August 2013

Bi- and Multi Level Game Theoretic Approaches in Mechanical Design

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BI- AND MULTI LEVEL GAME THEORETIC APPROACHES IN MECHANICAL DESIGN

By

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A Dissertation Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

in

Engineering

at

The University of Wisconsin – Milwaukee

August 2013

ABSTRACT

BI- AND MULTI LEVEL GAME THEORETIC APPROACHES IN MECHANICAL DESIGN

by

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The University of Wisconsin-Milwaukee, 2013
Under the Supervision of Professor Anoop Dhingra

This dissertation presents a game theoretic approach to solve bi and multi-level optimization problems arising in mechanical design. Toward this end, Stackelberg (leader-follower), Nash, as well as cooperative game formulations are considered. To solve these problems numerically, a sensitivity based approach is developed in this dissertation. Although game theoretic methods have been used by several authors for solving multi-objective problems, numerical methods and the applications of extensive games to engineering design problems are very limited. This dissertation tries to fill this gap by developing the possible scenarios for multi-objective problems and develops new numerical approaches for solving them.

This dissertation addresses three main problems. The first problem addresses the formulation and solution of an optimization problem with two objective functions using the Stackelberg approach. A computational procedure utilizing sensitivity of follower's solution to leader's choices is presented to solve the bi-level optimization problem numerically. Two mechanical design problems including flywheel design and design of

high speed four-bar mechanism are modeled based on Stackelberg game. The partitioning of variables between the leader and follower problem is discussed, and a variable partitioning metric is introduced to compare various variable partitions.

The second problem this dissertation focuses on is modeling the multi-objective optimization problem (MOP) as a Nash game. A computational procedure utilizing sensitivity based approach is also presented to find Nash solution of the MOP numerically. Some test problems including mathematical problems and mechanical design problems are discussed to validate the results. In a Nash game, the players of the game are at the same level unlike the Stackelberg formulation in which the players are at different levels of importance.

The third problem this dissertation addresses deals with hierarchical modeling of multi-level optimization problems and modeling of decentralized bi-level multi-objective problems. Generalizations of the basic Stackelberg model to consider cases with multiple leaders and/or multiple followers are missing from the literature. Three mathematical problems are solved to show the application of the algorithm developed in this research for solving hierarchical as well as decentralized problems.

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Anoop Dhingra for his continuous guidance, patience, support and friendship throughout my dissertation work. I appreciate my dissertation committee members: Dr Ron Perez, Dr Ilya Avdeev, Dr Matthew McGinty and Dr Wilkistar Otieno for their support, insights and suggestions. I express my sincere thanks to Professor Hans Volkmer for his help with a mathematical proof of my algorithm. I would like to thank my good friends in Design Optimization group. I also thank the University of Wisconsin-Milwaukee for providing me financial support to do my research. I never will forget the good memories that I had here.

I also would like to thank my father and mother for their influence that shaped me the person I am today.

Finally, my special thanks to my dear wife, Zeinab Salari Far. Without her support and patience this success was not possible.

Dedicated to my dear wife Zeinab and my Parents who always encouraged me and supported me to continue my education.

I should mention my little cute son, Parsa. He has given me energy to work on my research.

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Chapter 1

INTRODUCTION

The topic of multiple objective optimization problems comes from the field of multiple criteria decision making. Multiple criteria decision making deals with methods and algorithms to analytically model and solve problems with multiple objective functions. Multi-objective optimization (MOO) problems requiring a simultaneous consideration of two or more conflicting objective functions frequently arise in design. This dissertation addresses solutions to multi-objective problems arising in the context of mechanical design.

1.1 Multiple Objective Optimization Problems

Multiple criteria decision making has two aspects, namely, multi-attribute decision analysis and multiple objective optimization. Multiattribute decision analysis is applicable to problems in which the decision maker is dealing with a small number of alternatives in an uncertain environment. This aspect helps in resolving public policy problems such as nuclear power plant location, location of an airport, location of a waste processing facility, etc. This aspect has been covered in detail by Keeney and Raiffa (1993). The second aspect of multiple criteria decision making deals with the application of optimization techniques in solving these problems. Techniques for solving multiple criteria (objective) optimization have been developed since early 1970s.

Solutions to multi-objective problems where all objective functions are simultaneously minimized generally do not exist. Therefore, optimization techniques generally look for the best compromise solution amongst all objectives. Since modeling the decision maker's preferences is a primary goal of multi-objective optimization, to find the best compromise solution, there should be some procedure to obtain preference information from the decision maker along with selection of a suitable optimization scheme. Hwang and Masud (1979) classified the optimization techniques into three groups according to the timing of requesting the preference information: (i) Articulation of the decision maker's preferences prior to optimization, (ii) Progressive articulation of preferences (during or in sequence with optimization), and (iii) A posteriori articulation of preferences (after optimization problem has been solved).

Marler and Arora (2004) did a comprehensive survey on the multi-objective optimization methods available on literature. They divided the methods based on how the decision makers articulate their preferences including priori articulation, posteriori articulation and no articulation of preferences.

1.1.1 Methods with a Priori Articulation of Preferences

In these methods, preferences are dictated by the decision maker before the optimization problem is solved. The difference between the methods is based on the different utility functions they may use. Some of the methods which are based on an apriori articulation of preferences are discussed below:

Weighted Sum Method:

The weighting method is a conventional approach to solve multi-objective optimization problems. In this method, a weight is assigned to each objective function and the summation of weighted objective functions is considered as the overall objective function. Steuer (1989) related the weights to the preference of decision maker. Many works have been done to select the weights. Saaty (1977) provided an eigenvalue method to determine the weights. This method involves the pairwise comparison between the objective functions. This provides a comparison matrix with eigenvalues which are the weights. Yoon and Hwang (1995) developed the ranking method to select the weights. In this method, the objective functions are ranked by importance. The least important objective function gets a weight of one and the integer weights with increments are assigned to objective functions that are more important. There are some constraints in applying weighted sum method. For example, Messac et al. (2000) proved that it is impossible for this method to obtain points on non-convex portions of the Pareto optimal frontier set. Also, Papalambros and Wilde (1988) stated that this approach can mislead concerning the nature of optimum design.

Lexicographic Method:

In the Lexicographic method, the objective functions are ranked in order of importance by the decision maker. The optimization problem of objective function deemed most important is solved and the optimum solution is obtained. The second most important objective function can be optimized by considering that the optimum value of the previous objective function should not be changed. This procedure is repeated until all objective functions have been considered. Rentmeesters et al. (1996) showed that the optimum solution of lexicographic method does not satisfy the constraint qualification of Kuhn-Tucker optimality conditions. The authors developed other optimality conditions for the lexicographic approach.

Goal Programming Method:

The basic idea in goal programming is to establish a goal level for each objective function. The overall objective is to minimize the deviation of each objective function from its own goal level. Charnes and Cooper (1961), Lee (1972) and Ignizio (1980) developed the goal programming method. Lee and Olson (1999) reviewed the applications of goal programming method. Although the method has the wide range of applications, there is no guarantee that the solution obtained with this method is a Pareto optimal solution. Weighted goal programming method, in which weights are assigned to the deviation of each objective function from its goal, was developed by Charnes and Cooper (1977).

Bounded Objective Function Method:

In this approach, only the most important objective function is minimized and the other objective functions are considered as constraints. The lower and upper bounds are set for the other objective functions. Haimes et al. (1971) developed ε-constraint method in which only the upper limits are considered. Miettinen (1999) showed that if exists a solution to ε-constraint, then the solution is a weakly Pareto optimal solution. If the solution is unique, then it is Pareto optimal. Chankong and Haimes (1983) proved that if the problem is convex and objective functions are strictly convex, then the solution is unique. Ehrgott and Ryan (2002) improved ε-constraint by allowing the objective functions, which are in constraints, to be violated and penalizing any violation in the objective function.

There are some other methods such as weighted min-max, physical programming and weighted product method in the literature which are based on a priori articulation of preferences.

Utility Theory:

An approach to solving multiple objective optimization problems is correlating the objective functions with value functions; these functions are comparable, and combining these value functions yields a problem with single objective function. Figure of Merit (FOM) is an approach to evaluate the multiple objective functions. A more analytic approach for the evaluation of attributes (objective functions) is Utility analysis developed by Von Neumann (1947), Savage (1954) and Keeney and Raiffa (1993). Thurston (1991) compared FOM approach by utility analysis. Thurston (1994) applied Utility function in optimization of a design problem. The author defined an overall utility function for the design problem of single utility function for each objective function. For each single utility function one single scaling constant has been defined which shows the relative merits of the utility functions. These scaling constants can be obtained by tools such as the Analytic Hierarchy Process (AHP) developed by Saaty (1988) or fuzzy analysis developed by Zadeh (1975). To construct the utility function for each objective function, Thurston (1994) used the lottery questions to assess a set of points on each single utility function. The best fit of these points shows the form of the utility function.

1.1.2 Methods with a Posteriori Articulation of Preferences

The methods using posteriori articulation of preferences first look for a set of Pareto optimal solutions and then according to the decision maker preference, the best compromise solution will be selected from the Pareto optimal set. The advantage of this method is that the solution set is independent of the decision maker's preferences. These methods are constructed with the target of obtaining Pareto points and then selecting the optimal solution amongst these Pareto optimal points.

Algorithms using posteriori articulation of preferences to solve MOLP's can be divided into two categories: (1) Algorithms finding all efficient extreme points. (2) Algorithm finding just efficient points. Steuer (1976) showed that all algorithms are in the first category consist of three phases. Phase one and two find an initial extreme and an initial efficient point respectively. Phase three searches for all efficient extreme points. The algorithms in this category differ in their approaches in phase three. Steuer (1975) developed two computer codes, ADBASE and ADEX to obtain all efficient extreme points. Most works in the area of posteriori articulation of preferences have been done in category two. Some of the methods with a Posteriori Articulation of Preferences are discussed in below:

Normal Boundary Intersection (NBI):

Das and Dennis (1998) developed NBI method. The weighted sum method has a shortcoming of not being able to find Pareto optimal points in non-convex problems. But NBI approach uses a scalarization method to produce Pareto optimal set for non-convex problems. However, the method may also produce non-Pareto optimal points. It means that it does not provide a sufficient condition for the Pareto optimality of the solutions. Das and Dennis (1998) applied NBI to a three-bar truss design problem with five objective functions and four design variables.

Normal Constraint (NC):

Messac et al. (2003) improved NBI method to eliminate non-Pareto optimal solutions from the optimal solution set. In normal constraint method, first it determines the ideal point and its components for each objective function. A plane passing through the ideal points is called the utopia hyper plane. The objective functions are normalized based on the ideal solution. NC method uses the normalized function value to tackle with disparate function scales. This part is different than NBI method.

The other approaches available in the literature are Evolutionary algorithms, Genetic algorithms and Directed search domain.

The weighted sum method, goal programming method and lexicographic method are some of the common approaches in the literature to solve multiple objective optimization problems. In all these methods, the optimum solution is dependent to the preferences of the decision maker. For example, in weighted sum method, by changing the weights of the objective functions, the optimum solution may change. Also, there is no guarantee that the optimum solution of these methods is a Pareto optimal solution. Game theory method is not sensitive to preferences of the decision maker and also it can provide the Pareto optimal solution, for cooperative game.

The other methods are attempting to change the multi-objective function problem to a single objective problem and solve it, but game theoretic models consider each objective function individually. This makes the game theory an interesting topic to do research. This thesis studies game theoretic models which can be applied in mechanical design. In next section, game theory as a tool for solving multi objective problems is reviewed.

1.2 Game Theory Approaches in Design

In game theory, the multi-objective optimization problem is treated as a game where each player corresponds to an objective function being optimized. The notion of designers as players in a game has been demonstrated by several authors (Vincent, 1983; Rao, 1987; Lewis and Mistree, 1997; Badhrinath and Rao, 1996; Hernandez and Mistree, 2000; Shiau and Michalek, 2009). The players control a subset of design variables and seek to optimize their individual payoff functions.

The objective functions of players are often conflicting and the designers may not have the capability of finding a compromise solution. In this situation, game theory can be an appropriate tool to model interactions between designers. There are three types of games that can be uses in the context of design: cooperative game, non-cooperative (Nash) game, and an extensive game. In a cooperative game, the players have knowledge of the strategies chosen by other players and collaborate with each other to find a Pareto-optimal solution. If a cooperation or coalition among the players is not possible, the players make decision by making assumptions about unknown strategies selected by other players. In extensive games, the players make decisions sequentially. The extensive games can be non-cooperative game but it is considered separately in this research. In the next three sections, these three types of games will be discussed in some detail.

1.2.1 Non-Cooperative Games in Design

In a non-cooperative game, each player has a set of variables under his control and optimizes his objective function individually. The player does not care how his selection affects the payoff functions of other players. The players bargain with each other to obtain an equilibrium solution, if one exists. In the literature, this solution is called Nash equilibrium solution (Mcginty (2012)). Vincent (1983) first proposed the use of a non-cooperative game in design. Two designers play in a non-cooperative game and end up to the solution. Vincent showed that the Nash solution is usually not on the Pareto optimal set. Rao (1987) also discussed the Nash game with two designers as players. The case in which there is more than one intersection for rational reaction sets has been studied by Rao. Rao and Hati (1980) extended the idea of two-designer game to define a

Nash equilibrium solution to n-player non-cooperative game. Finding an intersection of rational reaction sets for the all players is difficult, so the Nash solution is usually empty.

Several approaches have been proposed over the years for the computation of Nash solutions in game-theoretic formulations. These include methods based on Nikaido–Isoda function (Contreras et al. 2004), rational reaction set with DOE-RSM approach (Lewis and Mistree 1998) and monotonicity analysis (Rao et al. 1997). Recently, Deutsch et al. (2011) modeled the interaction between an inspection agency and multiple inspectees as a non-cooperative game and obtained all possible Nash equilibria. Their model employs a n-person player game where there is one player (inspection agency) on one side and multiple players (the inspectees) on other side of the game. Explicit closed-form solutions were presented to compute all Nash equilibria.

For some problems arising in mechanical design such as the pressure vessel problem considered in Rao et al. (1997), closed form expressions for Nash equilibria can be obtained using the principles of montonicity analysis (Papalambros and Wilde, 2000). However, in general, numerical techniques are needed to find the solution. A design of experiments based approach (Montgomery 2005) coupled with response surface methodology (Myers and Montgomery 2002) has been proposed by Lewis and Mistree (1998), Marston (2000), and Hernandez and Mistree (2000). This approach has been used by the authors to obtained Nash solutions for non-cooperative games as well as Stackelberg games. Lewis and Mistree (2001) discussed modeling interactions of multiple decision makers. They used statistical techniques such as design of experiments and second-order response surface for numerical approach.

1.2.2 Extensive Games in Design

Extensive design games refer to situations in which the designers make the decisions sequentially. Extensive games with two players have been used in engineering design and are called Stackelberg games. There are two groups of players in this game. One is called Leader which dominates the other group called follower. The leader makes its decision first and according to its decision, the follower optimizes its objective function. Rao and Badhrinath (1997) modeled the conflicts between designer's and manufacturer's objective functions using a Stackelberg game. They construct parametric solution of rational reaction of follower and substitute this solution in the leader's problem to find its optimum solution. In both design examples presented in the paper, the Stackelberg's solution that they obtained was Pareto optimal, although the Stackelberg's solution in general is not Pareto optimal. Lewis and Mistree (1997) showed application of the Stackelberg game in the design of a Boeing 727, while Hernandez (2000) showed the application in design of absorption chillers. Lewis and Mistree (1998) compared the solution of Stackelberg game with cooperative game and Nash solution (non-cooperative game) in design of a pressure vessel and a passenger aircraft. Shiau and Michalek (2009) developed an engineering optimization method by considering competitor pricing reactions to the new product design. Nash and Stackelberg conditions are imposed on three product design cases for price equilibrium.

One critical point in solving a bi-level problem as a Stackelberg game is obtaining the rational reaction set (RRS) of the follower. For simple problems, RRS can be obtained by solving the optimization problem of follower parametrically. It gives an explicit equation for RRS. Rao and Badhrinath (1996) and J.R.Rao and coauthors (1997) applied this approach. The other way to construct RRS of follower is using response surface methodology (RSM) which gives an approximation of RRS. Lewis and Mistree (1997), Lewis (1998) and Hernandez (2000) applied RSM in their problems for solving Stackelberg game.

1.2.3 Cooperative Games in Design

A cooperative game means that all designers or some designers (which form a coalition) cooperate. In this game, the players have knowledge of the strategies chosen by other players and collaborate with each other to find a Pareto-optimal solution. In Nash and Stackelberg game, the players do not cooperate. It is not unusual that players improve their non-cooperative solution by cooperating. This approach has been discussed by Vincent (1983), S.S.Rao (1987), Rao and Badrinath (1996) and Marston (2000). Also, a model for such a game in the context of imprecise and fuzzy information was presented by Dhingra and Rao (1995). This cooperative fuzzy game theoretic model was used to solve a four bar mechanism design problem. The solution of cooperative games is Pareto optimal.

If Player 1 and 2 cooperate, then there may be two approaches to get the cooperative solution. The first approach deals with obtaining the Pareto optimal frontier set. All points, which are in this set, are Pareto optimal in point of view of player 1 and 2. There are several techniques to get Pareto optimal frontier set for players. These include the NSGA-II method developed by Deb (2002) based on genetic algorithms. The TPM is a population-based stochastic approach for finding Pareto optimal frontier set. Das and Dennis (1998) developed NBI method. Shukla and Deb (2007) compared these different approaches and discussed the limitations of each method. NBI and TPM have some difficulties when the Pareto optimal set is discontinuous or non-uniformly spaced.

A problem with these approaches is that a single solution still needs to be selected from the pareto-optimal set for implementation. These methods do not yield a single solution from the pareto-optimal set termed the cooperative solution.

The other approach for obtaining the cooperative solution is by defining a bargaining function. In bargaining function, the players will collaborate to maximize the difference of their objective functions from the worst value that they can get in the game. In the literature, this solution is called Nash bargaining solution (Mcginty (2012)). In this research, whenever it talks about Nash it means Nash equilibrium game (Non-cooperative game).

1.3 Summary

Although game theoretic methods have been used by several authors for solving multi-objective problems, applications of extensive games to engineering design problems are limited. The limited applications of Stackelberg games to design problems are based on using response surface methodologies to construct rational reaction sets (RRS). This research presents an alternate approach for obtaining Nash and Stackelberg solutions that utilize sensitivity based formulation. The sensitivity of optimum solution to problem parameters has been explored by Sobieski et al. (1982) and Hou et al. (2004). This idea is adapted herein to construct the RRS for Nash and Stackelberg solutions. Generalizations of the basic leader-follower model to consider cases with multiple leaders and/or multiple followers are also missing from the literature. This thesis is an attempt to address these identified shortcomings in the existing literature.

1.4 Dissertation Organization

This dissertation has been divided into five main chapters. Chapter 2 discusses terminology associated with solving MOO problems.

Chapter 3 discusses game theoretic mathematical models for solving bi-level optimization problems using Stackelberg game and Nash game approaches. A sensitivity based approach is developed to numerically solve optimization problem modeled as a Stackelberg game. Also, an algorithm is developed to solve the optimization problem modeled as a Nash game. A convergence proof of the proposed algorithm is also presented.

Chapter 4 develops the sensitivity based approach to numerically solve the multi-objective optimization problems modeled as a Nash game. It also considers a bi-level problem with one leader and three followers where the followers have a Nash game among themselves and the interaction between the followers and the leader is a Stackelberg game. When solving a bi-level optimization problem using as a Stackelberg game, it is necessary to capture the sensitivity of leader's solution to follower's variables. Previous work in this area has used design of experiment techniques (DOE) to get the rational reaction set for the follower. This chapter provides an introduction to design of experiments (DOE) and response surface method (RSM). Two examples are presented to demonstrate the benefit of using the proposed sensitivity based over the DOE-RSM method.

Chapter 5 presents two mechanical design problems as an application of the technique which has been presented in chapter 3. The first problem is the flywheel design optimization problem which has been modeled by a bi-level optimization problem. The variable partitioning between the leader and the follower is an issue in this problem and is discussed in detail. A criterion is proposed to identify the best variable partitioning. The design of high speed four-bar mechanism is the second design optimization problem discussed in this chapter. The dynamic and kinematic performances of the mechanism are considered simultaneously. The problem is modeled and solved as a multi-level design optimization problem as a Stackelberg game.

Chapter 6 addresses generalization of the basic Stackelberg model (one leader-one follower problem) to both hierarchical as well as decentralized problems. Towards this end, problems with one leader and several followers are considered where the followers could be arranged in a hierarchical or decentralized manner. Finally, problems with several followers and several leaders are also studied in this research. For decentralized approach with multiple objective functions in leader and the follower two different scenarios are studied. Two numerical examples are solved for these two scenarios.

Finally, Chapter 7 summarizes the main finding of this research.

CHAPTER 2

BASIC CONCEPTS IN MULTI-OBJECTIVE OPTIMIZATION

Multi-objective optimization, also known as multi criteria optimization, deals with a simultaneous consideration of two or more conflicting objective functions in a design problem. When the problem has one objective function, the optimum solution is easy to obtain. It involves optimizing the objective function subject to the constraints present in the problem, but when the problem has more than one objective function, the solution approach is not as in simple as in the single objective function case. This dissertation deals with multi-objective, multi-level design optimization problems and develops new computational approaches for solving such problems.

2.1 Techniques for Solving Multi-Objective Optimization Problems

There are several approaches for solving multi-objective optimization problems. These include the weighted sum method, scalarization techniques, methods to find Pareto optimal frontier, game theory methods, etc. Some of these methods were explained in chapter 1. The method that is considered in this research is using game theory to solve multi-objective optimization problems. In the game theory approach, each player corresponds to an objective function. The players compete/collaborate with each other to improve their respective payoff (objective function value). There are three main types of games: (1) Non-cooperative game. (2) Cooperative game. (3) Sequential game (Leader-Follower). Figure 1.1 shows theses types of games and the techniques which exist in the literature for solving the problems. For example, it can be seen that there are two approaches to get the cooperative solution including Pareto optimal frontier set and maximizing a multiplication function. There are four techniques discussed in the literature, NSGA-II, TPM, NBI and Naïve and slow, to get Pareto optimal frontier set.

In cooperative games, the players have knowledge of the other player's moves and they work (cooperate) together to find the best possible solution. Some times because of process or information barriers, coalition among the players is not possible. So the players can not cooperate. The non-cooperative (Nash) solution is a solution for this case. Besides the cooperative and non-cooperative models, the players can also make their decision sequentially. This sequential interaction may be advantageous when the influence of one player on another is strongly uni-directional. Leader-Follower (Stackelberg) game can be used when one or more objective functions (Leader) make their decision first. Once the leader makes its decision, the follower makes its decision. There is an assumption that the follower will behave rationally. This thesis focuses more on solving multi-objective optimization problems using the Stackelberg game approach. This is because in certain types of design problems, the decisions are made in a sequential manner.

There are some definitions needed to better understand the concepts discussed in subsequent chapters. These definitions and associated terminology are given in the next section.

2.2 Definitions and Terminology

The general form for a multi-objective optimization problem can be stated as selecting values for each of *n* decision variables, $x = (x_1, x_2, ..., x_n)$, in order to optimize *p* objective of functions, $f_1(x), f_2(x), ..., f_p(x)$ subject to constraints. By assuming all objective functions are to be minimized, the problem can be stated mathematically by:

$$min \;\; F(x) = [f_1(x), f_2(x), ..., f_p(x)] \quad subject\ to \;\; x \in X \tag{2.1}$$

where

$$X = \left\{ x \in \mathbb{R}^n \,\middle|\; g_j(x) \le 0, h_k(x) = 0, \; j = 1, 2, ..., m, k = 1, 2, ..., q \right\}$$

where $g_j(x)$ are m inequality constraints and $h_k(x)$ are q equality constraints and X is the set of feasible solutions for problem in Eq. (2.1).

A solution, $x^s \in X$, which minimizes each of the objective functions simultaneously is called a Superior solution. Since at least two of the p objective functions are conflicting, a superior solution to problem shown in Eq. (2.1) rarely exists. The definition of Superior solution mathematically is given below.

Superior Solution: A solution x^s to problem shown in Eq. (2.1) is said to be superior if and only if $x^s \in X$ and $f_i(x^s) \le f_i(x)$ for $i = 1, ..., p$ for all $x \in X$.

The outcome associated with a superior solution is the ideal. The definition of ideal is as follows.

Ideal: The ideal for problem defined in Eq. (2.1) is a point in the outcome space, $F^I = (f_1^I, ..., f_p^I)$, such that f_i^I for $i = 1, ..., p$ is the optimum objective function value for the problem:

$$\text{Min} \;\; f_i^I(x) \qquad \text{subject to} \quad x \in X.$$

Suppose there are two objective functions, then Fig 2.2 shows the Ideal point. Z_1^* and Z_2^* are the optimum value of objective functions f_1, f_2 respectively when they are considered separately. Point Z^* is the Ideal point which minimizes f_1, f_2 simultaneously.

It was mentioned before that the Ideal point rarely exists. It is clear from Fig. 2.2 that the Ideal point is not in the feasible space.

Pareto Solution (Efficient Solution): A Pareto solution x^P to problem in Eq. (2.1) is a feasible solution, $x^P \in X$, for which there does not exist any other feasible solution, $x \in X$, such that $f_i(x) \le f_i(x^P)$ for all $i = 1,\dots,p$ and $f_i(x) < f_i(x^P)$ for at least one $i = 1,\dots,p$.

Often, the optimum solutions may not be Pareto optimal solution but they satisfy other criteria which are making them significant for practical applications. For example, weakly Pareto optimal criteria can be defined as follows:

Weakly Pareto Solution: A point, $x^* \in X$, is weakly Pareto optimal if and only if there does not exist another feasible solution, $x \in X$, such that $f_i(x) < f_i(x^*)$ for all $i = 1,\dots,p$.

Typically, there will be many Pareto solutions to a multi-objective problem. To determine what solution should be selected requires further information from the decision maker concerning his preferences. One way to present this information is the use of a value function over the multiple objectives of the problem.

Value Function: A function υ, which associates a real number $\upsilon(F(x))$ to each $x \in X$, is said to be a value function which represents a particular decision maker's preference provided that:

1) $F(x^1) \sim F(x^2)$ if and only if $\upsilon(F(x^1)) = \upsilon(F(x^2))$ for $x^1, x^2 \in X$;

2) $F(x^1) > F(x^2)$ if and only if $\upsilon(F(x^1)) > \upsilon(F(x^2))$ for $x^1, x^2 \in X$

where $F(x^1) \sim F(x^2)$ denotes that decision maker is indifferent between outcomes $F(x^1)$ *and* $F(x^2)$. $F(x^1) > F(x^2)$ denotes that the decision maker prefers outcomes (x^1) over outcomes (x^2).

