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We consider a full set of harmonics for the Stokes wave in deep water in the absence of viscosity, and examine the role that higher harmonics play in modifying the classical Benjamin-Feir instability. Using a representation of the wave coefficients due to Wilton, a perturbation analysis shows that the Stokes wave may become unbounded due to interactions between the N^{th} harmonic of the primary wave train and a set of harmonics of a disturbance. If the frequency of the n^{th} harmonic is denoted $\omega_n = \omega\,(1 \pm \delta)$ then instability will occur if

$$0 < \delta < \frac{\sqrt{2}\; k\, n^n s_n}{(n-1)!} \tag{0.1}$$

subject to the disturbance initially having sufficiently large amplitude. We show that, subject to initial conditions, all lower harmonics will contribute to instability as well, and we identify the frequency of the disturbance corresponding to maximum growth rate.

1. Introduction

In Benjamin-Feir's seminal work identifying criteria for the instability of gravity waves on deep water [1], a bound was established on the frequency of the disturbance as a function of wave steepness. This result was obtained through a perturbation analysis, with the wave train represented by a second order Stokes wave of the form

$$\eta\,(x,t) = a\cos\left(kx - \omega t\right) + \tfrac{1}{2}a^2 k \cos\left[2(kx - \omega t)\right]$$

A disturbance was introduced in the form of two sidebands with slightly different wave numbers and amplitude varying in time,

$$\begin{aligned}\tilde{\eta}_{1,2}\,(x,t) &= \varepsilon_{1,2}\cos\left[k(1\pm\kappa)x - \omega\,(1+\delta)\,t + \gamma_{1,2}(t)\right] \\ &\quad + \varepsilon_{1,2}ak\cos\left[kx + k(1\pm\kappa)x - \omega t - \omega\,(1+\delta)\,t + \gamma_{1,2}\,(t)\right]\end{aligned}$$

with the positive sign corresponding to the index $i = 1$ and the negative sign to $i = 2$. The phase functions $\gamma_1\,(t)$ and $\gamma_2\,(t)$ were necessary to insure that the dispersion relation holds to second order for the sidebands.

Benjamin and Feir demonstrated that interactions between the disturbance and the first harmonic of the primary wave can lead to resonance effects so long as $0 < \delta < \sqrt{2}\,ak$, (i.e., the frequency of the disturbance lies sufficiently close to the fundamental frequency of the wave train). This result followed by obtaining a set of three coupled ordinary differential equations in $\varepsilon_1\,(t)$, $\varepsilon_2\,(t)$, and $\theta\,(t) = \gamma_1\,(t) + \gamma_2\,(t)$ and demonstrating that the amplitudes $\varepsilon_1\,(t)$ and $\varepsilon_2\,(t)$ may become unbounded if δ is in the stated range.

In this work, we consider the fully nonlinear Stokes wave

$$\eta\,(x,t) = H = \sum_{n=1}^{\infty} a_n \cos\left(k_n x - \omega_n t\right) = \sum_{n=1}^{\infty} a_n \cos\zeta_n$$

and a disturbance represented by the infinite sum

$$\tilde{\eta}_i = \sum_{n=1}^{\infty} \varepsilon_{n,i} \cos \zeta_{n,i} + \sum_{n=1}^{\infty} \varepsilon_{n,i} a_n \cos (\zeta_n + \zeta_{n,i}) \ .$$

Out of all possible interactions between the disturbance modes and the primary wavetrain, we show that critical mode interactions with the primary wavetrain may be identified for *every* harmonic of the disturbance. Characterizing each of these interactions leads to a coupled system of equations quite similar to those considered by Benjamin and Feir. This leads to a range of frequencies for the n^{th}harmonic of the sidebands where contributions to instability may occur if $0 < \delta < \sqrt{2}kn^n s_n/(n-1)!$, where s_n is a parameter related to wave steepness ak. The value of δ corresponding to marginal stability decreases with n, so that if instability corresponding to a particular harmonic is identified, all lower harmonics will contribute to instability as well, providing that certain initial conditions are satisfied. The initial conditions require $|\varepsilon_{n,i}(t_0)|$ to be sufficiently large at some fixed time t_0.

2. Formulation of the problem

We consider a two-dimensional Stokes wave on water of finite depth, modeled as an inviscid fluid, where $z = 0$ represents the mean surface level, and $z = \eta(x,t)$ the free surface. Then Laplace's equation governs the motion, with nonlinearities captured through the kinematic and dynamic conditions at the surface. We seek the velocity potential $\phi(x,z,t)$ and $\eta(x,t)$ satisfying

$$\begin{aligned} &\phi_{xx} + \phi_{zz} = 0 \\ &\left.\begin{aligned} \phi_z &= \eta_t + \varepsilon \phi_x \eta_x \\ \phi_t + \eta &+ \tfrac{1}{2}\varepsilon\left(\phi_x^2 + \phi_z^2\right) = 0 \end{aligned}\right\} \text{on } \ z = 1 + \varepsilon\eta \\ &\nabla\phi \to 0 \qquad\qquad\qquad\qquad \text{as } \ z \to -\infty \end{aligned}$$

We represent the free surface as

$$\eta(x,t) = H = \sum_{n=1}^{\infty} a_n \cos(k_n x - \omega_n t) = \sum_{n=1}^{\infty} a_n \cos \zeta_n$$

And the potential as

$$\phi(x,y,t) = \Phi = \sum_{n=1}^{\infty} \omega_n k_n^{-1} a_n e^{k_n z} \sin \zeta_n$$

where $\omega_n^2 = g k_n$.

