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AN OPTIMAL CONTROL ALGORITHM FOR RAMP METERING OF URBAN FREEWAYS

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Abstract

An urban freeway is treated as a dynamic process. A state model for the freeway is obtained with sectional traffic densities as states and entrance flow rates as controls. A linear programming problem is solved to obtain the optimal freeway densities and entrance flow rates under steady-state conditions, and a state regulator is used to minimize the deviations in traffic densities from these optimal steady-state values.

Introduction

It is well established by theoretical and experimental work that a plot of the steady-state flow rate, y, (in vehicles per hour) as a function of the traffic density, x, (in vehicles per mile) for a long uniform section of freeway or street with no exit or input points is of the general form shown in Figure 1.[1] It follows that the density on any freeway will eventually exceed the point of maximum flow rate if access is uncontrolled under heavy demand. Consequently, maximum effective use of any freeway facility, or surface street network, whether of good or bad design and construction, can be realized only by controlling its loading. With the high cost of construction of urban freeways, even a modest increase in the efficiency of their operation will result in a considerable economic gain.

A method called ramp metering[2,3] is being used in some cities to keep the freeway density below the critical value by controlling entrance ramp flow rates. However, existing ramp metering systems base their control action only on conditions in the immediate vicinity of the individual entrance ramps. For an alternative this paper describes a freeway control algorithm which coordinates the entrance ramp flow rates used by the individual controllers to:

(1) maximize the number of vehicles served under conditions of over-demand.
(2) balance the lengths of the ramp queues created by access control of the freeway.
(3) suppress the effects of random disturbances in the traffic flow.

Such a system would be implemented by placing vehicle presence detectors at appropriately selected locations along the freeway to sense vehicle density. This information would be transmitted to a digital processor which would set the ramp metering rates according to the algorithm to be described.

Development of a State Model of a Freeway

The dynamic behavior of a freeway as required for on-line control is dominated by two mechanisms: (i) conservation of vehicles, and (ii) drivers in the traffic stream reacting to increasing density by reducing their speed. Mechanism (i) can be expressed mathematically by

$$\frac{\partial x}{\partial t} + \frac{\partial y}{\partial z} = 0 \qquad (1)$$

where z is the position coordinate along the freeway. Equation (1) can be interpreted as saying that for a short section of freeway the rate of change of flow with respect to position along the freeway, $\partial y/\partial z$, is proportional to the difference between the flows into and out of the section. If this difference is nonzero, a change in density in the section as a function of time will be observed.

At any given point the flow and density are related by

$$y = vx \qquad (2)$$

where v is the speed of the vehicles. A number of relationships have been proposed for quantitatively describing (ii). Based on the data taken from the Lodge Freeway[1] the linear model of Greenshields appears to be the most realistic for these purposes. Thus the speed will be given by

$$v = v_f\left(1 - \frac{x}{x_j}\right) \qquad (3)$$

where v_f is the free speed, the limiting value of speed as density approaches zero. The jam density, at which all vehicles will come to a halt, is denoted by x_j. Typically x_j is approximately 40% of the bumper to bumper density.

Equation (2) is a per lane relationship. If at same z the lane densities are assumed to be uniform, (2) combined with (3) can be modified so that

$$y = \ell(z)\, v_f\left(1 - \frac{x}{x_j}\right)x, \quad \text{for } x \leq x_j \qquad (4)$$

where y and $\ell(z)$ are the total flow rate and the number of lanes in each direction at z.

Entrance and exit ramps are assumed to cause a discontinuity in the freeway stream flow by the amount of the ramp flow. The effects of the ramps will enter the model by the presence in $\partial y/\partial z$ of terms of the form

$$-\sum_j y_i^j \delta(z^j) + \sum_k y_o^k \delta(z^k)$$

where $\delta(z^j)$ is the Dirac impulse function, z^j is the location of the jth entrance ramp, and y_i^j and y_o^k are the flows of input ramp j and output ramp k respectively.