Given the value function υ, problem defined in Eq. (2.1) can be changed to the following problem:

$$\text{Max} \quad \upsilon(F(x)) \qquad \text{subject to } x \in X \tag{2.2}$$

Solving problem (2.2) means finding the solution which maximizes the value function over all feasible solutions. Such a solution is called a ***best compromise solution.*** Problem shown in Eq. (2.2) has changed the multi-objective problem to a single objective problem. It means that if a value function can be defined, there would not be any need for multi-objective optimization techniques. But, a value function is difficult to obtain for a multi-objective problem since it requires the preference structure of decision maker to be defined, which is not easily possible. The value function is a kind of utility function discussed in chapter 1.

Bargaining function: This is a function providing the cooperative solution for the players who collaborate with each other in the game. The expression for this function is below:

$$\text{Max} \quad Z = \prod_i (f_i - f_{wi}) \tag{2.3}$$

where f_i is the objective function of the players and f_{wi} is the worst value for objective function i. Suppose player 1 can control x_1 and x_2. If player 1 solves its problem, then the optimum values will be x_{11}^* and x_{21}^*. By plugging in these values in player 2's objective function, then player 2's payoff $f_2(x_{11}^*, x_{21}^*) = f_{w2}$ will be the worst value for $f_2(x_1, x_2)$. Similarly, the worst value for $f_1(x_1, x_2)$ can be obtained. The bargaining function in Eq. (2.3) maximizes the distance of each player's payoff from the worst value. Maximizing Eq. (2.3) gives the cooperative solution for players 1 and 2. It can be shown that solutions which maximize Eq. (2.3) are Pareto-optimal.

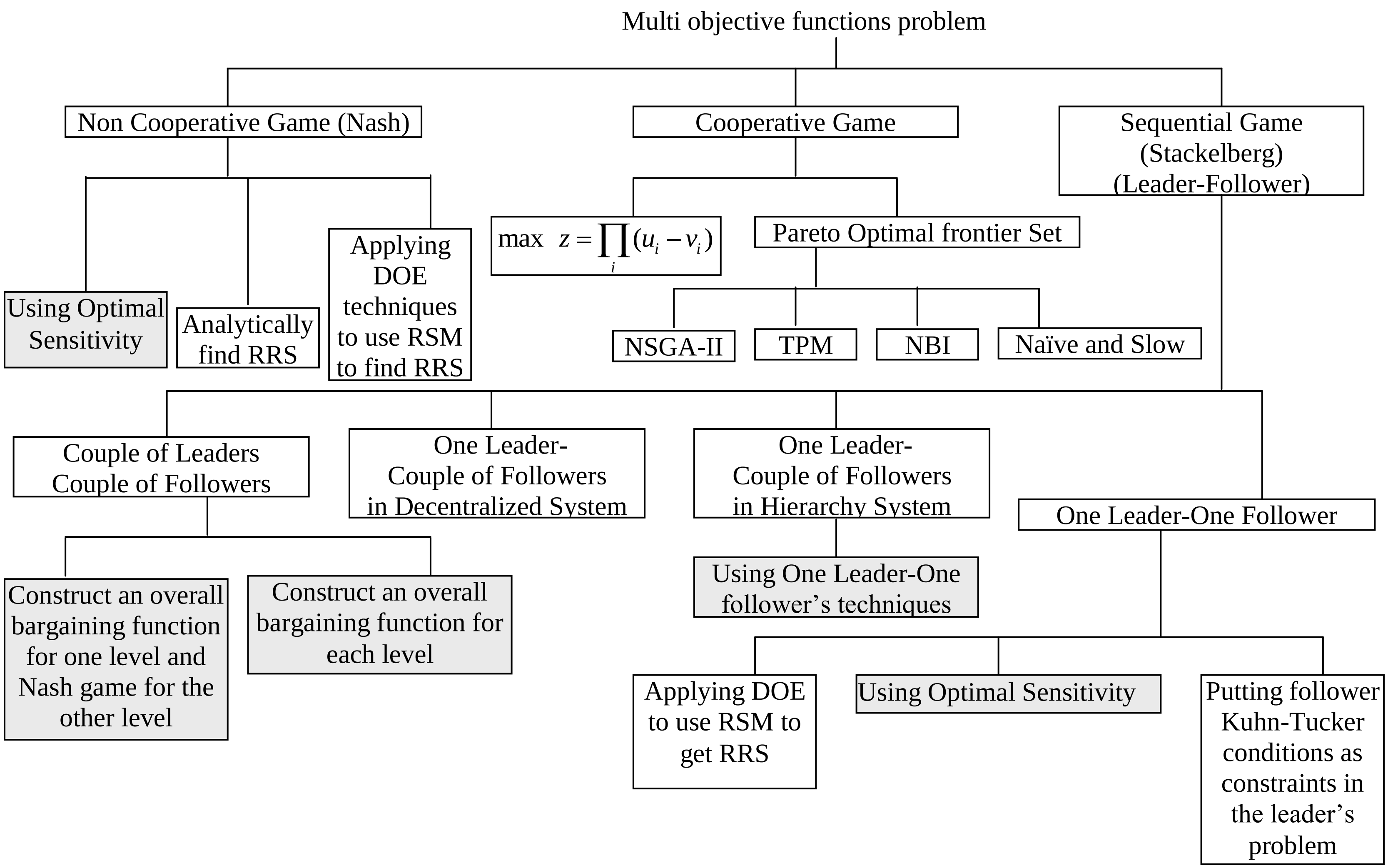


Figure 2.1 Flowchart for Possible Cases for Solving Multi-Objective Problems with Game Approach

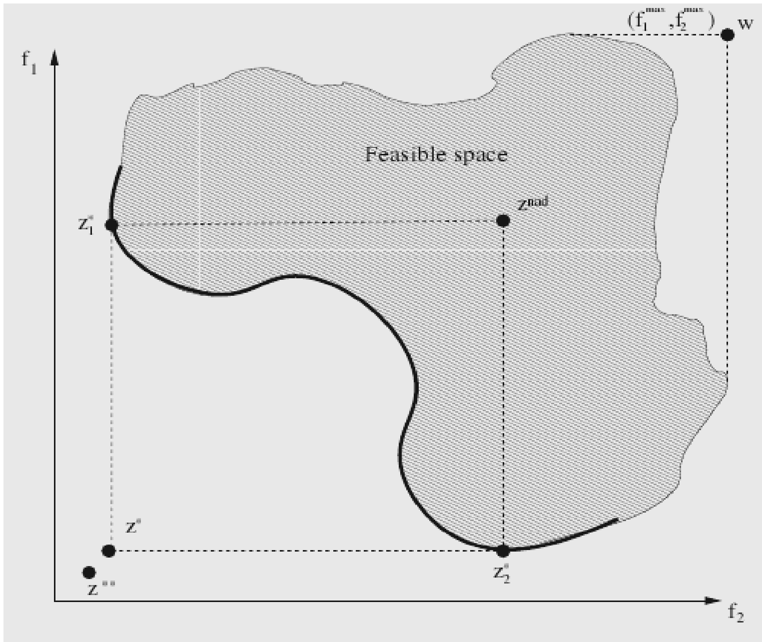


Figure 2.2 Ideal Point

CHAPTER 3

MODELING MULTI-OBJECTIVE PROBLEMS USING GAME THEORY

This chapter discusses mathematical models for solving multi-objective and multi-level design optimization problems using game theory. For multi-level design optimization, the Stackelberg game has been discussed and for multi-objective problem where all objective functions are in the same level, the problem has been modeled as a Nash game. A sensitivity based approach is developed to numerically solve the Stackelberg and Nash game formulations.

3.1 Game Theoretic Models in Design

Consider two players, A and B, who can select strategies x_1 and x_2 where $x_1 \in X_1 \subset R^{n_1}$ and $x_2 \in X_2 \subset R^{n_2}$. Here X_1 and X_2 are the set of all possible strategies each player can select. U is defined as the set of strategies which are feasible for the two players. The objective (cost or loss) functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ account for the cost of players 1 and 2, respectively. The game theory models deal with finding the optimum strategy (x_1, x_2) which corresponds to the decision protocol of the specific game model. The goal of each model is to minimize the objective (loss) function for each player.

An optimum strategy pair (x_1^*, x_2^*) is said to be stable if neither of the players have an incentive to revise their strategy. When the optimum strategy has this property, it is defined to be a Pareto solution for that problem.

The minimum values that the objective functions f_1 and f_2 can reach within the feasible set (x_1, x_2) is defined as

$$\begin{aligned} L_1 &= \inf_{(x_1,x_2)\in U} f_1(x_1,x_2), \\ L_2 &= \inf_{(x_1,x_2)\in U} f_2(x_1,x_2) \end{aligned} \tag{3.1}$$

It is expected that there is no solution (x_1^{**}, x_2^{**}) that simultaneously satisfies $f_1(x_1^{**}, x_2^{**}) = L_1$ and $f_2(x_1^{**}, x_2^{**}) = L_2$. The shadow minimum is defined as $L := (L_1, L_2)$.

The various models and corresponding solutions for the two players A and B can be classified into four categories: (1) Conservative solution (2) Nash solution (3) Cooperative or Pareto solution (4) Stackelberg solution.

The conservative solutions are used when two players do not cooperate. Player one, assumes that player two decides on the strategy x_2 which is least advantageous for player one's objective function. Then player one selects x_1 from the feasible set, which corresponds to the minimum value for f_1. A similar approach is used by player two to find its conservative solution. The strategy (x_1, x_2) that satisfies the above description is called the conservative solution. The mathematical form of the conservative model is defined as

$$f_1^*(x_1) = \sup_{x_2 \in X_2} f_1(x_1, x_2),$$

$$f_2^*(x_2) = \sup_{x_1 \in X_1} f_2(x_1, x_2) \tag{3.2}$$

where $\sup f_1$ corresponds to the value for x_2 which gives the largest f_1 based on the selection of x_1 and $\sup f_2$ corresponds to the value for x_1 which gives the largest f_2 based on the selection of x_2.

$$T_1 = \inf_{x_1 \in X_1} f_1^*(x_1),$$

$$T_2 = \inf_{x_2 \in X_2} f_2^*(x_2) \tag{3.3}$$

where $\inf f_1^*(x_1)$ corresponds to the value for x_1 which gives the smallest f_1 and $\inf f_2^*(x_2)$ corresponds to the value for x_2 which gives the smallest f_2. The conservative strategies for the two players are T_1, T_2. Player A knows that he can not get a value worse than T_1 and will reject any strategy x_1 for a given value x_2 for which $f_1(x_1, x_2) \geq T_1$.

The Nash or non-cooperative solution (x_1^N, x_2^N), has the property:

$$f_1(x_1^N, x_2^N) = \min_{x_1 \in X_1} f_1(x_1, x_2^N)$$

and

$$f_2(x_1^N, x_2^N) = \min_{x_2 \in X_2} f_2(x_1^N, x_2) \tag{3.4}$$

Finding the Nash solution is often difficult since it is a fixed point on a nonlinear map as shown below,

$$(x_1^N, x_2^N) \in X_1^N(x_2^N) \times X_2^N(x_1^N) \tag{3.5}$$

where

$$X_1^N(x_2) := \{x_1^N \in X_1 : f_1(x_1^N, x_2) = \min_{x_1 \in X_1} f_1(x_1, x_2)\} \tag{3.6}$$

$$X_2^N(x_1) := \{x_2^N \in X_2 : f_2(x_1, x_2^N) = \min_{x_2 \in X_2} f_2(x_1, x_2)\} \tag{3.7}$$

where $X_1^N(x_2)$ and $X_2^N(x_1)$ are called rational reaction sets for players 1 and 2 respectively. The term rational reaction set is discussed in the next section.

The Cooperative or Pareto solution (x_1^P, x_2^P) is expected to yield a better result than the solution related to non cooperative solution. It is likely that the players can improve on the Nash solution by cooperating with each other. A pair (x_1^P, x_2^P) is a Pareto solution if there is no other pair (x_1, x_2) such that,

$$f_1(x_1, x_2) < f_1(x_1^P, x_2^P)$$

and

$$f_2(x_1, x_2) < f_2(x_1^P, x_2^P) \tag{3.8}$$

The set of Pareto solution is usually large and it requires some other selection criteria within the Pareto solutions.

The core solution (x_1^C, x_2^C) is the same as Pareto solution with these two additions,

$$f_1(x_1^C, x_2^C) < T_1 \quad \text{and} \quad f_2(x_1^C, x_2^C) < T_2 \tag{3.9}$$

where T_1 and T_2 have been defined in Eq. (3.3).

The Stackelberg game is a special case of a bi-level game where one player dominates the other player. Suppose player A is the leader (or dominant) and player B is the follower. Player A knows the optimum strategy (solution) of Player B. When player A chooses a strategy (its design variables) player B can see the choices made by Player A. Player B solves its problem and finds the optimum solution with respect to player B. Player A can now adjust its strategy based on choices made by player B.

The model of the Stackelberg solution when player A is the leader can be written as follows,

$$\begin{aligned} &\mathit{minimize}: f_1(x_1, x_2) \\ &(x_1, x_2) \in U \\ &\\ &\mathit{subject\ to}: x_2 \in X_2^N(x_1). \end{aligned} \tag{3.10}$$

On the other hand, when B is the leader, the problem is:

$$\begin{aligned} &\mathit{minimize}: f_2(x_1, x_2) \\ &(x_1, x_2) \in U \\ &\\ &\mathit{subject\ to}: x_1 \in X_1^N(x_2). \end{aligned} \tag{3.11}$$

where $X_1^N(x_2), X_2^N(x_1)$ are given by Eqs. (3.6) and (3.7).

3.1.1 Rational Reaction Set for Stackelberg and Nash Solutions

Consider two players, 1 and 2, with objective functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$. They select strategies x_1 and x_2 from a set of possible strategies where $x_1 \in X_1 \subset R^{n_1}$ and $x_2 \in X_2 \subset R^{n_2}$ respectively. Here X_1 and X_2 are the set of all possible strategies each player can select. The game theory approach deals with finding the optimum strategy (x_1, x_2) which results in highest possible payoff for each player. Three game theoretic models that have been used in the context of engineering design include non-cooperative (Nash) game, cooperative game, and the Stackelberg game in which one player dominates other player(s). In the Stackelberg method, the leader and the follower have different objective functions and each player has control over specific variables. The leader chooses optimum values for its variables by solving its problem, then the follower observes those values and solves its problem and finds optimum values for its variables. From an implementation view point, the bi-level optimization problem is solved by using backward induction. It begins with follower's problem. Assuming the value of leader's decision variables are fixed, the follower's objective function is optimized. By varying follower's variables, the optimum values of follower's variables as a function of leader's variables are obtained. Then these functions are substituted in the leader's problem and the leader optimizes its objective function to obtain optimum values of leader's variables.

The Stackelberg game can be used to model the behavior of decision makers (players) when they operate in a hierarchical manner. Let $\left\{f_f, f_l\right\}$ be a set of objective functions for follower and leader respectively. The follower and leader's problems are given by Eqs. (3.12) and (3.13) respectively:

$$\begin{aligned} &\min f_f = f_f(x_f, x_l) \\ &\text{by varying} \quad x_f \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\min f_l = f_l(x_f, x_l) \\ &\text{by varying} \quad x_l \end{aligned} \tag{3.13}$$

where subscripts f and l correspond to follower and leader objective function and variables respectively. The follower can determine its set of optimum solution(s) based on the choices made by the leader. This solution set is called rational reaction set (RRS) for the follower. The RRS for the follower is defined as follows:

$$f_f\left(x_f^R, x_l\right) = \min_{x_f \in X_2} f_f\left(x_f, x_l\right) \rightarrow \quad x_f^R\left(x_l\right) \tag{3.14}$$

where $x_f^R\left(x_l\right)$ is the optimum solution of the follower (player 2) which varies depending on the strategy x_l chosen by the leader (player 1). It implies that the optimum values of follower's variables are given as a function of leader's variables (Eq.(3.15)):

$$x_f^* = x_f^R\left(x_l\right) \tag{3.15}$$

This RRS of the follower is substituted in Eq. (3.13) to solve the leader's problem and find optimum values of the leader's variables. Next, by substituting these optimum values in Eq. (3.15), the optimum values of follower variables can be obtained.

The Nash game is a non-cooperative game where each player determines its set of optimum solutions based on the choices made by other player(s). This set of solutions for each player is the rational reaction set (RRS). The RRS for players 1 and 2 are defined as follows:

$$f_1\left(x_1^N, x_2\right) = \min_{x_1 \in X_1} f_1\left(x_1, x_2\right) \rightarrow \quad x_1^N\left(x_2\right) \tag{3.16}$$

$$f_2\left(x_1, x_2^N\right) = \min_{x_2 \in X_2} f_2\left(x_1, x_2\right) \rightarrow \quad x_2^N\left(x_1\right) \tag{3.17}$$

where x_1^N is the optimum solution of player 1 which varies depending on the strategy x_2 chosen by player 2. The function $x_1^N\left(x_2\right)$ would be RRS for player 1. Similarly, $x_2^N\left(x_1\right)$ is the RRS of player 2. The intersection of these two sets, if it exists, is the Nash solution for the non-cooperative game. Therefore, $\left(x_1^N, x_2^N\right)$ is a Nash solution if

$$\left(x_1^N, x_2^N\right) = x_1^N\left(x_2\right) \otimes x_2^N\left(x_1\right) \tag{3.18}$$

When the Stackelberg and Nash problems are solved numerically, it is very difficult to obtain explicit expressions for $x_f^R\left(x_l\right)$, $x_1^N\left(x_2\right)$ and $x_2^N\left(x_1\right)$. The numerical approach which exists in the literature for generating the RRS is based on design of experiments (DOE) combined with response surface methodology (RSM). The RSM utilizes DOE (design of experiments) techniques to construct various experiments for the players that one is interested in finding the RRS. Then a response surface is fitted to the experiment outcomes to find an approximation to the RRS.

This thesis presetns a new method based on sensitivity information to approximate the RRS for the players. The proposed method uses Taylor series to approximate the RRS. For example, for the Stackelberg problem, the x_f (optimum solution of the follower) can be written as:

$$x_f = x_f^* + \frac{dx_f^*}{dx_l}\Delta x_l = x_f^* + \frac{dx_f^*}{dx_l}(x_l - x_l^1) \tag{3.19}$$

where x_f^* is the optimum solution of the follower's problem corresponding to x_l^1, and $\frac{dx_f^*}{dx_l}$ denotes how the optimum solution of the follower's problem is varying with leader's variable, x_l. Eq. (3.19) needs $\frac{dx_f^*}{dx_l}$ which denotes the sensitivity of optimal solution of follower's problem to the leader's variable. x_f and x_l can be the vectors, but here for the sake of simplicity they are represented as scalar variables. To find $\frac{dx_f^*}{dx_l}$, the sensitivity information for the follower problem is needed. In the next section, it is discussed how the sensitivity information will be obtained.

3.2 Optimum Sensitivity Derivatives

An optimization problem with inequality constraints can be represented as

$$\begin{aligned} &\text{Min } f(x,p) \quad x \in R^n \\ &\quad \text{by varying } x \\ &\text{subject to } \; g_j(x,p) \le 0 \quad j = 1,2,\ldots n_g \end{aligned} \tag{3.20}$$

where $x \in R^n$ is the variable vector which is unknown and p denotes the vector of problem parameters. The integer n_g is the number of inequality of the constraints. There are no equality constraints. Hou et al. (2004) showed the general case of this problem when there are equality constraints as well. The problem parameters can be a vector, for simplicity, they are taken as a scalar. Let the optimum solution to problem given by Eq. (3.20) be x^*. It is known that x^* satisfies Kuhn-Tucker necessary conditions,

$$\begin{aligned} &\frac{dL}{dx}(x^*, p) = 0 \\ &g_j(x^*, p) = 0 \qquad j \in \bar{n}_g \\ &\lambda_j \geq 0 \end{aligned} \tag{3.21}$$

where $\bar{n}_g$ denotes active constraints and the Lagrangian (L) defined as

$$L = f + \sum_{j=1}^{\bar{n}_g} \lambda_j g_j \tag{3.22}$$

Rewriting Eq. (3.21):

$$\begin{aligned} &\frac{dL}{dx}\left(x^*, p\right) = \frac{\partial f\left(x^*, p\right)}{\partial x} + \sum_{j=1}^{\bar{n}_g} \lambda_j \frac{\partial g_j\left(x^*, p\right)}{\partial x} = 0 \\ &g_j\left(x^*, p\right) = 0 \qquad j \in \bar{n}_g \\ &\lambda_j \geq 0 \end{aligned}$$

The values of λ_j can be obtained by

$$\lambda = -(G^T G)^{-1} G^T \nabla f \tag{3.23}$$

where $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \end{bmatrix}$, are Lagrange multipliers and $G = [\nabla g_1 \ \nabla g_2 \ldots]$ contains gradient information for active constraints.

The optimum sensitivity derivatives can be obtained by differentiating Eq. (3.21) with respect to p as

$$\frac{\partial^2 L}{\partial x^2}\frac{dx^*}{dp}+\frac{\partial^2 L}{\partial x\partial p}=0=\left[\frac{\partial^2 f\left(x^*,p\right)}{\partial x^2}+\sum_{j=1}^{\bar{n}_g}\lambda_j\frac{\partial^2 g_j\left(x^*,p\right)}{\partial x^2}\right]\frac{dx^*}{dp}+ \\ \frac{\partial^2 f\left(x^*,p\right)}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g}\lambda_j^1\frac{\partial^2 g_j\left(x^*,p\right)}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g^1}\frac{\partial\lambda_j}{\partial p}\frac{\partial g_j\left(x^*,p\right)}{\partial x} \tag{3.24}$$

$$\frac{dg_j\left(x^*,p\right)}{dp}=\frac{\partial g_j\left(x^*,p\right)}{\partial x_1}\frac{dx^*}{dp}+\frac{\partial g_j\left(x^*,p\right)}{\partial p}=\left(\frac{\partial g_j}{\partial x}\right)^T\frac{dx^*}{dp}+\frac{\partial g_j}{\partial p}=0 \tag{3.25}$$

where $\frac{\partial^2 L}{\partial x^2}=\frac{\partial^2 f}{\partial x^2}+\sum_{j\in\bar{n}_g}\lambda_j\frac{\partial^2 g_j}{\partial x^2}$, $\quad\frac{\partial^2 L}{\partial x\partial p}=\frac{\partial^2 f}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g}\lambda_j\frac{\partial^2 g_j}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g}\frac{\partial\lambda_j}{\partial p}\frac{\partial g_j}{\partial x}$

Rearranging Eqs. (3.24) and (3.25)

$$\left[\frac{\partial^2 f\left(x^*,p\right)}{\partial x^2}+\sum_{j=1}^{\bar{n}_g}\lambda_j\frac{\partial^2 g_j\left(x^*,p\right)}{\partial x^2}\right]\frac{dx^*}{dp}+\sum_{j=1}^{\bar{n}_g}\frac{\partial\lambda_j}{\partial p}\frac{\partial g_j^1\left(x^*,p\right)}{\partial x}= \\ -\left[\frac{\partial^2 f\left(x^*,p\right)}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g}\lambda_j\frac{\partial^2 g_j\left(x^*,p\right)}{\partial x\partial p}\right] \tag{3.26}$$

$$\left(\frac{\partial g_j}{\partial x}\right)^T\frac{dx^*}{dp}=-\frac{\partial g_j}{\partial p}=0 \tag{3.27}$$

Eqs. (3.26) and (3.27) can be written in a matrix form as

$$\begin{bmatrix}\frac{\partial^2 L}{\partial x^2} & \frac{\partial g}{\partial x}\\ \left(\frac{\partial g}{\partial x}\right)^T & 0\end{bmatrix}\begin{bmatrix}\frac{dx^*}{dp}\\ \frac{d\lambda}{dp}\end{bmatrix}=-\begin{bmatrix}\frac{\partial^2 f}{\partial x\partial p}+\sum_{j=1}^{\bar{n}_g}\lambda_j\frac{\partial^2 g_j}{\partial x\partial p}\\ \frac{\partial g}{\partial p}\end{bmatrix} \tag{3.28}$$

$\frac{dx^*}{dp}$ which indicates how the optimum solution (in Eq. (3.20)) varies with p. By solving this system of equations, the vector x can be written as

$$x = x^* + \frac{dx^*}{dp}\Delta p \tag{3.29}$$

Now, compare Eq. (3.29) with Eq. (3.19). The follower variables x_f can be replaced by Eq. (3.29) in leader problem. Whenever the leader variables are updated in leader optimization problem, i.e. Δx_l changes, the follower variables x_f will be updated in the leader problem. The next section develops the sensitivity based approach to numerically solve the Stackelberg game model.

3.3 Sensitivity Based Algorithm for Obtaining Stackelberg Solutions

Let us assume that the optimization problem of players 1 (leader) and 2 (follower) for Stackelberg game model can be written as follows:

$$\begin{array}{ll} \text{Min} & f_L\left(x_{lead}, x_f\right) \quad x \in R^n \\ & \text{by varying } x_{lead} \end{array}$$

$$\text{subject to} \quad g_j^L\left(x_1, x_2\right) \le 0 \quad j = 1, 2, \ldots n_g^1 \tag{3.30}$$

$$\begin{array}{ll} \text{Min} & f_f\left(x_{lead}, x_f\right) \quad x \in R^n \\ & \text{by varying } x_f \end{array}$$

$$\text{subject to} \quad g_j^f\left(x_1, x_2\right) \le 0 \quad j = 1, 2, \ldots n_g^2 \tag{3.31}$$

where $x_{lead} \in R^{n_1}, x_f \in R^{n_2}, x = (x_{lead}, x_f)$. The leader (Player 1) has control over x_{lead} and the follower's (player 2) strategy $\left(x_f\right)$ acts as parameter vector in player 1's problem. n_g^1 and n_g^2 denotes the number of inequality constraints of players 1 and 2 respectively. First, the optimization problem of the follower shown in Eq. (3.31) will be solved by assuming an initial value for $x_{lead} = x_{lead}^{1}$. Let the optimum value corresponding to $x_{lead} = x_{lead}^{1}$ be x_f^{*1}. As the value of x_{lead}^{1} changes, the optimum value of x_f^{*1} will change. This change can be approximated by using a first order Taylor series expansion as shown in Eq. (3.32)

$$x_f = x_f^{*1} + \frac{dx_f^{*1}}{dx_{lead}} \partial x_{lead} = x_f^{*1} + \frac{dx_f^{*1}}{dx_{lead}} \left(x_{lead} - x_{lead}^{1} \right) \tag{3.32}$$

where x_f^{*1} is optimum vector of follower's problem corresponding to initial values of leader's variables. The term $\frac{dx_f^{*1}}{dx_{lead}}$ is the sensitivity information of the follower problem and can be obtained by solving the system of equations explained in the previous section. ∂x_{lead} is the difference of leader's variables from the initial value$\left(x_{lead}^{1} \right)$. x_f is the updated new optimum vector for follower' problem corresponding to new values of leader's variables. This function can be substituted in the leader's problem in Eq. (3.30). Then, the leader solves its problem. Let the optimum solution be x_{lead}^{*1}. Compare this value with x_{lead}^{1}. If there is the significant difference between the two values, substitute $x_{lead} = x_{lead}^{*1}$ and repeat the steps until convergence occurs. Fig. 3.1 shows the flowchart of this algorithm. Chapter 5 presents an application of this algorithm to two mechanical design problems including flywheel design and design of high speed four bar mechanisms.