We introduce perturbations,

$$\eta = H + \varepsilon\tilde{\eta} \quad \phi = \Phi + \varepsilon\tilde{\phi} \tag{2.1}$$

and seek conditions that allow the disturbance $\tilde{\eta}$ to exhibit unbounded growth. By linearity of Laplace's equation, we have

$$\nabla^2\tilde{\phi} = \tilde{\phi}_{xx} + \tilde{\phi}_{yy} = 0 \tag{2.2}$$

$\nabla\tilde{\phi} \to 0$ as $z \to -\infty$.

The kinematic and dynamic boundary conditions may be written as

$$\eta_t + \eta_x (\phi_x)_{z=\eta} - (\phi_z)_{z=\eta} = 0 \tag{2.3}$$

$$g\eta + (\phi_t)_{z=\eta} + \tfrac{1}{2}\left(\phi_x^2 + \phi_z^2\right)_{z=\eta} = 0 \tag{2.4}$$

We conduct a standard perturbation analysis by substituting (2.1) into (2.3) and (2.4), setting $O(\varepsilon)$ terms to zero, and expanding the resulting expressions about $z = H$. Then,

$$\tilde{\eta}_t + \tilde{\eta}_x (\Phi_x)_{z=H} + \tilde{\eta}\left(-\Phi_{zz} + H_x \Phi_{xz}\right)_{z=H} + \left(-\tilde{\phi}_z + H_x \phi_x\right)_{z=H} = 0 \tag{2.5}$$

$$g\tilde{\eta} + \tilde{\eta}\left(\Phi_x \Phi_{xz} + \Phi_z \Phi_{zz} + \Phi_{tz}\right)_{z=H} + \left(\tilde{\phi}_t + \Phi_x \tilde{\phi}_x + \Phi_z \tilde{\phi}_z\right)_{z=H} = 0 \tag{2.6}$$

Consider the analytic continuation of $\tilde{\phi}$ over the neighborhood of the free surface, neglect terms greater than $O\left(a_n^2\right)$, and Taylor expand all expressions about the mean surface level $z = 0$. Upon substituting for Φ and H, the kinematic condition (2.5) may be written

$$\begin{aligned}
&\tilde{\eta}_t + \tilde{\eta}_x \left(\sum_{n=1}^{\infty} \omega_n a_n \cos\zeta_n + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \omega_n k_n \cos\zeta_n \cos\zeta_m\right) \\
&\quad + \tilde{\eta}\left(-\sum_{n=1}^{\infty} \omega_n k_n a_n \sin\zeta_n - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m k_n k_m \omega_m \sin\zeta_n \cos\zeta_m\right) \\
&\quad - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \omega_m k_m^2 \cos\zeta_n \sin\zeta_m \\
&\quad - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q k_m k_q^2 \omega_q \cos\zeta_n \sin\zeta_m \cos\zeta_q \\
&\quad - \left(\tilde{\phi}_z + \sum_{n=1}^{\infty} a_n k_n \sin\zeta_n \tilde{\phi}_x\right)_{z=0} \\
&\quad - \left(\sum_{n=1}^{\infty} a_n \cos\zeta_n \tilde{\phi}_{zz} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m k_m \omega_m \cos\zeta_n \sin\zeta_m \tilde{\phi}_{xz}\right)_{z=0} \\
&\quad - \tfrac{1}{2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \cos\zeta_n \cos\zeta_m \tilde{\phi}_{zzz}\right)_{z=0} \\
&\quad - \tfrac{1}{2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q k_q \cos\zeta_n \cos\zeta_m \sin\zeta_q \tilde{\phi}_{xzz}\right)_{z=0} = 0
\end{aligned} \tag{2.7}$$

The dynamic condition (2.6) may be written

$$\begin{aligned}
&g\tilde{\eta} + \tilde{\eta}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \omega_n \omega_n k_m \cos\zeta_n \cos\zeta_m \right.\\
&\quad \left. + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \omega_n \omega_n k_m \sin\zeta_n \sin\zeta_m - \sum_{n=1}^{\infty} \omega_n^2 a_n \cos\zeta_n\right) \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q k_q \omega_m \omega_q k_m k_q \cos\zeta_n \cos\zeta_m \cos\zeta_q \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q \omega_m \omega_q k_q^2 \cos\zeta_n \cos\zeta_m \cos\zeta_q \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q \omega_m \omega_q k_m k_q \cos\zeta_n \sin\zeta_m \sin\zeta_q \\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty} a_n a_m a_q k_q \omega_m \omega_q k_q^2 \cos\zeta_n \cos\zeta_m \sin\zeta_q \\
&\quad + \left\{\tilde{\phi}_t + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m k_m \omega_m^2 k_m \cos\zeta_n \cos\zeta_m + \sum_{n=1}^{\infty} a_n \omega_n \cos\zeta_n \tilde{\phi}_x \right.\\
&\quad + \sum_{n=1}^{\infty} a_n \omega_n sn\zeta_n \tilde{\phi}_z + \sum_{n=1}^{\infty} a_n \cos\zeta_n \left(\tilde{\phi}_{tz} + O(a_n)\right) \\
&\quad \left. + \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \cos\zeta_n \cos\zeta_m \left(\tilde{\phi}_{tzz} + O(a_n)\right)\right\}_{z=0} = 0
\end{aligned} \tag{2.8}$$

