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Difference of Two Weighted Composition Operators on Bergman Spaces

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Definition

- $H(\mathbb{D})$: All Analytic functions on $\mathbb{D}$
- **Weighted Bergman space**
  $A^p_\alpha = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty \right\}$, where
  $dA_\alpha(z) = (\alpha + 1)\left(1 - |z|^2\right)^\alpha dA(z)$, $\alpha > -1$
- $A^2_0 = A^2$ (Bergman Space)

Definitions

- φ : **Analytic** from $\mathbb{D} \to \mathbb{D}$

Definition

The composition operator with symbol φ:

$$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D}), \quad \boldsymbol{C_\varphi(f) = f \circ \varphi}$$

- u : **Measurable** from $\mathbb{D} \to \mathbb{C}$

Definition

The weighted composition operator with weight u **and symbol** φ:

$$uC_\varphi : H(\mathbb{D}) \to \text{All measurable functions on } \mathbb{D}, \quad \boldsymbol{uC_\varphi(f) = u(f \circ \varphi)}$$

Introduction

Question

When is $uC_\varphi - vC_\psi$ compact? (Assume $\boldsymbol{u}$ and $\boldsymbol{v}$ are **analytic**)

It is known that:

Theorem (Z.Čučković and R. Zhao, 2007)

- $1 < p \leq q < \infty$

*Then uC_φ is **compact** from A^p_α into A^q_β if and only if*

$$\lim_{|z|\to 1_-} \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\overline{z}\,\varphi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |u(w)|^q \, dA_\beta(w) = 0.$$

A theorem of Moorhouse

Question

When is $C_\varphi - C_\psi$ compact?

Theorem (J. Moorhouse, 2005)

*$C_\varphi - C_\psi$ is **compact** on A^2_α if and only if **both***

$$\lim_{|z|\to 1_-} |\sigma(z)|\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0, \quad \lim_{|z|\to 1_-} |\sigma(z)|\frac{1-|z|^2}{1-|\psi(z)|^2} = 0.$$

Here $\sigma(z) = \frac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}$, $z \in \mathbb{D}$

Note: $|\sigma|$ is often referred to as the **Cancellation Factor.**

Connection between the Difference operator and Weighted Composition operators

The next theorem links the two types of operators.

Theorem (E. Saukko, 2011)

- $1 < p \leq q < \infty$

*Then $C_\varphi - C_\psi$ is compact from A^p_α into A^q_β if and only if σC_φ and σC_ψ are **both** compact from A^p_α into $L^q(A_\beta)$.*

Another Version of Saukko's theorem

Theorem (Another Version)

$C_\varphi - C_\psi$ is compact from A^p_α into A^q_β if and only if each of the following holds:

(a)

$$\lim_{|z|\to 1_-} \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{z}\,\varphi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |\sigma(w)|^q \, dA_\beta(w) = 0,$$

(b)

$$\lim_{|z|\to 1_-} \int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{z}\,\psi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |\sigma(w)|^q \, dA_\beta(w) = 0$$

Answering the Original Question

Question

When is $uC_\varphi - vC_\psi$ compact? (Assume $\boldsymbol{u}$ and $\boldsymbol{v}$ are **analytic**)

Definition

For $\gamma \in \mathbb{R}$, $\boldsymbol{M(\gamma)}$ is defined as follows:

$$M(\gamma) = \{f : \|f(z)(1 - |z|^2)^\gamma\|_{L^\infty} < \infty\}$$

Compactness of $uC_\varphi - vC_\psi$

- $0 < p \le q < \infty$
- $\frac{2+\alpha}{p} \le \frac{2+\beta}{q}$
- $u, v \in \boldsymbol{M}(\frac{2+\beta}{q} - \frac{2+\alpha}{p})$

Theorem (Acharyya and Wu, 2017)

*$uC_\varphi - vC_\psi : A^p_\alpha \to A^q_\beta$ is **compact** if and only if each of the following holds:*

(a)

$$\lim_{|z|\to 1_-} |\sigma(z)| \left(|u(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z|\to 1_-} (1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z)-v(z)| \left(\frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Compactness of $uC_\varphi - vC_\psi$

Theorem (Acharyya and Wu, 2017)

*$uC_\varphi - vC_\psi : A^p_\alpha \to A^q_\beta$ is **compact** if and only if each of the following holds:*

(a)

$$\lim_{|z|\to 1_-} |\sigma(z)| \left(|u(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z|\to 1_-} (1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z)-v(z)| \left(\frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Proof: "$\Longrightarrow$" Suppose $uC_\varphi - vC_\psi : A^p_\alpha \to A^q_\beta$ is compact.
Let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Note that $k_a, \varphi_a k_a \to 0$ weakly. Thus

$$\lim_{|a|\to 1_-} \|uC_\varphi(k_a) - vC_\psi(k_a)\|_{q,\beta} = 0, \quad \lim_{|a|\to 1_-} \|uC_\varphi(\varphi_a k_a) - vC_\psi(\varphi_a k_a)\|_{q,\beta} = 0$$

