7-15-2013

Field Localization and the Nambu-Jona-Lasinio Mass Generation Mechanism in an Alternative 5-Dimensional Brane Model

Preston Jones
California State University, Fresno, jonesp13@erau.edu

Gerardo Muñoz
California State University, Fresno

Douglas Singleton
California State University, Fresno

Triyanta
Institutr Teknologi Bandung

Field localization and the Nambu–Jona-Lasinio mass generation mechanism in an alternative 5-dimensional brane model

Preston Jones* and Gerardo Muñoz†

Physics Department, California State University Fresno,

2345 East San Ramon Avenue M/S 37, Fresno, California 93740-8031

Douglas Singleton‡

Physics Department, California State University Fresno,

2345 East San Ramon Avenue M/S 37, Fresno, California 93740-8031 and

Department of Physics, Faculty of Mathematics and Natural Sciences,

Institut Teknologi Bandung, Jalan Ganesha 10 Bandung 40132, Indonesia

Triyanta§

Department of Physics, Faculty of Mathematics and Natural Sciences,

Institut Teknologi Bandung, Jalan Ganesha 10 Bandung 40132, Indonesia

(Dated: March 13, 2018)

Abstract

We consider a 5-dimensional brane world model with a single brane which is distinct from the well known Randall-Sundrum model. We discuss the similarities and differences between our brane model and the Randall-Sundrum brane model. In particular we focus on the localization of 5D fields with different spins – spin 0, spin 1/2, spin 1 – to the brane, and a self-consistent mass generation mechanism. We find that the brane model studied here has different (and in some cases superior) localization properties for fields/particles with different spins to the brane, as compared to the original 5-dimensional brane models. In addition this alternative 5D brane model exhibits a self generation mechanism which recalls the self-consistent approach of Nambu and Jona-Lasinio.

*Electronic address: pjones@csufresno.edu

†Electronic address: gerardom@csufresno.edu

‡Electronic address: dougs@csufresno.edu

§Electronic address: triyanta@fi.itb.ac.id

I. INTRODUCTION

One of the most active areas of recent research has been the work on large [1–3] and infinite [4–8] extra dimensions commonly referred to as "brane" world models. A key motivation for these extra dimensional brane models is to address the problem of why the energy scale of gravity (the Planck scale) should be 16 orders of magnitude larger than the energy scale of particle physics of the Standard Model (the TeV scale). A good feature of these large and infinite extra dimensional theories is that they can be probed with experiments that are in the reach of current technology. This is in distinction from traditional Kaluza-Klein theories or string theory where the direct, experimental impact of the extra spatial dimensions is effectively non-existent at energy scales accessible with current or reasonable extrapolations of near future technology. The possible experimental probes of these large and infinite extra dimensional theories span the gamut from unique particle accelerator signatures, such as the possible creation of mini-black holes at the LHC [9], or deviations of Newton's inverse square law of gravity at micro-meter and smaller distances [10].

An important requirement of these theories with large and infinite extra dimensions is to explain why at energy scales probed to the present the world appears to have only three spatial dimensions. The answer given by these theories is that all matter particles/fields and all gauge particles/fields should be bound to a 3+1 dimensional membrane (or "brane" for short) in the higher dimensional space-time [6, 7]. In this way the world, as probed by matter particles of spin 0 or spin 1/2 or by force carrying gauge particles of spin 1, would still appear effectively 3+1 dimensional up to some energy scale.

However, the original infinite extra dimensional brane metric of [4, 5, 8] where not able to localize all types of particles to the 3+1 brane in a simple manner. One could localize spin 0 fields on the brane but only at expense of not localizing spin 1/2 fields [11]. One could choose the parameters of the 5D metric [4, 5, 8], such as the sign of the of "warp" factor, so as to localize the spin 1/2 fields but then spin 0 fields would not be localized [11]. And finally gauge bosons of spin 1 were not localized for any choice of parameters of the 5D metric [12]. One could localize all the fields of various spin in these 5D brane models by introducing additional non-gravitational interactions, but this spoiled the simplicity of the model.

In this paper we will investigate a simple variant of the 5D brane metrics of [4, 5, 8] which appears similar to these original single-brane models. As far as we can find the fact that this alternative 5D warped metric is a different space-time has not been noted in the literature up to now nor have its properties been investigated. Additionally this alternative 5D metric has several key physical distinctions with respect to the usual warped metric of [4, 5, 8]: (i) It does not require a fine-tuning between the brane tension and the bulk cosmological constant as is the case with the usual 5D brane metric. (ii) It has different localization properties for fields of different spins (i.e. spin 0, spin 1/2 and spin 1) to the brane. In particular we find that for this alternative 5D brane metric spin 1 fields can be localized with a decreasing warp factor. (iii) In the case of massive scalar fields localized to the brane one finds that the masses are generated by a self-consistent manner reminiscent of the mass generation mechanism of Nambu and Jona-Lasinio [13].

II. THREE 5D BRANE METRICS FOR TWO 5D BRANE SPACE-TIMES

Following [3–5, 8] we take the general gravitational action for the 5-dimensional brane world model, including a 5D cosmological constant, as

$$S_g = -\frac{1}{2\kappa^2}\int d^5x\sqrt{g}R_5 + \frac{\lambda}{\kappa^2}\int d^5x\sqrt{g} + S_{matter}, \tag{1}$$

where R_5 is the 5D Ricci scalar, g is the determinant of the 5D metric, λ is a 5D cosmological constant and S_{matter} is the action for any matter in the system. In the Randall-Sundrum model [3–5, 8] S_{matter} was a 4D thin brane tension given by $\sigma\delta(z)$ i.e. a delta-function brane tension σ. For the alternative 5D metrics which is the focus of this paper we will find that S_{matter} has, not only a delta-function brane tension, but also a bulk energy-momentum. (An excellent review of the standard brane models with the above type of action given in (1) can be found in [14]). To include some additional field, Φ, one would expand the above action as $S = S_g + S_\Phi$ where S_Φ would be the action for the additional scalar, spinor or gauge field. However, before studying the behavior of various matter and gauge fields in the 5D space-time we will examine in some detail three 5D brane metrics. Two of the metrics are just the Randall-Sundrum 1-brane metric. The third metric, which is the focus of this paper, at first appears to be some version of the Randall-Sundrum 1-brane metric will be shown to be different from the Randall-Sundrum 1-brane metric.

We begin with the original form of the 5D brane metric [3–5, 8] which has the form

$$ds^2_{[y]} = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{2}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ is the flat 4D metric. The constant k is fixed [3, 11, 14] from the gravitational field equations with a fine tuning to the cosmological constant, λ. We will discuss this condition in the following section. An alternative version of the Randall-Sundrum model [15] reverses the sign on the constant $k \to -k$ and in the components of the stress-energy tensor.

A coordinate transformation for the extra dimension puts the metric in a more obviously conformally flat form. The coordinate transformation to these conformally flat coordinates is $dy^2 = e^{-2k|y|}dz^2$ [16–18] and the metric and extrinsic curvature in the new system of coordinates is

$$ds^2_{[z]} = \Omega^2(z)\eta_{AB}dx^A dx^B = \frac{1}{(k|z|+1)^2}\eta_{AB}dx^A dx^B \tag{3}$$

where the warp factor (i.e. $\Omega(z) = \frac{1}{(k|z|+1)}$ no longer has an exponential form but is rather an inverse power of $|z|$. This z coordinate system is a well known alternative way to write the metric (2). It is particularly useful in writing a Schrödinger-like equation for the gravitational perturbations of the 5D space-time. We will show in the following section that the two metrics (2) (3) are alternative descriptions of the same space-time.