Neglecting cross terms in (2.7) and (2.8) we obtain for the kinematic condition,

$$\begin{aligned}&\tilde{\eta}_t + \tilde{\eta}_x \sum_{n=1}^{\infty} \omega_n a_n \cos\zeta_n - \tilde{\eta} \sum_{n=1}^{\infty} \omega_n k_n a_n \sin\zeta_n \\ &\quad - \left(\tilde{\phi}_z + \tilde{\phi}_x \sum_{n=1}^{\infty} a_n k_n \sin\zeta_n + \tilde{\phi}_{zz} \sum_{n=1}^{\infty} a_n \cos\zeta_n\right)_{z=0} = 0\end{aligned} \tag{2.9}$$

and for the dynamic condition,

$$\begin{aligned}&g\tilde{\eta} - \tilde{\eta} \sum_{n=1}^{\infty} \omega_n^2 a_n \cos\zeta_n \\ &\quad + \left\{\tilde{\phi}_t + \tilde{\phi}_x \sum_{n=1}^{\infty} a_n\omega_n \cos\zeta_n + \tilde{\phi}_z \sum_{n=1}^{\infty} a_n\omega_n \sin\zeta_n + \tilde{\phi}_{tz} \sum_{n=1}^{\infty} a_n \cos\zeta_n\right\}_{z=0} = 0\end{aligned} \tag{2.10}$$

Equations (2.9) and (2.10) generalize the boundary conditions obtained by Benjamin and Feir [1] insofar as terms corresponding to $n = 1$ produce the $O(a)$ terms they obtained by a similar perturbation analysis. In that work, the $O(a)$ terms were used to specify the form of the disturbance, while $O\left(a^2\right)$ terms were used to produce coupled equations for the amplitude of the sideband modes. Neglecting cross terms in (2.7) and (2.8) eliminated terms which would correspond to these $O\left(a^2\right)$ terms in Benjamin and Feir's work.

3. Conditions for instability due to a particular harmonic

While equations (2.9) and (2.10) were obtained by ignoring certain cross terms, the form of the equations generalized the $O(a)$ terms obtained by Benjamin and Feir [1]. We now retain precisely the collection of cross-terms in (2.7) and (2.8) to produce $O\left(a_n^2\right)$ terms that will generalize the $O\left(a^2\right)$ in that work. In this way, the kinematic condition may be represented as

$$\begin{aligned}\tilde{\eta}_t - \left(\tilde{\phi}\right)_{z=0} &= \sum_{n=1}^{\infty} a_n \left\{k_n\omega_n \sin\zeta_n\,\tilde{\eta} - \omega_n \cos\zeta_n\,\tilde{\eta}_x + \left(k_n \sin\zeta_n\tilde{\phi}_x + \cos\zeta_n\tilde{\phi}_{zz}\right)_{z=0}\right\} \\ &+\tfrac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left(2k_n^2\omega_n \sin 2\zeta_n\,\tilde{\eta} - k_n\omega_n\left(1 + 2\cos 2\zeta_n\right)\tilde{\eta}_x\right) \\ &+\tfrac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left\{k_n \sin 2\zeta_n \left(2k_n\tilde{\phi}_x + \tilde{\phi}_{xz}\right) + k_n \cos 2\zeta_n\tilde{\phi}_{zz} + \tfrac{1}{2}\left(1 + \cos 2\zeta_n\right)\tilde{\phi}_{zzz}\right\}_{z=0}\end{aligned} \tag{3.1}$$

and the dynamic condition represented as

$$\begin{aligned}g\tilde{\eta} + \left(\tilde{\phi}_t\right)_{z=0} &= \sum_{n=1}^{\infty} a_n \left(\omega_n^2 \cos\zeta_n\,\tilde{\eta} - \left(\omega_n \cos\zeta_n\,\tilde{\phi}_x + \omega_n \sin\zeta_n\,\tilde{\phi}_z + \cos\zeta_n\,\tilde{\phi}_{tz}\right)_{z=0}\right) \\ &+\tfrac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left(k_n\omega_n^2\left(1 - \cos 2\zeta_n\right)\tilde{\eta} + \left(\omega_n \sin 2\zeta_n \left(k_n\tilde{\phi}_z + \tilde{\phi}_{zz}\right)_{z=0}\right)\right) \\ &+\tfrac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left(\left(1 + \cos 2\zeta_n\right)\left(k_n\omega_n\tilde{\phi}_x + \omega_n\tilde{\phi}_{xz} + \tfrac{1}{2}\tilde{\phi}_{zzt}\right) + k_n \cos 2\zeta_n\,\tilde{\phi}_{zt}\right)_{z=0}\end{aligned} \tag{3.2}$$

Now, assume that the disturbance consists of two sideband modes together with the products of their interaction with the basic wave train,

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2 \quad , \quad \tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2.$$

We define the arguments

$$\zeta_{n,1} = k_n\left(1 + \kappa\right)x - \omega_n\left(1 + \delta\right)t - \gamma_{n,1}(t) \tag{3.3}$$

$$\zeta_{n,2} = k_n\left(1-\kappa\right)x - \omega_n\left(1-\delta\right)t - \gamma_{n,2}\left(t\right), \tag{3.4}$$

where $\gamma_{n,1}(t)$ and $\gamma_{n,2}(t)$ are required so that each harmonic satisfies the dispersion relation to $O\left(\delta^2\right)$. We denote frequencies and wave numbers for the sidebands by

$$\omega_{n,1} = \omega\left(1+\delta\right) \tag{3.5}$$

$$\omega_{n,2} = \omega\left(1-\delta\right), \tag{3.6}$$

and

$$k_{n,1} = k_n\left(1+\kappa\right) \tag{3.7}$$

$$k_{n,2} = k_n\left(1-\kappa\right) \tag{3.8}$$

where κ and δ are small constants.