3.4 Sensitivity Based Algorithm for Obtaining Nash Solutions

Assume that the optimization problem for players 1 and 2 for Nash game model can be written as follows:

$$\text{Min} \quad \begin{array}{l} f_1(x_1, x_2) \quad x \in R^n \\ \text{by varying } x_1 \end{array}$$

$$\text{subject to} \quad g_j^1(x_1, x_2) \le 0 \quad j = 1, 2, \ldots n_g^1 \tag{3.33}$$

and for player 2

$$\text{Min} \quad \begin{array}{l} f_2(x_1, x_2) \quad x \in R^n \\ \text{by varying } x_2 \end{array}$$

$$\text{subject to} \quad g_j^2(x_1, x_2) \le 0 \quad j = 1, 2, \ldots n_g^2 \tag{3.34}$$

where $x_1 \in R^{n_1}, x_2 \in R^{n_2}, x = (x_1, x_2)$ and $n = n_1 + n_2$. Player 1 has control over x_1 and player 2's strategy (x_2) acts as parameter vector in player 1's problem. n_g^1 and n_g^2 denote the number of inequality constraints in problems for players 1 and 2 respectively.

Now, the problem modeled as a Nash game (Eqs. (3.33) and (3.34)) is ready to be solved numerically using the sensitivity based approach.

Assume an initial value for $x_2 = x_2^1$, and solve optimization problem of player 1 in Eq. (3.33). Let the optimum value corresponding to $x_2 = x_2^1$ be x_1^{*1}. As the value of x_2^1 changes, the optimum value x_1^{*1} will change. One can linearize this change by using a first order Taylor series expansion as shown in Eq. (3.35). Here $\frac{dx_1^*}{dx_2}$ is calculated by performing a sensitivity analysis as explained in section 3.2.

$$x_1 = x_1^{*1} + \frac{dx_1^*}{dx_2} \Delta x_2 = x_1^{*1} + \frac{dx_1^*}{dx_2}\left(x_2 - x_2^1\right) \tag{3.35}$$

Similarly, the optimum solution for player 2 can be found by assuming an initial value $x_1 = x_1^1$. Eq. (3.36) shows how the optimum solution of player 2 varies as x_1 changes from x_1^1. Here the value of $\frac{dx_2^*}{dx_1}$ is computed using sensitivity analysis.

$$x_2 = x_2^{*1} + \frac{dx_2^*}{dx_1}\Delta x_1 = x_2^{*1} + \frac{dx_2^*}{dx_1}\left(x_1 - x_1^1\right) \tag{3.36}$$

Consider now Eqs. (3.35) and (3.36). One can solve this system of equations for x_1 and x_2. Let the solution be denoted as x_1^{N1} and x_2^{N1}. Compare these values with x_1^1 and x_2^1 respectively. If there is the significant difference between the two solutions, substitute $x_1 = x_1^{N1}$ and $x_2 = x_2^{N1}$ and repeat the steps till convergence occurs. Fig. 3.2 shows the flowchart of this algorithm. The next section presents a convergence of this algorithm. Chapter 4 presents application of this algorithm for solving some mathematical and mechanical design problems.

3.4.1 Convergence Proof

This section discusses convergence proof for algorithms presented in Figs. 3.1 and 3.2. Assume players 1 and 2 have real-valued functions $x = x(y)$ and $y = y(x)$ as their rational reaction sets respectively. The intersection of these two functions provides the Nash solution. Let $g(x, y) = x - x(y)$ and $h(x, y) = y - y(x)$. The solution of $g(x, y) = h(x, y) = 0$ gives the Nash solution. In order to find a solution of

$$g(x, y) = h(x, y) = 0 \tag{3.37}$$

consider the following approximation method. One picks an initial guess $\left(x_1, y_1\right)$. Then it one tries to find x_1^*, y_1^* such that

$$g(x_1^*, y_1) = h(x_1, y_1^*) = 0 \tag{3.38}$$

One linearizes g at (x_1^*, y_1) and h at (x_1^*, y) and determines the next approximation (x_2, y_2) as the solution of

$$\begin{aligned} &g_x(x_1^*, y_1)(x - x_1^*) + g_y(x_1^*, y_1)(y - y_1) = 0, \\ &h_x(x_1, y_1^*)(x - x_1) + h_y(x_1, y_1^*)(y - y_1^*) = 0. \end{aligned} \tag{3.39}$$

Then we iterates on this procedure. It can be noticed that these two equations are equal two Eqs. (3.35) and (3.36) where $x = x_1$ and $y = x_2$. This is exactly the same thing shown in flowchart Fig. 3.2.

The local convergence of the method can be proved as follows. To simplify the notation, it is assumed that the system shown in Eq. (3.37) has the solution (0,0).

Theorem 1. Suppose that g, h are defined and twice continuously differentiable in a neighborhood of $\left(0,0\right)$. Suppose that

$$g\left(0,0\right) = h\left(0,0\right) = 0, \qquad g_x\left(0,0\right) \neq 0, h_y\left(0,0\right) \neq 0, \tag{3.40}$$

and

$$g_x\left(0,0\right) h_y\left(0,0\right) - g_y\left(0,0\right) h_x\left(0,0\right) \neq 0. \tag{3.41}$$

Then the method converges in the following sense. If (x_1, y_1) is chosen sufficiently close to (0, 0) then the sequence of iterates (x_n, y_n) is well-defined and converges to (0,0).

Proof. By using Eq. (3.40) and the implicit function theorem to the equation $g(x, y) = 0$ at $(0,0)$, one obtains $a > 0$ and a twice continuously differentiable function $G(y)$ on $y \in [-a, a]$ with $G(0) = 0$ such that

$$g(G(y), y) = 0 \qquad \textit{for} \;\; y \in [-a, a]$$

Similarly, there is $b > 0$ and a twice continuously differentiable function $H(x)$ on $x \in [-b, b]$ with $H(0) = 0$ such that

$$h(x, H(x)) = 0 \qquad \textit{for} \; x \in [-b, b]$$

Replacing a, b by $\min \{a, b\}$, one may assume that $a = b$. By assumption in Eq. (3.41) $G'(0) H'(0) \neq 1.$ Therefore, one can make $a > 0$ so small that

$$\left|1 - G'(x) H'(y)\right| \geq c > 0 \quad \textit{for all} \; x, y \in [-a, a]. \tag{3.42}$$

Moreover, set

$$\begin{aligned} k_1 &:= \max \left|G'(y)\right| \qquad y \in [-a, a] \\ k_2 &:= \max \left|G''(y)\right| \qquad y \in [-a, a] \\ L_1 &:= \max \left|H'(x)\right| \qquad x \in [-a, a] \\ L_2 &:= \max \left|H''(x)\right| \qquad x \in [-b, b] \end{aligned} \tag{3.43}$$

Let $\left(x_1, y_1\right) \in \left[-a, a\right]^2$. Then $x_1^* = G(y_1)$ and $y_1^* = H(x_1)$ are well defined. The linear system shown in Eq. (3.39) to find x_2, y_2 changes to:

$$\begin{aligned} &x - G\left(y_1\right) = G^{'}(y_1)(y - y_1) \\ &y - H(x_1) = H^{'}(x_1)(x - x_1) \end{aligned} \tag{3.44}$$

The solution of this system of equation is

$$\begin{aligned} x_2 &= \frac{G^{'}(y_1)\left(H(x_1) - H^{'}(x_1)x_1\right) + G(y_1) - G^{'}(y_1)y_1}{1 - G^{'}(y_1)H^{'}(x_1)}, \\ y_2 &= \frac{H^{'}(x_1)\left(G(y_1) - G^{'}(y_1)y_1\right) + H(x_1) - H^{'}(x_1)x_1}{1 - G^{'}(y_1)H^{'}(x_1)} \end{aligned} \tag{3.45}$$

Note that the denominators are nonzero because of assumption in Eq. (3.42). By definition of k_2, L_2 in Eq. (3.43),

$$\begin{aligned} &\left|G(y) - G^{'}(y)y\right| \le \frac{k_2}{2} y^2 \qquad \mathit{for}\ y \in \left[-a, a\right], \\ &\left|H(x) - H^{'}(x)x\right| \le \frac{L_2}{2} x^2 \qquad \mathit{for}\ x \in \left[-a, a\right] \end{aligned} \tag{3.46}$$

it follows that

$$\left|x_2\right| \le c^{-1}\left(k_1 \frac{1}{2} L_2 x_1^2 + \frac{1}{2} k_2 y_1^2\right)$$

in the same way

$$\left|y_2\right| \le c^{-1}\left(L_1 \frac{1}{2} K_2 y_1^2 + \frac{1}{2} L_2 x_1^2\right)$$

Therefore, there exists a constant M such that

$$\max\{|x_2|,|y_2|\} \le \mathrm{M}\left(\max\{|x_1|,|y_1|\}\right)^2$$

If (x_1, y_1) is selected so close to $(0,0)$ such that

$$\max\{|x_1|,|y_1|\} \le min\left\{a, \frac{1}{2\mathrm{M}}\right\},$$

then

$$\max\{|x_2|,|y_2|\} \le \frac{1}{2}\max\{|x_1|,|y_1|\}.$$

By iterating it can be obtained

$$\max\{|x_{n+1}|,|y_{n+1}|\} \le \frac{1}{2}\max\{|x_n|,|y_n|\}$$

For all $n = 1, 2, 3, \ldots$. This shows that the sequence (x_n, y_n) converges to $(0,0)$. The sequence of numerical iterates converge to the optimum solution.

Let the solution after convergence occurs be (x_1^*, x_2^*). It is shown next that (x_1^*, x_2^*) is the Nash solution for players 1 and 2.

Proposition 1: If (x_1^*, x_2^*) is the solution obtained from the algorithm shown in Fig. 3.2, then (x_1^*, x_2^*) is the Nash solution for players 1 and 2.

Proof: Since (x_1^*, x_2^*) is the solution obtained after convergence criterion has been met, one can write

$$\begin{aligned} \text{Min} \quad & f_1\left(x_1, x_2^*\right) = f_1\left(x_1^*, x_2^*\right) \\ & x_1 \in \mathrm{X}_1 \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \text{Min} \quad & f_2\left(x_1^*, x_2\right) = f_2\left(x_1^*, x_2^*\right) \\ & x_2 \in \mathrm{X}_2 \end{aligned} \tag{3.48}$$

where X_1 and X_2 are feasible solution sets of players 1 and 2 respectively.

According to the definition of a Nash solution, any solution which satisfies Eqs. (3.16) and (3.17) is a Nash solution. Comparing Eqs. (3.47) and (3.48) with Eqs. (3.16) and (3.17), it is obvious that

$$\left(x_1^*, x_2^*\right) = \left(x_1^N, x_2^N\right)$$

3.5 Summary

This chapter sequentially develops the mathematical model for numerically solving Stackelberg and Nash game problems. The sensitivity based approach is a new contribution to the literature presented to numerically solve the optimization problem for both Stackelberg and Nash games. Chapter 4 presents application of this algorithm on several problems. Chapter 5 presents application of Stackelberg game modeling in the context of two mechanical design problems. These two problems are solved numerically using the sensitivity based approach developed in this chapter.

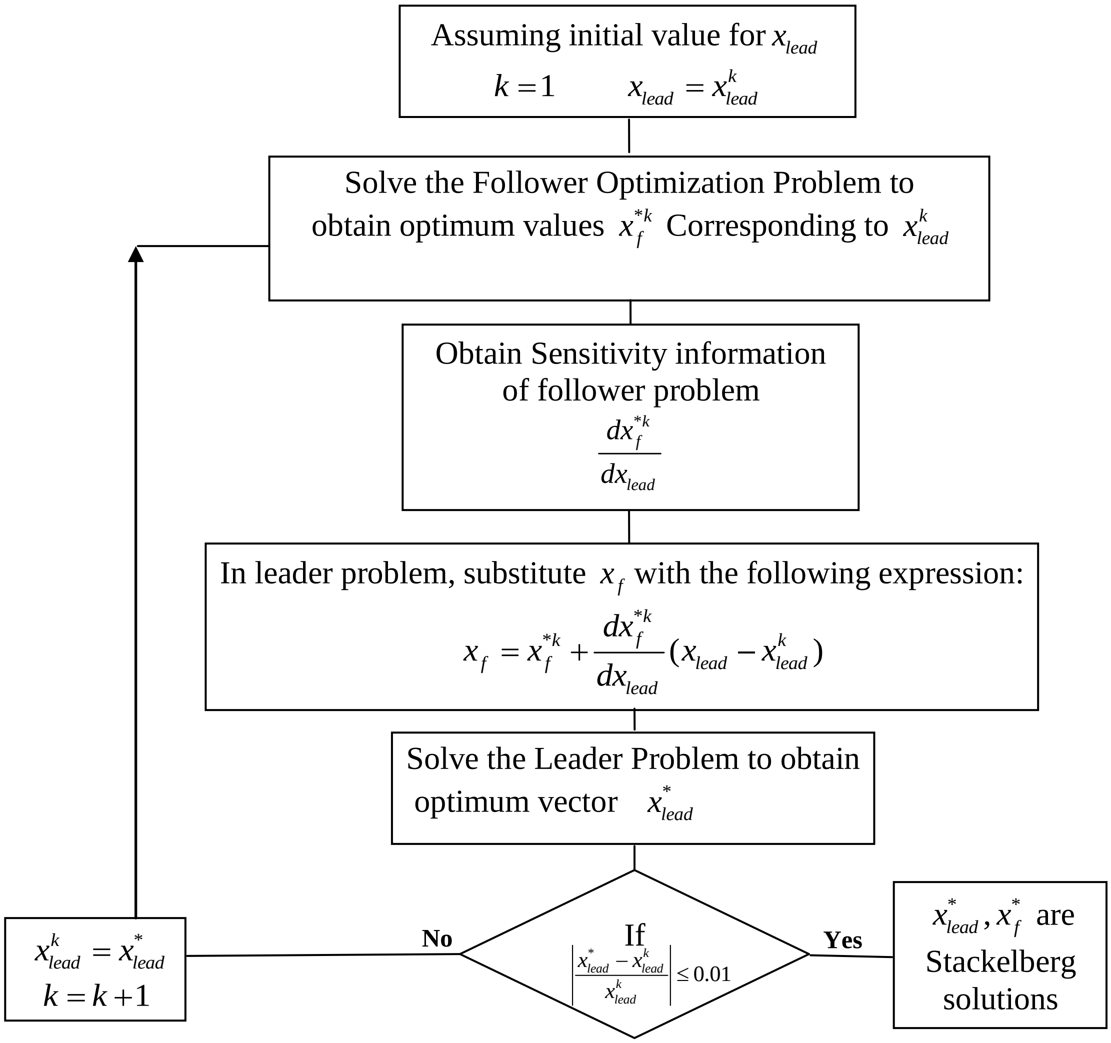


Figure 3.1 Computational Procedure for Obtaining Stackelberg Solution Using Sensitivity Method.

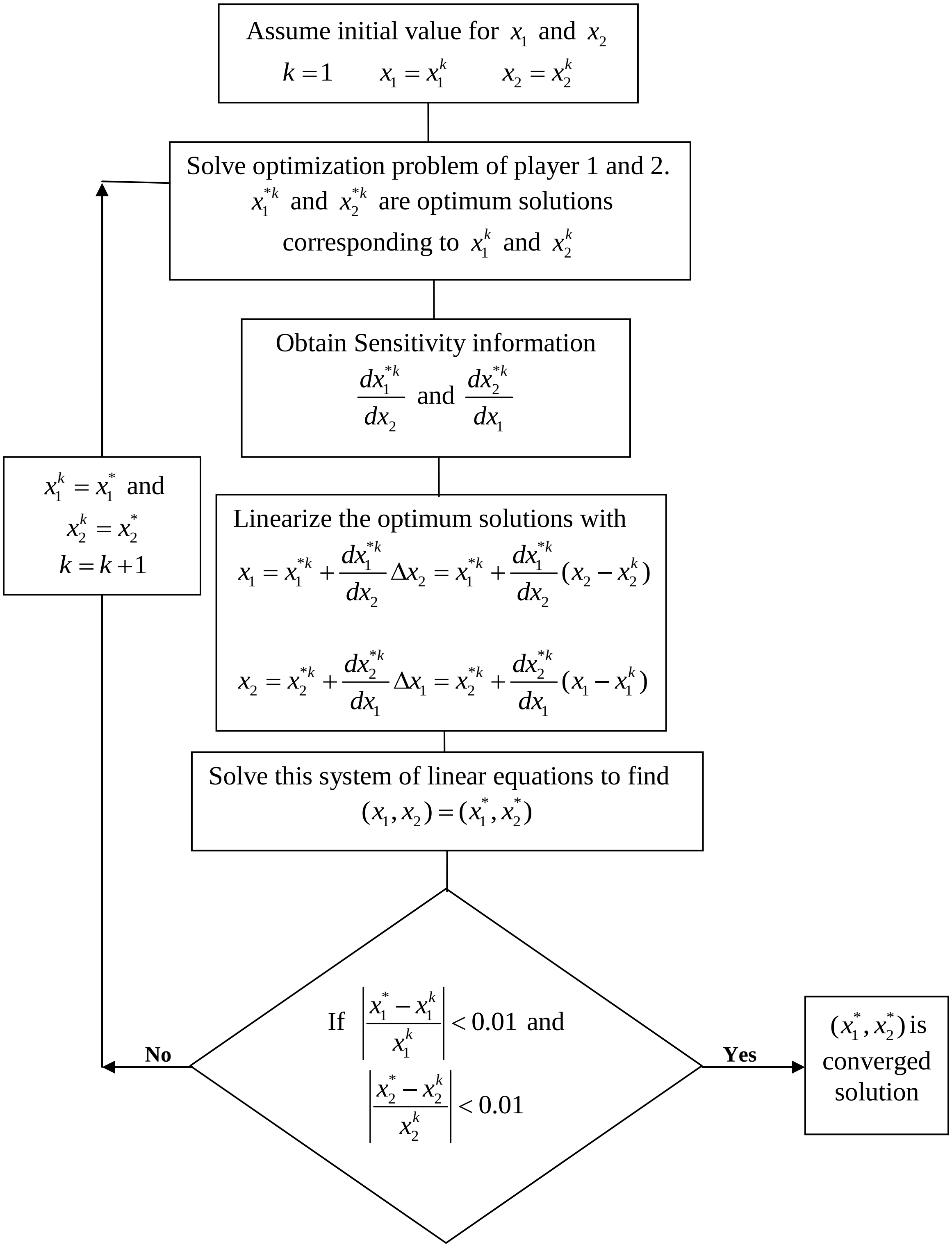


Figure 3.2 Computational Procedure for Obtaining Nash Solution Using Sensitivity Method.

CHAPTER 4

GENERATING RRS USING DOE-RSM AND SENSITIVITY BASED APPROACHES

4.1. Introduction

The sensitivity based approach was presented in chapter 3 to determine Nash solution(s) in multi-objective problems modeled as a non-cooperative game. The proposed approach provides an approximation to the rational reaction set (RRS) for each player. An intersection of these sets yields the Nash solution for the game. An alternate approach for generating the RRS based on design of experiments (DOE) combined with response surface methodology (RSM) was also mentioned. In this chapter, the two approaches for generating RRS are compared on three problems to find Nash and Stackelberg solutions. Three examples are presented to demonstrate the versatility of the sensitivity based method for obtaining Nash and Stackelberg solutions in multilevel optimization problems. Results for three example problems with two or more objectives, and isolated as well as non-isolated Nash solutions are presented. It is shown that the sensitivity based approach for constructing the RRS is computationally more efficient than RSM-DOE techniques because of (i) its lower computational burden, (ii) its ability to find all Nash solutions, and (iii) on one example problem, yielding better Nash solutions than those reported in the literature.

There are two points about the Nash game that might be interesting to mention: (i) there might be a bargaining Nash game in which more than one Nash solution exists. For example, assume the intersection of RRS of players 1 and 2 shown in Eqs. (3.16) and

(3.17) has more than one solution. Then it is possible that one of the solutions is better on all objective functions for players 1 and 2 than the other Nash solutions. (ii) If there is a solution $\left(x_1^N, x_2^N\right)$, then a question arises is how can it be verified that it is a Nash solution. Suppose player 1 knows about the strategy of the player 2 , x_2^N , then player one asks itself: Can I improve my objective function by switching from x_1^N to other strategy? If every player would answer No to this question, then $\left(x_1^N, x_2^N\right)$ is the Nash solution. But if any player answers Yes to the question, then the solution is not a Nash solution. This comes from the definition of Nash equilibria.

4.2. DOE-RSM Method

Design of Experiments (DOE) is studied in statistics and has been widely applied to engineering problems. The independent variables, governing variables, are set at specific values which are called levels. By identifying a minimum and a maximum for each variable, a two-level experiment can be set up. The outcomes, which are called outcomes from the experiments, are regressed over the independent variables to build an empirical model of the system. There are two major categories of experiments: full factorial designs and fractional factorial designs.

Full Factorial Design: To construct an approximation model that can model the interaction between design variables, a full factorial approach is needed to investigate all possible interactions of design variables. In full factorial designs, lower and upper bounds of the design variables in the optimization problem are defined: If there are n design variables and each design variable is defined at only upper and lower bounds (two levels), then the total number of trials required to implement the experiment is 2^n . The experiment is called a 2^n full factorial experiment. If the midpoints of the design variables are also included, then there are 3 levels and the total number of trials needed to carry out the three-level full factorial design would be 3^n. A 3^3 full factorial experiment is shown in Fig. 4.9.

Fractional Factorial Design: It can be seen that the number of trials required in implementing a full factorial experiment increases rapidly with an increase in the number of design variables. This can be very time consuming and resource intensive. A full factorial design is used for five or fewer variables. For large number of variables, a fraction of full factorial design can be used. This is called fractional factorial design. It is used for screening the important design variables. The fractional factorial experiment can reduce the number of trials required to complete the experiment. For a 3^n full factorial design, a $\left(\frac{1}{3}\right)^P$ fraction of full factorial design can be considered as fractional factorial design. Assuming $P=1$ in a 3^3 full factorial design, the fractional factorial design is one-third of full factorial design. It is shown in Fig. 4.10 (Montgomery 2005).

When a first-order model (linear regression) suffers lack of fit due to interaction between variables, a second-order model can significantly improve the model approximation. A general second-order model is defined as

$$y = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n}\sum_{i=1}^{n} \underset{i<j}{a_{ij} x_i x_j} \tag{4.1}$$

where x_i and x_j are the design variables and a are the parameters which should be determined.

Central Composite Design (CCD): By adding center and axial points, CCD can help construct a second-order model. Fig. 4.11 shows a CCD for 3 design variables (with two levels) (Montgomery 2005). The design involves 2^n (n=3) full factorial points, $2n$ axial points and 1 center points. CCD presents an alternative to 3^n trials need for construction of second-order models. It reduces the number of experiments compared to a full factorial design (15 in case of CCD compared to 27 for a full factorial design). The number of center point can be repeated in order to improve the precision of the experiment. If the purpose of replicated points is to obtain model with lower error, it is better to have more than 4 or 5 replications.

4.3. Numerical Examples

Three examples are presented to compare the sensitivity based approach and the DOE-RSM method.

4.3.1 Bilevel Problem with Three Followers

Consider a bilevel problem with one leader and three followers where the followers have non-cooperative game among themselves. The leader has control over variables $x=(x_1,x_2)$ and the followers one, two and three control variables $y_1=(y_{11},y_{12})$, $y_2=(y_{21},y_{22})$, $y_3=(y_{31},y_{32})$ respectively. A Stackelberg-Nash solution to this problem using genetic algorithm has been presented by Liu (1998). The problem is as follows:

$$min \quad f_{lead}\left(x, y_1, y_2, y_3\right) = \frac{3\left(y_{11}+y_{12}\right)^2 + 5\left(y_{21}+y_{22}\right)^2 + 10\left(y_{31}+y_{32}\right)^2}{2x_1^2 + x_2^2 + 3x_1x_2}$$

subject to:

$$x_1 + 2x_2 \le 10$$
$$x_1, x_2 > 0$$

(4.2)

$$min \quad f_1\left(y_1\right) = y_{11}^2 + y_{12}^2$$

subject to:

$$y_{11} + y_{21} + y_{31} \ge x_1$$
$$y_{12} + y_{22} + y_{32} \ge x_2$$
$$y_{11} \ge 1, y_{12} \ge 2,$$

$$min \quad f_2\left(y_2\right) = y_{21} + y_{22} + \frac{y_{11}}{y_{21}} + \frac{y_{12}}{y_{22}}$$

subject to:

$$y_{21}, y_{22} > 0,$$

(4.3)

$$min \quad f_3(y_3) = \frac{(y_{31}-y_{21})^2}{y_{31}} + \frac{(y_{32}-y_{22})^2}{y_{32}}$$

subject to:

$$2y_{31} + 3y_{32} = 5$$
$$y_{31}, y_{32} > 0$$

This is a leader follower problem with three players in the follower level. The sensitivity based approach presented in chapter 3 and Fig. 3.2 will be applied first to find a Nash solution amongst the followers. Next, the algorithm shown in Fig. 3.1 is used to find a Stackelberg solution between the leader and the set of followers. The algorithm converges after 4 iterations to $f_{lead} = 1.510,\ f_1 = 7.697,\ f_2 = 6.061,\ f_3 = 0.483$. To verify that the converged solution is indeed a Nash solution for followers 1 to 3, the approach discussed at the beginning of this chapter will be applied. It is verified that this is a Nash solution for the followers 1 to 3. Table 4.1 compares the results obtained using the proposed algorithm with those reported by Liu (1998). It can be seen from Table 1 that both approaches yield the same value for leader's objective function $(f_{lead} = 1.510)$ but the sensitivity based approach yields a better solution for all three followers (f_1, f_2, f_3) compared to those reported by Liu. It shows that the sensitivity based approach improves the Nash solution. The computational time required to obtain this solution is 2.08 sec versus 9 minutes reported by Liu using a genetic algorithm.

