ORF (Matrix Based)

```
function optTau = ComputeOptRebFreqMatrix(MU,S,T,delT,N,W,nu_p)

MUC = repmat(MU,N+1,1)+repmat(MU',1,N+1);
XI_PAS = zeros(1, T/delT + 1);
MUCS = MUC + S; m = 0;
for t = 0:delT:T
    m = m+1;  ES=exp(S*t);
    WEMU=W.*exp(MU*t); WEMUC=WEMU'*WEMU;
    X = sum(WEMU); X_P = sum(MU.*WEMU);
    Y=sum(sum(WEMUC.*(ES - 1)));
    Y_P = sum(sum(WEMUC.*(ES.*MUCS - MUC)));
    XI_PAS(m) = log(X) -  0.5*log(1 + Y/ X^2 );
    X_P = (X_P - 0.5*((Y_P*X - 2*X_P*...
```

```
        Y)/(X^2+Y)))/X;
    if t==0
        maxXI_HYD = nu_p*T; optTau = 0;
    else
        k = floor(T/t); tp = mod(T,t);
        tp_idx = 1+round(tp/delT);
        %compute hybrid portfolio growth
        %using growth map theorem
        XI_HYD = k*XI_PAS(m) +  XI_PAS(tp_idx);
        if(XI_HYD > maxXI_HYD)
            maxXI_HYD = XI_HYD; optTau = t;
        end
        if (X_P - nu_p) < 0
            %found stable reb freq
            break;
        end
    end
end
```