Discretization of the Freeway Model

As indicated by (1) and (4) a freeway is a nonlinear, distributed parameter system. The analysis which follows will employ spatial discretization of (1). Instead of discretizing into sections of uniform length, the boundaries between sections of the freeway are assumed to be chosen so that all exit ramps, entrance ramps, and changes in number of lanes occur at the boundaries of sections. Additional section boundaries may be added at points of pronounced change in geometric features of the freeway which could be expected to affect the flow of traffic. With these assumptions as to the location of the section boundaries, for each section (1) can be approximated by

$$\frac{dx^k}{dt} = \frac{1}{d^k} [y^{k-1} - \delta_o^{k-1} y_o^{k-1} - y^k + \delta_i^k y_i^k \quad (5)$$

where

y^k = the flow rate at the downstream boundary of section k

x^k = the density in section k corresponding to y^k

y_i^k = the input rate of the entrance ramp of section k

y_o^k = the flow rate of the output ramp of section k

δ_i^k = 1 if an entrance ramp is located in section k, 0 if no entrance ramp is located in section k

δ_o^k = 1 if an exit ramp is located in section k, 0 if no exit ramp is located in section k

d^k = the physical length of section k

y^o = flow rate from uncontrolled section of the freeway into section 1

k = 1, 2, ... n

Since the entrance and exit ramps always occur at a section boundary, the convention has been adopted to assign them to sections such that an entrance (exit) ramp always is at the upstream (downstream) end of the section. y^n is the outflow at the end of the freeway onto the city street system or outbound into the intercity portion of the freeway network. Assuming that, at any exit ramp, a known fraction $f^k(t)$, of the total flow will leave the freeway, the exit flow at ramp k is

$$y_o^k = f^k y^k \quad (6)$$

Using (4) and (6) in (5), it becomes

$$\frac{dx^1}{dt} = \frac{1}{d^1} [y^o - \ell^1 v_f^1 (x^1 - \frac{[x^1]^2}{x_j^1}) + \delta_i^1 y_i^1] \quad (7a)$$

$$\frac{dx^k}{dt} = \frac{1}{d^k} [\ell^{k-1} v_f^{k-1} (x^{k-1} - \frac{[x^{k-1}]^2}{x_j^{k-1}})$$

$$(1 - \delta_o^{k-1} f^{k-1}) - \ell^k v_f^k (x^k - \frac{[x^k]^2}{x_j^k}) +$$

$$\delta_i^k y_i^k] \quad (7b)$$

k = 2, 3, ... n

where y^o is the inflow from the uncontrolled portion of the freeway at the upstream end. In compact form:

$$\frac{dx}{dt} = F(x) + B y_i \quad (8)$$

where

F(x) is a vector whose components are the terms on the right hand side of (7) involving x^k

$$B = \text{diag} [\frac{\delta_i^1}{d^1}, \frac{\delta_i^2}{d^2}, \ldots \frac{\delta_i^n}{d^n}].$$

In (8) the symbol $\delta_i^1 y_i^1$ is understood to include the contribution of both y^o and $\delta_i^1 y_i^1$ in (7a).

Optimal Control of the Freeway

It is reasonable to assume that after the freeway is under control, the entrance flow rates as well as the traffic densities along the freeway will finally approach some steady state values. Therefore, the control vector $y_i(t)$ is divided into a steady state component, called a reference component, which is constant during each control interval and a time varying component. The reference component is selected on the basis of maximizing the total number of vehicles served and also balancing the entrance queues using a steady state model of the system which considers the nonlinearities.

For this reference value of control vector there will be a corresponding steady state density in each section of the freeway. A local linearization of the freeway model is performed about this reference value of density. The varying component of the control vector is determined using standard linear regulator techniques based on the linearized model and a quadratic performance functional of deviations from reference density and reference ramp flow rates.

The philosophy used here is first to find an optimal steady state density vector and then regulate the entrance ramp rates to keep the state of the system near this vector.

The steady state model can be derived by setting dx^k/dt to zero in (5), eliminating $\delta_o^k y_o^k$ using (6), and setting $y_i^k = y_{ir}^k$. The result is

$$y_r^1 = \delta_i^1 y_{ir}^1 \quad (9a)$$

$$y_r^k = (1 - \delta_o^{k-1} f^{k-1}) y_r^{k-1} + \delta_i^k y_{ir}^k \quad (9b)$$