4.3.2 Design of a Pressure Vessel

Consider next the problem dealing with the design of a thin-walled pressure vessel with three design variables; the radius R, the length L, and the thickness T (see Fig. 4.1). This problem has been used as a test problem in the literature by several researchers (Rao et al. 1997, Lewis and Mistree 1998). The two objective functions include maximizing the volume (VOL) and minimizing the weight (WGT) of the vessel. Player 1 (VOL) wishes to maximize the volume by controlling variables R and L whereas player 2 (WGT) minimizes the weight with control over variable T. The vessel is under internal pressure P. The problem constraints include: (i) the circumferential stress should not exceed the tensile stress, and (ii) some additional geometric constraints due to space limitations. These constraints are given in Eqs. (4.4)-(4.7).

$$\sigma_{circ} = \frac{PR}{T} \leq S_t \tag{4.4}$$

$$5T - R \leq 0 \tag{4.5}$$

$$R + T - 40 \leq 0 \tag{4.6}$$

$$L + 2R + 2T - 150 \leq 0 \tag{4.7}$$

The mathematical form of the problems for players VOL and WGT are given in Eqs. (4.8) and (4.9) respectively.

$$Min \quad f_1 = -V(R,L) = -\rho\left[\frac{4}{3}\pi R^3 + \pi R^2 L\right] \tag{4.8}$$

by varying R, L

subject to Eqs. (4.4)-(4.7)

$$R_l \le R \le R_u$$
$$L_l \le L \le L_u$$

For player WGT:

$$Min \quad f_2 = W(R,T,L) = \rho\left[\frac{4}{3}\pi(R+T)^3 + \pi(R+T)^2 L - \left(\frac{4}{3}\pi R^3 + \pi R^2 L\right)\right]$$

by varying T

subject to Eqs. (4.4)-(4.7) (4.9)

$$T_l \le T \le T_u$$

where ρ is the cylinder density and $R_l, R_u, L_l, L_u, T_l, T_u$ denote the lower and upper bounds on radius, length and thickness of the vessel respectively. The problem's constants are given in Table 4.2.

The Nash solution of the non cooperative game between players VOL and WGT is found by applying the algorithm shown in Fig 3.2. It should be noted that changing the initial point for the radius results in a different Nash solution; this means that there are infinite Nash solutions for this problem. Figs. 4.2 and 4.3 show these solutions as a function of vessel radius (R). The Nash solution(s) for this problem have also been derived analytically by Rao et al. (1997) and are given by Eq. (4.10). It may be noted that the entire positive dimensional set of Nash solutions given by Eq. (4.10) is reproduced using the sensitivity based approach as shown in Figs. 4.2 and 4.3.

$$\frac{S_t\left(150-L_u\right)}{2\left(P+S_t\right)} \leq R^N \leq \frac{40S_t}{P+S_t}$$
$$L^N = 150-2R^N\left[\frac{P}{S_t}+1\right] \tag{4.10}$$
$$T^N = \frac{PR^N}{S_t}$$

Marston (2000) presents a DOE based approach to approximate the RRS for the minimization WGT problem in Eq. (4.9). It is needed to design an experiment in the variables R and L, substitute these variables in the follower problem. The two factors R and L each have three levels with a five repeated points in the center of the DOE block. Table 4.3 shows the design data for this problem. For each (R, L) combination, an optimum solution for T is obtained. The third column of the Table 4.3 is the optimum solution for T corresponding to the set of (R, L). The response surface regression of optimum T over (R, L) yields the following approximation function of RRS.

$$T(R,L) = -0.0002+0.1112*R \tag{4.11}$$

where $T(R,L)$ approximates the optimum vector of WGT problem for varying values of R and L. It can be seen that the variable L does not appear in RRS of WGT. Repeating the above steps for the VOL problem yields the RRS for variables R and L as follows:

$$R(T) = 9*T \tag{4.12}$$

$$L(T) = 150-20*T \tag{4.13}$$

Next, these three RRS are used to obtain Nash and Stackelberg solutions.

Nash Solution: The intersection of three RRS functions in Eqs. (4.11)-(4.13) yields the Nash solution of the game. The Nash equilibrium for this case is (R=28.4 in, L=86.9 in, T=3.16 in), which is a unique solution. However, it should be noted that this problem has infinite Nash solutions. The RSM-DOE based method is unable to provide all Nash solutions to this problem. However, as shown in Figs. 4.2 and 4.3, the sensitivity based approach is able to generate all possible Nash solutions. These solutions match the analytical results given by Eq. (4.10).

Stackelberg Solution: With players VOL and WGT as leader and follower respectively, the Stackelberg game problem is solved by substituting Eq. (4.11), which is RRS of the follower problem, into the leader's problem. Both Marston (2000) and Rao et al. (1997) obtained the Stackelberg solution of (R=35.99 in, L=70 in, T=4 in).

The sensitivity based method outlined herein yields the same results as those obtained Marston and Rao et al. However, the sensitivity based approach is able to accomplish this at a much lower computational burden. Table 4.4 compares the number of optimization problems that were solved using each method to obtain the final solution. It can be seen from Table 4.4 that the DOE-RSM method requires the follower problem to be solved 14 times and the leader problem once. The sensitivity based approach requires the leader and follower problems to be solved 2 times each. It should be noticed that the number of iterations needed to get convergence depends on the initial values and the convergence criteria which have been selected. For example, by changing the criteria of convergence from 0.01 to 0.5, the algorithm converges after solving the leader and follower problems one time. To conclude, the DOE-RSM method requires a total of 15 optimization problems to be solved to obtain the optimum solution whereas the sensitivity based method obtains the solution by solving only 4 optimization problems.

Finally, a word of caution about sensitivity of final solution to numerical perturbation in regression coefficients obtained using the DOE method. The DOE- based method yields a regression coefficient of 0.1112 for “R” in Eq. (4.11). If this coefficient is changed slightly to 0.1111 and this new RRS function is substituted in the leader problem to solve a Stackelberg game, the optimum solution will be $\left(\mathrm{R}=7 \text{ in},\ \mathrm{L}=134.4 \text{ in},\ \mathrm{T}=0.78 \text{ in}\right)$. This solution is significantly different from the correct solution to this problem.

A justification for why this problem is sensitive to coefficient of the radius (R) is proposed next. The leader’s objective function $\left(f_1\right)$ monotonically increases with respect to R. So the leader attempts to increase R provided the constraints in Eqs. (4.4)-(4.7) are satisfied. Fig. 4.4 shows the stress constraint of the leader’s problem as a function of the radius (Eq. (4.4)) when the RRS of the follower problem is substituted in leader’s constraints. The vertical axis is the stress constraint of the leader’s problem. It can be seen from Fig. 4.4 that the stress constraint is satisfied for all values of the radius. Therefore, the leader will choose the upper bound of the radius value (R=36) to optimize its objective function.

If the RRS of the follower problem is changed to $T(R,L) = -0.0002+0.1111*\mathrm{R}$, Fig. 4.5 shows that the stress constraint will become active for R=7. Then, the optimum solution for the leader is R=7 which is quite different from the previous case. This small change in coefficient from 0.1112 to 0.1111 is quite likely depending on the software (Matlab vs Minitab) used for regression as well as the regression model (linear vs quadratic). The sensitivity based approach proposed herein is not prone to these solution instabilities due to numerical perturbations.

4.3.3 Two-Bar Truss Problem

This problem has been considered by Azarm and Li (1990). The two-bar truss problem shown in Fig. 4.6 is subject to a vertical load of 100 kN at point C. The variables are the cross-sectional areas of the bars $x_1, x_2,$ and the y-coordinate of joint C. The problem constraints include limitations on the stress in the elements, which should not exceed 100,000 kN/m^2, and the bounds on vertical coordinate (y). The objective function is to minimize the volume of the truss. The problem formulation is as follows:

$$
\begin{aligned}
&\textit{Minimize} \quad f(x_1, x_2, y) = x_1\left(16 + y^2\right)^{0.5} + x_2\left(1 + y^2\right)^{0.5} \\
&\text{subject to:} \\
&\qquad 20\left(16 + y^2\right)^{0.5} - 100,000\, y x_1 \le 0 \\
&\qquad 80\left(1 + y^2\right)^{0.5} - 100,000\, y x_2 \le 0 \\
&\qquad 1 \le y \le 3 \\
&\qquad x_1, x_2 > 0
\end{aligned}
\tag{4.14}
$$

Azarm and Li (1990) decomposed the problem in two levels. Level one is the follower problem, with two players, players 1 and 2, who have control over variables x_1 and x_2 respectively. The follower problems are given below:

$$
\begin{aligned}
&\underset{x_1}{\textit{minimize}} \quad f_1(x_1, y) = x_1\left(16 + y^2\right)^{0.5} \\
&\text{subject to:} \\
&\qquad 20\left(16 + y^2\right)^{0.5} - 100,000\, y x_1 \le 0 \\
&\qquad x_1 > 0
\end{aligned}
\tag{4.15}
$$

$$\underset{x_2}{\text{minimize}} \quad f_2(x_2, y) = x_2\left(1+y^2\right)^{0.5}$$

subject to: (4.16)

$$80\left(1+y^2\right) - 100,000\, y x_2 \le 0$$

$$x_2 > 0$$

The leader problem is given as:

$$\underset{y}{\text{minimize}} \quad f(x_1, x_2, y) = f_1(x_1, y) + f_2(x_2, y)$$

subject to: (4.17)

$$1 \le y \le 3$$

This problem can be modeled as a Stackelberg game with two players in the follower level. Using the principles of monotonicity analysis, it can be verified that the constraints are active at optimum solution of the follower problems when they are optimized individually. So the optimum solutions of follower problems are as follows:

$$x_1^*(y) = 20\left(16+y^2\right)^{0.5} / \left(100,000 y\right) \tag{4.18}$$

$$x_2^*(y) = 80\left(1+y^2\right)^{0.5} / \left(100,000 y\right) \tag{4.19}$$

Since Eqs. (4.18) and (4.19) show the variation of optimum solution of x_1^* and x_2^* with respect to y, they are the closed-form function of RRS for the followers. It can be noticed that these RRS are non linear functions of y. By substitution of these RRS in the leader problem, the optimum solution of the leader can be obtained. The optimum solution of $\left(x_1 = 4.48, x_2 = 8.96, y = 2\right)$ was reported for this problem by Azarm and Liu (1990). Using the sensitivity based approach and the algorithm shown in Fig. 3.1 to solve the Stackelberg game formulation, a solution identical to that reported by Azarm and Li is obtained after 4 iterations.

Discussed next is the implementation of the algorithm for this example: Here $x_{lead} = y, x_f = (x_1, x_2)$. Set $k = 1$, the convergence criteria=0.01 and initial value of $y = 1.2$. Solve optimization problems for the followers in Eqs. (4.15) and (4.16) assuming $y = 1.2$. The optimum solution would be $x_1^{*1} = 6.96$ and $x_2^{*1} = 10.41$. In leader problem, Eq. (4.17), substitute x_1, x_2 by these approximations:

$$x_1 = 6.96 + \frac{dx_1^{*1}}{dy}(y - 1.2) \tag{4.20}$$

$$x_2 = 10.41 + \frac{dx_2^{*1}}{dy}(y - 1.2) \tag{4.21}$$

These two functions are approximations of RRS for the follower 1 and 2 respectively. The terms $\frac{dx_1^{*1}}{dy}$ and $\frac{dx_2^{*1}}{dy}$ are sensitivity information of the followers' optimization problem obtained by solving Eq. (3.28). Now, the leader problem will be solved and the optimum solution would be $y^{*1} = 1.8976$. Compute convergence criteria: $\left|\frac{y^{*1} - y^1}{y^1}\right| = \left|\frac{1.89 - 1.2}{1.2}\right| = 0.58$. Since it does not meet convergence limit (0.001), the second iteration is started. k is updated to 2 and $y^k = y^{*1} = 1.8976$. Using this new updated value for y, the optimum values of the followers would be $x_1^{*2} = 4.66$ and $x_2^{*2} = 9.04$. The new approximations would be:

$$x_1 = 4.66 + \frac{dx_1^{*2}}{dy}(y - 1.89) \tag{4.22}$$

$$x_2 = 9.04 + \frac{dx_2^{*2}}{dy}(y - 1.89) \tag{4.23}$$

The new optimum value of y can be obtained by substituting these new approximations in the leader's problem and optimizing it. That would be $y^{*2} = 1.9760$. The convergence criterion is 0.05 which does meet the specified limit and a third iteration is needed. For $k = 3$, $y^k = 1.98$, $x_1^{*3} = 4.49$, $x_2^{*3} = 8.95$ and $y^{*2} = 1.9981$. The convergence criterion is 0.011 which is still more than the limit (0.01). So, the fourth iteration will be performed. For $k = 4$, $y^k = 1.9981$, $x_1^{*3} = 4.49$, $x_2^{*3} = 8.95$ and . The convergence criteria is less than 0.01 and the iteration stops. The optimum solution would be $x_1^* = 4.49$, $x_2^* = 8.95$ and $y^k = 1.9981$. It may be noted that the number of iterations depends on the limit of convergence criterion set for the algorithm. For example, if the limit decreases from 0.01 to 0.1, then the algorithm converges after 3 iterations but with less accurate solution. If the limit is set to 0.001, then 5 iterations are needed to get convergence for this problem and the solution would be $x_1^* = 4.47$, $x_2^* = 8.96$ and $y^k = 2.0000$.

To solve this problem using DOE based method, an experiment was designed for the follower problem. Since there is a single factor (y), the interval of y ([1, 3]) was divided into 10 even spaces. A regression analysis of the results of follower's experiment yielded the following approximation of follower's RRS.

$$x_1(y) = 9.44 - 2.42y \tag{4.24}$$

$$x_2(y) = 11.80 - 1.26y \tag{4.25}$$

By substituting these approximations in the leader problem, an optimum solution with $(x_1 = 2.17, x_2 = 8.01, y = 3)$ is obtained. Fig. 4.7 shows analytical and RSM approximation of $x_1(y)$, Eqs. (4.18) and (4.24). It shows how the sensitivity based approach and RSM method converges to different solution. Also, Fig. 4.8 showes the leader objective function in Eq. (4.17) where RSM approximation of $x_1(y)$ and $x_2(y)$, Eqs. (4.24) and (4.25) are substituted. From the Fig. 4.8, it can be noticed that the minimum value of the leader objective function occurs at $y = 3$. A quadratic approximation with 11 experiments yielded the following optimum $(x_1 = 7.9868, x_2 = 11.0933, y = 1)$ whereas a 21 experiment quadratic regression yielded the following approximate RRS:

$$
\begin{aligned}
x_1 &= 13.9855y^2 - 7.3489y + 1.2872 \\
x_2 &= 14.9663y^2 - 4.8282y + 0.9004
\end{aligned}
\tag{4.26}
$$

and an optimum of $(x_1 = 7.9238, x_2 = 11.0385, y = 1)$. It can be seen that there is a significant difference between the optimum solutions obtained using the 3 RSM formulations and the exact solution of the Stackelberg problem. Because of the nonlinear nature of the RRS, the DOE based method is unable to converge to the correct Stackelberg solution for this problem. It seems the reason that DOE method does not converge to correct solution but sensitivity based approach does would be existence of updating the x_1 and x_2 in each iteration. In the RSM method, this updating does not exist and linear fixed functions in Eqs. (4.24) and (4.25) are used for x_1 and x_2.

4.4 Conclusions

In this chapter, the approach proposed in chapter 3 is tested on three example problems for which solutions are available in the literature. It is seen that the proposed sensitivity based approach is (i) computationally less intense, and less prone to numerical errors than a RSM-DOE approach, (ii) is able to approximate non linear RRS, (iii) can find all Nash solutions where the Nash solution is not a singleton, and (iv) for one example problem, is able to improve the Nash solution that was reported in literature. Further extensions of the proposed approach to hierarchical systems with multiple leaders and multiple followers are presented in chapter 6 .

Table 4-1 Comparison of Results for Example 1 (from Liu 1998).

	Liu's results	Sensitivity based approach
f_l	1.510	1.510
f_1	12.323	10.821
f_2	6.225	6.061
f_3	0.835	0.483
$x^* = \left(x_1^*, x_2^*\right)$	(5.768,2.116)	(5.379,2.310)
$y_1^* = \left(y_{11}^*, y_{12}^*\right)$	(2.885,2.000)	(2.612,2.000)
$y_2^* = \left(y_{21}^*, y_{22}^*\right)$	(1.699,1.414)	(1.616,1.414)
$y_3^* = \left(y_{31}^*, y_{32}^*\right)$	(1.183,0.878)	(1.149,0.900)

Table 4-2 Pressure Vessel Problem Parameters.

P	S_t	ρ	L_l	L_u	R_l	R_u	T_l	T_u
3890 lb	35000 lb	0.283 lbs/in^3	0.1 in	140 in	0.1 in	36 in	0.5 in	6 in

Table 4-3 Experimental Design to Obtain RRS for the Follower (WGT).

R(in)	L(in)	T(R,L) (in)
20.25	55	2.2506
4.5	10	0.5001
36	100	4.0000
4.5	100	0.5001
36	10	4.0000
20.25	55	2.2506
20.25	55	2.2506
20.25	55	2.2506
4.5	55	0.5001
36	55	4.0000
20.25	55	2.2506
20.25	100	2.2506
20.25	10	2.2506
20.25	55	2.2506

For each row, one follower optimization problem has been solved to calculate the optimum value of T corresponding to that set of R and L.

Table 4-4 Number of Optimization Problems Solved for Example 2.

Method	WGT Problem	VOL Problem	Total
DOE-RSM	14	1	15
Optimal Sensitivity	2	2	4

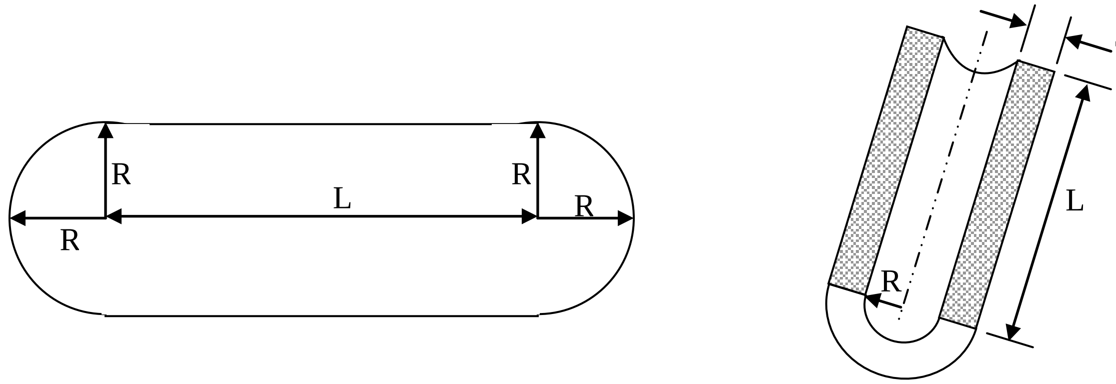


Figure 4.1 Thin-Walled Pressure Vessel.

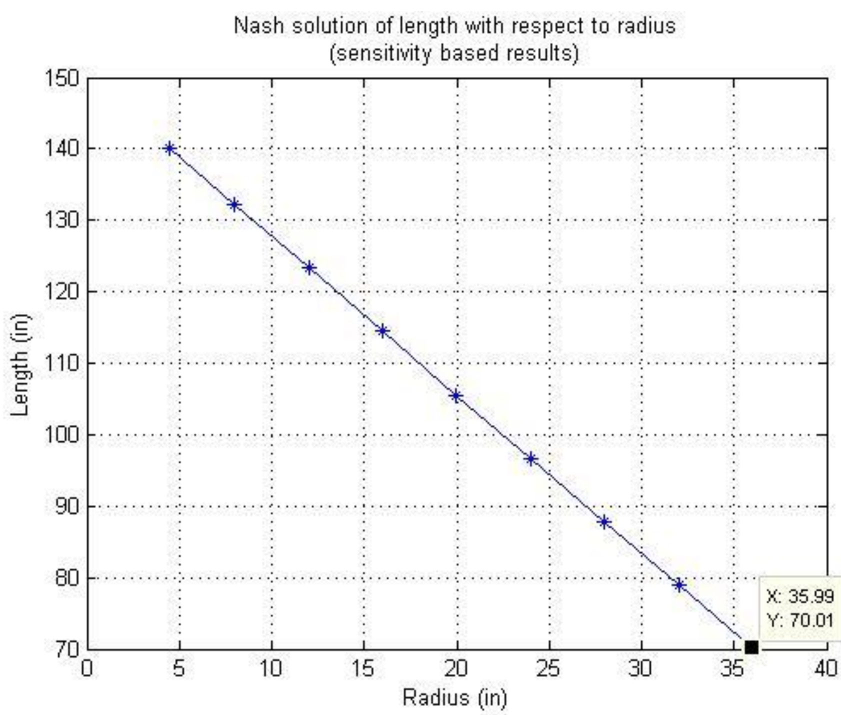


Figure 4.2 Nash Solution Length vs Radius for Pressure Vessel Problem.

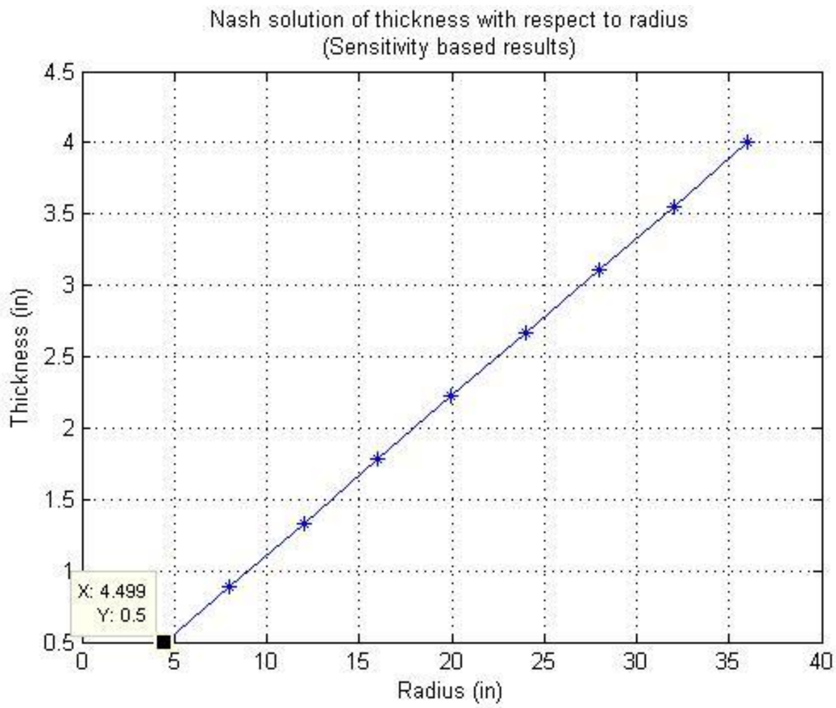


Figure 4.3 Nash Solution Thickness vs Radius for Pressure Vessel Problem.

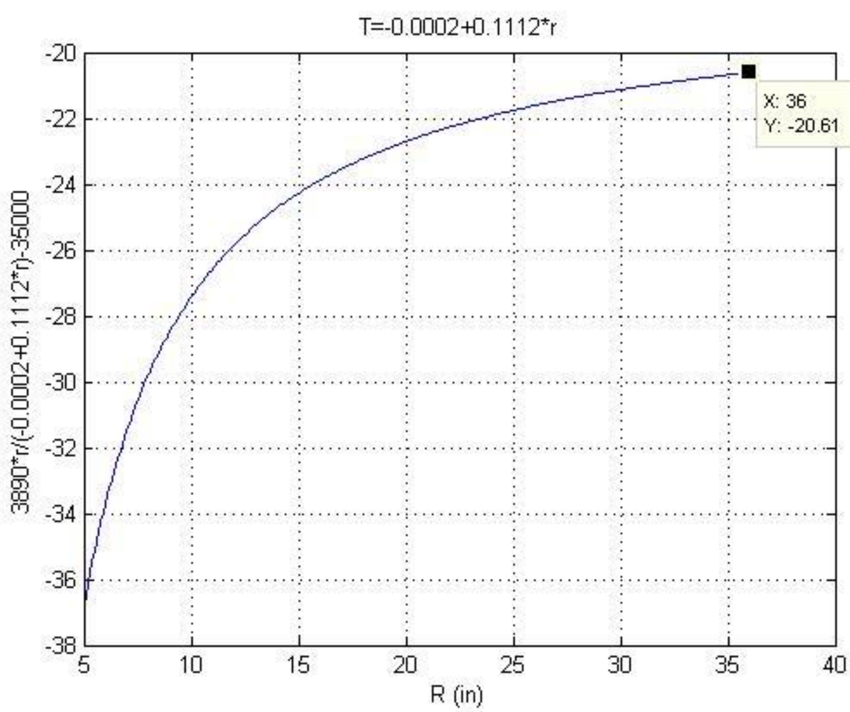


Figure 4.4 Stress Constraint of Player VOL.

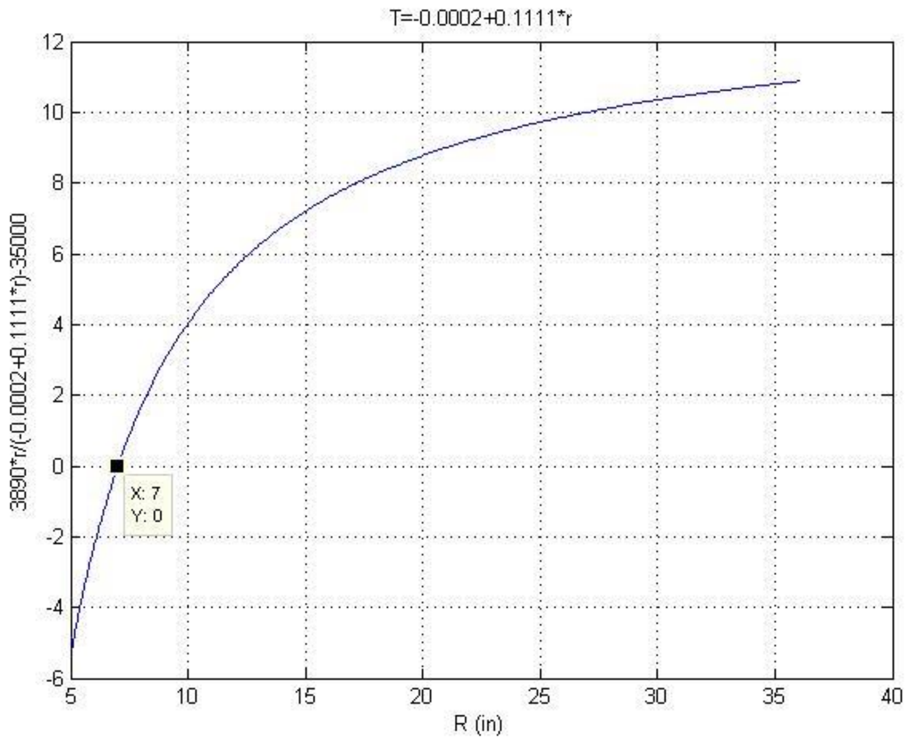


Figure 4.5 Stress Constraint of Player VOL.