A third system of coordinates, which appears to be a hybrid of the y system of (2) and the z coordinate system of (3), was suggested by various authors [8, 16, 19–24]. This third system of coordinates, which we unimaginatively call the r coordinate system, is also conformally flat but with an exponential warp factor rather than an inverse power of the coordinate as in the case of the metric form (3). This metric has the form

$$ds^2_{[r]} = a^2(r)\eta_{AB}dX^A dX^B = e^{-2k|r|}\eta_{AB}dX^A dX^B \tag{4}$$

where the warp factor is again exponential $a(r) = e^{-k|r|}$. There is a coordinate transformation which relates (2) with (4) given by $dy^2 = e^{-2k|r|}dr^2 \to e^{-k|r|} = 1 - k|y|$ and $e^{-k|r|}dX^\mu = e^{-k|y|}dx^\mu \to dX^\mu = \frac{e^{-k|y|}}{1-k|y|}dx^\mu$. Since there is a transformation relating the metrics in (2) and (4) one might be tempted to conclude that (as is the case for metrics (2) and (3)) the two metrics are the same. However, one hint that (2) and (4) are different comes from the fact that the transformation which relates the two metrics is singular at $|y| = 1/k$. Further for the transformation relating the y coordinate system with the r coordinate to be global one would need to require that the differential be exact, $dX^\mu = A(x_\mu, y)dx_\mu + B(x_\mu, y)dy$ and that the derivative obey $\partial_y A = \partial_{x_\mu} B = 0$. This condition is not satisfied in general since $\partial_y A \neq 0$. The conclusion is that the transformation between y and r coordinates is local but not global. However, on any bulk coordinate foliation with $y = y_c$ a constant $\partial_y A = 0$ and the coordinate transformation is global. On the other hand for the coordinate transformation between the y and z coordinates is global since one can obtain an exact differential. In the following section we will find other ways in which the 5D brane space-time given by (4) is different from the original 5D brane metrics of (2) and (3). We will also high light some important physical distinctions between the different 5D brane world metrics. In particular the localization properties of the space-time given by the r coordinate is different than the space-time given by the y and z coordinate metrics. In addition the r-metric has a self consistent mass generation mechanism given in [13].

III. EINSTEIN EQUATIONS AND THE STRESS ENERGY TENSORS

We now study the energy-momentum tensor of the metrics (2), (3), and (4) by feeding them through the 5D Einstein equations. To facilitate the study of the three metrics we will write the metrics in a generic form as

$$ds^2 = a^2(|x^5|)\eta_{\mu\nu}dx^\mu dx^\nu - b^2(|x^5|)dx^5 dx^5 \tag{5}$$

where $x_5 = r, y, z$ depending on whether one is dealing with metric (4), (2) or (3). The ansatz functions a, b are functions of the absolute value of the fifth coordinate. The ansatz functions a, b can be changed depending on which of the three metrics one is dealing with (e.g. for the metric (4) $a = b = e^{-k|r|}$).

We first calculate the Ricci scalar for the three metrics. In this way we can see that (2) and (3) are the same space-time, while (4) is a different space-time. For the general ansatz (5) the Ricci scalar is

$$R = 4\frac{2aa'b' - 3a'^2 b - 4\delta\left(|x^5|\right) aa'b - 2aa''b}{a^2b^3} \, , \tag{6}$$

where the prime means differentiation with respect to x_5. By applying (6) to each of the three metrics above we find

$$R_{[y]} = -20k^2 + 16k\delta\left(y\right), \tag{7a}$$

$$R_{[z]} = -20k^2 + 16k\delta(z) + 16k^2\delta(z)\,z, \tag{7b}$$

$$R_{[r]} = -12k^2 e^{2kr} + 16k\delta(r)\,e^{2kr}. \tag{7c}$$

The three different metrics/coordinate systems are indicated via the bracketed subscripts. Since $\delta(z)\,z = 0$ one can see that the Ricci scalars for the y and z coordinates are the same, indicating that the y and z metrics represent the same space-time. However $R_{[r]}$ is different than $R_{[y]}$ and $R_{[z]}$ indicating that the metric (4) is a different space-time than the space-time given by the metrics (2) or (3).

Next, we obtain the energy-momentum tensor connected with the metrics (2), (3), and (4) by feeding them through the 5D Einstein equations

$$G_{AB} + g_{AB}\lambda = \kappa^2 T_{AB} \ . \tag{8}$$

Since all three metrics are symmetric about the location of the brane at $x_5 = 0$ we will only focus on one side of the brane, namely $x_5 > 0$ (this avoids the unnecessary complication of writing the down step functions whenever there is a first derivative of $a(|x_5|)$ and/or $b(|x_5|)$. The second derivatives of $a(|x_5|)$ and/or $b(|x_5|)$ will give rise to $\delta(|x_5|)$ i.e. the brane energy density. The components of the Einstein tensor for the brane coordinates $(\mu, \nu = 0, 1, 2, 3)$ for the general metric (5) are

$$G_{\mu\nu} = \frac{3}{b^3}\eta_{\mu\nu}\left(a'^2 b + 2aa'b\delta\left(x^5\right) + aa''b - aa'b'\right) \ , \tag{9}$$

For the bulk coordinates one finds

$$G_{55} = -6\left(\frac{a'}{a}\right)^2 \ . \tag{10}$$

In terms of the three coordinate systems – y, z, and r – the Einstein equations on the brane become

$$\eta_{\mu\nu}e^{-2k|y|}\left(6k^2 + \lambda_{[y]}\right) - 6k\eta_{\mu\nu}\delta(y) = \kappa^2 T_{\mu\nu}, \tag{11a}$$

$$\eta_{\mu\nu}\frac{1}{(k|z|+1)^2}\left(6k^2 + \lambda_{[z]}\right) - 6k\eta_{\mu\nu}\delta(z) = \kappa^2 T_{\mu\nu}, \tag{11b}$$

$$\eta_{\mu\nu}\left(3k^2 + e^{-2k|r|}\lambda_{[r]}\right) - 6k\eta_{\mu\nu}\delta(r) = \kappa^2 T_{\mu\nu}, \tag{11c}$$

where $g_{\mu\nu}\delta(y) = \eta_{\mu\nu}\delta(y)$ and $g_{\mu\nu}\delta(z) = \eta_{\mu\nu}\delta(z)$. These delta functions in the 4D energy-momentum tensor indicate that the matter sources are thin branes. Such thin branes can be obtained as the limit of "thick" brane solutions [25] [26] [27] [28]. The bulk Einstein equations for the three systems are,

$$-\left(6k^2+\lambda_{[y]}\right)=\kappa^2 T_{55}, \tag{12a}$$

$$-\frac{1}{\left(k\left|z\right|+1\right)^2}\left(6k^2+\lambda_{[z]}\right)=\kappa^2 T_{55}, \tag{12b}$$

$$-6k^2-e^{-2k|r|}\lambda_{[r]}=\kappa^2 T_{55}. \tag{12c}$$

From (11a), (11b), (12a), and (12b) one notices that it is possible to reduce the energy-momentum tensor to almost vacuum, with the exception of the non-zero brane tension $-6k\eta_{\mu\nu}\delta\left(y\right)/\kappa^2$ or $-6k\eta_{\mu\nu}\delta\left(z\right)/\kappa^2$, by fine tuning the constant k to the 5D cosmological constant via the condition $\lambda_{[y]}=\lambda_{[z]}=-6k^2$. Looking at the energy-momentum components for the r coordinate system – (11c) and (12c) – one can see that a similar fine-tuning is not possible in this case. The simplest choice for this metric is to take the 5D cosmological constant as vanishing i.e. $\lambda_{[r]}=0$. With this choice the energy momentum tensor for the r coordinate system becomes

$$3k\eta_{\mu\nu}\left(k-2\delta(|r|)\right)=\kappa^2 T_{\mu\nu}, \tag{13a}$$

$$-6k^2=\kappa^2 T_{55}. \tag{13b}$$

Thus the fifth component of the energy-momentum is a bulk constant $T_{55}=-6k^2/\kappa^2$. The other components of the energy-momentum tensor , $T_{\mu\nu}$, are composed of a bulk constant term, $3k^2$, a constant term, $-6k\delta(|r|)$ which, by the delta function, is confined to the brane. For observers confined to the brane the effective 4D energy-momentum tensor will appear as an effective 4D cosmological constant term (i.e. $T^{4D}_{\mu\nu}=\Lambda_{4D}\eta_{\mu\nu}$ with $\Lambda_{4D}=3k^2-6k$). The sign of this effective 4D cosmological constant, Λ_{4D} can be negative (for $0<k<2$) or positive (for $k<0$ or $k>2$). For $k=2$ effective 4D cosmological constant vanishes, but one still has a constant bulk term. The difference in the energy-momentum tensors for the three metrics again indicates that the y and z metrics represent the same space-time, while the r metric is a related, but different space-time.