Then, for $i = 1, 2$, we define

$$\tilde{\eta}_i = \sum_{n=1}^{\infty} \varepsilon_{n,i}\cos\zeta_{n,i} + \sum_{n=1}^{\infty} \varepsilon_{n,i}k_n a_n \cos\left(\zeta_n+\zeta_{n,i}\right) + O\left(k_n^2 a_n^2 \varepsilon_{n,i}\right) \tag{3.9}$$

and

$$\begin{aligned}\tilde{\phi}_i = &\sum_{n=1}^{\infty} k_{n,i}^{-1} e^{k_{n,i}z}\left\{\varepsilon_{n,i}\left(\omega_{n,i}L_{n,i} + \dot{\gamma}_{n,i}M_{n,i}\right)\sin\zeta_{n,i} + \dot{\varepsilon}_{n,i}N_{n,i}\cos\zeta_{n,i}\right\} \\ &+ \sum_{n=1}^{\infty} \omega_n a_n \varepsilon_{n,i} D_{n,i} e^{\left|k_n - k_{n,i}\right|z}\sin\left(\zeta_n - \zeta_{n,i}\right)\end{aligned} \tag{3.10}$$

We further assume that $\varepsilon_{n,i}$and $\gamma_{n,i}$ are slowly varying functions of time, such that their derivatives have the properties

$$\dot{\varepsilon}_{n,i} = O\left(w_n k_n^2 a_n^2 \varepsilon_{n,i}\right), \qquad \dot{\gamma}_{n,i} = O\left(w_n k_n^2 a_n^2\right). \tag{3.11}$$

Now, among the infinitely many products arising from the nonlinear interaction between these disturbance modes and the basic wave train, there will be components with arguments

$$\left.\begin{aligned} 2\zeta_n - \zeta_{n,1} = \zeta_{n,2} + \left(\gamma_{n,1}+\gamma_{n,2}\right) \\ 2\zeta_n - \zeta_{n,2} = \zeta_{n,1} + \left(\gamma_{n,1}+\gamma_{n,2}\right)\end{aligned}\right\} \tag{3.12}$$

suggesting that resonance may be induced between the sidebands by the interaction of particular harmonics. This may contribute to instability if the sum of the time-dependent phase functions approaches a constant,

$$\theta_n = \gamma_{n,1} + \gamma_{n,2} \to \text{const. as} \quad t \to \infty \tag{3.13}$$

Indeed, upon substitution of (3.9) and (3.10) into (2.9) and (2.10), a variety of cross terms involving interaction between the primary wave and the disturbance may be identified so that (3.12) applies. For example,

$$\sin 2\zeta_n \cos\zeta_{n,1} = \tfrac{1}{2}\left[\sin\left(2\zeta_n - \zeta_{n,1}\right) + \sin\left(2\zeta_n + \zeta_{n,1}\right)\right] \sim \tfrac{1}{2}\sin\left(\zeta_{n,2} + \theta_n\right)$$

and similarly

$$\sin 2\zeta_n \cos\zeta_{n,2} = \tfrac{1}{2}\left[\sin\left(2\zeta_n - \zeta_{n,2}\right) + \sin\left(2\zeta_n + \zeta_{n,2}\right)\right] \sim \tfrac{1}{2}\sin\left(\zeta_{n,1} + \theta_n\right).$$

Upon substitution, there exists a set of interactions so that, after equating coefficients, the kinematic boundary condition reduces to the pair of equations

$$\begin{aligned}&\varepsilon_{n,1}\left\{\omega_{n,1}\left(1-L_{n,1}\right) + \dot{\gamma}_{n,1}\left(1-M_{n,1}\right)\right\}\sin\zeta_{n,1} + \dot{\varepsilon}_{n,1}\left(1-N_{n,1}\right)\cos\zeta_{n,1} \\ &\quad = \omega_n k_n^2 a_n^2\left\{\tfrac{5}{4}\varepsilon_{n,1}\sin\zeta_{n,1} + \tfrac{5}{8}\varepsilon_{n,2}\sin\left(\zeta_{n,1}+\theta_n\right)\right\}\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\varepsilon_{n,2}\left\{\omega_{n,2}\left(1-L_{n,2}\right)+\dot{\gamma}_{n,2}\left(1-M_{n,2}\right)\right\}\sin\zeta_{n,2}+\dot{\varepsilon}_{n,2}\left(1-N_{n,2}\right)\cos\zeta_{n,2}\\&\quad=\omega_n k_n^2 a_n^2\left\{\tfrac{5}{4}\varepsilon_{n,2}\sin\zeta_{n,2}+\tfrac{5}{8}\varepsilon_{n,1}\sin\left(\zeta_{n,2}+\theta_n\right)\right\}\end{aligned}\quad. \tag{3.15}$$

for $n = 1, 2, 3, ...$ Furthermore, there exists a set of interactions so that the dynamic boundary condition reduces to the pair of equations

$$\begin{aligned}&\varepsilon_{n,1}\left\{\omega_{n,1}^{-1}\left(gk_{n,1}-\omega_{n,1}^2 L_{n,1}\right)-\dot{\gamma}_{n,1}\left(1+M_{n,1}\right)\right\}\cos\zeta_{n,1}+\dot{\varepsilon}_{n,1}\left(1+N_{n,1}\right)\sin\zeta_{n,1}\\&\quad=-\omega k_n^2 a_n^2\left\{\tfrac{3}{4}\varepsilon_{n,1}\cos\zeta_{n,1}+\tfrac{3}{8}\varepsilon_{n,2}\cos\left(\zeta_{n,1}+\theta_n\right)\right\}\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}&\varepsilon_{n,2}\left\{\omega_{n,2}^{-1}\left(gk_{n,2}-\omega_{n,2}^2 L_{n,2}\right)-\dot{\gamma}_{n,2}\left(1+M_{n,2}\right)\right\}\cos\zeta_{n,2}+\dot{\varepsilon}_{n,2}\left(1+N_{n,2}\right)\sin\zeta_{n,2}\\&\quad=-\omega k_n^2 a_n^2\left\{\tfrac{3}{4}\varepsilon_{n,2}\cos\zeta_{n,2}+\tfrac{3}{8}\varepsilon_{n,1}\cos\left(\zeta_{n,2}+\theta_n\right)\right\}\end{aligned} \tag{3.17}$$