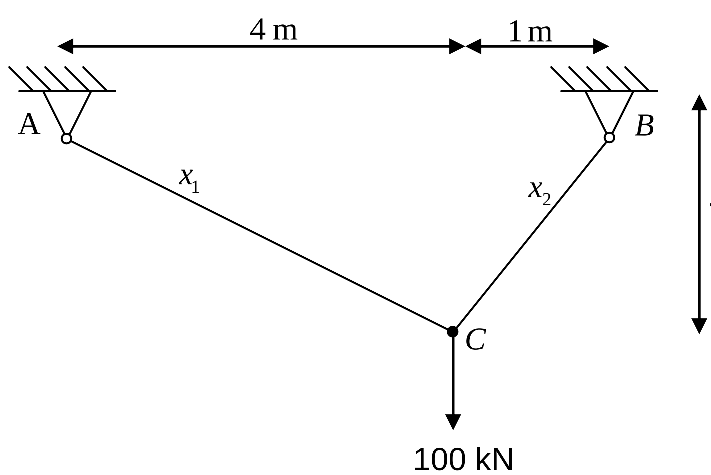


Figure 4.6 Two-Bar Truss Problem.

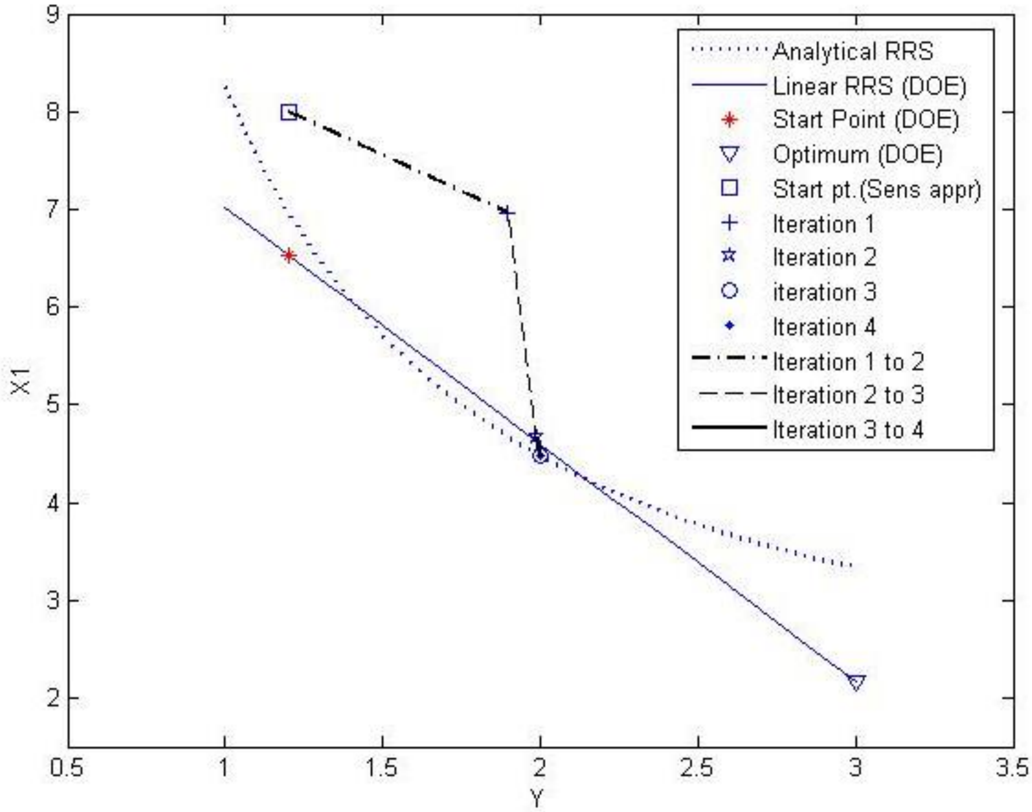


Figure 4.7 The Analytical and RSM Approximation RRS for x_1.

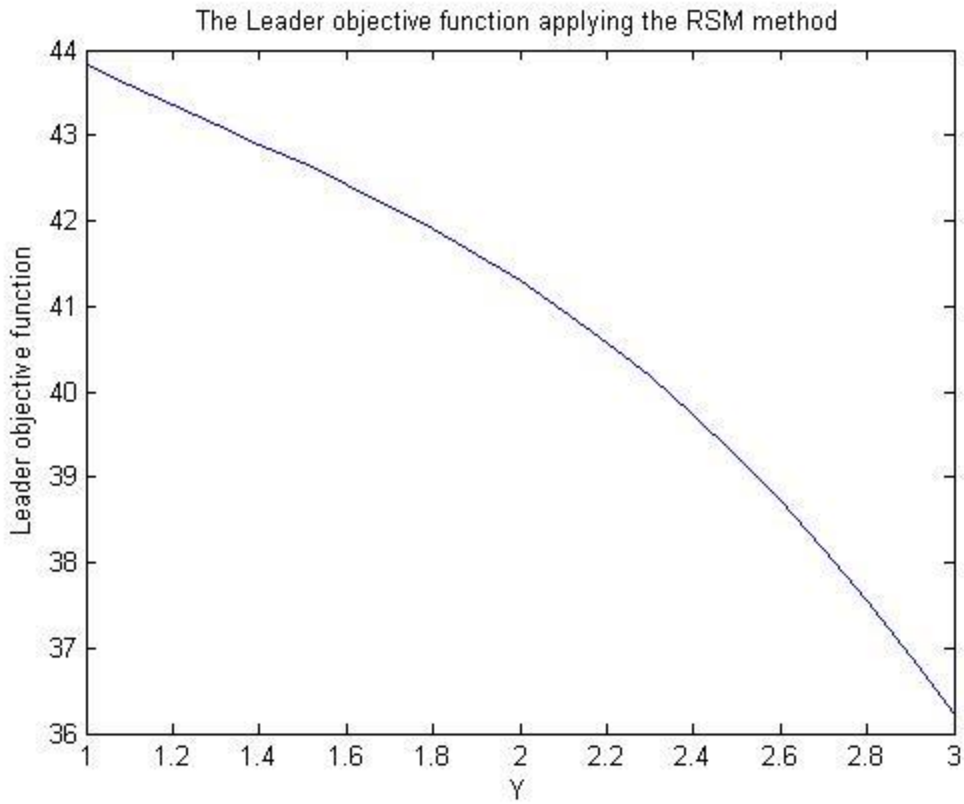


Figure 4.8 The Leader Objective Function Applying RSM Method.

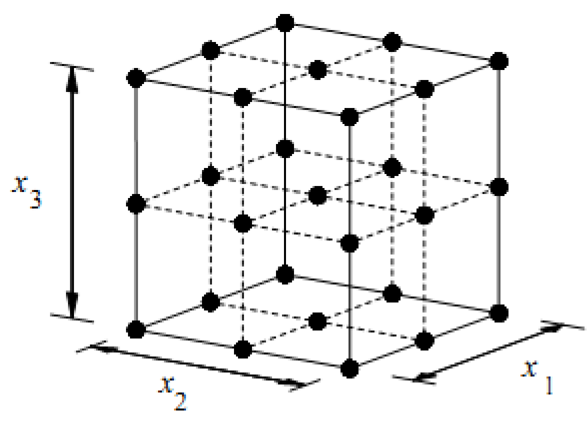


Figure 4.9 A 3^3 Full Factorial Design (27 points).

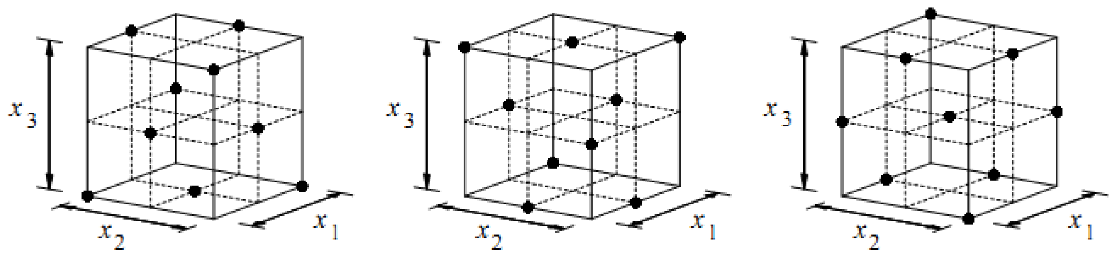


Figure 4.10 Three One-Third Fraction of the 3^3 Design.

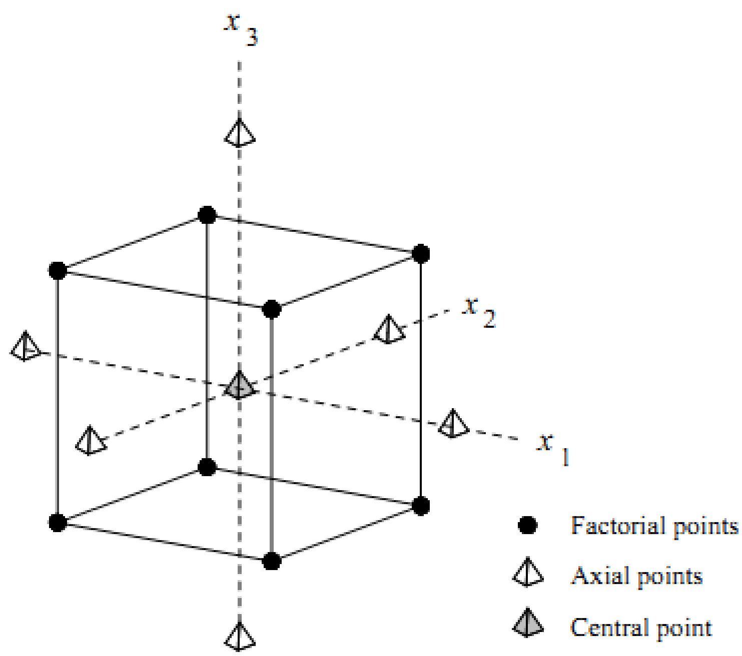


Figure 4.11 Central Composite Design for 3 Design Variables at 2 Levels.

CHAPTER 5

APPLICATION OF STACKELBERG GAMES IN MECHANICAL DESIGN

This chapter illustrates the application of a bilevel, leader-follower model for solving two mechanical design problems (i) optimum design of flywheels and (ii) optimum design of high speed mechanisms. Both optimization problems are modeled as a Stackelberg game. The first problem deals with design of flywheels wherein the objective is to maximize the energy stored in the flywheel while simultaneously minimizing the manufacturing costs. This model consists of two conflicting objective functions. The second problem considers the design of a high speed mechanism as a multi objective optimization problem wherein the kinematic and dynamic criteria are optimized simultaneously. The partitioning of variables between the leader and follower problem is discussed, and a variable partitioning metric is introduced to compare various variable partitions. The sensitivity based approach discussed in chapter 3 is applied for exchanging information between follower and leader problems.

5.1 Optimum Design of Flywheels

High speed rotating disks are commonly used as flywheels as well as rotors in turbines and compressors. Flywheels are used to store energy in many power generation applications and help smoothen torque fluctuations. The flywheel problem has been considered by Sandgren and Ragsdell (1983) and Bhavikatti and Ramakrishnan (1980) who used Fourier series and 5th order polynomial functions respectively to find the optimum shape of the flywheel for a single objective function. A solution to this problem for the two-objective function case has been considered by Pakala (1994) and Rao (1997). However, considerations pertaining to updating of the follower's solution as well as variable partitioning are not addressed in these works. For simple problems where explicit expressions for objective and constraint functions are available, a gradient based approach for variable updating has been presented by Azarm and Li (1990). However, this approach is not helpful when explicit expressions are not available for objectives and constraints, such as the design problem considered herein. The sensitivity based variable updating approach presented in previous chapter is general, and is used along with proposed variable partitioning metric to solve the flywheel problem using a Stackelberg game based approach.

5.1.1 Design Problem Formulation

The two objective functions used for the flywheel problem include maximizing the kinetic energy stored in the flywheel while simultaneously minimizing the manufacturing cost. The proposed overall objective is to determine optimum flywheel shape that maximizes the kinetic energy stored in the flywheel while minimizing the overall manufacturing cost. The flywheel shape is one of the most important determinants of the amount of energy stored in the flywheel as well as the induced stresses. A uniform cross-section for flywheel is quite uneconomical because all the material is not fully stressed. Besides the hyperbolic cross-section profile proposed by Stodola, mathematical programming techniques have been applied by Bhavikatti and Ramakrishnan (1980), Sandgren and Ragsdell (1983) and Dems and Turant (2009) to optimize the cross section of rotating disks. In Bhavikatti's approach, the disk is approximated by a number of rings and the optimum profile is obtained by smoothening the stepped shape. Sandgren used polynomial and Fourier series approximations. Polynomial approximations as well as Fourier series allow for a continuous form representation with a limited number of variables. A use of Fourier series for thickness representation affords the ability to represent any general function with a relatively few terms, and allows for exact value of derivatives of thickness function with respect to radius (r) to be used in the computation of radial and tangential stresses.

The flywheel has a specified inside radius R_i and an outside radius R_o. The thickness is different at any radius, and this relation is defined by function $f(r)$. The general cross sectional shape of the flywheel is shown in Fig 5.1.

The rotational speed ω is fixed and the total kinematic energy stored in the flywheel can be calculated. The constraints include allowable limits on induced stress, and flywheel mass and thickness. To insure that the stresses do not exceed the allowable limit, N points over the cross section from R_i to R_o are defined and the stress values at each of these N points are calculated. The maximum stress value amongst these N points should be less than the allowable stress.

The total mass is calculated using numerical integration, and should be less than a maximum allowable value. There is one constraint on the cross-section thickness; the maximum thickness across the whole of profile should be less than a specified value.

5.1.2 Thickness Function

The flywheel thickness is a function of its radius. A Fourier series representation is used to define the thickness (t) as a function of the radius (r) as:

$$t(r) = s_0 + \sum_{i=1}^{n} a_i \sin\left(\pi i \frac{r - R_i}{R_o - R_i}\right) + \sum_{i=1}^{n} b_i \cos\left(\pi i \frac{r - R_i}{R_o - R_i}\right). \quad R_i \le r \le R_o \tag{5.1}$$

In this work, n=3 is used in the thickness function. It assumes a 3-term approximation because it gives a good control over the general form of the flywheel profile without an undue increase in the problem complexity. The problem variables are s_0, a_i and b_i. Therefore, a total of 7 variables are used to determine the profile shape of the flywheel. With the thickness function (the coefficients of Fourier series) specified, the values of objective functions and constraints can be obtained.

An upper limit on thickness is one of the constraints. The maximum profile thickness should be less than allowable thickness. Eq. (5.2) shows this constraint:

$$max(t(r)) \le t_{allowable} \qquad R_i \le r \le R_o \tag{5.2}$$

5.1.3 Mass and Kinetic Energy

The kinetic energy stored in the flywheel is the first objective function of the optimization problem. Since it is of interest to store as much the energy in the flywheel as possible for a given flywheel weight, this objective function will be maximized. The kinetic energy stored in the flywheel is given as:

$$KE = \int_{R_i}^{R_o} \frac{1}{2} V^2 dM = \int_{R_i}^{R_o} \frac{1}{2} (r\omega)^2 \rho 2\pi r\, t(r)\, dr = \rho \int_{R_i}^{R_o} \pi\omega^2 r^3 t(r) dr \tag{5.3}$$

where ρ is the mass density of the flywheel disk, and ω is the angular velocity. The flywheel thickness $t(r)$ is a function of the flywheel radius at each point. The limits of integration are the inner radius (R_i) and the outer radius (R_o).

An upper limit ($M_{\max}$) is placed on the overall flywheel mass. The mass constraint is defined as:

$$M = \rho\int_{R_i}^{R_o} 2\pi r t(r) dr \le M_{\max} \tag{5.4}$$

5.1.4 Stress Analysis

The stress analysis begins with formulating the governing equations for the flywheel. Assuming that the tangential forces have negligible effect on stresses compared to centrifugal forces, a force balance on a flywheel stress element yields the following governing equations of equilibrium:

$$\frac{d}{dr}(t(r) r\sigma_r) - t(r)\sigma_\theta + \rho\omega^2 r^2 t(r) = 0 \tag{5.5}$$

$$(\sigma_\theta - \sigma_r)(1+\nu) + r\frac{d\sigma_\theta}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \tag{5.6}$$

where σ_r, σ_θ are radial and tangential stresses respectively, and ν is the Poisson's ratio. Defining the stress function $\phi = t(r) r\sigma_r$:

$$\sigma_r = \frac{\phi}{t(r) r} \tag{5.7}$$

Substituting Eq. (5.7) in Eq. (5.5) and solving for σ_θ yields:

$$\sigma_\theta = \frac{1}{t(r)}\left[\frac{d\phi}{dr} + \rho\omega^2 r^2 t(r)\right] \tag{5.8}$$

Next, substituting Eq. (5.7) and Eq. (5.8) into Eq. (5.6), the resulting second order of differential equation will be as follows:

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} - \phi + (3+\nu)\rho\omega^2 t(r) r^3 - \frac{r}{t(r)}\frac{dt(r)}{dr}\left(r\frac{d\phi}{dr} - \nu\phi\right) = 0 \tag{5.9}$$

The derivation of this differential equation is discussed in Timoshenko and Goodier (1951). This equation can be solved numerically, but it needs two initial conditions. Consider the inner and outer radii of flywheel; the radial stress at these points is zero, i.e. $\sigma_r = 0$ at $r = R_i$ and $r = R_o$. This provides two boundary conditions for ϕ:

$$\phi = 0 \text{ at } \quad r = R_i \tag{5.10}$$

$$\phi = 0 \text{ at } \quad r = R_o \tag{5.11}$$

Now there is a two-point boundary value problem with a second order differential equation. By solving this equation, the values of ϕ and $\frac{d\phi}{dr}$ will be obtained as a function of r. The number of points will depend on how many points (r) are defined for solving the ODE. Next, using Eq. (5.7) and Eq. (5.8), the radial and tangential stress at each r can be obtained. An application of distortion energy theory helps find the total stress acting at each point (r) as:

$$\sigma_{total} = \left[(\sigma_r(r) - \sigma_\theta(r))^2 + \sigma_r{}^2(r) + \sigma_\theta{}^2(r)\right]^{\frac{1}{2}} \tag{5.12}$$

The maximum value of this stress should be less than the allowable stress. Eq. (5.13) expresses this constraint.

$$max\left[(\sigma_r(r) - \sigma_\theta(r))^2 + \sigma_r{}^2(r) + \sigma_\theta{}^2(r)\right]^{\frac{1}{2}} \le \sigma_a \qquad R_i \le r \le R_o \tag{5.13}$$

Pakala (1994) and Sandgren (1983) used this equation for the stress constraint.

5.1.5 Manufacturing Objective Function

The manufacturing cost is the second objective function and is quantified by the deviation of the thickness function from a straight line profile. Large deviations in thickness function from a straight line lead to higher manufacturing costs. By minimizing these deviations, the manufacturing costs can be reduced. To define this objective function, the disk is divided into N equal parts and the thickness $t(r)$ for each part can be obtained. Suppose for $r = r_i$ the thickness is $t(r_i)$ and for adjacent section r_{i+1} the thickness is $t(r_{i+1})$. The difference of these values is Δt_i. The summation of absolute value of these Δt_i is defined as the second objective function as:

$$f_2 = \sum_{i=1}^{N-1} \left| t(r_{i+1}) - t(r_i) \right| = \sum_{i=1}^{N-1} \Delta t_i \tag{5.14}$$

5.1.6 The Optimization Problem

The optimization problem has two objective functions, seven variables and three constraints. The objectives are to maximize the kinematic energy stored in the flywheel while keeping the manufacturing costs low. This problem will be set up as a bi-level (leader-follower) model. One of the objective functions, energy stored in the flywheel, is considered as the leader whereas the second one, manufacturing cost, is treated as the follower. All constraints are associated with the leader problem. A three term Fourier series representation is used to define the shape function. The problem variables are the 7 coefficients (constant terms + 3 sin/cos terms) of this Fourier series. For the problem, it is not clear which variables should be associated with the leader problem and which variables with the follower problem. Some of the 7 coefficients are assigned as variables for the leader problem and the remaining variables are assigned to the follower problem. Since the variable partitioning is not unique, several variable partitions can be explored for the entire set of variables. There are a total of 126 different variable partitions. Each partitioning can be used to set up and solve a bi-level Stackelberg problem, and has its own solution.

Now there are two objective functions given by Eq. (5.3) and Eq. (5.14), and three nonlinear constraints on thickness Eq. (5.2), mass Eq. (5.4) and stress Eq. (5.13). The variables are Fourier coefficients of thickness function. ($s_0, a_1, a_2, a_3, b_1, b_2, b_3$).

5.1.7 Partitioning the variables

In some bi-level problems, the partitioning of decision variables for leader and follower problems is obvious. However, for our problem, it is not clear which of the variables should be associated with the leader's problem and which variables should be associated with the follower's problem. Since a total of 126 partitions of seven variables are possible and each combination will yield a possible optimal solution, what is needed is a criterion to compare these results to identify the best variable partitioning. One criterion is developed and proposed in this work as a variable partitioning metric (VPM):.

$$VPM = \frac{(f_l - f_{wl})(f_f - f_{wf})}{(f_{bl} - f_{wl})(f_{bf} - f_{wf})} \tag{5.15}$$

where f_l and f_f are the values of leader and follower objective functions when the optimization problem is solved using the Stackelberg approach. f_{bl}, f_{bf}, f_{wl} and f_{wf} denote best and worst values of leader and follower objective function, respectively. These values are obtained as follows. The leader optimization problem with seven variables is considered and solved. The optimum value of objective function for this problem is f_{bl}. If the optimum objective vector of this problem is substituted in the follower objective function (Eq. (5.14)), the corresponding follower objective function value would be f_{wf}. Similarly, by solving the follower optimization problem (Eq. (5.14)) the value of f_{bf} can be obtained and by substituting the optimum vector of follower's problem into leader's objective function (Eq. (5.3)), the value f_{wl} can be calculated.

5.1.8 Numerical Results

The first step involved in solving the bi-level optimization problem is to solve two single objective optimization problems separately. Each single objective problem is solved by ignoring the other objective function as follows:

Maximize

$$f_1(s_0, a_1, a_2, a_3, b_1, b_2, b_3) \tag{5.16}$$

subject to

$$\begin{aligned} &M \leq M_{allowable} \\ &\sigma_{total} \leq \sigma_{allowable} \\ &t(r)_{\max} \leq t_{allowable} \end{aligned} \tag{5.17}$$

and

Minimize

$$f_2(s_0, a_1, a_2, a_3, b_1, b_2, b_3) \tag{5.18}$$

subject to

same constraints given by Eq. (5.17)

The first objective function is to maximize the kinetic energy stored in flywheel rotating at 630 radians per second. The flywheel thickness is controlled by seven variables $\left(s_0, a_1, a_2, a_3, b_1, b_2, b_3\right)$. The maximum allowable mass is 80.5 kg and the stress can not exceed 250 Mpa. The inner and outer flywheel radii were set as 2.54 and 30 cm respectively. The maximum profile thickness is limited to 12 cm. The optimization problem is given as:

Minimize

$$-K.E.\,(s_0, a_1, a_2, a_3, b_1, b_2, b_3) \tag{5.19}$$

subject to

$$mass_{total} - 80.5 \leq 0 \tag{5.20}$$

$$\sigma_t - 2.5\times 10^8 \leq 0 \tag{5.21}$$

$$t(r)_{\max} - 0.12 \leq 0 \tag{5.22}$$

Similarly, by changing the objective function to manufacturing cost, the optimum solution to the second optimization problem is obtained. Table 5.1 shows the results for these two single objective optimization problems.

When only the objective function corresponding to the leader is considered, the optimized objective function value is 1.95×10^6 (also denotes as f_{bl}). Using this optimum vector in follower's objective function yields a follower objective function value of 1.6219 (f_{wf}). The second row in Table 5.1 corresponds to an optimization of the follower objective function with optimized of value 0.0324 (f_{bf}) and the corresponding leader objective function value is 0.3535×10^6 (f_{wl}). Figure 5.2 shows the flywheel profile for both these solutions.

In the second step, the bi-level problem is formulated as a Stackelberg game. The leader and follower objective functions are given by Eq. (5.3) and Eq. (5.14) respectively. All constraints (Eqs. (5.20), (5.21) and (5.22)) are associated with the leader's problem, and no constraints are imposed on the follower's problem. There are seven variables, $\left(s_0, a_1, a_2, a_3, b_1, b_2, b_3\right)$. The leader can pick up some of the Fourier coefficients as variables and the remaining coefficients will act as variables for the follower. Suppose the variables s_0, a_2, b_3 are assigned to the leader, then the follower will have a_1, a_3, b_1, b_2 as its variables. The optimization problem for leader and follower can be formulated as:

Leader Problem:

Minimize $-K.E.(s_0, a_1, a_2, a_3, b_1, b_2, b_3)$

by varying (s_0, a_2, b_3)

subject to (5.23)

$$mass_{total} - 80.5 \le 0$$

$$\sigma_t - 2.5\times10^8 \le 0$$

$$t(r)_{\max} - 0.12 \le 0.$$

Follower problem:

Minimize $f_2(s_0, a_1, a_2, a_3, b_1, b_2, b_3) = \sum_{i=1}^{N-1} \left|t(r_{i+1}) - t(r_i)\right| = \sum_{i=1}^{N-1} \Delta t_i$

by varying (a_1, a_3, b_1, b_2) (5.24)

The follower problem is solved at first. There are four variables (a_1, a_3, b_1, b_2) and three parameters (s_0, a_2, b_3). Assuming that

$$s_0 = s_0^1, a_2 = a_2^1, b_3 = b_3^1, \tag{5.25}$$

where s_0^1, a_2^1, b_3^1 are known assumed values for s_0, a_2, b_3. The follower problem is unconstrained because all constraints are considered in the leader's problem. After solving the follower's problem, the optimum values for a_1, a_3, b_1, b_2 are obtained.

Next, let $x = [a_1, a_3, b_1, b_2]$, and $p = [s_0, a_2, b_3]$. Substituting these values into system of equations given by Eq. (3.28), the sensitivity information $\dfrac{dx^*}{dp}$ is obtained. Here, x, p are vectors. This sensitivity information is used to construct Eq. (3.29) for vector x. After substituting this expression in the leader problem (Eq. (5.23)), the leader problem has s_0, a_2, b_3 as variables and the constraints are given by Eqs. (5.20), (5.21) and (5.22).