D.3 Compute ORF Function

```
close all; clear;

MU = [0.24 0.20 0.15];
ExpSigma_0 = [0.3000    0.2646    0.1732];
ExpCorrC = [1.0000    0.2520    0.1925;
    0.2520    1.0000   -0.2182;
    0.1925   -0.2182    1.0000];
rf = 0.1;
save portfolio_params

T = 100;
delT=0.001;
eps=0.000001;%error margin to converge

%compute active log optimal portfolio params
[gr_opt MUrf Srf MURFC MURFSC Wrft]  = compute_active_portfolio_params()

%Compute the inflection points for the given horizon T
[tau_ao T_INFL_VEC K_VEC]  = ...
    ComputeInflectionSet(T, 0.9, eps, MUrf, Srf, Wrft, MURFSC, MURFC)

%Now compute the profile of ORF for the horizon length
T_VEC=[];TAU_O_VEC=[];TAU_AO_VEC=[];XH_VEC=[];XA_VEC=[];
i=1;
k=1;%reb divisor
for t=0:delT:T
    %if horizon is long enough simply use the asymp orf
```

```
    if t >= max(T_INFL_VEC)
        TAU_O_VEC(i)=tau_ao;%orf = aymp orf if the horizon is long enough
        k=floor(t/tau_ao);
    else
        %search for the reb divisor to apply
        while t > T_INFL_VEC(k)
            k=k+1
        end
        TAU_O_VEC(i)=t/k;%orf = horizon/reb divisor
    end
    T_VEC(i)=t;
    TAU_AO_VEC(i)= tau_ao;%just for plotting purpose

    %hybrid portfolio ELPV
    XH_VEC(i)= ...
        k*passive_log_port_val_function(TAU_O_VEC(i), MUrf, Srf, Wrft);

    %active portfolio ELPV
    XA_VEC(i)=gr_opt*t;

    %passive portfolio ELPV
    XP_VEC(i) = passive_log_port_val_function(t, MUrf, Srf, Wrft);
    i=i+1;
end
cd('C:\Publications\MyPhDThesis');

figure
plot(T_VEC,TAU_O_VEC,'-b',T_VEC,TAU_AO_VEC,'--g','LineWidth',2)
h = legend('ORF','Asymptotic ORF',2, 'Location','Best');
title('ORF Profile');
xlabel('HORIZON T')
ylabel('REB FREQ (YEAR)')
xlim([0 T])

figure
plot(T_VEC,XA_VEC,'--b',T_VEC,XP_VEC,'-.g',T_VEC,XH_VEC,'c','LineWidth',2)
hold on
h = legend('active','passsive','hybrid',3, 'Location','Best');
set(h,'Interpreter','none')
title('Expected Log of Portfolio Value');
xlabel('HORIZON T')
ylabel('ELPV')
xlim([0 T])
```

```
function [gr_opt MUrf Srf MURFC MURFSC Wrft] = ...
    compute_active_portfolio_params()

load portfolio_params
```

```
S = corr2cov(ExpSigma_0, ExpCorrC);

W = S \ (MU - rf)'; %solve for optimal weights
w_rf = 1 - sum(W); %compute the weight of the risk free asset

Wrf = [w_rf; W];

[N x] = size(W);

MUrf = [rf MU];

%compute portfolio mean
mu_opt = MUrf*Wrf

%compute portfolio std dev
sig_opt = sqrt(sum(sum((W*W').*S)))

%compute portfolio growth rate
gr_opt = mu_opt -0.5*sig_opt^2;

%construct the correlation matrix with risk-free asset
S_sz = size(S);
Srf = [zeros(S_sz(1),1) S];
S_sz = size(Srf);
Srf = [zeros(1,S_sz(2));Srf];

%the following invariants will be used in portfolio value calculation
MURFC = repmat(MUrf,N+1,1) + repmat(MUrf',1,N+1);
MURFSC = MURFC + Srf;
Wrft = Wrf';
```

```
%This function returns the asymptotic orf, inflection points and the
%rebalance divisor set for any given input investment horizon
%and error margin to converge. Tk0 = first inflection point guess
function [tau_ao T_INFL_VEC K_VEC]  = ...
    ComputeInflectionSet(T, Tk0, eps, MUrf, Srf, Wrft, MURFSC, MURFC)

format long;
f_asymp_tau = @(t )asymp_tau_function(t, MUrf, Srf, Wrft, MURFSC, MURFC);
tau_ao = fzero(f_asymp_tau,1.5);

T_INFL_VEC=[];K_VEC=[];
k = 0; Tk = Tk0;%initial guess
Dk=0;

while T > Tk
    k = k+1
    Dkm1 = Dk;%last iteration difference
```

```
    Tkm1 = Tk;%last iteration inflection point

    f = @(t)inflection_function(t,k,MUrf, Srf, Wrft);
    Tk = fzero(f,Tkm1)
    Dk=Tk-Tkm1;
    T_INFL_VEC(k)=Tk;K_VEC(k)=k;
    if abs(Dkm1 - Dk) <= eps
       return;
    end
end
```

```
function DIFF = asymp_tau_function(t,MUrf, Srf, Wrft, MURFSC, MURFC)

DIFF = ...
 passive_log_port_val_dvt_function(t,MUrf, Srf, Wrft, MURFSC, MURFC) - ...
    passive_log_port_val_function(t, MUrf, Srf, Wrft)/t;
```

```
function DIFF = inflection_function(t, k, MUrf, Srf, Wrft)

DIFF = k*passive_log_port_val_function(t/k, MUrf, Srf, Wrft)...
    - (k+1)*passive_log_port_val_function(t/(k+1), MUrf, Srf, Wrft);
```

```
function EXP_LOG_PAS_PORT_FW = ...
    passive_log_port_val_function(t, MUrf, Srf, Wrft)

if t==0
    EXP_LOG_PAS_PORT_FW = 0;
    return;
end

[EXP_PAS_PORT VAR_PAS_PORT ESRF WEMU WEMUC] = ...
    passive_port_val_function(t, MUrf, Srf, Wrft);


EXP_LOG_PAS_PORT_FW = log(EXP_PAS_PORT) -...
    0.5*log(1 + VAR_PAS_PORT/ EXP_PAS_PORT^2 );
```

```
function EXP_LOG_PAS_PORT_FW_DVT = ...
    passive_log_port_val_dvt_function(t,MUrf, Srf, Wrft, MURFSC, MURFC)