for $n = 1, 2, 3, ...$ By adding the coefficients of $\cos\zeta_{n,i}$ in (22) and coefficients of $\sin\zeta_{n,i}$in (23); then, adding coefficients of $\sin\zeta_{n,i}$ in (22) and $\cos\zeta_{n,i}$ in (23), we obtain a system of four equations,

$$\left.\begin{aligned}\frac{d\varepsilon_{n,1}}{dt}&=\left(\frac{1}{2}\omega k_n^2 a_n^2\sin\theta_n\right)\varepsilon_{n,2}\\ \frac{d\varepsilon_{n,2}}{dt}&=\left(\frac{1}{2}\omega k_n^2 a_n^2\sin\theta_n\right)\varepsilon_{n,1}\\ \frac{d\gamma_{n,1}}{dt}&=\frac{1}{2}\left(\frac{gk_{n,1}}{\omega_{n,1}}-\omega_{n,1}\right)+\omega_{n,1}k_{n,1}^2 a_n^2\left(1+\frac{\varepsilon_{n,2}}{2\varepsilon_{n,1}}\cos\theta_n\right)\\ \frac{d\gamma_{n,2}}{dt}&=\frac{1}{2}\left(\frac{gk_{n,2}}{\omega_{n,2}}-\omega_{n,2}\right)+\omega_{n,2}k_{n,2}^2 a_n^2\left(1+\frac{\varepsilon_{n,1}}{2\varepsilon_{n,2}}\cos\theta_n\right)\end{aligned}\right\} \tag{3.18}$$

The last two equations of (3.18) may be added to give an equation for $\theta_n = \gamma_{n,1} + \gamma_{n,2}$,

$$\frac{d\theta_n}{dt}=\omega_n k_n^2 a_n^2\left\{1+\frac{\varepsilon_{n,1}^2+\varepsilon_{n,2}^2}{2\varepsilon_{n,1}\varepsilon_{n,2}}\cos\theta_n\right\}-\omega_n\delta^2 \tag{3.19}$$

The resulting three equations in $\varepsilon_{n,1}$, $\varepsilon_{n,2}$, and θ_n may be reduced to a single equation in $\varepsilon_{n,1}$ by introducing the parameters

$$T_n = k_n^2 a_n^2 \omega t \tag{3.20}$$

and

$$\alpha_n = 1 - \frac{\delta^2}{k_n^2 a_n^2}. \tag{3.21}$$

It may then be shown [1],

$$\frac{d\varepsilon_{n,1}^2}{dT_n}=\frac{d\varepsilon_{n,2}^2}{dT_n}=\varepsilon_{n,1}\varepsilon_{n,2}\sin\theta_n\quad, \tag{3.22}$$

$$\varepsilon_{n,1}\varepsilon_{n,2}\cos\theta_n+\alpha_n\varepsilon_{n,1}^2=\rho_n=const, \tag{3.23}$$

and

$$\varepsilon_{n,1}^2-\varepsilon_{n,2}^2=2\alpha_n\rho_n\left(1-v_n\right)=const. \tag{3.24}$$

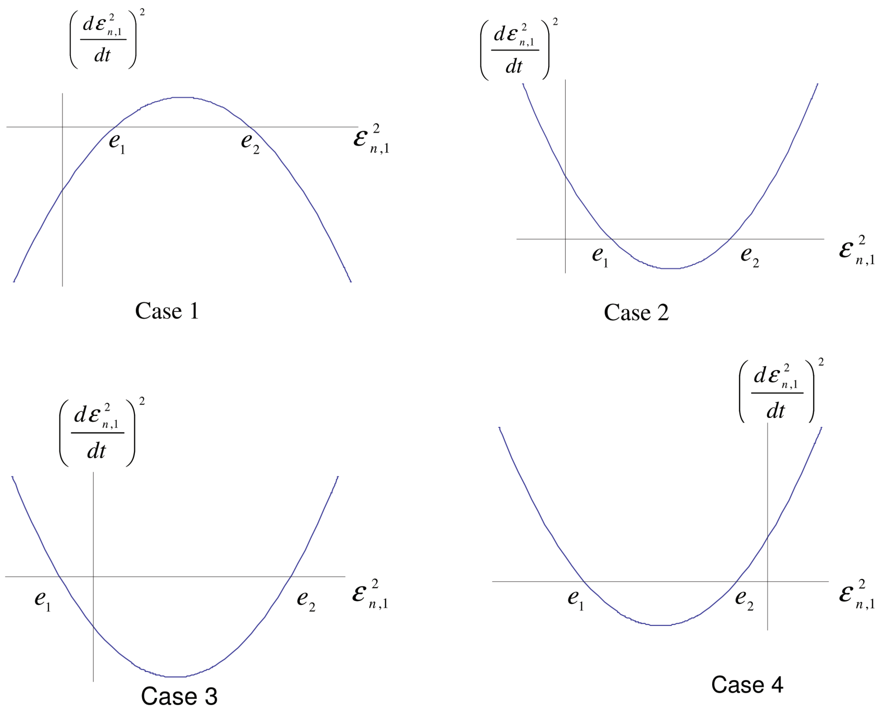

FIGURE 1. Growth rate of sidebands correspo$e_1 e_2 \left(\frac{d\varepsilon^2_{n,1}}{dt}\right)^2$nding to n^{th} harmonic.