The optimum solution to the leader problem is s_0^*, a_2^*, b_3^*. If the difference of these values and the values s_0^1, a_2^1, b_3^1 is within the allowable limit of 1%, the iterations terminate and these values are taken as the solution of the optimization problem Eq. (5.23) and Eq. (5.24) using the Stackelberg approach. If the convergence criterion is not met, then substitute s_0^*, a_2^*, b_3^* in the follower problem and repeat this procedure until convergence criteria is met. The results of the optimization problem are given in the third row of Table 5.1.

The bi-level optimization problem converged after 7 iterations. On comparing the values of objective functions using the Stackelberg formulation with the values given in Table 5.1, It can be seen that the value of leader's objective function was 1.95×10^{6} when only the leader was considered and it decreased to 1.45×10^{6} both the objectives were considered with follower in the Stackelberg game. Similarly the value of follower's objective function was 0.0324 when it was considered individually. When it was considered along with leader's function, the optimum value increased to 0.1675. Figure 5.3 compares the profile shapes of single and multi-objective solutions. The horizontal axis shows the length of the flywheel fixed at 0.3 m and the vertical axis is the thickness of flywheel in meter. It can be seen from Fig. 5.3 that when only the manufacturing cost is considered, the profile shape has less deviation than other two cases.

The value of the variable partitioning metric (VPM) given by Eq. (5.15) for this variable partitioning is 0.6240. As mentioned before, VPM was defined the criterion to compare different variable partitionings. The VPM can take any value between zero and one. The higher the value of the VPM, the better is that variable partitioning case. When the coefficient partitioning is changed the different optimum solutions are obtained. Table 5.2 shows some of the partitions which were tried; each partitioning case has its own optimum solution and corresponding optimum vector.

The results in the second row were obtained after 31 iterations. The value of VPM is 0.6926 which is better than the value associated with the solution in table 5.1. The third row has two variables for the leader and 5 variables for the follower. The values of a_2, b_1, b_3 compared to the other coefficients are negligible, and can be ignored in the profile function of the flywheel. Both values of leader and follower objective function values are worse compared to the second row; also the value of the VPM criterion for this partition is the smallest one amongst the five partitioning cases indicated in Table 4.2. The fourth row yields the highest value for VPM and is selected as the best partitioning case. The last two rows are other possible options for partitioning the coefficients. Figures 5.4 and 5.5 compare flywheel profile shapes for cases 2-6.

The bold graph in Figure 5.4 is for the case which has the best criterion value (0.832) the two other graphs are related to second and third rows of the table 5.2 which have criterion values of 0.6926 and 0.6027 respectively. These two graphs have more deviation in the flywheel profile than the bold one. Similarly in Fig 5.5, flywheels corresponding to rows 5 and 6 (with lower VPM values) show more deviation than shape corresponding to row 4. This example illustrated the application of sensitivity based approach to a complex problem where in partition in problem variables was not obvious.

5.2 Optimum Design of High-Speed 4-bar Mechanisms

This section considers the design of a high speed mechanism as a multi objective optimization problem wherein the kinematic and dynamic criteria are optimized simultaneously. The kinematic criteria include minimization of the structural error and a minimization of deviation of the transmission angle from its ideal value. The dynamic criterion used is minimization of the peak torque required to drive the input link over a cycle. A Stackelberg (leader-follower) game theoretic approach is used to solve the multiobjective problem. Three variants, wherein both the kinematic and the dynamic criteria are treated as the leader, are considered. The design variables include mechanism dimensions. The computational procedure using sensitivity information is used for approximating rational reaction sets needed for capturing exchange of information between the leader and the follower problems.

5.2.1 Introduction

The design of high speed mechanisms requires a simultaneous consideration of both kinematic dynamic criteria. The kinematic criteria involve minimizing the difference between the desired and generated motion while keeping the transmission angle close to its ideal value. The dynamic performance criteria include minimizing the peak driving torque required over a cycle while reducing the shaking forces transmitted to the frame. In the literature, little research has been done wherein both kinematic and dynamic criteria are considered simultaneously. Most of the research has focused either on consideration of only kinematic criteria (Cabrera et al. (2011), Acharyya and Mandal (2009)) or on the consideration of dynamic criteria (Rao (1986)). Various optimization techniques have been used in these works including, genetic algorithms, goal programming, fuzzy methods, and evolutionary algorithms. However, these works considered only a single-objective function. Recently, some works have appeared where multiple objective functions are considered (Khorshidi et al. (2011), McDougall and Nokleby (2010), Nariman et al. (2009), and Yan and Yan (2009)).

This section considers the design of planar high speed mechanism as a multiple objective problem wherein the dynamic and kinematic criteria are simultaneously considered as objective functions in the context of a Stackelberg game. The kinematic criteria are affected by link dimensions and orientations, whereas the dynamic criteria will be sensitive to link dimensions and orientations as well as counterweights added to all moving links. An example problem dealing with the design of a path generating four-bar mechanism is presented. The computational procedure utilizes sensitivity of follower's solution to leader's choices to generate rational reaction sets for the follower problem. The example shows that the proposed approach is able to simultaneously improve both kinematic and dynamic performance measures of the mechanism under consideration.

5.2.2 Mechanism Design Problem Formulation

Consider the design of a high speed path generating four-bar mechanism wherein both kinematic and dynamic criteria need to be considered simultaneously to improve the overall design. The kinematic criteria consist of two objective functions: (i) minimize the difference between the desired motion and the actual motion generated by the mechanism; (ii) minimize the deviation of the transmission angle from its ideal value $(90°)$ over the entire range of motion. The dynamic criteria include (i) minimization of input driving torque required over a cycle and/or (ii) minimization of shaking forces transmitted to the frame.

5.2.2.1 Kinematic Criteria and Constraints:

A four bar mechanism shown in Fig. 5.6 is to be designed to generate a desired path with rotation of the input link. The coordinates of the path described by the coupler point P are given as

$$X_{gi} = X_{O_A} + r_2 \cos(\theta_{2i} + \alpha) + r_5 \cos(\theta_{3i} + \alpha) - r_6 \sin(\theta_{3i} + \alpha) \tag{5.26}$$

$$Y_{gi} = Y_{O_A} + r_2 \sin(\theta_{2i} + \alpha) + r_5 \sin(\theta_{3i} + \alpha) + r_6 \cos(\theta_{3i} + \alpha) \tag{5.27}$$

where $$\theta_{2i} = \theta_{2s} + \bar{\theta}_{2i} \tag{5.28}$$

and (X_{O_A}, Y_{O_A}) are the coordinates of the ground pivot O_A, α is the angular orientation of the ground link, $r_j\,(j=1,2,\ldots,6)$ are the link lengths, and θ_{2s} is the starting position of the input link, and θ_{2i} and θ_{3i} are the angular orientation of link 2 and 3 at the i-th design position. Suppose one whole cycle of input link rotation is divided by N design positions $(i=1,2,\ldots,N)$. The corresponding desired values of the path coordinates is given as (X_{di}, Y_{di}). The first objective function minimizes the path error over the entire range of motion:

$$f_1 = \varepsilon = \sum_{i=1}^{N} \varepsilon_i^2 = \sum_{i=1}^{N} \left[(X_{di} - X_{gi})^2 + (Y_{di} - Y_{gi})^2 \right] \tag{5.29}$$

where

$$X_{di} = 0.4 - \sin 2\pi (t_i - 0.34) \tag{5.30}$$

$$Y_{di} = 2.0 - 0.9 \sin 2\pi (t_i - 0.5) \tag{5.31}$$

$$t_i = \frac{i-1}{N} \quad (i = 1,2,\ldots,N) \tag{5.32}$$

In this research, a value of $N = 15$ is used. The coordinated input link orientations are determined using

$$\bar{\theta}_{2i} = 2\pi t_i \tag{5.33}$$

The minimization of f_1 is achieved by varying the link lengths r_1 to r_6 and the ground coordinates X_{O_A}, Y_{O_A}, and α. The second kinematic criterion is to minimize the deviation of transmission angle (γ) from its ideal value $(90°)$ over the entire cycle.

$$f_2 = \delta = (\gamma_{\max} - 90)^2 + (\gamma_{\min} - 90)^2 \tag{5.34}$$

where the minimum and maximum values of γ can be obtained by

$$\cos\gamma_{\min} = \frac{r_3^2 + r_4^2 - (r_1 - r_2)}{2r_3 r_4} \tag{5.35}$$

$$\cos\gamma_{\max} = \frac{r_3^2 + r_4^2 - (r_1 + r_2)^2}{2r_3 r_4} \tag{5.36}$$

The constraints on the design problem include:

1. The mechanism should satisfy the loop closure equation at each design position. This is enforced through an equality constraint of the form:

$$2r_2 r_4 \cos(\theta_{2i} - \theta_{4i}) - 2r_1 r_4 \cos\theta_{4i} + 2r_1 r_2 \cos\theta_{2i} + r_3^2 = r_1^2 + r_2^2 + r_4^2 \qquad i = 1, 2, \ldots, N. \tag{5.37}$$

2. The path error at each design point should be less than a specified small quantity Δ,

$$\varepsilon_i \le \Delta, \qquad i = 1, 2, \ldots, N. \tag{5.38}$$

3. The following two constraints enforce the restriction to have input link as a crank:

$$r_1 + r_2 < r_3 + r_4 \tag{5.39}$$

$$(r_3 - r_4)^2 < (r_1 - r_2)^2 \tag{5.40}$$

4. The value of transmission angle over the entire cycle is constrained as

$$\frac{1}{6}\pi \le \gamma \le \frac{5}{6}\pi \tag{5.41}$$

5.2.2.2 Dynamic criteria and constraints:

The rigid links are assumed to have general shape and the revolute joints are frictionless. Each link has a length r_i, $i = 1, 2, 3, 4,$ and each moving link has a mass m_i

and a moment of inertia I_i with respect to the center of mass which is defined by r_{gi} and ϕ_i as shown in Fig. 5.6. The free body diagram for each link, including the ground link, is shown in Fig. 5.7. For each link, two force equilibrium equations and one moment equilibrium equation can be written resulting in the following system of equations:

$$F_{O2x} = F_{23x} - F_{12x} \tag{5.42}$$

$$F_{O2y} = F_{23y} - F_{12y} \tag{5.43}$$

$$T_S + T_{O2} - F_{32x} r_2 \sin(\theta_{2i}) + F_{32y} r_2 \cos(\theta_{2i}) - F_{O2x} r_{g2} \sin(\theta_{2i} + \phi_2) + F_{O2y} r_{g2} \cos(\theta_{2i} + \phi_2) = 0 \tag{5.44}$$

$$F_{O3x} = F_{34x} - F_{23x} \tag{5.45}$$

$$F_{O3y} = F_{34y} - F_{23y} \tag{5.46}$$

$$T_{O3} + F_{34x} r_3 \sin(\theta_{3i}) - F_{34y} r_3 \cos(\theta_{3i}) - F_{O3x} r_{g3} \sin(\theta_{3i} + \phi_3) + F_{O3y} r_{g3} \cos(\theta_{3i} + \phi_3) = 0 \tag{5.47}$$

$$F_{O4x} = -F_{34x} - F_{14x} \tag{5.48}$$

$$F_{O4y} = -F_{34y} - F_{14y} \tag{5.49}$$

$$T_{O4} - F_{34x} r_4 \sin(\theta_{4i}) + F_{34y} r_4 \cos(\theta_{4i}) - F_{O4x} r_{g3} \sin(\theta_{4i} + \phi_4) + F_{O4y} r_{g4} \cos(\theta_{4i} + \phi_4) = 0 \tag{5.50}$$

This system of equations consists of nine equations in nine unknowns including the x and y components of four bearing reactions $(F_{12}, F_{23}, F_{34}, F_{14})$ and the input torque (T_s). All inertia forces (F_{Oix}, F_{Oiy}) and couples (T_{Oi}) are known. The shaking force (SF) is vector summation of forces acting on the ground link.

$$SF = F_{21} + F_{41} \tag{5.51}$$

The dynamic analysis is performed at every five degree rotation of the input link. This results in 72 evaluations during each cycle of rotation. The ultimate objective is to design a mechanism which requires minimum driving torque. So in dynamic analysis, the input torque (T_s) is used as the objective function.

5.2.3 The Optimization Problem

There are three objective functions, nine variables including $r_1, r_2, r_3, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and 34 constraints in Eqs. (5.37)-(5.41). The objective functions include minimizing the path error over the entire range of motion Eq. (5.34), the deviation of transmission angle Eq. (5.34) and input torque over a cycle (T_s). The bi-level optimization problem has two objective functions. Based on which pair of objective functions is selected, the common variables between the two objective functions are determined. For example, if the deviation of transmission angle and input torque are considered, then the effective variables would be $r_1, r_2, r_3, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and the common variables are r_1, r_2, r_3, r_4. The variables $x_{OA}, y_{OA}, r_5, r_6, \alpha$ will not show up in the deviation of transmission angle's problem.

This problem will be set up as a bi-level (leader-follower) model. One of the objective functions, input torque, is considered as the leader whereas the second one, the deviation of transmission angle, is treated as the follower. Eqs. (5.39)-(5.41) are associated with the follower problem for the constraints. The constraints associated for leader problem are the structural error at each design position Eq. (5.38) and the equality constraint given by Eq. (5.37).

There are four common variables, r_1, r_2, r_3, r_4, that can be considered to be common between leader and follower problems. Since the variable partitioning is not unique, several variable partitions can be explored for the entire set of variables. According to the flowchart in Fig. 3.1, the follower's problem is solved first. The follower solves its problem based on some initially selected values for the leader's variables. Then the rational reaction set of the follower's variables is approximated as in Eq. (3.32). The follower's variables are substituted in the leader's problem using this approximation for the RRS. Now, the leader solves its problem. These steps are repeated until convergence occurs as shown in Fig. 3.1.

5.2.4 Partitioning the variables

In many Stackelberg formulations, the design variables which belong to each leader and follower problem are known, but for the problem at hand, there are 4 common variables (r_1, r_2, r_3, r_4). The leader and follower have the freedom to pick amongst these four variables which will be under their control. This results in several possible combinations for partitioning the variables. Each combination yields a potential solution to the optimization problem. The criteria discussed in Eq. (5.15) can be used to compare the results and select the best choice.

5.2.5 Numerical Results

Consider first the synthesis problem for a 4-bar mechanism where the objectives are to minimize the peak driving torque over a cycle as the input link goes through a complete rotation while simultaneously minimizing the deviation of the transmission angle from its ideal value of $90°$. The maximum value for the input torque obtained over the entire cycle is the leader's objective function. The input torque is obtained by solving Eqs. (5.42)-(5.50) in 5^o increments over a 360^o cycle. The follower minimizes deviation of transmission angle from the ideal value $(90°)$ over the entire range of motion (Eq. (5.34)). The common variables between these two objective functions are r_1, r_2, r_3, r_4. It is assumed that the leader has control over variables r_1, r_2 and the follower has the control over r_3, r_4. The leader will optimize its problem by varying $r_1, r_2, x_{OA}, y_{OA}, r_5, r_6, \alpha$. The bi-level optimization problem is given as:

<u>Bi-Level Problem 1</u>

Level 1 (leader): Minimize T_S

by varying $(r_1, r_2, x_{OA}, y_{OA}, r_5, r_6, \alpha)$

subject to

$$\begin{aligned} &2r_2 r_4 \cos(\theta_{2i} - \theta_{4i}) - 2r_1 r_4 \cos\theta_{4i} + 2r_1 r_2 \cos\theta_{2i} + r_3^2 = r_1^2 + r_2^2 + r_4^2 \\ &\varepsilon_i \leq 0.1, \qquad i = 1, 2, \ldots, 10. \end{aligned} \tag{5.52}$$

where ε_i is the structural error at each design position.

The follower's problem is:

Level 2 (follower): Minimize $f_2(r_1, r_2, r_3, r_4) = \delta = (\gamma_{\max} - 90)^2 + (\gamma_{\min} - 90)^2$ (5.53)

by varying r_3, r_4

subject to Eqs. (5.39)-(5.41)

where $\gamma_{\max}$ and $\gamma_{\min}$ can be obtained by Eqs. (5.35) and (5.36).

The follower problem is solved at first; there are two variable (r_3, r_4) and two parameters (r_1, r_2). Assuming that

$$r_1 = r_1^1, r_2 = r_2^1 \tag{5.54}$$

where r_1^1, r_2^1 are initial values for r_1, r_2. After solving the follower's problem, the optimum values for r_3 and r_4 can be obtained.

Next, for the follower's variables, the approximation of rational reaction set (RRS) is obtained. Using the sensitivity based approach explained in chapter 3, an approximation to the RRS for follower's variables is constructed as follows:

$$r_3 = r_3^* + \frac{\partial r_3}{\partial r_1}(r_1 - r_1^1) + \frac{\partial r_3}{\partial r_2}(r_2 - r_2^1) \tag{5.55}$$

$$r_4 = r_4^* + \frac{\partial r_4}{\partial r_1}(r_1 - r_1^1) + \frac{\partial r_4}{\partial r_2}(r_2 - r_2^1) \tag{5.56}$$

where r_3^* and r_4^* are optimum values for r_3, r_4 corresponding to $r_1 = r_1^1, r_2 = r_2^1$. $\frac{\partial r_3}{\partial r_1}$, $\frac{\partial r_3}{\partial r_2}$, $\frac{\partial r_4}{\partial r_1}$ and $\frac{\partial r_4}{\partial r_2}$ is the sensitivity information which is obtained from the follower problem. Then Eqs. (5.55) and (5.56) are substituted in the leader's problem and the leader optimizes its problem and obtains the optimum vector $\left(r_1^*, r_2^*\right)$. This optimum vector will be compared with $r_1 = r_1^1, r_2 = r_2^1$. If the difference is not significant, then optimum vector would be the solution for the game, otherwise this loop continues until r_1^* and r_2^* are relatively unchanged with respect to $r_1 = r_1^1, r_2 = r_2^1$.

Before solving the bi-level problem 1, optimization problems with single objective function are solved to get an idea about the best and worst possible values of leader and follower objective functions. This involves considering only one objective problem and finding the optimum solution, and repeating this procedure for the second function. These two optimization problems are given below:

Problem 1: Minimize T_S

by varying $(r_1, r_2, r_3, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha)$ (5.57)

subject to

Eqs. (5.37)-(5.41)

Problem 2: Minimize

$$\delta = (\gamma_{\max} - 90)^2 + (\gamma_{\min} - 90)^2$$

by varying $(r_1, r_2, r_3, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha)$ (5.58)

subject to

Eqs. (5.37)-(5.41)

Table 5.3 shows the results of these two problems. f_{bl} is optimum value of optimization problem Eq. (5.57). If this optimum vector is substituted in objective function of Eq. (5.58) the corresponding value would be f_{wf}. Similarly by solving problem given by Eq. (45), f_{bf} and f_{wl} are obtained.

Next, using the solution procedure shown in Fig. 3.1, the Stackelberg solution obtained is given in Table 5.4. There are several possible variable partitionings between the leader and follower problem. Table 5.4 shows three of the selected partitions which were tried. In first row of Table 5.4, T_S is treated as leader by having control over $r_1, r_2, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and deviation of transmission angle is follower with r_3, r_4 as its variables. In second row, leader and follower have control over $r_1, r_2, r_3, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and r_4 respectively. The leader and follower in third row control $r_1, r_2, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and r_3 respectively. It can be noted from Table 5.4 that when the number of variables which follower has control over decrease from r_3, r_4 to r_4, the optimum value of the follower will be increased from 1240 to 1380 which is to be expected. On the other hand, the leader's optimum value is getting better. The value of the variable partitioning metric (VPM) for the second partitioning is greater than the value associated with the other partitionings. It means that according to the VPM criteria, the partition which has variables $r_1, r_2, r_3, x_{OA}, y_{OA}, r_5, r_6, \alpha$ for leader and r_4 for follower is better than others.

Fig. 5.8 shows the variation of input torque over a whole 360° cycle for the starting solution and the Stackelberg solution. It may be noted the peak value of the input driving torque has been improved significantly. Fig. 5.9 shows the variation of the follower objective function over the whole cycle at the start point and for the Stackelberg solution. Once again, it can be seen that the deviation of the transmission angle from its ideal angle (90°) has been improved significantly over the entire range of motion.

A second variant for the problem when the leader objective function is a minimization of structural error Eq. (5.29) and the follower's objective is a minimization of the deviation of transmission angle is also considered. The optimization problem is given as:

<u>Bi-Level Problem 2</u>

Level 1 (leader): Minimize ε

subject to Eqs. (5.37) and (5.38)

and (5.59)

Level 2 (follower): Minimize $\delta = (\gamma_{\max} - 90)^2 + (\gamma_{\min} - 90)^2$

subject to Eqs. (5.39)-(5.41)

Since dimensions r_3, r_4 subtend the transmission angle, control of at least one of these two variables is given to the follower and leader will have control over the rest of variables. Table 5.5 shows the results of the optimization algorithm for two different partitioning. The last column of the table is VPM corresponding to each partitioning. It can be noticed that where the leader and follower have control over $r_1, r_2, r_4, x_{OA}, y_{OA}, r_5, r_6, \alpha$ and r_3 the VPM has the highest value. Table 5.6 shows the results of these two problems when they are considered individually. Fig. 5.10 shows the path function of the mechanism for the desired and generated function over a whole cycle when the problem considered by Stackelberg game. Fig. 5.11 shows the deviation of the transmission angle from 90° for the starting point and Stackelberg solution.

The third scenario for the bi-level optimization problem would be the case when the leader is the minimization of path error Eq. (5.29) and the follower is minimizing the input the maximum input torque over a whole cycle of crank rotation T_S. The optimization problem is given as:

<u>Bi-Level Problem 3</u>

Level 1 (leader): Minimize ε

subject to Eqs. (5.37) and (5.38)

and (5.60)

Level 2 (follower): Minimize T_s

subject to Eqs. (5.37)-(5.41)

The control of $r_1, r_2, r_3, x_{OA}, y_{OA}, r_5, r_6$ has been given to the leader and the follower has the control over r_4 and α. Table 5.7 shows the results for this bi-level optimization problem. Fig. 5.12 shows the path generated by the mechanism over a whole cycle. Fig. 5.13 shows the variation of input torque over a whole 360° cycle for the starting solution and the Stackelberg solution. It can be seen from Figs. 5.12 and 5.13 that a significant improvement in both the kinematic and dynamic performance measures is achieved simultaneously.

5.3 Summary

In this chapter, the flywheel design problem is modeled by a Stackelberg game. The concept of variable partitioning metric (VPM) in a Stackelberg game was considered. The VPM can be used to compare different variable partitioning cases when it was not clear which variables should be associated with leader's objective function and which variables are used with the follower's objective function. The solution procedure used sensitivity information from the follower problem for variable updating while solving the leader's problem.

In this chapter, an integrated approach to synthesizing high speed mechanisms for three kinematic and dynamic criteria was also studied. A multi-objective formulation was presented and the Stackelberg game approach was implemented to solve the bi level optimization problem. Three different bi-level game optimization problems were set up and solved numerically. For numerical solution, the sensitivity based approach was applied for approximating the rational reaction sets of the follower's variables. The numerical examples showed that the proposed approach yields a significant improvement in both the kinematic and dynamic performance measures simultaneously. The concept of partitioning the variables between leader and follower problem was discussed and a criteria, variable partitioning metric, was applied to compare and rank different variable partitionings.

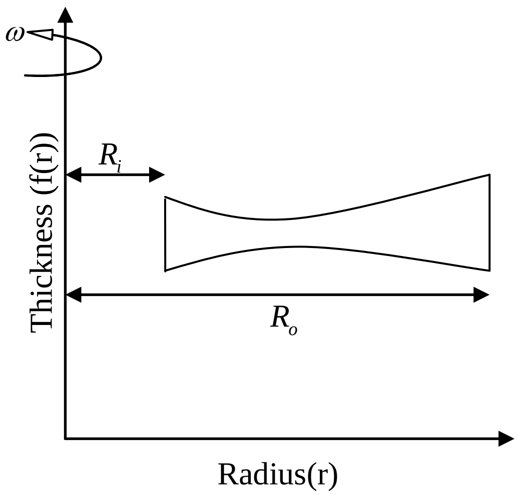


Figure 5.1 General Shape of the Flywheel.

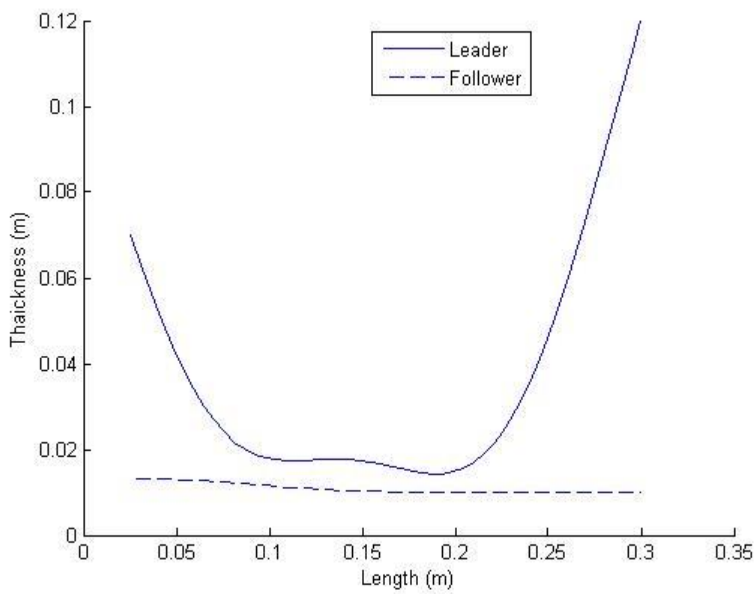


Figure 5.2 Profile Shape of Flywheel for Follower and Leader Problem.

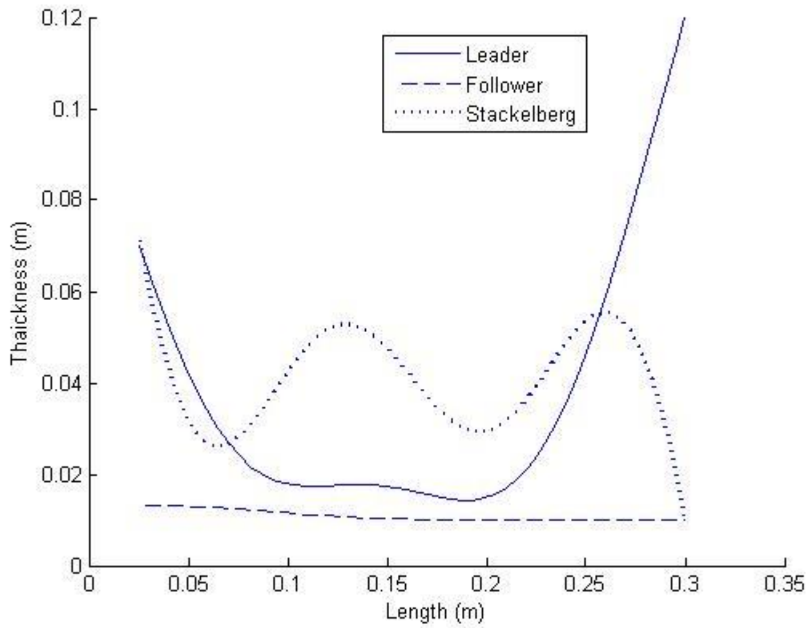


Figure 5.3 Flywheel Profile for Single Objective and Stackelberg Solutions.