The energy-momentum tensor in (13a) (13b) can be split into a constant term on the 4D brane plus a constant part in the 5D bulk as follows

$$\begin{aligned} T_{AB} &= T^{brane}_{AB}+T^{bulk}_{AB} \\ &= \frac{-6k\delta(r)}{\kappa^2}diag[1,-1,-1,-1,0]+\frac{3k^2}{\kappa^2}diag[1,-1,-1,-1,-2]\ , \end{aligned} \tag{14}$$

where $diag$ indicates a 5×5 diagonal matrix. The first term, proportional to $\delta(r)$, represents a constant 4D energy-momentum tensor which is confined to the brane. The second term, proportional to $\frac{3k^2}{\kappa^2}$, represents a constant term in the 5D bulk, but with the complication that the "pressure" term in the rr direction is twice that of the other three spatial directions. This difference between the pressures in the three spatial coordinates of the brane and the bulk spatial dimension could have been anticipated since there should be some difference between the nature of energy densities and pressures of the matter sources on the brane and in the bulk. If the matter sources had the same energy densities and pressures on the brane as in the bulk there would be no difference between the brane and bulk, and one would not have a warped brane geometry.

We now briefly discuss what kind of field sources could give rise to the split, constant energy-momentum tensor as in (14). The most obvious choice is that the T_{AB} could be generated by the condensate of some field(s), which for simplicity we will take to be scalar fields. In regard to the first term in (14), $T_{AB}^{brane} = \frac{-6k\delta(r)}{\kappa^2} diag[1, -1, -1, -1, 0]$, one can obtain such an energy momentum tensor from a scalar field condensate like the Standard Model Higgs, which is confined to the brane. In a realistic model this scalar field and its condensate would be confined to some finite thickness region near $r = 0$ rather than an infinitesimal thin region implied by $\delta(r)$. An energy-momentum tensor having the general form $T_{AB}^{brane} = F(x)\delta(r) diag[1, -1, -1, -1, 0]$ (where $F(x)$ is some function of the 4D coordinates of the brane) can be obtained via any field and field condensate which is localized to the brane [29]. In our case above $F(x) = const$. Obtaining a field theory source which gives T_{AB}^{bulk} from (14) requires a bit more thought due to the fact that the rr component is twice that of the other spatial components. An energy-momentum tensor of the form T_{AB}^{bulk} can be obtained from a ghost, scalar field, $\phi(x, r)$, having some self-interaction $V(\phi)$. The Lagrangian is of the form

$$\mathcal{L}_{5D} = -\frac{1}{2}(\partial_A \phi)(\partial^A \phi) - V(\phi) \ . \tag{15}$$

The unusual negative sign in front of the kinetic term indicates that this is a ghost field. While such fields are problematic, it has been argued [30–32] that this can be handled as long as the ghost fields are bulk fields (as is the case here). Also, effective ghost fields can arise in a natural way in the context of Weyl gravity [33]. Here our aim is simply to find some field theory source which can give the energy-momentum tensor associated with the brane metric of (4). The energy-momentum tensor associated with (15) is

$$T^{\phi}_{AB} = \frac{\partial \mathcal{L}_{5D}}{\partial \phi^{,A}} \phi_{,B} - g_{AB}\mathcal{L}_{5D} \ , \tag{16}$$

where as usual $\phi_{,A} = \partial_A \phi$. Using $\mathcal{L}_{5D}$ from (15) as well as the 5D metric (4) in (16) gives

$$T_{\mu\nu} \ = \ \eta_{\mu\nu}\left(-\frac{1}{2}(\partial_r\phi)^2 + e^{-2k|r|}V(\phi)\right) \tag{17}$$

$$T_{rr} \ = \ \left(-\frac{1}{2}(\partial_r\phi)^2 - e^{-2k|r|}V(\phi)\right) \ . \tag{18}$$

The above forms for $T_{\mu\nu}$ and T_{rr} take into account the fact that equations (11c) (12c) and (16) allow at most an r-dependent field. In order to obtain some energy-momentum tensor of the form T^{bulk}_{AB} from (14) we require that $(\partial_r\phi)^2 = \frac{3k^2}{\kappa^2}$ and $e^{-2k|r|}V(\phi) = \frac{9k^2}{2\kappa^2}$ as one moves into the bulk i.e. away from $r = 0$. The condition $(\partial_r\phi)^2 = \frac{3k^2}{\kappa^2}$ can be met by $\phi(r) = \frac{\sqrt{3}k}{\kappa}r$. Note that this form of the ghost field ϕ implies that it vanishes on the brane $r = 0$. The condition $e^{-2k|r|}V(\phi) = \frac{9k^2}{2\kappa^2}$ can be met by taking the potential to be of the appropriate form. Taking into account the behavior of $\phi(r)$ off the brane, i.e. $\phi(r) = \frac{\sqrt{3}k}{\kappa}r$, the potential has the form $V(\phi) = \frac{9k^2}{2\kappa^2}\exp[2\kappa\phi/\sqrt{3}]$. Such exponential potentials are called Liouville potentials and arise in string theory as well as in some quintessence models [34] [35] [36]. It is also possible to show that the above scalar field solution and potential solve the field equation for the scalar field namely

$$-\frac{1}{\sqrt{g}}\partial_A\left(\sqrt{g}g^{AB}\partial_B\phi\right) = -\frac{\partial V}{\partial \phi} \ ,$$

where $g = e^{-10kr}$ is the determinant of the metric. The linear character of the scalar field solution, i.e. $\phi \propto r$, is reminiscent of the linear potential which is postulated to lead to the confinement of quarks. In a similar way one might think to use this scalar field solution to localize other fields to the brane by coupling them to ϕ. However coupling ordinary matter/fields to a ghost field can lead to problems. Thus we avoid coupling this ghost field directly to any ordinary fields.

Thus it is possible to construct a field theory source (albeit with a ghost scalar field) that gives the energy-momentum tensor (14) associated with the brane metric (4). Before leaving this topic of possible field theory sources that might give an energy-momentum tensor of the form in (14) we recall that in reference [37] a brane model was given with the "less" warped metric of

$$ds^2 = e^{2f(r)}dt^2 - dx_i dx^i - dr^2 \ . \tag{19}$$

"Less" warped means that the warp factor $e^{2f(r)}$ sits only in front of the time component of the metric; in the Randall-Sundrum metric (2) the warp factor sits in front of all the coordinates except the extra spatial dimension; in the alternative brane metric (4) the warp factor sits in front of all the coordinates including the extra spatial dimension. For the metric in (19) it is possible to give an exact field theory sources in terms of a 5D $U(1)$ gauge boson with a vector potential of the form $A^B(x) = (a(r), 0, 0, 0, 0)$. This form of the 5D vector potential in combination with the metric (19) was shown to solve the 5D "Maxwell" equations and yield an energy-momentum tensor of the form $T_{AB} = K[1, -1, -1, -1, 1]$ where K is some constant. While not precisely of the form required to support the alternative 5D metric (4), this energy-momentum tensor does have the feature in common with the energy-momentum tensor of (14) that the pressure in the extra spatial direction, r, is different from that in the three spatial dimensions on the brane. Thus in addition to the example of the ghost scalar field theory source in (15) that would lead to the energy-momentum tensor in (14) it might also be possible to expand on the simple $U(1)$ Abelian gauge source of [37] to find some non-ghost source which would yield (14). For example one might try some combination of 5D $U(1)$ gauge field coupled to regular scalar field, or one could try a 5D non-Abelian gauge field or a 5D Born-Infeld $U(1)$ field.