We then obtain

$$\left(\frac{d\varepsilon^2_{n,1}}{dT_n}\right)^2 = \left(1-\alpha_n^2\right)\varepsilon^4_{n,1} + 2\alpha_n v_n \rho_n \varepsilon^2_{n,1} - \rho_n^2. \tag{3.25}$$

Equation (3.25) shows that the growth rate of the sidebands is expressible as a quadratic in $\varepsilon^2_{n,1}$. We first consider the case $\alpha_n^2 = 1$, when (3.25) reduces to a linear function in $\varepsilon^2_{n,1}$. Since (3.21) implies $\alpha_n < 1$, we consider $\alpha_n = -1$. Since (3.25) must be satisfied for arbitrary initial values of $d\varepsilon_{n,1}/dT_n$ and $\varepsilon^2_{n,1}$, it can be shown that $\rho_n v_n < 0$ so that (3.25) may be solved directly, with $\varepsilon^2_{n,1} \sim T_n$ so that this case must correspond to instability. We now suppose $\alpha_n^2 \neq 1$ and consider four cases for the form of the quadratic function (3.25), as shown in Figure 1.

In the case $1-\alpha_n^2 < 0$, unbounded sideband growth cannot occur, as there exists only a finite interval $e_1 \leqslant \varepsilon^2_{n,1} \leqslant e_2$ where the growth rate $\left(d\varepsilon^2_{n,1}/dT_n\right)^2 > 0$. If the initial conditions at time $T_n = T_{n,0}$ place $\varepsilon^2_{n,1}(T_{n,0})$ within this interval, then the amplitude $\varepsilon^2_{n,1}(T_n)$ must be greater at a slightly greater time T. Then $\varepsilon^2_{n,1}(T_n)$ will increase until it reaches the value e_2, when $\left(d\varepsilon^2_{n,1}/dT_n\right)^2 = 0$ and further growth is suppressed. If the initial conditions at $T_n = T_{n,0}$ place $\varepsilon^2_{n,1}(T_{n,0})$ outside this interval, then the amplitude $\varepsilon^2_{n,1}(T_n)$ will decrease at slightly greater time T, and unbounded growth will not occur.

We therefore consider the case $1-\alpha_n^2 \geqslant 0$ which, from (3.21) corresponds to

$$0 \leqslant \delta < \sqrt{2}k_n a_n, \tag{3.26}$$

providing the frequency range for the classical Benjamin-Feir instability in the case $n = 1$, when $k_1 = k$ and $a_1 = a$. If a higher harmonic satisfies (3.26), then (3.25) shows there exists an infinite interval $e_1 < \varepsilon_{n,1}^2$ where the growth rate $\left(d\varepsilon_{n,1}^2/dT_n\right)^2 > 0$. For each harmonic, if $1 - \alpha_n^2 > 0$, three cases arise depending on the number of intercepts on the positive $\varepsilon_{n,1}^2$ axis (cases 2-4 in Figure 1). Arguing as in case 1, the initial condition $\varepsilon_{n,1}^2(T_{n,0}) > e_2$ insures unbounded growth in cases 2 and 3, while any initial condition leads to unbounded growth in case 4. With these conditions, the wave is unstable regardless of the sign of $\varepsilon_{n,1}(T_{n,0})$, and the asymptotic growth rate of both sidebands is equal from (3.24).

We can characterize the conditions leading to unbounded growth due to the interaction between the primary wave and the n^{th} harmonic of the disturbance by noting that (3.25) may be written in the form

$$\left(\frac{d\varepsilon_{n,1}^2}{dT_n}\right)^2 = \left(1-\alpha_n^2\right)\left\{\left(\varepsilon_{n,1}^2 - A_N\right)^2 - B_n^2\right\}, \tag{3.27}$$

where

$$A_n = -\frac{\alpha_n v_n \rho_n}{1-\alpha_n^2} \quad \text{and} \quad B_n = \frac{\rho_n\left(1-\alpha_n^2+\alpha_n^2 v_n^2\right)^{1/2}}{\left|1-\alpha_n^2\right|},$$

so that in Figure 1

$$e_2 = \max\left\{A_n + B_n, A_n - B_n\right\}, \tag{3.28}$$

and the condition for unbounded growth in cases 2,3, and 4 may be written as

$$\varepsilon_{n,1}(T_{n,0}) > e_2 \tag{3.29}$$

If (3.26) and (3.29) both hold, then unbounded growth of the n^{th} harmonic of the disturbance will occur. Furthermore, the sign of $\varepsilon_{n,1}(T_n)$ will not change from its sign at initial time T_0, and inspection of (3.18) shows that $\varepsilon_{n,1}(T_n)$ and $\varepsilon_{m,1}(T_{m,0})$ must have the same sign. Therefore, it is not possible for unbounded sideband growth to cancel, when a pair of sidebands for a single harmonic is considered.

4. Conditions for instability: Collective effects of all harmonics

In Section 3, conditions were given on the disturbance and the initial conditions so that interaction between the primary wave and the n^{th} harmonic of the disturbance produce unbounded growth of the sideband amplitudes $\varepsilon_{n,1}(T_n)$ and $\varepsilon_{n,2}(T_n)$. These conditions were dependent on the coefficients of the Stokes wave, $\eta(x,t) = \sum_{n=1}^{\infty} a_n \cos\zeta_n$. Longuet-Higgins showed that the Stokes coefficients are decreasing up to very high orders of n [2,3]. In that representation, each coefficient is represented by a series where the leading term agrees with a formula of Wilton [5],

$$a_n = \frac{n^n}{n!} s_n, \tag{4.1}$$

with s_n a parameter related to the wave height. The first few terms of the Stokes wave may then be written

$$y = s_1 \cos kx + 2s_2 \cos 2kx + \tfrac{27}{6} s_3 \cos 3kx + \tfrac{64}{24} s_4 \cos 4kx + \tfrac{3125}{120} s_5 \cos 5x + ...$$