The dependent variables which are subject to control are y_r^k for k = 1, 2, ... n. By recursive substitution, explicit expressions for y_r^k are obtained as:

$$y_r^k = (1 - \delta_o^{k-1} f^{k-1})(1 - \delta_o^{k-2} f^{k-2})$$

$$\ldots (1 - \delta_o^1 f^1) \delta_i^1 y_{ir}^1 + (1 - \delta_o^{k-1} f^{k-1})$$

$$(1 - \delta_o^{k-2} f^{k-2}) \ldots (1 - \delta_o^2 f^2) \delta_i^2 y_{ir}^2$$

$$+ \ldots + \delta_i^k y_{ir}^k, \quad (10)$$

k = 2, 3, ..., n

The optimal values for y_{ir} are obtained by solving the following linear programming problem:

Maximize $\sum_{k=1}^{n} (c_q^k q_i^k + c_d^k d_i^k) y_{ir}^k$ subject

to the constraints

$$y_r^k \leq y_m^k, \; k = 1, 2, \ldots n. \quad (11)$$

$0 \leq y_{ir}^k \leq y_{im}^k$, for all k such that

$$q_i^k > 0. \quad (12)$$

$$0 \le y_{ir}^k \le \min(y_{im}^k, d_i^k) \text{ for all k such}$$

$$\text{that } q_i^k = 0. \tag{13}$$

where q_i^k and d_i^k are the queue length and demand rate respectively at entrance ramp k, c_q and c_d are two constant weighting vectors, y_m and y_{im} are the maximum allowable flow rate and entrance flow rate respectively and y_r^k are expressed in terms of y_{ir} by (9a) and (10).

Note that the object function of the linear programming problem always gives higher priorities to entrance ramps with longer queues and higher demands, so it is designed to balance the queues at the entrance ramps while maximizing the freeway service.

Once y_{ir} has been determined by linear programming, the corresponding value for x_r can be found by using (9a) and (10) to find y_r. The reference density for each section can be calculated from the flow-density characteristic, using the lower of the two densities which is possible for the y_r^k, i. e.

$$x_r^k = \frac{1}{2}\left[x_j^k - \sqrt{(x_j^k)^2 - \frac{4x_j^k\, y_r^k}{\ell^k\, v_f^k}}\right] \tag{14}$$

In the absence of any random disturbances in the traffic flow on the freeway maintaining the ramp metering rates at y_{ir} should keep the density at x_r. Since random accelerations and decelerations of vehicles in the traffic stream are certain to occur, it is necessary to superimpose a variable component on the ramp metering rates to regulate the densities to x_r. For this purpose let

$$x(t) = x_r + e(t) \tag{15}$$

and

$$y_i(t) = y_{ir} + w(t) \tag{16}$$

where e(t) and w(t) are perturbation vectors. Substituting (15) and (16) into (8), expanding $F(x_r + e)$ around x_r by Taylor series expansion, and neglecting the second order terms in e (F is quadratic in x so that the partial derivatives of F with respect to x of order higher than two are zero), one has:

$$\begin{aligned} \dot{e}(t) &= A\, e(t) + B\, w(t) \\ e(0) &= x(0) - x_r \end{aligned} \tag{17}$$

where

$$A = \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} & \frac{\partial F^1}{\partial x^n} \\ \frac{\partial F^n}{\partial x^1} & \frac{\partial F^n}{\partial x^2} & \frac{\partial F^n}{\partial x^n} \end{bmatrix}_{x=x_r} = [a^{ij}] \tag{18}$$

with

$$a^{kk} = \frac{-\ell^k v_f^k}{d^k}\left[1 - \frac{2x_r^k}{x_j^k}\right] \tag{19a}$$

$$a^{k,k-1} = \frac{\ell^{k-1} v_f^k}{d^k}\left[1 - \delta_o^{k-1} f^{k-1}\right]\left[1 - \frac{2x_r^{k-1}}{x_j^{k-1}}\right] \tag{19b}$$

$$a^{k\ell} = 0, \text{ for } \ell \ne k, k-1 \tag{19c}$$

Fortunately, system (17) is completely controllable.[4] Due to this complete controllability it is well known that an optimal control which minimizes the performance functional

$$J(w) = \frac{1}{2}\int_0^\infty [\langle e(t), Qe(t)\rangle + \langle w(t), Rw(t)\rangle]\, dt \tag{20}$$

(Q is a positive semidefinite and R is a positive definite matrix) exists, is unique, and is given by the equation:

$$w^*(t) = -R^{-1} B^T \hat{K} e^*(t) \tag{21}$$

where $\hat{K}$ is the constant n x n positive definite matrix which is the solution of the algebraic Riccati equation:

$$-\hat{K}A - A^T\hat{K} + \hat{K}BR^{-1}B^T\hat{K} - Q = 0 \tag{22}$$

the * denotes the optimality and T means transpose. The optimal entrance flow rate y_i^* is obtained by:

$$y_i^*(t) = y_{ir} + w^*(t) = y_{ir} - R^{-1}B^T\hat{K}e^*(t) \tag{23}$$

The vector $e^*(t)$ is obtained by:

$$e^*(t) = x(t) - x_r \tag{24}$$

where the vector x(t) is measured by the density detectors along the freeway.