Before moving on to the localization of fields onto the brane, located at $r = 0$ for the r-metric, we give a short explanation of the physical reason for the difference between the space-time represented by the y, z-metrics and the space-time given by the r-metric. Although all three metrics appear to have infinitely large extra dimensions since the extra spatial dimensions run from $y, z, r = -\infty$ to $y, z, r = +\infty$ only the y and z metric have infinite *proper* distance into the bulk. For example, consider a path in the y metric (2) which goes from $y = 0$ to $y = +\infty$ perpendicular to the 4D brane at $y = 0$. The proper length of this path, using (2), is $s = \int ds = \int_0^\infty dy = \infty$. In a similar manner for the z metric one finds that the proper distance into the bulk for the path $z = 0$ to $z = +\infty$ going perpendicular from the brane is infinite – $s = \int ds = \int_0^\infty \frac{dz}{k|z|+1} = \infty$. However, for the r metric from (4) one finds that the infinitesimal proper distance for a path going from $r = 0$ to $r = \infty$ perpendicular to the brane at $r = 0$ is $ds = e^{-k|r|} dr$. Integrating this gives

$$s = \int ds = \int_0^\infty e^{-kr} dr = \frac{1}{k} \ , \tag{20}$$

where in (20) we have dropped the absolute value sign since we are integrating over positive r.

Note also that the proper distance s in (20) would be infinite if one considered an increasing warp factor rather than a decreasing warp factor i.e if one let $k \to -k$. For the y, z metrics the proper distance into the bulk is infinite regardless of the choice of the sign of k. Thus the space-time represented by (4) is a single brane at $r = 0$ with either a finite (for $k > 0$) or infinite (for $k < 0$) proper distance into the bulk dimension.

In some sense the r coordinate metric of (4) for increasing warp factor is "more" of an infinite extra dimension metric that the Randall-Sundrum space-time given by (2) or (3). Although the Randall-Sundrum metrics have an infinite proper distance into the bulk it takes only finite proper time for a massive particle to "fall" from the brane at $y, z = 0$ to infinity $y, z = \infty$. In [15] [38] it was shown that the proper time for a massive particle to fall from the brane to infinity was $\tau = \pi/2k$. Following [39] one can straightforwardly calculate the proper time for a massive particle to "fall" over the r-metric brane from $r = 0$ to $r = \infty$. For an increasing warp factor ($k < 0$) this yields $\tau = \infty$ while for a decreasing warp factor ($k > 0$) this yields $\tau = 1/2k$, a finite proper time. Thus both in terms of proper distance from $r = 0$ to $r = \infty$ and in terms of proper time to fall from $r = 0$ to $r = \infty$ the r metric of (2), for $k < 0$, yields infinite results. One the other hand for a decreasing warp factor, $k > 0$, both the proper distance and proper time are finite.

IV. LOCALIZATION OF FIELDS OF VARIOUS SPINS

We now discuss the localization of fields of different spins (spin 0, spin 1/2 and spin 1) for the space-time represented by the metric in (4) i.e. the r coordinate system. As mentioned in the Introduction, it is important that most matter fields and fundamental interaction fields, with the possible exception of gravity, should be well confined to the 3 + 1-dimensional brane. The localization results for spin 0 [6–8] spin 1/2 [11] and spin 1 gauge bosons [12] are well known for the 5D space-time given by the metrics (2) and (3). The localization of fields of various spins is an unresolved issue for the original Randall-Sundrum model. Summarizing briefly the previous results for localization in the 5D Randall-Sundrum model one finds: (i) spin-0 fields are localized if $k > 0$ (i.e. a decreasing warp factor for our conventions here) but not localized if $k < 0$ (i.e. an increasing warp factor for our conventions here) [11]; (ii) spin-1 fields are not localized for either $k > 0$ or $k < 0$ [12]; (iii) spin-1/2 fields behave exactly opposite to spin-0 fields – they are not localized if $k > 0$ (i.e. a decreasing warp factor) but localized if $k < 0$ (i.e. an increasing warp factor) [11]. One attempt to address this issue was to consider 6D [40] [41] [42] [43] [44] and higher dimensional brane models [45] [46] [47]. These higher dimensional models did exhibit better localization behavior as compared to the 5D model.

The localization condition for fields onto the brane is that the action integral over the extra dimension should be finite [42, 47]. This association between a finite action integral and field localization can be considered in terms of the Wick rotated propagator [48] $\int Dxe^{-S(x,it)}$ or time independent $\int Dxe^{-S(x)}$. The propagator and the associated field will vanish unless the action integral is finite. The space-time background represented by the r coordinates in (4) has qualitatively different behavior in regard to the localization of fields to the brane as compared to the space-time background represented by the y/z metrics [11, 22].

A. Scalar field

We first consider a complex scalar field in r-metric space-time (4). The action for a complex scalar field can be written as [11, 20, 46],

$$S_0 = \int d^5x\sqrt{g}g^{MN}\partial_M\Phi^*\,\partial_N\Phi, \tag{21}$$

where the subscript 0 indicates the spin of the field. Notice that we are assuming that the 5D scalar field Φ is massless in that there is no term like $M^2\Phi^*\Phi$. Applying the principle of least action leads to the equation of motion for the scalar field,

$$\partial_M\left(\sqrt{g}\;g^{MN}\partial_N\Phi\right) = 0. \tag{22}$$

In terms of the general form of the brane metric given in (5) the scalar field equation of motion (22) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{1}{a^2b}\partial_5\left(\frac{a^4}{b}\partial_5\Phi\right) = 0, \tag{23}$$

where $\sqrt{g} = a^4b$ and $g^{55} = -\frac{1}{b^2}$. Decomposing the field, $\Phi\left(x^\mu, x^5\right) = \varphi\left(x^\mu\right)\chi_0\left(x^5\right)$, the equations for the field on the brane can be written in terms of the separation constant m,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi = -m^2\varphi. \tag{24}$$

The equations of motion for the field in the bulk are as follows,

$$\partial_5^2\chi_0 + 4\frac{a'}{a}\partial_5\chi_0 - \frac{b'}{b}\partial_5\chi_0 = -\frac{b^2}{a^2}m^2\chi_0. \tag{25}$$

Recalling the ansatz functions for the r-coordinate metric from (4)

$$a(r) = b(r) = e^{-k|r|}, \tag{26}$$

we find the following form of the scalar field equation in the bulk (25)

$$\left(\partial_r^2 - 3k\partial_r + m^2\right)\chi_0 = 0. \tag{27}$$

1. Scalar massless modes: $m = 0$

When $m = 0$ the solution to (27) is,

$$\chi_0(r) = c_0 + b_0\, e^{3\,kr}, \tag{28}$$

Previous efforts in studying scalar field zero modes have focused on the constant field solution which is obtained by setting $b_0 = 0$. Here we will keep the non-constant zero mode solution of (28) ($b_0\, e^{3\,kr}$) since it might have potentially useful localization properties under the change $k \to -k$ i.e. going from a decreasing warp factor to an increasing warp factor.

The action for the scalar field (21) in the r metric can be expanded as

$$S_0 = \int\limits_0^\infty dr\sqrt{g}\chi_0^*\chi_0 g^{\mu\nu}\int d^4x\partial_\mu\varphi^*\partial_\nu\varphi + \int\limits_0^\infty dr\sqrt{g}g^{rr}\partial_r\chi_0^*\partial_r\chi_0\int d^4x\,\varphi^2\ . \tag{29}$$

For the scalar field to be localizable means that the two integrals over the bulk dimension r must be finite i.e. the two integrals

$$N_0 = \int\limits_0^\infty dr\ a^2(r)b(r)\chi_0^*(r)\chi_0(r) \quad , \quad M_0^2 = \int\limits_0^\infty dr\ \frac{a^2(r)}{b(r)}\left(\partial_r\chi_0^*\partial_r\chi_0\right), \tag{30}$$

must be finite. Here N_0 is the 4-dimensional normalization and M_0 is the 4-dimensional mass of the brane wave function φ for the spin-0 field. Looking at (29) and taking into account that we want $\varphi(x^\mu)$ to behave like a 4D scalar field we should require that the finite values of N_0 and M_0 be

$$N_0 = 1 \quad \text{and} \quad M_0 = m. \tag{31}$$

For the background of the r system of coordinates we have $a(r) = b(r) = e^{-kr}$ so the integrals (30) be written out further as

$$N_0 = \int\limits_0^\infty dr\ e^{-3kr}\chi_0^*(r)\chi_0(r) \quad , \quad M_0^2 = \int\limits_0^\infty dr\ e^{-kr}\left(\partial_r\chi_0^*(r)\partial_r\chi_0(r)\right). \tag{32}$$

In the case of the scalar zero-modes $m = 0$ and $k > 0$ (i.e. a decreasing warp factor according to (4)) it is clear that in order for N_0, M_0 to be finite we need to take the constant mode in (28) by setting $b_0 = 0$ so that $\chi(r) = c_0$. With this choice we find that the integrals in (32) become $N_0 = \frac{(c_0)^2}{3k}$ and $M_0 = 0$. The result $M_0 = 0$ is as expected since we are dealing with the case when the 4D mass is zero, $m = 0$. Also from the first normalization condition in (31) we find that we need $c_0 = \sqrt{3k}$.