```

```
[EXP_PAS_PORT VAR_PAS_PORT ESRF WEMU WEMUC] = ...
    passive_port_val_function(t, MUrf, Srf, Wrft);

EXP_PAS_PORT_DVT = sum(MUrf.*WEMU);
VAR_PAS_PORT_DVT = sum(sum(WEMUC.*(ESRF.*MURFSC - MURFC)));

EXP_LOG_PAS_PORT_FW_DVT = (EXP_PAS_PORT_DVT - 0.5*...
    ((VAR_PAS_PORT_DVT*EXP_PAS_PORT - 2*EXP_PAS_PORT_DVT*...
    VAR_PAS_PORT)/(EXP_PAS_PORT^2+...
    VAR_PAS_PORT)))/EXP_PAS_PORT;
```

```
function [EXP_PAS_PORT VAR_PAS_PORT ESRF WEMU WEMUC] = ...
    passive_port_val_function(t, MUrf, Srf, Wrft)

WEMU=Wrft.*exp(MUrf*t);
WEMUC=WEMU'*WEMU;
ESRF=exp(Srf*t);

EXP_PAS_PORT = sum(WEMU);
VAR_PAS_PORT = sum(sum(WEMUC.*(ESRF - 1)));
```

Appendix E

Matlab Simulation Programs

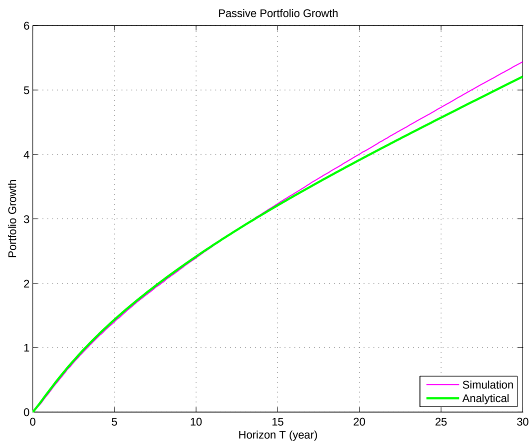
```
close all; clear; format long; warning('off','all');

% inputs for simulation parameters
NUM_MC_PATH = 1;% Number of MC paths to be simulated
delH = 0.1;
HORIZON = 1;
START_HOR = delH;
delT = 0.1;% smaller the better

%Portfolio parameters
MU = [0.1 0.24 0.20 0.15]; %first element is risk-free
ExpSigma = [0 0.3000    0.2646    0.1732];
ExpCorr = [1 0 0 0;
    0 1.0000    0.2520    0.1925;
    0 0.2520    1.0000   -0.2182;
    0 0.1925   -0.2182    1.0000];
N=size(MU,2);

%construct the covariance matrix with risk-free asset
ExpCov = corr2cov(ExpSigma, ExpCorr);

%solve for optimal weights
W = ExpCov(2:N,2:N)\(MU(:,2:N)-MU(1))';
%compute the weight of the risk free asset
W = [1 - sum(W); W];

%compute portfolio mean
mu_opt = MU*W;

%compute portfolio std dev
sig_opt = sqrt(sum(sum((W*W').*ExpCov)));

%compute portfolio growth rate
gr_opt = mu_opt -0.5*sig_opt^2;

NumSamples = round(HORIZON/delT) + 1;
%EPSVEC =[];
EPSVEC = zeros(NumSamples,NUM_MC_PATH*N*2);
%Generate correlated std random variables
```

```
for NRepl=1:NUM_MC_PATH % Number of MC paths

    randn('seed',NRepl);
    %Set seed for the RandNormal module.
    %Differnt see for different MC path.
    % Refer to Hull equation 17.16 in
    %'Basic numerical procedure' chapter
    % for price equation

    %Generate standard correlated normal variavles
    % Note: for standard normal variables, mean = vector of 0s
    % and std dev = vector of 1s that leads to the cov matrix which
    % is same as the correlation matrix

    EPS = mvnrnd(zeros(size(MU)), ExpCorr, NumSamples);
    %EPSVEC = [EPSVEC EPS -EPS];%optimize it later
    EPSVEC(:,(NRepl-1)*2*N+1:NRepl*2*N) = [EPS -EPS];
end % Number of MC paths

value = 1;
PRICE = ones(1,size(EPSVEC,2));%initial unit price vector for all paths
idx= 0;
numStockInitial = (value*W')./PRICE(1,1:N);
NUMSTOCK_VEC_INIT=repmat(numStockInitial,1,2*NUM_MC_PATH);

%NUMSTOCK_VEC=[];
NUMSTOCK_VEC=repmat(NUMSTOCK_VEC_INIT,NumSamples);
ELPV_ALL_REBFREQ = nan(NumSamples-1,NumSamples-1);

for T=START_HOR:delH:HORIZON %for each value of horizon

    idx = idx + 1;
    %NUMSTOCK_VEC = [NUMSTOCK_VEC; NUMSTOCK_VEC_INIT];
    %compute the next price row for all paths - keep just one price row,
    %the latest

    for c=1:size(EPSVEC,2)
        idx2=mod(c,N);
        if idx2==0
            idx2=N;
        end
        PRICE(1,c) = PRICE(1,c) * exp((MU(idx2) -...
            ExpSigma(idx2)^2/2)*delT + ...
            ExpSigma(idx2)*EPSVEC(idx,c)*sqrt(delT));

    end;
    f=0;
    VALUE_VEC = nan(round(T/delT),NUM_MC_PATH*2);
    for rebFreq=delT:delT:T
        f=f+1;