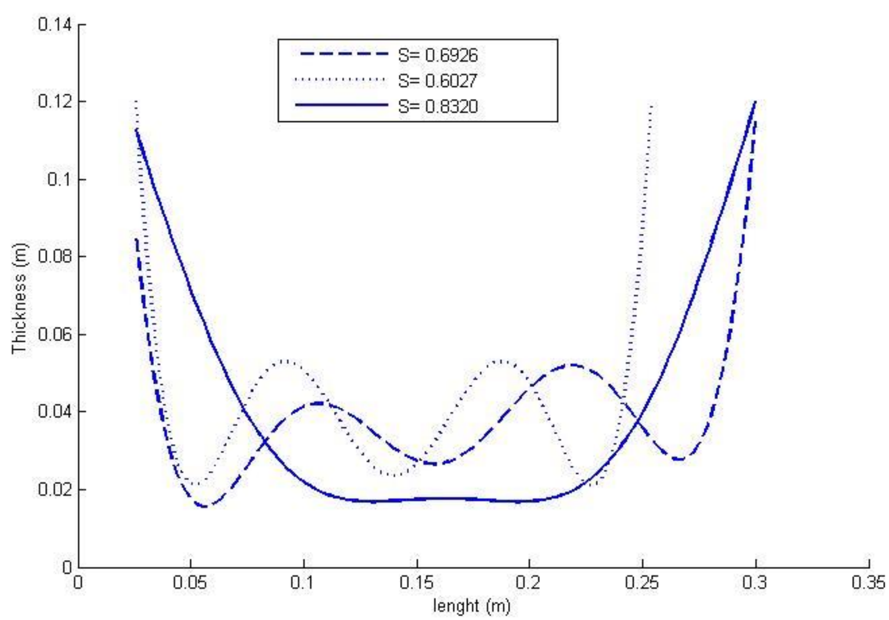


Figure 5.4 Flywheel Profile for Cases 2, 3, 4.

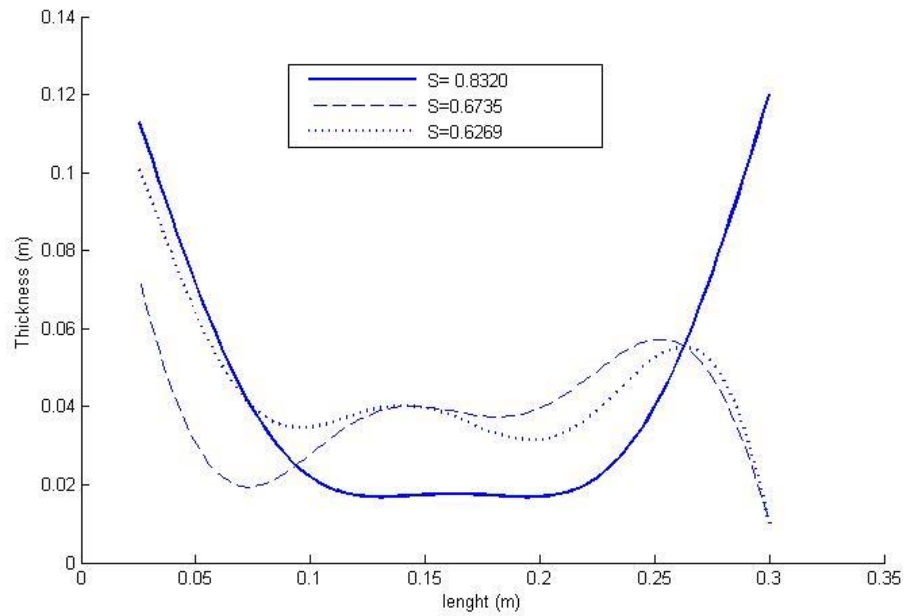


Figure 5.5 Flywheel Profile for Cases 4, 5, 6.

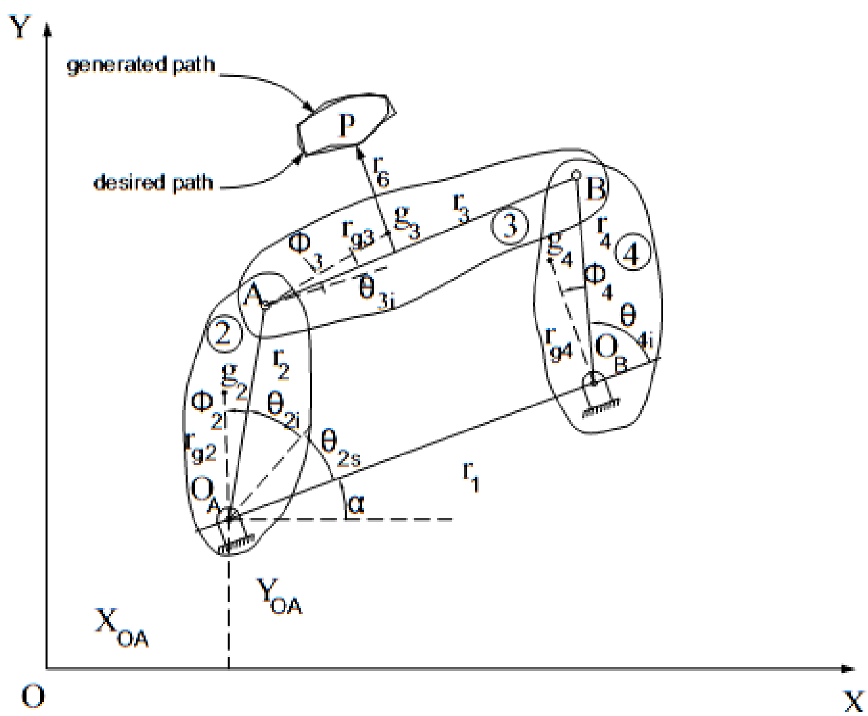
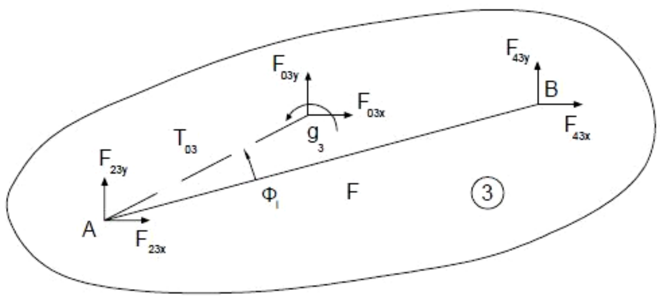
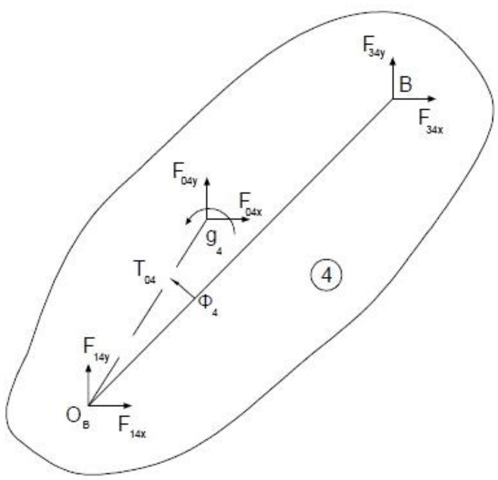
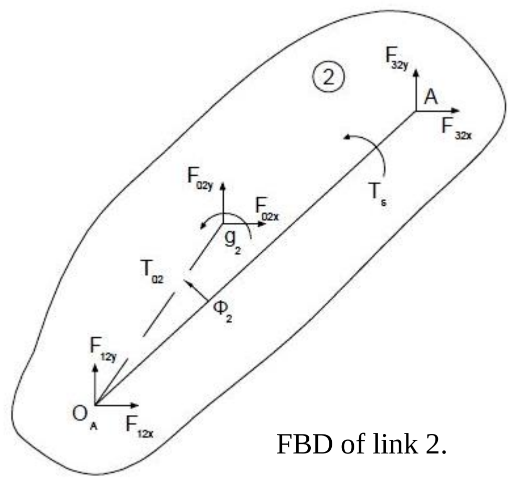


Figure 5.6 The Path Generating Four Bar Mechanism.



FBD of link 3.





FBD of link 2.

FBD of link 4.

FBD of the Ground Link

Figure 5.7 Free Body Diagrams of Four Bar Mechanism.

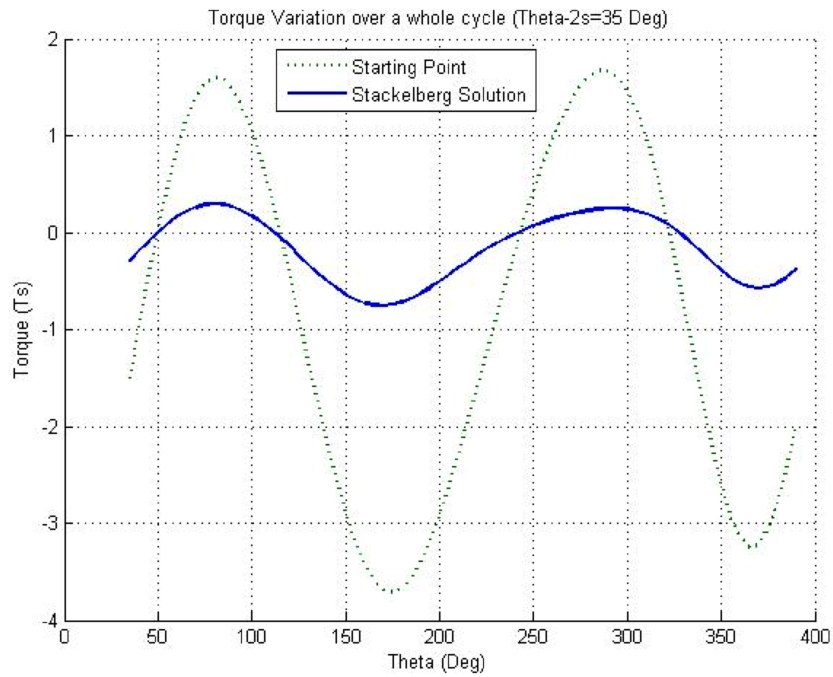


Figure 5.8 Input Torque Variation Over the Whole Cycle.

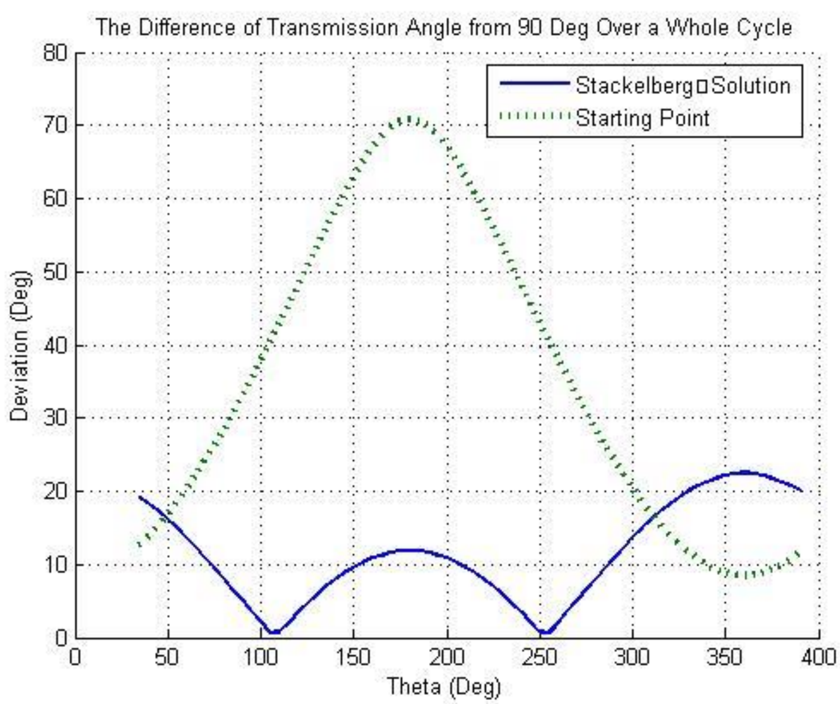


Figure 5.9 Transmission Angle Deviation from Ideal Value Over a Whole Cycle.

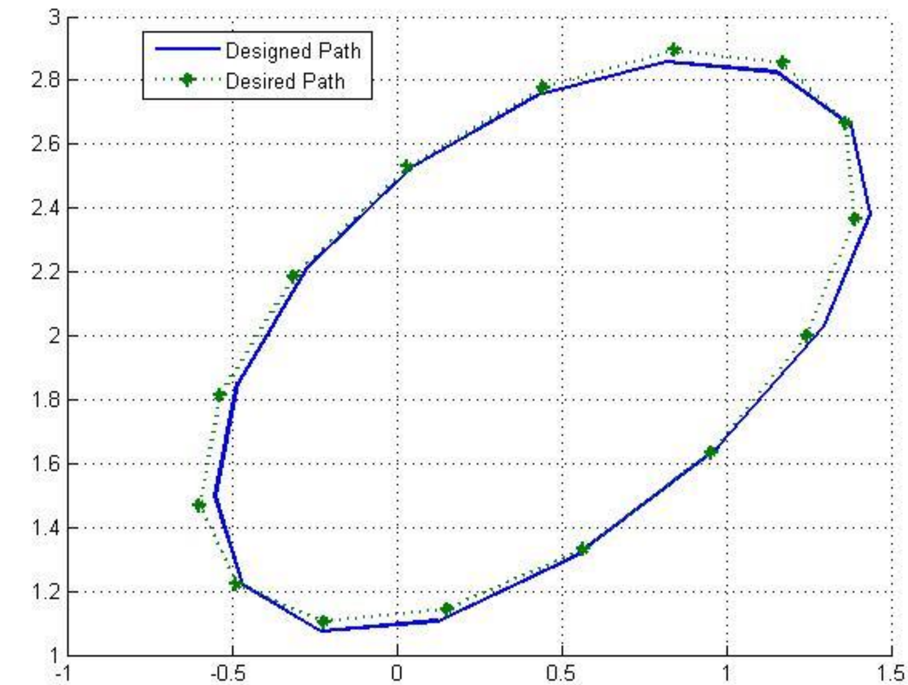


Figure 5.10 Desired versus Generated Path.

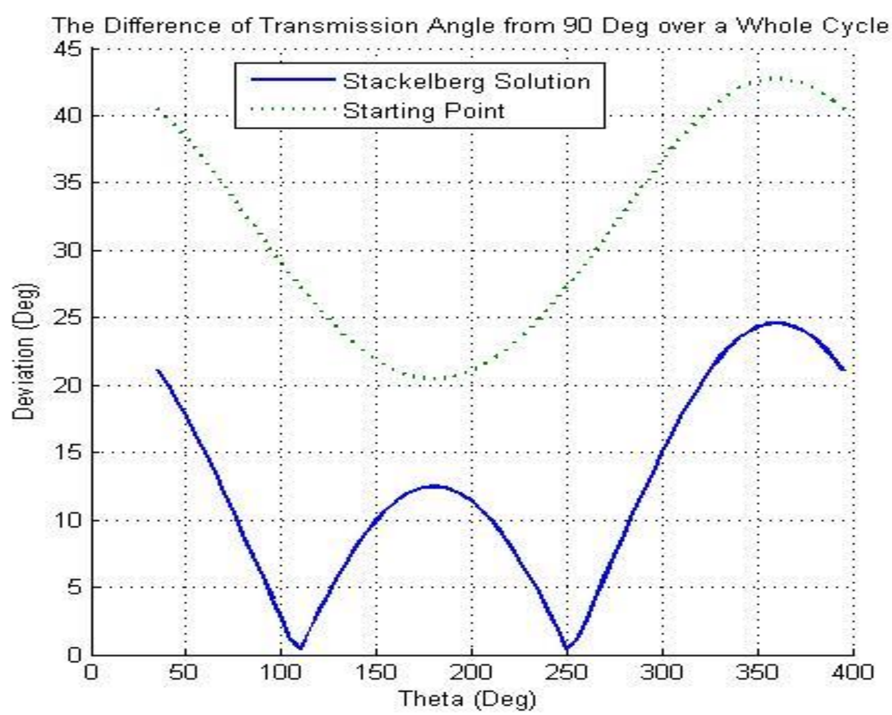


Figure 5.11 Deviation of Transmission Angle from Ideal Value.

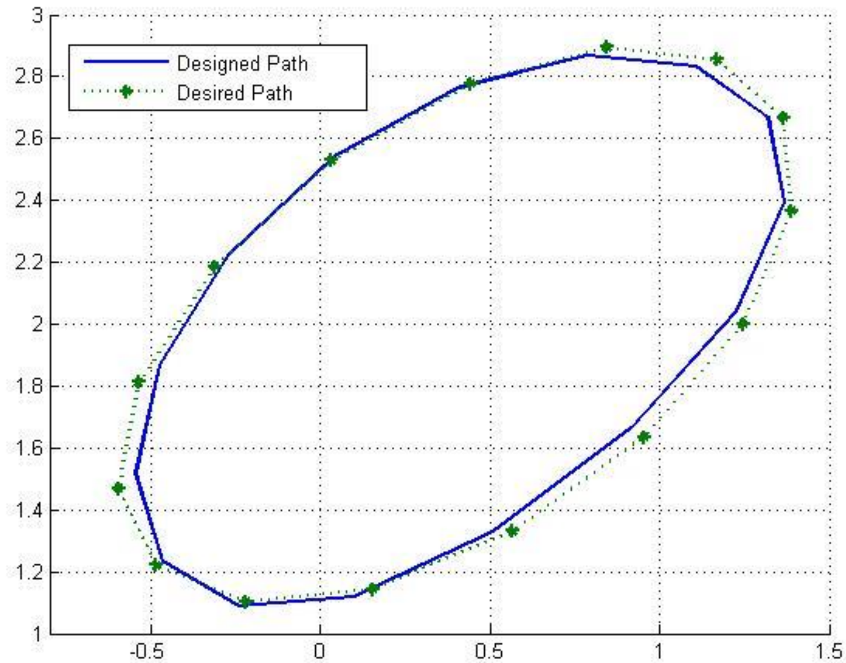


Figure 5.12 Desired Versus Generated Path.

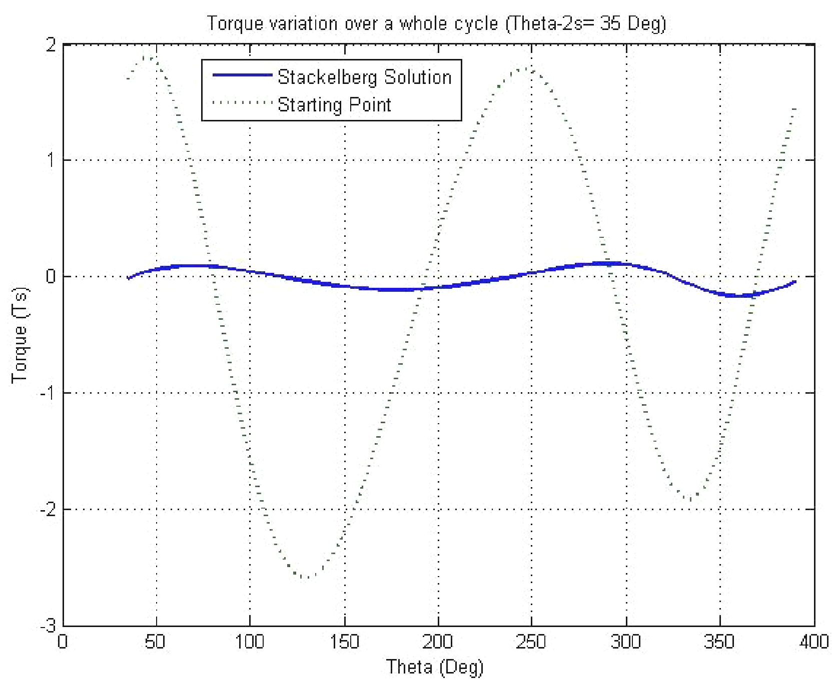


Figure 5.13 Input Torque Variation over the Whole Cycle.

Table 5-1 Optimum Solutions for Single Objective Optimizations and the Stackelberg Solution

Objective function	s_0	a_1	a_2	a_3	b_1	b_2	b_3	KE Joules	Manf.cost m
Kinetic Energy (Leader)	0.0906	-0.0824	0.0040	-0.0127	-0.0186	0.0045	-0.0064	1.95×10^6	1.6219
Manufacturing cost (Follower)	0.0106	0.0004	0.0001	-0.0001	0.0014	0.0010	0.0002	0.3535×10^6	0.0324
Stackelberg Solution	0.0417	-0.0015	-0.1020	-0.0004	0.0858	-0.0012	-0.0553	1.45×10^6	0.1675

Table 5-2 Optimum Solutions for Different Variable Partitions

Case	Leader Variables	Follower Variables	s_0	a_1	a_2	a_3	b_1	b_2	b_3	Kinetic Energy Joules	Manf. cost m	Variable Partitioning Metric VPM
1	s_0, a_2, b_3	a_1, a_3, b_1, b_2	0.0417	-0.0015	-0.1020	-0.0004	0.0858	-0.0012	-0.0553	1.45×10^6	0.1675	0.6240
2	s_0, a_1, b_1	a_2, a_3, b_2, b_3	0.6221	-0.9667	0.0171	0.1482	-0.0227	-0.5165	0.0082	1.65×10^6	0.2480	0.6926
3	s_0, a_1	a_2, a_3, b_1, b_2, b_3	0.7885	-1.2391	5.7×10^{-6}	0.1945	10^{-5}	-0.6685	5×10^{-7}	1.53×10^6	0.3156	0.6027
4	s_0, a_1, b_2	a_2, a_3, b_1, b_3	0.1171	-0.1153	0.0038	0.0152	0.0007	-0.0046	0.0010	1.85×10^6	0.2007	0.8320
5	s_0, a_2, b_1	a_1, a_3, b_2, b_3	0.0532	-0.0235	-0.0953	0.0026	0.0735	-0.0116	-0.0423	1.51×10^6	0.1431	0.6735
6	s_0, a_2, b_2	a_1, a_3, b_1, b_3	-0.0446	0.1500	-0.0907	-0.0322	0.0846	0.1000	-0.0392	1.43×10^6	0.1492	0.6269

Table 5-3 Objective Values for Single Objective Optimization.

Objective function	f_{bl}	f_{wl}
Torque (T_S)	0.1094	5.1857
Objective Function	f_{bf}	f_{wf}
Deviation of Transmission angle (δ)	233.62	1796

Table 5-4 Stackelberg Solution for Bi-Level Problem 1.

Leader Variables	Follower Variables	f_{leader} T_S	$f_{follower}$ δ
r_1, r_2, x_{OA}, y_{OA} r_5, r_6, α	r_3, r_4	0.796	804

Table 5-5 Stackelberg Solutions for Bi-Level Problem 2.

Leader Variables	Follower Variables	f_{leader} ε	$f_{follower}$ δ	VPM
r_1, r_2, x_{OA}, y_{OA} r_5, r_6, α	r_3, r_4	0.054	906.5	0.1017
$r_1, r_2, r_4, x_{OA}, y_{OA}$ r_5, r_6, α	r_3	0.0129	1043.1	0.3052

Table 5-6 Optimum Values for Single Objective Optimizations.

Objective function	f_{bl}	f_{wl}
Structural Error (ε)	0.0046	0.0668
Objective Function	f_{bf}	f_{wf}
Deviation of Transmission angle (δ)	233.6	1481.3

Table 5-7 Stackelberg Solution for Bi-level Problem 3.

Leader Variables	Follower Variables	f_{leader} ε	$f_{follower}$ T_s
$r_1, r_2, r_3, x_{OA}, y_{OA}, r_5, r_6$	r_4, α	0.0191	0.163

CHAPTER 6

GAME BASED APPROACHES IN HIERARCHICAL AND DECENTRALIZED SYSTEMS

This chapter presents application of game theory approach to solve two types of problem, hierarchical and decentralized bi-level multi-objective problem with multiple objective functions at the leader level and multiple players at the follower level. The sensitivity based approach is applied for numerical solutions. Two scenarios are studied in this chapter for modeling the decentralized bi-level multi-objective problem. The first scenario considers the cooperative game as an interaction between the players at upper (leader) level and the lower level (follower) individually. The interaction between the upper and lower level is considered as Stackelberg game. In the second scenario, the interaction in the lower level is modeled by Nash game. The sensitivity based method is used to provide an approximation to the rational reaction set (RRS) for each player. An alternate approach for generating the RRS based on design of experiments (DOE) combined with response surface methodology (RSM) is also explored. Two numerical examples are given to demonstrate the proposed algorithm for both scenarios. For the hierarchical approach, one numerical example is studied to show the application of the algorithm. For this example, there are three objective functions in three levels. The interaction between each level and its upper level is considered as a Stackelberg game.

6.1 Introduction

Bi-level decentralized decision-making problems with multiple decision makers at the upper and lower level frequently arise in manufacturing plants, logistic companies and any hierarchical organization. Fig. 6.1 shows the structure of a bi-level decentralized organization. Hierarchical problems are another type of multi-level problem in which there are several levels and with a decision maker at each level. Fig. 6.2 presents the structure of a Hierarchical problem.

This chapter applies sensitivity based approach to solve the hierarchical problem and decentralized bi-level problem. One numerical example for the hierarchical model and two test problems for decentralized bi-level are studied. This study shows that the sensitivity based approach for constructing the RRS is computationally more efficient than RSM-DOE techniques reported in the literature.