In addition to the localization of the constant scalar field zero modes for a decreasing warp factor the r-metric also appears to allow one to localize the non-constant zero modes (i.e. those modes obtained by choosing $c_0 = 0$ in (28)) for an *increasing* warp factor which is obtained by letting $k \to -k$ in (4), (28), and (32). Making the change $k \to -k$ and taking into account that now $\chi_0(r) = b_0 e^{-3kr}$ the integrals in (32) become $N_0 = \frac{b_0^2}{3k}$ and $M_0^2 = 9kb_0^2/5$. However having $M_0 \neq 0$ conflicts with $m = 0$ unless we take $b_0 = 0$ i.e. we get rid of the non-constant zero mode. Although in the end the result for the scalar field zero modes is the same as for the Randall-Sundrum metric - only the constant zero mode is localized and only for decreasing warp factor - the fact that the integrals in (32) are finite for an increasing warp factor with the non-constant zero modes already indicates potentially different and interesting localization behavior for the r-coordinate metric versus the usual Randall-Sundrum metric of (2) or (3). This is exactly what we find for the non-zero scalar modes which we study next.

2. *Scalar massive modes:* $m \neq 0$

We now turn to the case when $m \neq 0$. It is easy to show that (27) has the $m \neq 0$ solution

$$\chi_0 = c_0\, e^{\frac{3}{2}\, kr} e^{-\frac{1}{2}\, r\sqrt{9\, k^2 - 4\, m^2}} + b_0 e^{\frac{3}{2}\, kr} e^{\frac{1}{2}\, r\, \sqrt{9\, k^2 - 4\, m^2}}. \tag{33}$$

First it is clear that in order for the fields to be localized one must have $\frac{3}{2}|k| > m$ since otherwise the $\exp[\pm r\sqrt{9k^2 - 4m^2}/2]$ term in χ_0 from (33) will go from exponentially increasing/decreasing to oscillating in the r direction. As a result it is easy to see that N_0 and/or M_0 in (32) will diverge. Thus we have the interesting result that scalar particles are only confined to the brane of the r-coordinate metric up to some mass, m, which is set by, k, the degree of warping of the extra dimension. We have taken the absolute value of k since while k can be positive or negative, depending if one has a decreasing or increasing warp factor, the mass m should be positive definite.

Next for N_0 the e^{-3kr} term in the integral coming from the metric will always cancel the e^{3kr} factor coming from $\chi_0^*(r)\chi_0(r)$. Thus in order to have an overall decreasing exponential in the integrand of N_0 (and therefore a finite integral) we must select the first solution in (33) by setting $b_0 = 0$ i.e. only the solution

$$\chi_0(r) = c_0\, e^{\frac{3}{2}kr} e^{-\frac{1}{2}r\sqrt{9k^2-4m^2}}$$

is localizable. Under these conditions the behavior of N_0 is

$$N_0 = c_0^2 \int_0^\infty dr\ e^{-r\sqrt{9k^2-4m^2}} = \frac{c_0^2}{\sqrt{9k^2-4m^2}} \tag{34}$$

which again shows the requirement that $\frac{3}{2}|k| > m$ in order to localize these massive scalar modes. Also from the first equation in (31) we require $N_0 = 1$ or $c_0^2 = \sqrt{9k^2-4m^2}$. Note that since only k^2 appears in the integral (34) that N_0 will be finite even under the change $k \to -k$ i.e. from a decreasing warp factor to increasing warp factor. Next if we look at M_0 in (32) we see that it has the behavior

$$\begin{aligned} M_0^2 &= \frac{c_0^2\left(3k - \sqrt{9k^2-4m^2}\right)^2}{4} \int_0^\infty dr\ e^{2kr} e^{-\sqrt{9k^2-4m^2}r} \\ &= \frac{\sqrt{9k^2-4m^2}\left(3k - \sqrt{9k^2-4m^2}\right)^2}{4(\sqrt{9k^2-4m^2}-2k)}\ . \end{aligned} \tag{35}$$

When $k > 0$ (i.e. a decreasing warp factor) one needs the condition $2k < \sqrt{9k^2-4m^2}$ or $\frac{\sqrt{5}}{2}k > m$ in order for the scalar fields to be localized. This condition coming from the finiteness of M_0 is similar to the condition coming from the finiteness of N_0 which requires $\frac{3}{2}|k| > m$. The condition $\frac{\sqrt{5}}{2}k > m$ slightly lowers the mass scale at which particles are no longer localized to the brane relative to the condition $\frac{3}{2}|k| > m$.

We did not write the absolute value in the condition $\frac{\sqrt{5}}{2}k > m$ above since changing from decreasing to increasing warp factor (i.e. letting $k \to -k$) gives a different localization condition as we now show. Under the change $k \to -k$ (35) becomes

$$\begin{aligned} M_0^2 &= \frac{c_0^2\left(3k + \sqrt{9k^2-4m^2}\right)^2}{4} \int_0^\infty dr\ e^{-2kr} e^{-\sqrt{9k^2-4m^2}r} \\ &= \frac{\sqrt{9k^2-4m^2}\left(3k + \sqrt{9k^2-4m^2}\right)^2}{4(\sqrt{9k^2-4m^2}+2k)}\ . \end{aligned} \tag{36}$$

The integrals N_0 and M_0 are finite for both decreasing warp factor and increasing warp factor. Thus we found an interesting distinction between the 5D Randall-Sundrum metric and the r-coordinate metric (4) – massive scalar modes can be localized to the brane for both decreasing and increasing warp factors for the r-coordinate metric.

Another interesting feature is that one can fix the mass m *self-consistently* in a manner similar to the mass generation mechanism of Nambu and Jona-Lasinio [13]. In order for (24) (29) and (30) to be consistent with one another one needs $m^2 = M_0^2(m, k)$ where M_0 is a function of m and k. Applying $m^2 = M_0^2(m, k)$ to (35) and (36) and solving for m^2 yields

$$m^2 = \frac{k^2(11 \pm \sqrt{13})}{8} \ , \tag{37}$$

where the quadratic equations from (35) and (36) yield the same two, positive mass solutions.

B. Vector fields

Next we turn to a spin-1 vector gauge boson field in the r-metric space-time (4). The action for the vector field, $A^M(x^N)$, can be written as [11, 12, 46],

$$S_1 = -\frac{1}{4}\int d^5x\sqrt{g}g^{MN}g^{RS}F_{MR}F_{NS}, \tag{38}$$

where the subscript 1 again indicates the spin of the field and the 5D Faraday tensor is defined in the usual way as $F_{MN} = \partial_M A_N - \partial_N A_M$. Applying the principle of least action leads to the equation of motion for the vector field,

$$\frac{1}{\sqrt{g}}\partial_M\left(\sqrt{g}\ g^{MN}g^{RS}F_{NS}\right) = 0. \tag{39}$$

For the 5D vector gauge boson field $A_N(x^M)$ we take the following ansatz: $A_{x_5} = A_r = const.$ and $A_\mu(x^M) = a_\mu(x^\mu)c(x_5) = a_\mu(x^\mu)c(r)$. One can see that (39) has the constant solution

$$c(r) = c_1 \ \ \text{and} \ \ \partial^\mu f_{\mu\nu} = 0 \tag{40}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the 4D Faraday tensor, and the last equation in (40) are the 4D vacuum Maxwell equation. Unlike the 5D Randall-Sundrum metric which only has a constant solution [11] [12] the r space-time metric has the following non-constant solution

$$c(r) = \frac{c_1}{a(r)} = c_1 e^{k|r|} \ \ \text{and} \ \ \partial^\mu f_{\mu\nu} = 0 \ , \tag{41}$$

as can be verified by direct substitution into (39) (for this solution one also needs to use the 4D gauge freedom and impose the 4D Lorentz gauge $\partial_\mu a^\mu(x^\nu) = 0$). The existence of the non-constant solution (41) has interesting consequences for the localization of the gauge boson to the brane.