Implementation of the Control Algorithm

The data which the central controller must have in order to compute the optimal entrance ramp rates determined by this algorithm are x, d_i, q_i, x_j, v_f and f. These quantities can all be measured using suitably placed vehicle presence detectors. Densities are determined by accumulating the fraction of time a vehicle is indicated as present by the detector and multiplying this by the density of average length vehicles which would exist at bumper to bumper density. Demand is measured by counting vehicles passing a detector at a point far enough upstream on the entrance ramp that the queue will not reach it. The queues are measured from the difference between the counts of the demand measuring detectors and the counts of detectors placed at the downstream ends of the ramps. The fraction of the traffic stream leaving at each exit ramp f^k is measured by suitably placed detectors. Speeds are measured by dividing the time a vehicle presence is detected by average vehicle length. By curve fitting to a number of pairs of density and average vehicle speed measurements it is possible to determine values of x_j and v_f. These latter quantities change slowly so that the fact that it takes longer to arrive at an individual measurement of them is not serious.

The flow chart for the computer program which would implement the control algorithm is shown in Figure 2. The vectors x, x_j, v_f, d_i, f and q_i are monitored continuously. Whenever a density above a level judged to be critical is detected in any section of the freeway, the optimal ramp rates are computed and ramp metering is activated. New values of x, d_i, and q_i can be obtained approximately once per minute from the vehicle detectors. The ramp metering is continued until all queues are reduced to zero. New values of x_r and $\hat{K}$ are computed and used by the controller any time the measurements indicate a significant change in x_j or v_f and when a significant change in the relative demands or queue lengths occurs. The latter is considered to have occurred whenever the inequality

$$\left| \frac{c_q^k q_i^k(m+1) + c_d^k d_i^k(m+1)}{\sum_k [c_q^k q_i^k(m+1) + c_d^k d_i^k(m+1)]} - \frac{c_q^k(m)\; q_i^k(m) + c_d^k d_i^k(m)}{\sum_k [\, c_q^k q_i^k(m) + c_d^k d_i^k(m)]} \right| > \epsilon_3$$

m = index of the q_i and d_i measurements

ϵ_3 = suitably chosen threshold value

is satisfied for some value of k.

Conclusion

It has been demonstrated that well established techniques of optimal control can be applied to the optimization of ramp metering of urban freeways.

Acknowledgment

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References

1. Kreer, J. B. and Goodnuff, J. L., "On the Automatic Control of Freeway Density," IEEE Automotive Conference, Detroit, Michigan, September 1967.

2. May, A. D., "Improving Network Operations with Freeway Ramp Control," 43rd Annual Meeting of the Highway Research Board, Washington, D. C., January 13-17, 1964.

3. McCasland, W. R., Drew, D. R., and Wattleworth, J. A., "Houston Freeway Surveillance and Control Project: 1966 Progress Report." Program Review Meeting, Research and Development of Traffic Systems, Gaithersburg, Md., December 6-8, 1966.

4. Kalman, R. E., "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mex., vol. 5, pp. 102-199, 1960.

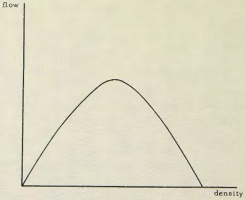

Figure 1. Steady State Flow Density Plot.

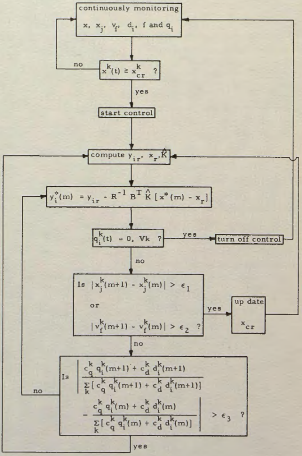

Figure 2. Flow Chart of Optimal Control Algorithm.