```

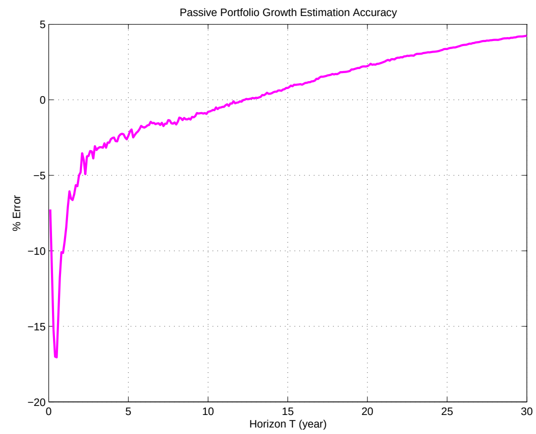
```
        for p=1:2*NUM_MC_PATH
            %get the price for the path
            S = PRICE(1,(p-1)*N+1:p*N);
            numStock = NUMSTOCK_VEC(f,(p-1)*N+1:p*N);
            %compute value vector
            value = sum(numStock .* S);
            VALUE_VEC(f,p)= value;

            %Matlab is not perfect in rounding. Apply some correction
            T = round(T/delT)*delT;
            rebFreq= round(rebFreq/delT)*delT;
            %check if it needs to be rebalanced at rebFreq
            if mod(T,rebFreq) == 0
                %rebalance! Really for next horizon!
                NUMSTOCK_VEC(f,(p-1)*N+1:p*N) = (value.*W')./S;
            end
        end
    end
    LVALUE_VEC=log(VALUE_VEC);
    LVALUE_VEC(imag(LVALUE_VEC) ~= 0) = NaN;

    %compute ELPV for each reb freq ignoring NaN values
    ELPV = nanmean(LVALUE_VEC, 2);
    ELPV_ALL_REBFREQ(1:round(T/delT),idx) = ELPV;
    %find orf and elpv at orf
    [elpv_orf I] = nanmax(ELPV, [],1);
    orf = delT.*I;
    T
    orf
    elpv_orf
end
save('Luen_Sim','ELPV_ALL_REBFREQ');
```

Appendix F

Simulation Results of Real Portfolio

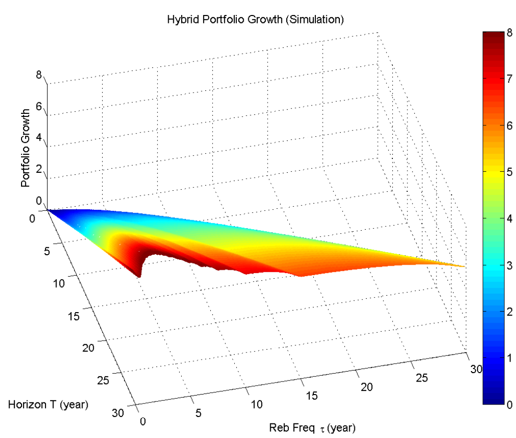


(a)