In regard to localization we first look at the constant solution from (40). Using the fact that $\sqrt{g} g^{MN} g^{RS} = a(r)\eta^{MN}\eta^{RS}$ the action in (38) reduces to

$$S_1 = -\frac{c_1^2}{4}\int_0^\infty dr\ e^{-kr}\int d^4x\ \eta^{\mu\nu}\eta^{\rho\sigma} f_{\mu\nu} f_{\rho\sigma}\ . \tag{42}$$

In order for the gauge boson to be localized to the brane at $r = 0$ one needs the integral over the fifth coordinate r to be finite and it is easy to see that for the decreasing warp factor this is in fact the case $\int_0^\infty dre^{-kr} = 1/k$. The reason for this improved localization of gauge bosons relative to the Randall-Sundrum metric (2) or (3), is that the r-metric (4) has an "extra" non-trivial metric component $\sqrt{g_{55}} = e^{-k|r|}$ which shows up in the integrand. This extra metric component makes the integral finite for the decreasing warp factor $a(r) = e^{-k|r|}$. However, for the increasing warp factor $a(r) = e^{k|r|}$ the integral over dr in (42) is infinite and the spin-1 gauge boson is not localized. This better localization behavior of spin-1 gauge boson for the constant solution (40) and decreasing warp factor is not such a big surprise since as pointed out in (20) the proper distance into the bulk is finite for decreasing warp factor. However the localization of spin 1 fields for decreasing warp factor is not trivial. In the next section we find that spinor fields are not localizable for the case of a decreasing warp factor i.e. when the proper distance in the r direction is finite. Thus the spinor case below will show that even if the warp factor is decreasing and the proper distance into the bulk is finite it is possible that the field modes can still be non-localizable. In light of this result for the spinor field the fact that the spin 1 field is localizable for a decreasing warp factor is a non-trivial result.

Next we look at the non-constant solution of (41). As before $\sqrt{g} g^{MN} g^{RS} = a(r)\eta^{MN}\eta^{RS}$. Now however,

$$\eta^{MN}\eta^{RS} F_{MR} F_{NS} = c^2(r)\eta^{\mu\nu}\eta^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} + \eta^{55}\eta^{\rho\sigma} F_{5\rho} F_{5\sigma}. \tag{43}$$

This looks promising since the first term contains the terms $c^2(r) = e^{2k|r|}$ which, if one lets $k \to -k$ (i.e. go from decreasing warp factor to increasing), can more than compensate for geometric $a(r)$ term. Specifically under the change $k \to -k$ the first term on the right hand side of (43) will contribute to the action, S_1, from (38) as

$$\int_0^\infty dr \quad a(r)c^2(r) \int d^4x \ \eta^{\mu\nu}\eta^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} =$$
$$\int_0^\infty dr \quad e^{-kr} \int d^4x \ \eta^{\mu\nu}\eta^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma} = \frac{1}{k} \int d^4x \ \eta^{\mu\nu}\eta^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma}.$$

Thus the integral over dr is finite and equal to $1/k$ so this first part of the action reduces to effective 4D electromagnetism. However as in the scalar case there is an inconsistency due to the second term in (43). In all of the above we have assumed that 4D Maxwell equations are valid i.e. $\partial_\mu f^{\mu\nu} = 0$. This equation also implies that the spin-1 gauge boson is massless. The second term in (43) generates a mass term unless one takes $c_1 = 0$ in (41). First we note that $F_{5\rho} = \partial_5 A_\rho = a_\mu(x^\nu)\partial_r(c(r))$ (recall that we have taken $A_5 = A_r = 0$). Thus the second term on the right hand side of (43) will give a term equal to

$$-\int_0^\infty dr \quad a(r)(\partial_r c(r))^2 \int d^4x a^\mu(x^\nu) a_\mu(x^\nu) = -\int_0^\infty dr \ k^2 e^{-kr} \int d^4x a^\mu(x^\nu) a_\mu(x^\nu)$$
$$= - \ k \int d^4x a^\mu(x^\nu) a_\mu(x^\nu) \tag{44}$$

which looks like an imaginary mass term $m = i\sqrt{2k}$ for the spin-1 gauge boson $a^\mu(x^\nu)$. Having a mass term (imaginary or real) is inconsistent with our choice $\partial_\mu f^{\mu\nu} = 0$. Thus the choice $\partial_\mu f^{\mu\nu} = 0$ forces us to only consider the solution which is constant in the bulk dimension (i.e. the solution in (40)) and discard the solution which is non-constant in the bulk dimension (i.e. the solution in (41)). One could try two things in order to avoid this inconsistency: (i) One could replace the massless Maxwell equations with the massive Proca equations $\partial_\mu f^{\mu\nu} = -m^2 a_\mu a^\mu$. This might provide a self-consistent Nambu Jona-Lasinio-like mechanism alternative to the Higgs mechanism for generating mass for gauge bosons. (ii) One could start with an initial massive 5D gauge boson instead of a massless 5D gauge boson by adding a term $\frac{1}{2}M^2 A_M A^M$ to (38). We leave these considerations for future investigations.

C. Spinor Fields

Finally we consider a 5D spin-1/2 spinor field, $\Psi(x^\mu, r)$, in the r-metric space-time (4). The action for can be written as [11, 46],

$$S_{1/2} = \int d^5x \sqrt{g} \bar{\Psi} i \Gamma^M D_M \Psi \ . \tag{45}$$

This action leads to the following equation of motion

$$i\Gamma^M D_M \Psi = (i\Gamma^\mu D_\mu + i\Gamma^r D_r)\Psi = 0 \ , \tag{46}$$

where Γ^M are the curved space-time Dirac gamma matrices which are related to the Minkowski space-time Dirac matrices, $\gamma^{\bar{M}}$, via $\Gamma^M = e^M_{\bar{M}}\gamma^{\bar{M}}$, with $e^{\bar{M}}_M$ and $e^M_{\bar{M}}$ being the funfbein and its inverse respectively. The funfbein connects the curved and flat space-times according to $g_{MN} = e^{\bar{M}}_M e^{\bar{N}}_N \eta_{\bar{M}\bar{N}}$ - here barred (unbarred) represented flat (curved) space-time indices. The covariant derivative is given by $D_M = \partial_M + \frac{1}{4}\omega^{\bar{M}\bar{N}}_M \sigma_{\bar{M}\bar{N}}$ where the Dirac tensor is defined via the commutator of Dirac gamma matrices as $\sigma_{\bar{M}\bar{N}} = \frac{1}{2}[\gamma_{\bar{M}}, \gamma_{\bar{N}}]$ and the spin connection $\omega^{\bar{M}\bar{N}}_M$ is given in terms of the funfbein and its derivatives by the following expression

$$\begin{aligned} \omega^{\bar{M}\bar{N}}_M &= \frac{1}{2}e^{N\bar{M}}(\partial_M e^{\bar{N}}_N - \partial_N e^{\bar{N}}_M) - \frac{1}{2}e^{N\bar{N}}(\partial_M e^{\bar{M}}_N - \partial_N e^{\bar{M}}_M) \\ &- \frac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e^{\bar{R}}_M \ . \end{aligned} \tag{47}$$

Calculating the spin connections for the general metric (5) obtains the non-zero elements as

$$\omega^{\bar{r}\bar{\nu}}_\mu = \delta^{\bar{\nu}}_\mu \frac{a'}{b} \tag{48}$$

where the primes denote differentiation with respect to r. Specializing to the metric (4) with $a(r) = b(r) = e^{-k|r|}$ one can obtain the covariant derivatives which take the form

$$D_\mu \Psi = \left(\partial_\mu + \frac{1}{2}\frac{a'}{b}\gamma_{\bar{r}}\gamma_{\bar{\mu}}\right) = \left(\partial_\mu + \frac{ik}{2}\gamma_{\bar{\mu}}\gamma_{\bar{5}}\right) \quad ; \quad D_r = \partial_r \ . \tag{49}$$