Compactness of $uC_\varphi - vC_\psi$

Apply the lemma:

Lemma

Suppose $0 < p < \infty$ *and* $0 < r < 1$. *There is a constant* $C > 0$ *such that for any* $z \in \mathbb{D}$ *and* $f \in A^p_\alpha$

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{2+\alpha}} \int_{\triangle(z,r)} |f(w)|^p dA_\alpha(w).$$

Also, use the elementary facts that $C_\varphi(\varphi_{\varphi(z)})(z) = 0$ and $\left|C_\psi(\varphi_{\varphi(z)})(z)\right| = |\sigma(z)|$, and a chain of inequalities to obtain

$$\lim_{|z|\to 1_-} \frac{|\sigma(z)||u(z)|(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} = 0,$$

$$\lim_{|z|\to 1_-} \frac{|u(z)-v(z)|(1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}}(1-|z|^2)^{\frac{2+\beta}{q}} = 0.$$

Compactness of $uC_\varphi - vC_\psi$

Similarly

$$\lim_{|z|\to 1_-} \frac{|\sigma(z)||v(z)|(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} = 0,$$

$$\lim_{|z|\to 1_-} \frac{|u(z)-v(z)|(1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}(1-|z|^2)^{\frac{2+\beta}{q}} = 0.$$

$\Longleftarrow$ (**has root in Moorhouse and Saukko's work:**)

It is sufficient to show that for any sequence $\{f_n\}$ in A^p_α with $\|f_n\|_{p,\alpha} \le 1$ and $f_n(z) \to 0$ as $n \to \infty$ uniformly on any compact set of $\mathbb{D}$, we have

$$\|(uC_\varphi - vC_\psi)(f_n)\|_{q,\beta} \to 0 \;\; \text{as} \;\; n \to \infty.$$

Partition the disk into E and E', with $E = \{z \in \mathbb{D} : |\sigma(z)| < \frac{2-\sqrt{3}}{2}\}$

Compactness of $uC_\varphi - vC_\psi$

We can write

$$(uC_\varphi - vC_\psi)(f_n) = (uC_\varphi - vC_\psi)(f_n)\chi_{E'} + (u-v)C_\psi(f_n)\chi_E + u(C_\varphi - C_\psi)(f_n)\chi_E.$$

Therefore we need to establish the following three statements.

$$\lim_{n\to\infty} \|(uC_\varphi - vC_\psi)(f_n)\chi_{E'}\|_{q,\beta} = 0,$$
$$\lim_{n\to\infty} \|(u-v)C_\psi(f_n)\chi_E\|_{q,\beta} = 0,$$
$$\lim_{n\to\infty} \|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{q,\beta} = 0.$$

The first two statements are true, due to the following lemma.

Compactness of $uC_\varphi - vC_\psi$

Lemma

Suppose $s, t > 0$, ω is a nonnegative locally bounded measurable function on $\mathbb{D}$, φ is a holomorphic self map of $\mathbb{D}$, and

$$\lim_{|z|\to 1_-} \omega(z)\frac{(1-|z|^2)^s}{(1-|\varphi(z)|^2)^t} = 0.$$

(a) *If $\beta > s-1$, then the measure $\varphi_*(\omega, A_\beta)$ is a compact $(2+\beta+t-s)$-Carleson measure.*

(b) *If $\beta > -1$ and $\omega \in M(\gamma)$ with $\gamma < 1+\beta$, then the measure $\varphi_*(\omega, A_\beta)$ is a compact $(2+\beta-\gamma+\epsilon(\gamma+t-s))$-Carleson measure for any $\epsilon \in (0, \min\{\frac{1+\beta-\gamma}{s-\gamma}, 1\})$ if $\gamma < s$, or $\epsilon \in (0,1)$ if $\gamma \geq s$.*

Compactness of $uC_\varphi - vC_\psi$

To prove the third statement

$$\lim_{n\to\infty} \|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{q,\beta} = 0,$$

we apply Fubini, the previous lemma, and the following lemma:

Lemma

Let $0 < p \le q < \infty$. There exists a constant $C > 0$, such that for all $a \in \mathbb{D}$, $z \in \triangle(a, \frac{2-\sqrt{3}}{2})$, and $f \in A^p_\alpha$ with $\|f\|_{p,\alpha} \le 1$

$$|f(z) - f(a)|^q \le C \frac{|\varphi_a(z)|^q}{(1-|a|^2)^{(2+\alpha)q/p}} \int_{\triangle(a,\frac{1}{2})} |f(w)|^p dA_\alpha.$$