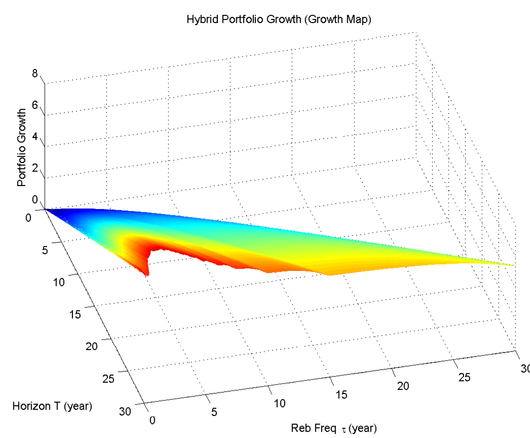


(b)

Figure F.1 Analysis of simulation results for passive strategy.

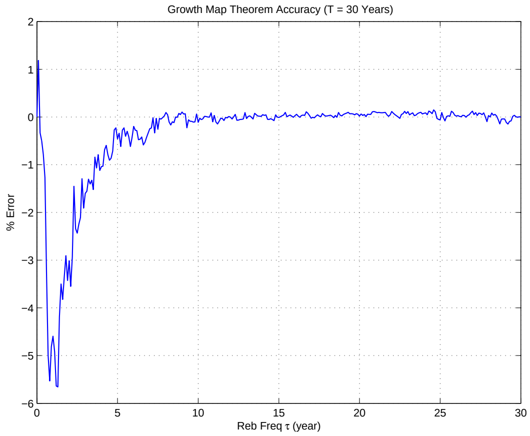


(a)

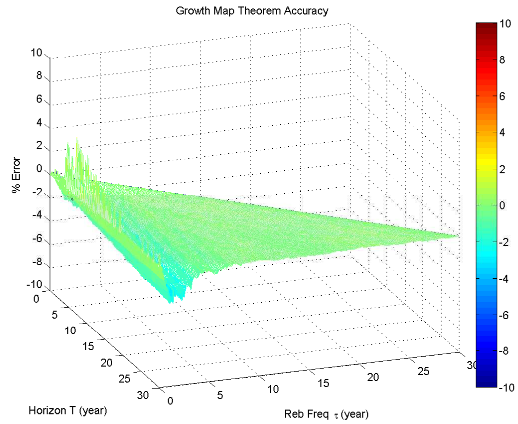


(b)

Figure F.2 Comparison of ELPV using growth map theorem with realized ELPV.

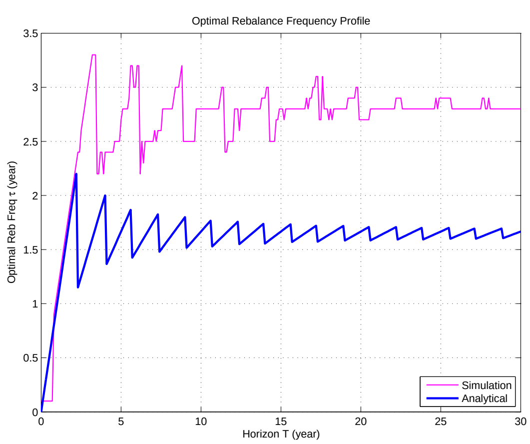


(a)

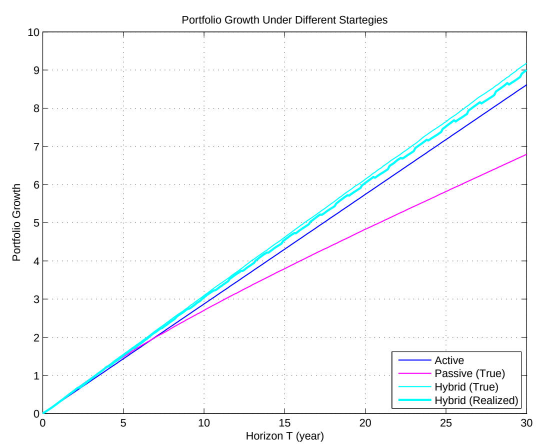


(b)

Figure F.3 Percentage error in estimating hybrid ELPV using growth map theorem.