We have replaced $\gamma_{\bar{r}}$ with the standard 4D $\gamma_{\bar{5}}$ taking into account the relationship between the two (i.e. $\gamma_{\bar{r}} = i\gamma_{\bar{5}}$ [49]) and we have used $\gamma_{\bar{r}}\gamma_{\bar{\mu}} = -\gamma_{\bar{\mu}}\gamma_{\bar{r}}$. We now separate the 5D spinor as $\Psi(x^M) = \psi(x^\mu)p(r)$ and we also separate $\psi(x^\mu)$ into left handed and right handed spinors as

$$\Psi(x^\mu, r) = \begin{pmatrix} \psi_R(x_\mu)p_R(r) \\ \psi_L(x_\mu)p_L(r) \end{pmatrix} \ , \tag{50}$$

where ψ_L and ψ_R are the usual two-component left-handed and right-handed Weyl spinors which satisfy the chiral conditions $\gamma_{\bar{5}}\psi_L = -\psi_L$ and $\gamma_{\bar{5}}\psi_R = \psi_R$. Finally we take $\psi_{R,L}$ to satisfy the following massive Dirac equation $i\gamma^\mu\partial_\mu\psi_L = m\psi_R$ and $i\gamma^\mu\partial_\mu\psi_R = m\psi_L$. Substituting all of this along with the covariant derivatives of (49) into the 5D Dirac equation (46) gives the following equations for $p_R(r)$ and $p_L(r)$

$$\partial_x p_R(r) - 2kp_R(r) = -mp_R(r) \quad ; \quad \partial_x p_L(r) - 2kp_L(r) = +mp_L(r) \ . \tag{51}$$

These equations have the following solution

$$p_R(r) = c_{1/2} e^{2kr - mr} \quad ; \quad p_L(r) = d_{1/2} e^{2kr + mr} \ , \tag{52}$$

where $c_{1/2}$ and $d_{1/2}$ are integration constants. Inserting p_R, p_L from (52) in the full 5D spinor from (50) and in turn inserting this into the action (45) we find that the first term (i.e. $\bar{\Psi} i \gamma^{\bar{\mu}} \partial_\mu \Psi$ the 4D kinetic energy term) is

$$c_{1/2}^2 \int_0^\infty e^{-2mr} dr \int d^4x \bar{\psi}_R i \gamma^\mu \partial_\mu \psi_R + d_{1/2}^2 \int_0^\infty e^{2mr} dr \int d^4x \bar{\psi}_L i \gamma^\mu \partial_\mu \psi_L \ . \tag{53}$$

In (53) the geometric factor e^{-4kr} is exactly canceled by the e^{4kr} coming from $p_{R,L}^2$. It is easy to see that while the first integral over r is convergent and so it would appear that one could localize the right handed spinors, ψ_R, the second integral is divergent and thus at least the left handed spinor, ψ_L are not localizable. Next turning the remaining two terms from the spinor action (45) which contain $\gamma_{\bar{r}} = i\gamma_{\bar{5}}$ (i.e. $\bar{\Psi}\gamma_{\bar{5}}(-2k + \partial_r)\Psi$) we find

$$\int d^5x \bar{\Psi} \gamma_{\bar{5}} (-2k + \partial_x) \Psi = c_{1/2} d_{1/2} \int_0^\infty dr \int d^4x (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \ , \tag{54}$$

which diverges for both left and right handed fields since $\int_0^\infty dr \to \infty$. Thus the integral over the extra dimension r for the 5D spinor action (45) diverges for both right handed and left handed fields. Note that in (54) one connects or mixes the right and left handed spinors which results in the e^{-mr} factor of p_R canceling the e^{mr} factor of p_L. The reason for this mixing is that in the chiral representation we use while γ_5 is diagonal, γ_0, is "anti"-diagonal so that $\Psi^\dagger \gamma_0$ moves the left handed components, $\psi_L p_L$, to be the first two-components of Ψ and moves the right handed components, $\psi_R p_R$, to be the first two-components of Ψ. The above results agree with the calculations in [46] in the massless limit, but appear to disagree in the massive limit. In the work [46] it is found that if $m \neq 0$ then the spinor fields are localized. The apparent difference in results comes first because here we are adding a 4D mass term while in [46] a 5D "mass" term is added. Further in [46] the 5D mass term added is not of the canonical form $-m\bar{\Psi}\Psi$ but rather $im\bar{\Psi}\Psi$. It is because the mass term added is imaginary that results in the localization of the spinor fields. Thus for our r-metric (4) spinor fields are not localized for either increasing or decreasing warp factors. One could localize the spinor fields by introducing a Yukawa coupling to the scalar field of the form

$$g\bar{\Psi}\Psi\Phi \ . \tag{55}$$

This type of localization method was first suggested in [49] and has been extensively used to localize otherwise non-localized fermions to the brane.

Although introducing a non-gravitational, Yukawa interaction between the 5D spinors and the 5D scalar in order to localize the spinors fields to the brane at $r = 0$, spoils the simplicity of the present brane model one might think to turn this apparent disadvantage into a positive – one could address the fermion family puzzle (i.e. why there appear to be three copies or three families of fundamental fermions) using the Yukawa coupling of (55). There have been various attempts to address the fermion family puzzle using brane worlds [50–53]. In particular attempts have been made to obtain a realistic mass spectrum and CKM matrix elements which mix fermions of different families [54–57].

V. DISCUSSION AND CONCLUSIONS

In this paper we examined in detail the structure of three different 5D brane metrics given by (2), (3), and (4). By investigating these three metrics in terms of their associated energy momentum tensors and their invariants like the Ricci Scalars given in (7a), (7b), and (7c) we found that the y and z coordinate metrics represented the same space-time – as was already known – but that the r coordinate metric (4), although having a form which appeared to be some combination of the y and z metrics, in fact represented a different space-time. As far as we found this difference between the space-time represented by the y and z metrics of (2) and (3) and the space-time represented by the r metric of (4) has not been discussed before in the literature. Unlike the space-time associated with the y and z metrics, the r metric of (4) could be either an infinite extra dimensions (if the warp factor was increasing i.e. $e^{2k|r|}$ with $k > 0$) or a finite extra dimension (if the warp factor was decreasing i.e. $e^{-2k|r|}$ with $k > 0$). For the decreasing warp factor, although the coordinate distance r into the bulk extra dimension was infinite, the proper distance was finite as discussed around equation (20). On the other hand for an increasing warp factor the proper distance into the bulk was infinite as in the case of the y and z metrics. Moreover the r - metric with an increasing warp factor could be considered as a "more" infinite extra dimensional metric as compared to the y, z metrics. For the y, z - metrics, even though the proper distance into the bulk was infinite, it took only a finite proper time $\tau = \pi/2k$ for a massive particle to fall from the brane $y, z = 0$ to $y, z = \infty$. However, as shown at the end of section III for the r-metric with *increasing* warp factor, the proper distance into the bulk was infinite and the proper time for a massive particle to fall from the brane at $r = 0$ to $r = \infty$ was infinite.

Next we analyzed the localization properties of fields with spin 0, 1/2, and 1 to the brane at $r = 0$ for the r-metric. For scalar and spinor fields we studied the localization of both massless and massive modes. The spin 0, scalar field case had massless modes localized to the brane for the decreasing warp factor but not for increasing warp factor. This was the same as the localization of massless modes in the y, z metrics. The massive modes on the other hand could be localized to the r-metric brane for either increasing or decreasing warp factor. This was an improvement over the localization of massive spin 0 modes for the y, z metric. In addition the massive mode scalar field had the interesting requirement that for either increasing or decreasing warp factor there was some maximum mass beyond which the massive modes would not be localized. This condition comes from the extra dimensional part of the scalar wave function (i.e. $e^{-\frac{1}{2}r\sqrt{9k^2-4m^2}}$ from (33)) and implies that one needs the scalar field mass to satisfy $3|k|/2 > m$ or $\sqrt{5}k/2 > m$. Another interesting feature of the massive modes for the scalar fields was that the mass was fixed via a self-consistent condition reminiscent of the self-consistent mass generation mechanism of [13]. Essentially the calculated mass, M_0, from (36) depended on both m and k and this mass was required to be equal to m yielding the self-consistency condition $m^2 = M_0^2$ giving rise to two masses, m as given in (37). Both masses in (37) do satisfy the condition that $3|k|/2 > m$. Thus for the increasing warp factor there are two localized modes which are both massive with the masses given by (37). For the decreasing warp factor one has the condition $\sqrt{5}k/2 > m$ and thus there are two localized modes: one massless mode and one massive mode given by the lower mass in (37).