Using Stokes' representation [4],

$$y = a\cos kx + \tfrac{1}{2}ak^2\cos 2kx + \tfrac{3}{8}a^3k^2\cos 3kx + \tfrac{1}{3}a^4k^3\cos 4kx + \tfrac{125}{384}a^5k^4\cos 5x + ...$$

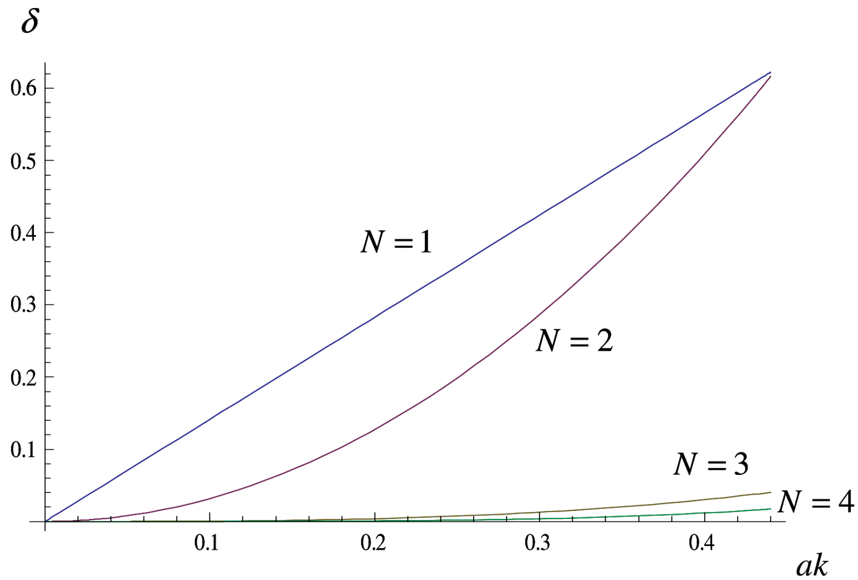

FIGURE 2. The maximum value of δ as a function of wave steepness ak so that all harmonics up to the N^{th} harmonic contribute to instability. $ak\delta N = 2N = 1N = 3N = 4$.

and comparing coefficients, we have

$$s_1 = a, \quad s_2 = \tfrac{1}{4}a^2k, \quad s_3 = \tfrac{1}{12}a^3k^2, \quad s_4 = \tfrac{1}{8}a^3k^2, \quad s_5 = \tfrac{24}{625}a^3k^2.$$

Now consider the collective contribution of all harmonics in (3.9) and (3.10). As noted in Section 3, values of $\delta\,(n)$ providing the marginal stability case for a particular harmonic decrease with n. Therefore, if a particular harmonic of the disturbance has frequency sufficiently close to the corresponding harmonic of the primary wave, unbounded sideband growth will occur for all lower order harmonics, so long as the initial condition (3.29) is met. Furthermore, if (3.29) holds for $n = M$, but not for $n = M + 1$, then unbounded growth will indeed occur, as the coefficients of the higher harmonics will decay in time and will not suppress the instability.

Now suppose that (3.26) holds for some harmonic. Then it must hold for all lower order harmonics, and we may consider the sequence of harmonics where (3.26) holds. For these harmonics, since only cases 2-4 in Figure 1 need be considered,

$$-1 < \alpha_n < 1$$

or, equivalently,

$$0 < \frac{\delta^2}{n^2k^2a_n^2} < 2$$

From (4.1),

$$0 < \frac{(n!)^2\,\delta^2}{n^2k^2n^{2n}s_n^2} < 2$$

which may be written

$$0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!}. \tag{4.2}$$

n	s_n	$a_n = \left(\frac{n^n}{n!}\right) s_n$	$\delta = \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)\,!}$
1	a	0.1	0.013884009
2	$a^2k/4$	0.000490874	0.000136306
3	$a^3k^2/12$	0.003681554	0.001533442
4	$a^4k^3/8$	1.26165×10^{-7}	7.0067×10^{-8}
5	$24a^5k^4/625$	9.28965×10^{-10}	6.4488×10^{-10}

TABLE 1. Sample coefficients of Stokes wave and disturbance frequency corresponding to marginal instability for the nth harmonic.

For the first few harmonics, the marginal stability condition is,

$$n = 1: \qquad 0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!} = \sqrt{2}ak$$

$$n = 2: \qquad 0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!} = 4\sqrt{2}\,k\,s_2 = \sqrt{2}\,a^2k^2$$

$$n = 3: \qquad 0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!} = \frac{27\sqrt{2}\,ks_3}{4} = \frac{9\sqrt{2}}{16}\,a^3k^3$$

$$n = 4: \qquad 0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!} = \frac{256\sqrt{2}\,ks_4}{6} = \frac{16\sqrt{2}}{3}\,a^4k^4$$

$$n = 5: \qquad 0 < \delta < \frac{\sqrt{2}\,k\,n^n s_n}{(n-1)!} = \frac{3125\sqrt{2}\,ks_4}{24} = 5\sqrt{2}\,a^5k^5$$

Since $ak \ll 1$, the value of δ corresponding to the marginal instability decreases with n. This is expected, since disturbance frequencies sufficiently close to that of the primary wave so that sideband amplitude $\varepsilon_{n,1}$ becomes unbounded, will be sufficiently small so that sideband amplitudes for all lower order harmonics are also unbounded. Therefore, (4.2) may be considered a generalization of the Benjamin-Feir criterion in the sense that, if it is satisfied, all harmonics up to order n contribute to unbounded growth. This conclusion is subject to initial conditions, represented by (3.29). Indeed, if (4.2) is satisfied for a (possibly infinite) set of harmonics, instability will occur only if (3.29) is satisfied for at least one value of n.