6.2 Decentralized Bi-level Model

Fig. 6.1 shows the structure of the decentralized bi-level system. Consider four players, 1 2, 3 and 4 with objective functions $f_1(x_1,x_2,x_3)$, $f_2(x_1,x_2,x_3)$, $f_3(x_1,x_2,x_3)$ and $f_4(x_1,x_2,x_3)$ respectively. Assume that players 1 and 2 are in the same level and this level functions as a leader. Players 3 and 4 are in the same level and this level functions as the follower. The optimization problem for these 4 players is modeled as below:

For player 1:

$$\text{Min} \begin{array}{l} f_1\left(x_1,x_2,x_3\right) \quad x \in R^n \\ \text{by varying } x_1 \end{array}$$

$$\text{subject to} \tag{6.1}$$

$$g_j^1\left(x_1,x_2,x_3\right) \le 0 \quad j = 1,2,\ldots n_g^1$$

for player 2:

$$\text{Min} \quad \begin{array}{l} f_2(x_1, x_2, x_3) \quad x \in R^n \\ \text{by varying } x_1 \end{array}$$

subject to (6.2)

$$g_j^2(x_1, x_2, x_3) \leq 0 \quad j = 1, 2, \ldots n_g^2$$

for player 3:

$$\text{Min} \quad \begin{array}{l} f_3(x_1, x_2, x_3) \quad x \in R^n \\ \text{by varying } x_2 \end{array}$$

subject to (6.3)

$$g_j^3(x_1, x_2, x_3) \leq 0 \quad j = 1, 2, \ldots n_g^3$$

for player 4:

$$\text{Min} \quad \begin{array}{l} f_4(x_1, x_2, x_3) \quad x \in R^n \\ \text{by varying } x_3 \end{array}$$

subject to (6.4)

$$g_j^4(x_1, x_2, x_3) \leq 0 \quad j = 1, 2, \ldots n_g^4$$

For this problem, two scenarios can be considered. In the first scenario, the interaction between players 1, 2 and 3, 4 is considered as a cooperative game. Then, the interaction between levels 1 and 2 is considered as a Stackelberg game. The second scenario, assumes cooperative game between players 1 and 2, and Nash game between player 3 and 4. Then the Stackelberg game between level 1 and 2 is modeled and solved.

To capture the cooperative behavior between players 1 and 2, the bargaining function in Eq. (6.5) is used:

$$f_B = \frac{(f_1 - f_{w1})(f_2 - f_{w2})}{(f_{b1} - f_{w1})(f_{b2} - f_{w2})} \tag{6.5}$$

where f_B is bargaining function and f_1, f_2 are the values of players 1 and 2 objective functions. f_{b1}, f_{b2}, f_{w1} and f_{w2} denote best and worst values of players 1 and 2 objective functions, respectively. These values are obtained as follows. If the player 1 optimization problem with three variables, x_1, x_2, x_3, is solved, then the optimum value of objective function for this problem is called f_{b1}. This is the best value player 1 can achieve.

The player 1 optimization problem in Eq. (6.1) is a minimization problem. If it changed to a maximization problem and is solved by varying x_1, x_2, x_3, then the optimum value of objective function is called f_{w1}. Similarly, by solving the player 2 optimization problem in Eq. (6.2) by varying x_1, x_2, x_3, the value of f_{b2} can be obtained. If player 2 optimization problem is changed to maximization problem, then the optimum value of objective function called f_{w2} is obtained by varying x_1, x_2, x_3.

To get the Nash solution and Stackelberg solution, the sensitivity based approach discussed on chapter 3 is applied. The flowchart shown in Fig. 3.2 is applied to get the Nash solution for player's 3 and 4 problem, and then the flowchart shown in Fig 3.1 is implemented to find the Stackelberg solution between level 1 and 2. This algorithm is applied to two numerical examples.

6.3 Decentralized Bi-level Model Example

6.3.1 Example 1

To demonstrate the proposed algorithm procedure for solving a decentralized bi-level optimization problem, two following examples are considered. In the first example, there are three objective functions in leader level, and there are two followers. Each follower has two objective functions. The mathematical model of this problem is as below:

Level 1

$$\underset{\text{by varying } x_0}{\text{Min}} \left(f_{11} = 2x_0 - x_1 + 2x_2,\ f_{12} = 2x_0 + x_1 - 3x_2,\ f_{13} = 3x_0 - x_1 + x_2 \right) \tag{6.6}$$

Level 2:

First Follower:

$$\underset{\text{by varying } x_1}{\text{Min}} \left(f_{21} = x_0 - x_1 - 4x_2,\ f_{22} = -x_0 + 3x_1 - 4x_2 \right) \tag{6.7}$$

Second Follower:

$$\underset{\text{by varying } x_2}{\text{Min}} \left(f_{31} = 7x_0 + 3x_1 - 4x_2,\ f_{32} = x_0 + x_2 \right) \tag{6.8}$$

Subject to:

$$\begin{aligned}
&x_0 + x_1 + x_2 \le 3 \\
&x_0 + x_1 - x_2 \le 1 \\
&x_0 + x_1 + x_2 \ge 1, \\
&-x_0 + x_1 + x_2 \le 1 \\
&x_2 \le 0.5, \\
&x_0, x_1, x_2 \ge 0
\end{aligned}$$

Two scenarios discussed in section 6.3 are applied for this example. In the first scenario, the interaction between the objective functions in level 1 is considered as cooperative game and a bargaining function explained in Eq. (6.5) is applied for this level. Table 6.1 shows the best and worst values for objective functions f_{11} to f_{31}. Eq. (6.9) shows the bargaining function for the players in level 1.

$$f_{B1} = \left(\frac{f_{11}-4}{4-(-1)}\right)\left(\frac{f_{12}-2}{2-(-1)}\right)\left(\frac{f_{13}-5}{5-(-1)}\right) \tag{6.9}$$

In this scenario, the interaction between the followers 1 and 2 is also considered as cooperative game, and followers 1 and 2 construct a bargaining function. This function is shown as below:

$$f_{B2} = \left(\frac{f_{21}-1}{1-(-2.5)}\right)\left(\frac{f_{22}-3}{3-(-3.5)}\right)\left(\frac{f_{31}-8.5}{8.5-(-0.5)}\right)\left(\frac{f_{32}-2}{2-(0)}\right) \tag{6.10}$$

where f_{B2} is the bargaining function of level 2 (follower level). The worst and best values of the followers' objective functions are obtained from Table 6.1. The new design optimization problem can be written as below:

Level 1: (6.11)

Max f_{B1}
by varying x_0

Level 2: (6.12)

Max f_{B2}
by varying x_1, x_2

Subject to:

$$x_0 + x_1 + x_2 \le 3$$
$$x_0 + x_1 - x_2 \le 1$$
$$x_0 + x_1 + x_2 \ge 1,$$
$$-x_0 + x_1 + x_2 \le 1$$
$$x_2 \le 0.5,$$
$$x_0, x_1, x_2 \ge 0$$

Then, the interaction between the level 1 and 2 would be a Stackelberg game. The objective function considered for level 2 would be f_{B2} which is defined in Eq. (6.10). The sensitivity based approach explained in chapter 3 has been used in this problem to approximate the RRS for the level 2 problem. Table 6.2 shows the results for this scenario. These results are same as the results that Ibrahim (2009) reported in his paper.

The other method which can be used to approximate the RRS of the bargaining functions of level 2 is applying the DOE-RSM method. An experiment was designed for the bargaining function of Level 2 and response surface method was applied on it. In this designed experiment, x_0 goes from zero to 1.5 in steps of 0.1. For each value of x_0, the optimization problem of level 2 shown in Eq. (6.12) can be solved to get the optimum solution for x_1^* and x_2^*. Then one can regress x_1^* and x_2^* over x_0. The results are the RRS for x_1^* and x_2^* as function of x_0. Eqs. (6.13) and (6.14) are the RRS for x_1^* and x_2^*.

$$x_1^* = -0.2721 * x_0 + 0.2978 \tag{6.13}$$

$$x_2^* = 0.5 \tag{6.14}$$

These two equations show that how the optimum solutions of x_1^* and x_2^* are varying with x_0. If these functions are plugged in the leader's problem shown in Eq. (6.11) the optimum solution for the leader would be $x_0^* = 0.2778$. Then the optimum solution for the follower would be $x_1^* = 0.22$ and $x_2^* = 0.5$. If these results are compared with the results reported in the literature by Ibrahim (2009), it can be seen that there is a significant difference between the results. It means that the DOE-RSM method can not provide the optimum results for this problem.

There is a second scenario which is also considered for this problem. In this scenario, followers 1 and 2 can construct their own bargaining functions as below:

$$f_{Bf1} = \left(\frac{f_{21} - 1}{1 - (-2.5)} \right) \left(\frac{f_{22} - 3}{3 - (-3.5)} \right) \tag{6.15}$$

$$f_{Bf2} = \left(\frac{f_{31} - 8.5}{8.5 - (-0.5)} \right) \left(\frac{f_{32} - 2}{2 - (0)} \right) \tag{6.16}$$

where f_{Bf1} and f_{Bf2} are the bargaining functions for followers 1 and 2 respectively.

The interaction between the bargaining function of follower 1 and 2 would be considered as a Nash game. The interaction between level 1 and 2 is the Stackelberg game. Table 6.3 shows the results for this scenario.

6.3.2 Example 2

The second numerical example considers a decentralized bi-level optimization problem. In this example, there are two objective functions in the leader level, and two players are in the follower level. The leader and each follower have two objective functions. The mathematical model of this problem is given below:

Level 1:

$$\text{Min} \quad \left(f_{11} = \frac{-x_0 - 4x_1 + 2x_2}{2x_0 + 3x_1 + x_2 + 2},\ f_{12} = \frac{-2x_0 + x_1 + 3x_2 + 4}{2x_0 - x_1 + x_2 + 5} \right) \tag{6.17}$$

by varying x_0

Level 2 :

Follower 1:

$$\text{Min} \quad \left(f_{21} = \frac{3x_0 - 2x_1 + 2x_2}{x_0 + x_1 + x_2 + 3},\ f_{22} = \frac{-7x_0 - 2x_1 + x_2 + 1}{5x_0 + 2x_1 + x_2 + 1} \right) \tag{6.18}$$

by varying x_1

Follower 2:

$$\text{Min} \quad \left(f_{31} = \frac{x_0 + x_1 + x_2 - 4}{x_0 - 3x_1 + 10x_2 + 6},\ f_{32} = \frac{2x_0 - x_1 + x_2 + 4}{-x_0 + x_1 + x_2 + 10} \right) \tag{6.19}$$

by varying x_2

Subject to

$$\begin{aligned} & x_0 + x_1 + x_2 \le 5, \quad -x_0 + x_1 + x_2 \le 1 \\ & x_0 + x_1 - x_2 \le 2, \quad x_0 - x_1 + x_2 \le 4 \\ & x_0 + x_1 + x_2 \ge 1, \quad x_0 + 2x_2 \le 4 \\ & x_0, x_1, x_2 \ge 0 \end{aligned} \tag{6.20}$$

Table 6.4 shows the best and worst values for objective functions when they are considered individually.

The interaction between the objective functions of the followers are considered as cooperative function by forming the bargaining function explained in Eq. (6.5). Eq. (6.21) shows this bargaining function.

$$f_{B2} = \left(\frac{f_{21} - 1.353}{1.353 - (-0.5)} \right) \left(\frac{f_{22} - 1}{1 - (-1.18)} \right) \left(\frac{f_{31} - (-0.026)}{-0.026 - (-0.75)} \right) \left(\frac{f_{32} - 1.125}{1.125 - 0.2727} \right) \tag{6.21}$$

Also, the leader can form a bargaining function shown in Eq. (6.22)

$$f_{B1} = \left(\frac{f_{11} - 0.667}{0.667 - (-0.733)}\right)\left(\frac{f_{12} - 1.25}{1.25 - 0}\right) \tag{6.22}$$

The new design optimization problem can be written as follows:

Level 1:

$$\begin{aligned} &\text{Max} \quad f_{B1} \\ &\quad \text{by varying } x_0 \end{aligned} \tag{6.23}$$

Level 2:

$$\begin{aligned} &\text{Max} \quad f_{B2} \\ &\quad \text{by varying } x_1, x_2 \end{aligned} \tag{6.24}$$

Subject to

$$\begin{aligned} &x_0 + x_1 + x_2 \le 5, \quad -x_0 + x_1 + x_2 \le 1 \\ &x_0 + x_1 - x_2 \le 2, \quad x_0 - x_1 + x_2 \le 4 \\ &x_0 + x_1 + x_2 \ge 1, \quad x_0 + 2x_2 \le 4 \\ &x_0, x_1, x_2 \ge 0 \end{aligned}$$

The interaction between level 1 and 2 is Stackelberg game. Table 6.5 shows the results of this problem.

6.4 Hierarchical Model

Consider three players, 1, 2 and 3, who select strategies x_1, x_2 and x_3 respectively, where $x_1 \in X_1 \subset R^{n_1}$, $x_2 \in X_2 \subset R^{n_2}$ and $x_3 \in X_3 \subset R^{n_3}$. Here X_1, X_2 and X_3 are the set of all possible strategies each player can select. Let U denote the set of strategies which are feasible for the three players. The objective functions $f_1(x_1, x_2, x_3)$,

$f_2(x_1, x_2, x_3)$ and $f_3(x_1, x_2, x_3)$ represent the cost function for players 1, 2 and 3 respectively. The hierarchical problem in three levels can be modeled as follows:

Level 1:

$$\begin{aligned}&\text{Min } \ f_1(x_1, x_2, x_3) \quad x \in R^n\\&\text{by varying } x_1\\&\text{subject to}\\&g_j^1(x_1, x_2, x_3) \le 0 \quad j = 1, 2, \ldots n_g^1\end{aligned} \tag{6.25}$$

For level 2:

$$\begin{aligned}&\text{Min } \ f_2(x_1, x_2, x_3) \quad x \in R^n\\&\text{by varying } x_2\\&\text{subject to}\\&g_j^2(x_1, x_2, x_3) \le 0 \quad j = 1, 2, \ldots n_g^2\end{aligned} \tag{6.26}$$

For level 3

$$\begin{aligned}&\text{Min } \ f_3(x_1, x_2, x_3) \quad x \in R^n\\&\text{by varying } x_3\\&\text{subject to}\\&g_j^3(x_1, x_2, x_3) \le 0 \quad j = 1, 2, \ldots n_g^3\end{aligned} \tag{6.27}$$

To solve this problem, the first step is to obtain the RRS for player 3 which is given by the following equation.

$$x_3^* = x_3^R(x_1, x_2) \tag{6.28}$$

where x_3^* is the optimum solution of player 3 which is varying with x_1, x_2. Then this function is substituted in optimization problem of players 1 and 2 shown in Eqs. (6.25) and (6.26). The second step is getting the RRS for the player 2 which can be represented as:

$$x_2^* = x_2^R\left(x_1\right) \tag{6.29}$$

By substituting Eq. (6.29) in optimization problem of player 1 in Eq. (6.25), this optimization problem can be solved and the optimum solution for x_1^* can be obtained. By substitution of x_1^* in Eq. (6.29), the optimum solution of player 2 will be calculated (x_2^*). To find the optimum solution of player 3, x_1^* and x_2^* is plugged in Eq. (6.28). The RRS shown in Eqs. (6.28) and (6.29) are obtained by sensitivity based approach discussed in chapter 3. This method is applied on a numerical example and is presented in next section.

6.5 Hierarchical Model Example

Consider a hierarchical problem with four levels. The player 1 in level one controls variables $x = \left(x_1, x_2\right)$ and players 2, 3 and 4 control variables $y_1 = \left(y_{11}, y_{12}\right)$, $y_2 = \left(y_{21}, y_{22}\right)$, $y_3 = \left(y_{31}, y_{32}\right)$ respectively. A Stackelberg-Nash solution for this problem using sensitivity based approach is presented in chapter 4. Liu (1998) also solved this problem by using genetic algorithm. The problem is as follows:

$$\text{Min} \quad f_1\left(x, y_1, y_2, y_3\right) = \frac{3\left(y_{11}+y_{12}\right)^2+5\left(y_{21}+y_{22}\right)^2+10\left(y_{31}+y_{32}\right)^2}{2x_1^2+x_2^2+3x_1x_2}$$

by varying x

(6.30)

subject to:

$$x_1+2x_2 \leq 10$$
$$x_1, x_2 > 0$$

$$\text{Min} \quad f_2\left(y_1\right) = y_{11}^2 + y_{12}^2$$

by varying y_1

subject to: (6.31)

$$y_{11}+y_{21}+y_{31} \geq x_1$$
$$y_{12}+y_{22}+y_{32} \geq x_2$$
$$y_{11} \geq 1, y_{12} \geq 2,$$

$$\text{Min} \quad f_3\left(y_2\right) = y_{21}+y_{22}+\frac{y_{11}}{y_{21}}+\frac{y_{12}}{y_{22}}$$

by varying y_2 (6.32)

subject to:

$$y_{21}, y_{22} > 0,$$

$$\text{Min} \quad f_4(y_3) = \frac{(y_{31}-y_{21})^2}{y_{31}}+\frac{(y_{32}-y_{22})^2}{y_{32}}$$

by varying y_3

subject to: (6.33)

$$2y_{31}+3y_{32}=5$$
$$y_{31}, y_{32} > 0$$

This is a leader follower system with player 1 as the leader for the players 2, 3 and 4. Also, player 2 is the leader for the players 3 and 4. Similarly, player 3 is the leader for the player 4. The solution procedure starts from player 4. Section 6.4 explained the steps should be done to solve this problem with sensitivity based approach. The first column of the Table 6.6 shows the results of this problem. The same example by considering non-cooperative game (Nash) between players 2, 3 and 4 and Stackelberg between level 1, player 1, and level 2 was modeled and solved in section 4.1.1. The second column of Table 6.6 shows the results for this Stackelberg-Nash problem.

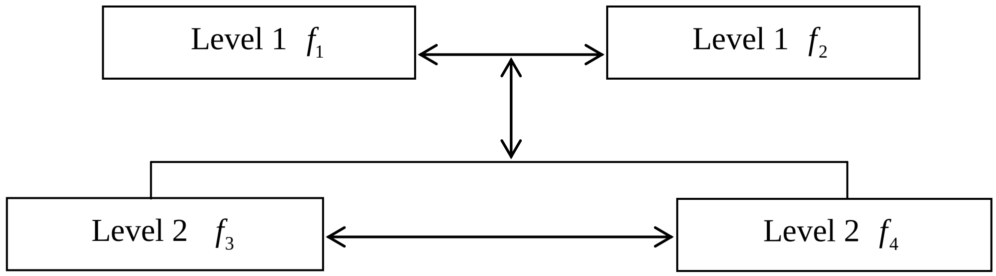


Figure 6.1 Decentralized Systems.

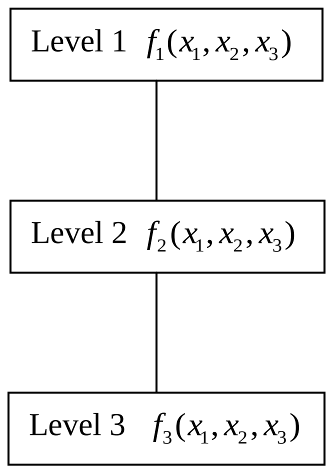


Figure 6.2 Hierarchical System with Three Levels.

Table 6-1 The Best and Worst Values of Objective Functions.

	f_{11}	f_{12}	f_{13}	f_{21}	f_{22}	f_{31}	f_{32}
Min f_{ij} (Best)	-1	-1	-1	-2.5	-3.5	-0.5	0
Max f_{ij} (Worst)	4	2	5	1	3	8.5	2

Table 6-2 Optimum Solution for Cooperative-Stackelberg Scenario.

x_1^*	x_2^*	x_3^*	f_{11}^*	f_{12}^*	f_{13}^*	f_{21}^*	f_{22}^*	f_{31}^*	f_{32}^*
0	0.5	0.5	-1	-1	-1	-2.5	-3.5	-0.5	0

Table 6-3 Optimum Solution for Nash-Stackelberg Scenario.

x_1^*	x_2^*	x_3^*	f_{11}^*	f_{12}^*	f_{13}^*	f_{21}^*	f_{22}^*	f_{31}^*	f_{32}^*
0.647	0.128	0.223	1.612	0.753	2.036	-0.373	-1.155	4.021	0.870

Table 6-4 The Best and Worst Values of Objective Functions.

	f_{11}	f_{12}	f_{21}	f_{22}	f_{31}	f_{32}
Min f_{ij} (Best)	-0.733	0	-0.5	-1.18	-0.75	0.2727
Max f_{ij} (Worst)	0.667	1.25	1.353	1	-0.026	1.125

Table 6-5 Optimum Solution for Cooperative-Stackelberg Scenario.

x_1^*	x_2^*	x_3^*	f_{11}^*	f_{12}^*	f_{21}^*	f_{22}^*	f_{31}^*	f_{32}^*
1	1	0	-0.571	0.5	0.2	-1	-0.4	0.5

Table 6-6 Hierarchical Model Solution.

	Hierarchical Solution	Stackelberg-Nash
f_1	1.5831	1.510
f_2	5	10.821
f_3	5.335	6.061
f_4	0.8736	0.483
$x^* = \left(x_1^*, x_2^*\right)$	(4.3007,2.8497)	(5.379,2.310)
$y_1^* = \left(y_{11}^*, y_{12}^*\right)$	(1.000,2.000)	(2.612,2.000)
$y_2^* = \left(y_{21}^*, y_{22}^*\right)$	(2.0068,1.4142)	(1.616,1.414)
$y_3^* = \left(y_{31}^*, y_{32}^*\right)$	(0.8736,1.0843)	(1.149,0.900)

CHAPTER 7

CONCLUSIONS

The overall objectives of this dissertation can be classified into three categories: (1) Modeling and solving the multi-objective design optimization problems with Stackelberg game. Towards this end, a new computational procedure utilizing the sensitivity of follower's solution to leader's choice is presented. (2) Non cooperative game, Nash game, can be used for modeling a multi-objective optimization problem. A new algorithm has been developed to find Nash solutions numerically. (3) Developing the numerical algorithm for solving of decentralized bi-level multi-objective optimization problems and hierarchical systems.

7.1 Stackelberg Game

The mathematical model of a bi-level optimization problem modeled as a Stackelberg game is developed. Solving a Stackelberg problem is quite different than modeling the problem with Stackelberg game. The available literature usually discusses the modeling the design optimization problems with Stackelberg game and less consideration has been given to numerical approaches. This research addresses the formulation and solution of a bi-level optimization problem using the Stackelberg approach. A computational procedure utilizing sensitivity of follower's solution to leader's choices is also presented to solve the bi-level optimization problem numerically. When the follower's problem is solved, optimum values of follower's variables are determined for given values of leader variables, which are treated as fixed parameter values. The optimum values for leader's variables are updated during each iteration. This requires an updating follower's optimum solution while the leader's variables are changing. The main contribution of this thesis is developing a new approach based on sensitivity information to feed back into the leader problem the variable updating information coming up from the follower problem. The most challenging part in solving a Stackelberg optimization problem is finding the rational reaction set for the follower. Once the RRS of the follower is known, the leader’s problem can be solved. The available method in the literature to approximate the RRS is applying DOE-RSM techniques. This thesis introduced a new technique for approximation the RRS of the follower.

The variables in the optimization problem are partitioned into two groups, variables associated with the leader and variables associated with the follower. For some Stackelberg problems, partitioning of variables between the leader and the follower is obvious, but for other problems such as the flywheel design problem considered herein, this choice is not clear. For problems where it is not obvious which variables should be associated with which objective function (leader or follower), an analytical criterion was proposed to compare various partitioning and rank them.

Two mechanical design problems including flywheel design and design of high speed 4-bar mechanism were modeled by the Stackelberg game. The sensitivity based approach was applied to solve the problems numerically. For the flywheel problem, two types of objective functions including minimizing manufacturing cost and maximizing the absorbed kinetic energy were considered as two players in Stackelberg game. The partitioning issue was discussed in this problem and the best partition was selected. For high speed mechanism, the dynamic and kinematic criteria were considered as objective functions. Three different bi-level game optimization problems were set up and numerically solved. The numerical results show that the Stackelberg game approach significantly improves both the kinematic and dynamic performance criteria simultaneously.

7.2 Non-Cooperative (Nash) Game

The sensitivity based approach can be applied to determine Nash solution(s) in multiobjective problems modeled as a non-cooperative game. This approach can provide an approximation to the rational reaction set (RRS) for each player. An intersection of these sets yields the Nash solution for the game. The other approach which exists in the literature to approximate the RRS is applying design of experiment (DOE) combined with response surface method (RSM). This thesis explored this method in some numerical examples and results were compared with sensitivity based approach. Minitab 16 was used to design the experiments (DOE) and apply the response surface method.

The DOE-RSM method was compared with sensitivity based approach on three example problems. It was seen that the proposed sensitivity based approach requires less computational effort than a RSM-DOE approach. The pressure vessel problem was tested for this purpose. The results of pressure vessel problem also showed that the sensitivity based approach is less prone to numerical errors than a RSM-DOE approach. The Nash solution in the pressure vessel problem was not a unique solution. The sensitivity based algorithm could find all Nash solutions, but RSM-DOE method could not produce all Nash solutions. For the two-bar truss problem, the closed-form functions of RRS for the followers were non linear functions. The sensitivity based approach was able to approximate these non linear RRS correctly, although the RSM-DOE method was not successful. Finally for one bi-level test problem (Liu's problem) the proposed approach in this research could improve the Nash solution that was reported in the literature.

7.3 Hierarchical and Decentralized Systems

There are four versions of the Stackelberg game, namely: (1) One objective function in leader and one in follower, (2) One leader and several followers arranged such that there is one follower at each level. This is a Hierarchical System. (3) One leader and several followers with all followers on the same level. This represents a decentralized system. (4) Several leaders and several followers. The sensitivity based approach was applied for Hierarchical system. For the decentralized system, the interaction between the followers was considered as Nash game and the interaction between the two levels was Stackelberg game. For systems with several leaders and followers, two scenarios were discussed. The first scenario considered the cooperative game between players of follower and the leader level individually, and then a Stackelberg game was set up between the two levels. The second scenario assumed a Nash game interaction between the players in the follower level and cooperative game in leader level, and then the Stackelberg game was applied between two levels. One numerical example for each scenario was tested and the results were checked with results reported in the literature.

7.4 Scope for Future Work

There was a big assumption in considering all types of models discussed, developed and implemented in this research. It was assumed that all the mathematical models, variables and parameters were deterministic.

The real world is full of uncertainty and this uncertainty needs to be considered in the modeling of the engineering problems. Future work could consider the uncertainty concept in modeling of multi-objective optimization problems. New numerical methods would be needed to solve these probabilistic problems.

The DOE-RSM method can approximate RRS for the follower's problem. Based on the experiment designed for the follower's problem, it provides a fixed function as an approximation for RRS. The pattern of this function is not getting updated while the leader's problem is doing iterations. This is a reason that DOE-RSM method in some problems can not converge to a correct solution. Updating the RRS can improve the efficiency of this method. One approach can be applying moving least square method to capture updating effect. To apply this method, it needs to provide more experiments. One technique can be adding a level to the experiment. For example if there is an experiment with two levels it can be expanded to an experiment with three levels but it increases the number of experiments from 2^n to 3^n. Other efficient approaches can be considered to provide desired number of experiments to the problem so that model updating can be done.

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