For the spin 1 gauge bosons the r-metric localized the massless modes to the brane for decreasing warp factor but not for increasing warp factor. This was an improvement over the usual y, z metrics where spin 1 gauge bosons were not localized either for increasing or decreasing warp factors. As in the scalar case it might be that considering massive modes would lead to localization for both types of warp factors. We leave this investigation of the massive spin 1 gauge bosons for future work. The fact that spin 1 gauge bosons are localized for a decreasing warp factor for the r-metric is in some sense an expected result since for decreasing warp factor the bulk dimension is not an infinite dimension – equation (20) shows that the proper distance into the bulk for decreasing warp factor is finite. One other point about the spin 1 gauge boson case for the r-metric in comparison to the y, z metrics – the r metric allowed for a non-constant extra dimensional part for the spin 1 gauge field as given in (41); for the y, z metrics the only solution for the extra dimensional part of the field was the constant solution given in (40) [11]. For the vector field we did not try to introduce a 4D mass term via $\partial_\mu f^{\mu\nu} = -m^2 a_\mu a^\mu$ nor did we try to add a 5D mass term via $\frac{1}{2}M^2 A_M A^M$ to (38). We leave this possibility for future work , but we note that these options may allow one to introduce an alternative to the Higgs mechanism for giving mass to spin-1 vector gauge bosons.

Finally, in the case of the spinor fields we found that they could not be localized for either decreasing warp factor or increasing warp factor. Thus to localize the spinor fields one would need to use some non-gravitational interaction as a means of localization such as the Yukawa coupling between the spinor and scalar fields, given in (55). The fact that spinors are not localized for either increasing or decreasing warp factors does show that the localization of spin 1 fields to the brane for decreasing warp is not a trivial result.

Acknowledgments PJ was supported in part by Capital One Bank through an endowed professorship. The work of DS was supported through a 2012-2013 Fulbright Scholars Grant.

---

[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **429**, 263 (1998).

[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B **436**, 257 (1998).

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).

[4] M. Gogberashvili, Int. J. Mod. Phys. D **11**, 1635 (2002).

[5] M. Gogberashvili, Int. J. Mod. Phys. D **11**, 1639 (2002).

[6] M. Gogberashvili, Mod. Phys. Lett. A **14**, 2025 (1999).

[7] M. Gogberashvili, Europhys. Lett. **49**, 396 (2000).

[8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[9] S. Giddings and S. Thomas, Phys. Rev. D **65**, 056010 (2002).

[10] E.G. Adelberger, B. R. Heckel, S. Hoedl, C.D. Hoyle, D. J. Kapner, and A. Upadhye, Phys. Rev. Lett. **98**, 131104, (2007).

[11] B. Bajc and G. Gabadadze, Phys. Lett. B **474**, 282 (2000).

[12] A. Pomarol, Phys. Lett. B **486**, 153 (2000).

[13] Y. Nambu and G. Jona-Lasinio Phys. Rev. **122**, 345 (1961).

[14] V. A. Rubakov, Phys. Usp. **44**, 871 (2001); Usp. Fiz. Nauk **171**, 913 (2001).

[15] W. Mück, K. S. Viswanathan, and I. V. Volovich, Phys. Rev. D **62**, 105019 (2000).

[16] Abdel Pérez-Lorenzana, “An Introduction to the brane world”, arXiv:hep-ph/0406279v2.

[17] Kazuya Koyama, JCAP **0409**, 010 (2004).

[18] Nima Arkani-Hamed, Savas Dimopoulos, Gia Dvali, Nemanja Kaloper, Phys. Rev. Lett. **84** 586-589 (2000).

[19] Andreas Karch and Lisa Randall, JHEP **0105**, 008 (2001).

[20] Rhys Davies and Damien P. George, Phys. Rev. D **76**, 104010 (2007).

[21] L. B. Castro, Phys. Rev. D **83**, 045002 (2011).

[22] Abdel Pérez-Lorenzana, J. Phys. Conf. Ser. **18**, 224 (2005).

[23] Csaba Csákia, Michael L. Graesserb, and Graham D. Kribsc, Phys. Rev. D **63**, 065002 (2001).

[24] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).

[25] K. A. Bronnikov and B. E. Meierovich, Grav. Cosmol. **9**, 313 (2003).

[26] K.A. Bronnikov and S.G. Rubin, Grav. Cosmol. **13**, 191 (2007).

[27] V. Dzhunushaliev, V. Folomeev, D. Singleton, and S. Aguilar-Rudametkin Phys. Rev. D **77**, 044006 (2008).

[28] V. Dzhunushaliev, V. Folomeev, and M. Minamitsuji, Rept. Prog. Phys. **73**, 066901 (2010).

[29] G. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).

[30] R. Koley and S. Kar, Mod. Phys. Lett. A **20**, 363 (2005).

[31] R. Koley and S. Kar, Phys. Lett. B **623**, 244 (2005); Erratum-ibid. B **631**, 199 (2005)

[32] R. Koley and S. Kar, Class. Quant. Grav. **24**, 79 (2007).

[33] M. Gogberashvili and D. Singleton, Mod. Phys. Lett. A **25**, 2131 (2010).

[34] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988).

[35] C. Wetterich, Nuc. Phys. B **302**, 668 (1988).

[36] T. Barreiro, E.J. Copeland, and N.J Nunes, Phys. Rev. D **61**, 127301 (2000).

[37] M. Visser, Phys. Lett. B **159**, 22 (1985)

[38] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, Class. Quant. Grav. **17**, 4437 (2000).

[39] P. Jones, G. Muñoz, M. Ragsdale, and D. Singleton, Am. J. Phys. **76**, 73 (2008).

[40] M. Gogberashvili and D. Singleton, Phys. Rev. D **69**, 026004 (2004).

[41] M. Gogberashvili and D. Singleton, Phys. Lett. B **582**, 95(2004).

[42] P. Midodashvili, Europhys. Lett. **69**, 478 (2004).

[43] P. Midodashvili and L. Midodashvili, Europhys. Lett. **66**, 640 (2004).

[44] M. Gogberashvili and P. Midodashvili, Phys. Lett. B **515**, 447 (2001).

[45] I. Oda, Phys. Rev. D **62**, 126009 (2000).

[46] Ichiro Oda, Phys. Lett. B **508**, 96 (2001).

[47] D. Singleton, Phys. Rev. D **70**, 065013 (2004).

[48] A. Zee, *Quantum field theory in a nutshell*, Princeton University Press, 12 (2003).

[49] V.A Rubakov and M.E. Shaposhnikov, Phys. Lett. B **125**, 136 (1983).

[50] Y. Liu, L. Zhao, X. Zhang, and Y. Duan, Nucl. Phys. B **785**, 234 (2007).

[51] Y. Liu, L. Zhao, and Y. Duan, JHEP **0704**, 097 (2007).

[52] Y. Liu, et al., Phys. Rev. D **80**, 065019 (2009).

[53] Y. Kodama, K. Kokubu, and N. Sawado, Phys. Rev. D **79**, 065024 (2009).

[54] M. Gogberashvili, P. Midodashvili, and D. Singleton, JHEP **0708**, 033 (2007).

[55] S. Aguilar and D. Singleton, Phys.Rev. D **73**, 085007 (2006).

[56] D. B. Kaplan and S. Sun, Phys. Rev. Lett. **108**, 181807 (2012).

[57] Z. Guo and B. Ma, JHEP **0909**, 091 (2009).