Values for the first five coefficients using wavelength $2\pi/k = 64$ cm and $a = 0.1$ are compiled in Table 1 using Wilton's formula for the coefficients. These parameters give $ak = 0.098 \ll 1$. The last column provides the marginal instability criterion given in (4.2).

Equation (4.1) shows that the instability criterion (4.2) depends on the wave steepness a^nk^{n-1}. In Figure 2, we illustrate values of $\delta\,(n)$ as a function of steepness for $1 \leqslant n \leqslant 4$ Note that the value of $\delta\,(n)$ decreases with n for all values of $ak < .44$. While $\delta\,(1) = \delta\,(2)$ for $ak = 0.44$, it must be remembered that the small amplitude assumption $ak << 1$ is in place, so that values of ak where this equality occurs are arguably outside the realm applicable to this analysis.

We now consider the growth of the amplitude of the disturbance. Following [1], we solve (3.27) directly for $d\varepsilon_{n,1}^2/dT_n$ by separating variables, and recalling the asymptotic

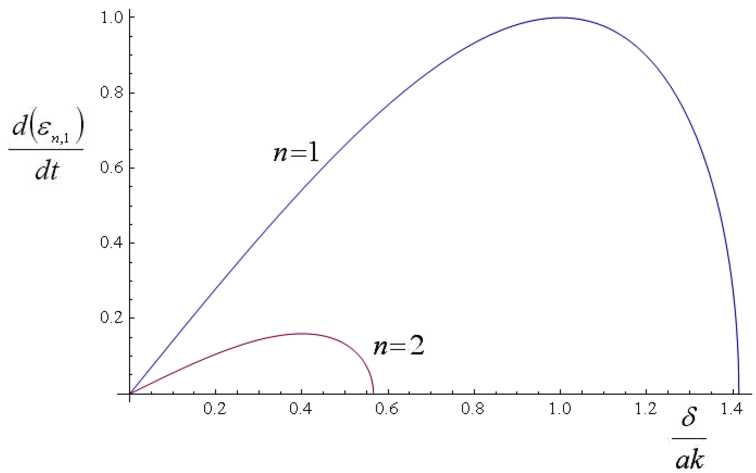

FIGURE 3. Growth rate $d\ln(\varepsilon_{n,1})/dt$ as a function of δ/ak for cases $n = 1, 2, 3$.

equivalence of $\varepsilon_{n,1}(T_n)$ and $\varepsilon_{n,2}(T_n)$. We obtain

$$\varepsilon_{n,1} \sim \exp\left[\frac{\delta}{2}\sqrt{2k_n^2 a_n^2 - \delta^2}\ \omega T_n\right]. \tag{4.3}$$

Applying (3.20),

$$\ln\varepsilon_{n,1} = \frac{\delta}{2}\omega\, a_n k_n \sqrt{2 - \left(\frac{\delta}{a_n k_n}\right)^2}\ t, \tag{4.4}$$

and then (4.1), with $k_n = nk$,

$$\ln\varepsilon_{n,1} = \frac{\delta}{2}\omega\left(\frac{n^n}{n!}s_n\right) nk\sqrt{2 - \left(\frac{n!\delta}{n^{n+1}ks_n}\right)^2}\ t. \tag{4.5}$$

If we select $s_n = a^n k^{n-1}$, noting that this parameter goes to zero with the wave height as required, we obtain

$$\ln\varepsilon_{n,1} = \frac{\omega a^2 k^2}{2}\left(\frac{n^{n+1}}{n!}\right)\left(\frac{\delta}{ak}\right)\sqrt{2 - \left(\frac{(n-1!}{n^n(ak)^{n-1}}\right)^2\left(\frac{\delta}{ak}\right)^2}\ t. \tag{4.6}$$

Figure 3 shows a graph of $d\ln(\varepsilon_{n,1})/dt$ versus δ/ak where the case $n = 1$ corresponds to Figure 1 in [1] and, following that development, we have employed the normalization $wa^2k^2/2 = 1$. For $n > 1$, the factor ak appears and Figure 1 uses $ak = 0.1 \ll 1$ to display the growth rate of the amplitude for the first two harmonics. The pattern persists for higher harmonics, and (4.6) gives the maximum growth rate occurring at

$$\delta = \frac{n\left(ak^{n-1}\right)}{(n-1)!}. \tag{4.7}$$

Note that in the case $n = 1$, (4.7) gives the maximum growth rate occurring at the known value of $\delta = ak$, while the maximum growth rate of the second harmonic occurs at $\delta = 2ak$, shown in Figure 3 for the case $ak = 0.1$.

Assuming appropriate initial conditions and sufficiently small δ so that a set of initial harmonics contribute to resonance, the collective asymptotic growth rate represented by the N^{th} partial sum will be approximated by the sum of the growth rates of the individual harmonics. From (4.5), we have

$$\sum_{n=1}^{N} \frac{d\varepsilon_{n,1}}{dt} = \sum_{i=1}^{N} \frac{\delta}{2}\omega a_n k_n \sqrt{2n^2 - \left(\frac{\delta}{a_n k_n}\right)^2} \;\; \exp\left\{\frac{\delta}{2}\omega a_n k_n \sqrt{2n^2 - \left(\frac{\delta}{a_n k_n}\right)^2}\right\} t. \quad (4.8)$$

which, for sufficiently large N, represents the asymptotic growth rate for the fully nonlinear Stokes wave.

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