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Weighted Composition Operators on Analytic Function Spaces: Some Recent Progress

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Embry - Riddle Aeronautical University, Daytona Beach, FL

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Definition

- $H(\mathbb{D})$: ALL Analytic functions : $\mathbb{D} \to \mathbb{C}$
- φ : **Fixed** Analytic function : $\mathbb{D} \to \mathbb{D}$

Definition

$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$

$$f \mapsto f \circ \varphi$$

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- u : **Fixed** Analytic function : $\mathbb{D} \to \mathbb{C}$

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$uC_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$

$$f \mapsto u.(f \circ \varphi)$$

is clearly a linear operator. This operator is called **the Weighted Composition operator with symbol φ and weight u**

A Generalization of C_φ

Weighted composition operator

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Linearity

u : Analytic

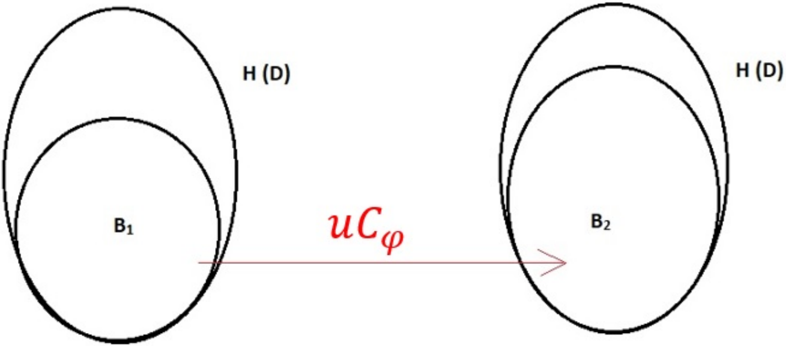
Fact

$uC_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$

$$f \mapsto u\,.(f \circ \varphi)$$

is a ***linear*** *operator.*

Let the fun begin!



Analytic Function Spaces - Examples

Weighted Bergman space $A_\alpha^p = L_\alpha^p \cap H(\mathbb{D}) \quad (\alpha > -1)$

Hardy space $H^p := \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} < \infty \right\}$

Dirichlet space $\mathcal{D} := \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} \left| f'(z) \right|^2 \frac{dA(z)}{\pi} < \infty \right\}$

Operator-theoretic behaviors - Examples

- C_φ is **ALWAYS bounded** on A^p_α **(and** H^p**)**
- C_φ is **bounded on** $\mathcal{D}$ **iff**

$$\frac{\mu_\varphi\left(S\left(\zeta,\delta\right)\right)}{\delta^2} = O\left(1\right),\ \left(|\zeta| = 1,\ 0 < \delta < 2\right)$$

- For **bounded, analytic** u, $uC_\varphi : A^2_\alpha \to A^2_\alpha$ is **compact iff**

$$\lim_{|z|\to 1} |u(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

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Some References

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- **Our operator of interest:** $C_\varphi - C_\psi$
- **More generally:** $uC_\varphi - vC_\psi$
- **Properties we investigate:** Bounded, Compact, Hilbert-Schmidt

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Hilbert-Schmidt operator

- X, Y : Separable Hilbert spaces of complex-valued functions
- $\{e_j\}$: Orthonormal basis in X

Definition

T is **Hilbert-Schmidt** if

$$\|T\|_{HS} = \left\{ \sum_{j=0}^{\infty} \|Te_j\|^2 \right\}^{\frac{1}{2}} < \infty$$

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Hilbert - Schmidt norm of $C_\varphi - C_\psi$ (on A^2_α)

- $\boldsymbol{\sigma(z)} = \dfrac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\,\psi(z)}, z \in \mathbb{D}$

Theorem (Choe, Hosokawa and Koo, 2010)

Consider $C_\varphi - C_\psi$ acting on A^2_α. Then

$$\|C_\varphi - C_\psi\|^2_{HS} \asymp \int_{\mathbb{D}} \frac{|\boldsymbol{\sigma^2(z)}|\, dA_\alpha(z)}{(1 - |\varphi(z)|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\boldsymbol{\sigma^2(z)}|\, dA_\alpha(z)}{(1 - |\psi(z)|^2)^{2+\alpha}}$$

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Operator Version

Theorem (Operator-theoretic version)

$C_\varphi - C_\psi$ *is* ***HS*** *on* $A^2_\alpha \iff \sigma C_\varphi$ *and* σC_ψ *are BOTH* ***HS*** *on* A^2_α.

Result extends for operators: $\boldsymbol{A^2_\alpha \to L^2(\mu)}$

Theorem (Acharyya and Wu, 2017)

$uC_\varphi - vC_\psi$ *is* ***HS*** $\iff \sigma u C_\varphi,\ \sigma v C_\psi,\ (1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\varphi$ *and* $(1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\psi$ *are* ***HS***

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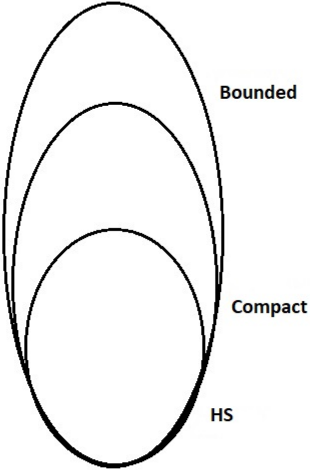
Compact? Bounded?



Bingo!

$$uC_\varphi - vC_\psi : A^p_\alpha \to L^q(\mu)$$

- $0 < p \le q < \infty$
- $\frac{2+\alpha}{p} \le \frac{2+\beta}{q}$
- $u, v \in M(\frac{2+\beta}{q} - \frac{2+\alpha}{p})$

Theorem (Acharyya and Wu, 2017)

*$uC_\varphi - vC_\psi$ is **compact** $\Longleftrightarrow$ σuC_φ, σvC_ψ, $(1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\varphi$ and $(1-|\sigma|^2)^{1+\frac{\alpha}{2}}(u-v)C_\psi$ are **compact***

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*Holds for **boundedness***

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General Methodology

- $C_\varphi - C_\psi$: **Symmetrical**
- $uC_\varphi - vC_\psi$: **Lacking symmetry**
- Choose your "testing functions" **intelligently!**

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Connection with other operators

- $D^k(f) = f^{(k)}$

Theorem (Acharyya and Ferguson, 2018)

*$u_0 C_{\varphi_0} + u_1 C_{\varphi_1} D + u_2 C_{\varphi_2} D^2 + \ldots + u_n C_{\varphi_n} D^n : A^p_\alpha \to H^\infty$ is **compact** $\Longleftrightarrow$ Each component is **compact**.*

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Operator Theory in Applications

- Quantum Mechanics

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- Quantum Mechanics
- Control Theory

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- Quantum Mechanics
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- Signal Processing

Thank You!