(a)



(b)

Figure F.4 Analysis of simulation results for hybrid strategy.

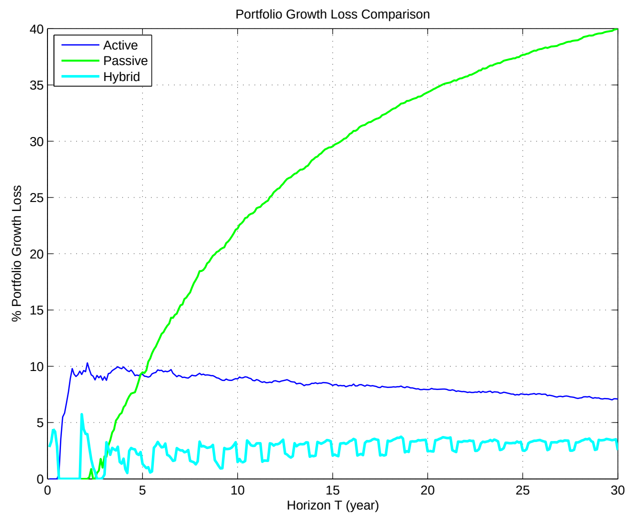


Figure F.5 Investor loss percentage in using optimal hybrid strategy.

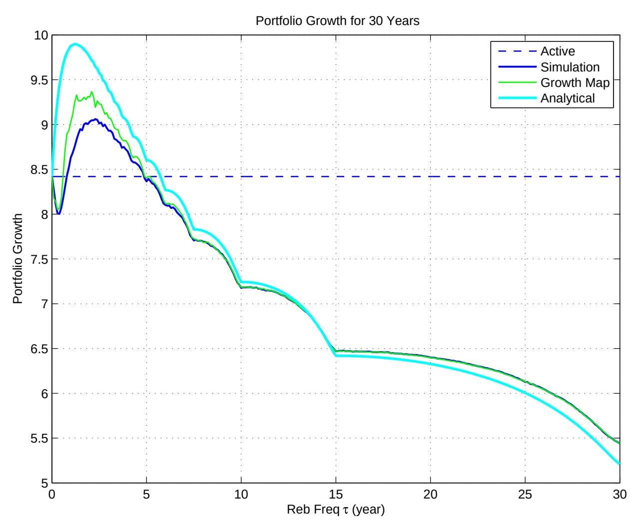


Figure F.6 ELPV for 30 years at various rebalancing frequencies.

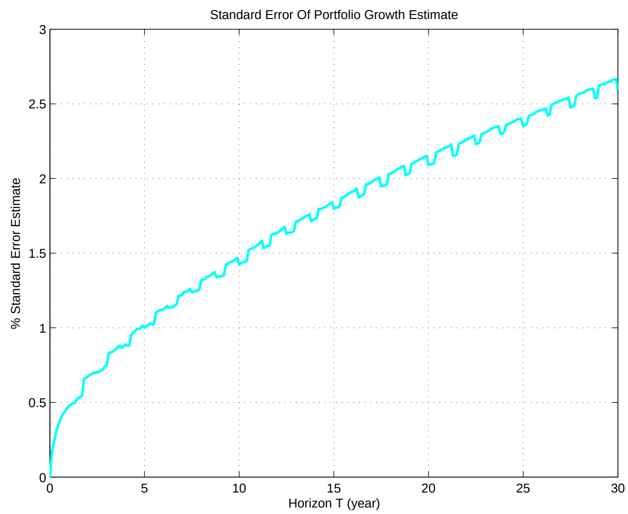


Figure F.7 Standard error of estimate of ELPV.

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