The theorem of Moorhouse

Theorem (Acharyya and Wu, 2017)

*$uC_\varphi - vC_\psi : A^p_\alpha \to A^q_\beta$ is **compact** if and only if each of the following holds:*

(a)

$$\lim_{|z|\to 1_-} |\sigma(z)| \left(|u(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z|\to 1_-} (1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z)-v(z)| \left(\frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Corollary (J. Moorhouse, 2005)

*$C_\varphi - C_\psi$ is **compact** on A^2_α if and only if **both***

$$\lim_{|z|\to 1_-} |\sigma(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0, \quad \lim_{|z|\to 1_-} |\sigma(z)| \frac{1-|z|^2}{1-|\psi(z)|^2} = 0.$$

Hilbert-Schmidt operator (definition)

- X : Separable Hilbert space
- $\{e_j\}$: Orthonormal basis

Definition

T is **Hilbert-Schmidt** if

$$\|T\|_{HS(X)} = \left\{ \sum_{j=0}^{\infty} \|Te_j\|^2 \right\}^{\frac{1}{2}} < \infty$$

Notational Simplicity: $\|T\|_{HS(X)} = \|T\|_{HS}$

Hilbert - Schmidtness of $C_\varphi - C_\psi$

Theorem (B.R. Choe, T. Hosokawa and H. Koo, 2010)

Let $\alpha \geq -1$. *Consider* $C_\varphi - C_\psi$ *acting on* A^2_α. *Then*

$$\|C_\varphi - C_\psi\|^2_{HS} \asymp \int_{\mathbb{D}} \frac{|\boldsymbol{\sigma^2(z)}|\, dA_\alpha(z)}{(1-|\varphi(z)|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\boldsymbol{\sigma^2(z)}|\, dA_\alpha(z)}{(1-|\psi(z)|^2)^{2+\alpha}}$$

Here $\alpha = -1$ *corresponds to* H^2, *and the comparability constants depend only on* α.

Hilbert - Schmidtness of $uC_\varphi - vC_\psi$

Theorem (Acharyya and Wu, 2017)

- $\mathbb{E}$: $\mathbb{D}$ *or* $\mathbb{T}$
- u, v : ***Measurable***
- $uC_\varphi - vC_\psi$ *acting from* $A^2_\alpha \to L^2(\mu)$

Then

$$\|uC_\varphi - vC_\psi\|^2_{HS} \asymp \int_{\mathbb{E}} |\sigma|^2 \left(\frac{|u|^2}{(1-|\varphi|^2)^{2+\alpha}} + \frac{|v|^2}{(1-|\psi|^2)^{2+\alpha}} \right) d\mu$$
$$+ \int_{\mathbb{E}} (1-|\sigma|^2)^{2+\alpha} |u-v|^2 \left(\frac{1}{(1-|\varphi|^2)^{2+\alpha}} + \frac{1}{(1-|\psi|^2)^{2+\alpha}} \right) d\mu$$

A Key Lemma:

Lemma (Acharyya and Wu, 2017)

- *For* $z, w \in \mathbb{D}$, *define* $\rho = \frac{z-w}{1-\bar{z}w}$
- $\alpha > -2$

Then

$$|A|^2 K_z^{(\alpha)}(z) + |B|^2 K_w^{(\alpha)}(w) + 2\Re\left(A\overline{B}K_w^{(\alpha)}(z)\right) \asymp$$
$$|\rho|^2 \left(|A|^2 K_z^{(\alpha)}(z) + |B|^2 K_w^{(\alpha)}(w)\right)$$
$$+(1-|\rho|^2)^{2+\alpha} |A+B|^2 \left(K_z^{(\alpha)}(z) + K_w^{(\alpha)}(w)\right).$$

Corollary

Theorem (Acharyya and Wu, 2017)

$$\|uC_\varphi - vC_\psi\|_{HS}^2 \asymp \int_{\mathbb{E}} |\sigma|^2 \left(\frac{|u|^2}{(1-|\varphi|^2)^{2+\alpha}} + \frac{|v|^2}{(1-|\psi|^2)^{2+\alpha}} \right) d\mu$$
$$+ \int_{\mathbb{E}} (1-|\sigma|^2)^{2+\alpha} |u-v|^2 \left(\frac{1}{(1-|\varphi|^2)^{2+\alpha}} + \frac{1}{(1-|\psi|^2)^{2+\alpha}} \right) d\mu$$

Corollary

Consider the following operators

$$\sigma u C_\varphi,\ \sigma v C_\psi,\ (1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\varphi \text{ and } (1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\psi$$

from A^2_α *or* H^2 *to* $L^2(\mu)$. *Then* $uC_\varphi - vC_\psi$ *is Hilbert-Schmidt if and only if all of the four operators are Hilbert-Schmidt.*

Questions?

Thank You!