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Traveling Wave Solutions to Kawahara and Related Equations

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Traveling wave solutions to Kawahara equation (KE), transmission line (TL), and Korteweg-de Vries (KdV) equation are found by using an elliptic function method which is more general than the tanh-method. The method works by assuming that a polynomial ansatz satisfies a Weierstrass equation, and has two advantages: first, it reduces the number of terms in the ansatz by an order of two, and second, it uses Weierstrass functions which satisfy an elliptic equation for the dependent variable instead of the hyperbolic tangent functions which only satisfy the Riccati equation with constant coefficients.

When the polynomial ansatz in the traveling wave variable is of first order, the equation reduces to the KdV equation with only a cubic dispersion term, while for the KE which includes a fifth order dispersion term the polynomial ansatz must necessary be of quadratic type.

By solving the elliptic equation with coefficients that depend on the boundary conditions, velocity of the traveling waves, nonlinear strength, and dispersion coefficients, in the case of KdV equation we find the well-known solitary waves (solitons) for zero boundary conditions, as well as wave-trains of cnoidal waves for nonzero boundary conditions. Both solutions are either compressive (bright) or rarefactive (dark), and either propagate to the left or right with arbitrary velocity.

In the case of KE with nonzero boundary conditions and zero cubic dispersion, we obtain cnoidal wave-trains which represent solutions to the TL equation. For KE with zero boundary conditions and all the dispersion terms present, we obtain again solitary waves, while for KE with all coefficients present and nonzero boundary condition, the solutions are written in terms of Weierstrass elliptic functions. For all cases of the KE we only find bright waves that are propagating to the right with velocity that is a function of both dispersion coefficients.

Keywords: Kawahara equation, KdV equation, transmission line equation, Jacobi and Weierstrass elliptic functions, elliptic function method.

I. INTRODUCTION

In recent years many methods have been used to find analytic solutions to nonlinear partial differential equations (PDEs). Among the multitude of papers, we shall only refer to two sets of studies which will pertain to this work. The first class of papers are: truncation procedure in the Painlevé analysis [54] in which authors define the Painlevé property that determines the Lax pairs of the Burgers, KdV, and the modified KdV equations; Hirota bilinear method [15] where multiple collisions of N solitons with varying amplitudes have been obtained for the KdV equation; the Prelle-Singer method [50] where a system of differential equations has been shown to have an elementary integral expressible in terms of exponentials, logarithms and algebraic functions; the factorization method [6] where traveling wave solutions of the standard and compound KdV-Burgers equations are found using factorizations; or the homogeneous balance method [52] where solitary wave solutions of two types of variant Boussinesq equations are obtained. Then, we can also enumerate the trial function method [25] where transformations of solutions obtained by the Weiss-Tabor-Carnevale method are used for investigation of Kuramoto-Sivashinsky equation; the nonlinear transformation method [49] where the authors constructed traveling wave solutions for nonlinear diffusion equations with polynomial nonlinearities; the well-known inverse scattering transform [1, 38] and Bäcklund transformation [35]; the first integral method [40] where Nizovtseva uses a first integral method which gives singular and kink profiles for the Allen-Cahn hyperbolic equation.

The second class of papers mentioned here are: the simplest equation method [24] where Kudrsyashov uses the general solutions of simplest nonlinear differential equations and takes into consideration all pos-

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sible singularities of Kuramoto- Sivashinsky equation, as well as the equation for description of nonlinear waves in a convective fluid; the G'/G expansion method [26] where it is shown to be equivalent to the tanh-method first developed by Malfliet and Hereman [29–31]; the automatic method of Parkes [41]; the method of Q functions [20, 27] used on Fisher equation and on a seventh order ODE; the generalized Riccati equation method [57] where a new generalized transformation is applied to Whitham-Broer-Kaup (WBK) equation; the sinh-cosh-method [53] where the author finds solitons, kinks, and periodic solutions of Benjamin-Bona-Mahony (BBM) equation; the modified tanh-method [9, 10] where the author uses a modified by a parameter Riccati equation; the algebraic method where algorithms using sophisticated Mathematica programs are used to find closed-form solutions in terms of Jacobi elliptic functions [5, 14]; the sech-method [32] to find solitons to a seventh order KdV equation. Then we also include the Jacobi elliptic function method [11] used on a double sine-Gordon, Hirota equation, and the coupled Schrödinger-KdV system; the work of Fu and Liu [12, 28] on Jacobi elliptic function expansion method; and Porubov [42–44] on traveling periodic solutions of a pair of coupled nonlinear Schrödinger equations obtained in terms of Weierstrass elliptic $\wp$ functions.

More importantly, if for the former class of papers the methods yield restrictive solutions involving elementary functions which generate solitary waves, singular solutions as rational functions, periodic trigonometric solutions, kinks and fronts, the latter studies involve finding analytical solutions of evolution equations in terms of Jacobi, Weierstrass or elliptic theta functions.

Motivated by the work of the authors of the second class of papers, an elliptic function method, which is easier to implement and more general than the hyperbolic tangent method is applied to a nonlinear dispersive PDE known as Kawahara equation (KE) to find periodic solutions in terms of Weierstrass $\wp$ elliptic functions, Jacobi elliptic or hyperbolic functions. This equation takes the form

$$u_t + \kappa u u_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0, \tag{1}$$

and was investigated numerically in a study of magneto-acoustic waves in a cold collision-free plasma [18]. The coefficients of this equations are: α, β - third and fifth order dispersive terms, κ - the strength of the nonlinearity (wave steepening) and are real constants. We may assume that $\beta > 0$, because by the transformations $u \to -u$, $x \to -x$, and $t \to -t$, we obtain the same equation as (1) with the dispersive terms reversed in sign [18]. Under certain circumstances, the third order dispersion coefficient α becomes very small, or even zero, so one should include the higher order dispersion β which will balance the nonlinear effect κ [13, 17].

We can write Eq. (1) in Hamiltonian form [7]

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta \mathcal{H}}{\delta u}\right), \tag{2}$$

where the Hamiltonian is

$$\mathcal{H} = -\frac{1}{6}\kappa u^3 + \frac{1}{2}\alpha {u_x}^2 + \frac{1}{2}\beta {u_{xx}}^2. \tag{3}$$

Using this Hamiltonian, KE (1) has the conserved energy density

$$\mathcal{I} = \int_{-\infty}^{\infty} \mathcal{H} dx. \tag{4}$$

By using the Fréchet derivative which corresponds to the Euler-Lagrange operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial}{\partial u_{xx}}, \tag{5}$$

then Eq. (2) becomes KE (1).

By applying a traveling wave ansatz $u(\xi) = u(x - ct)$, with c being the velocity of the unidirectional traveling wave in the x direction at time t, yields a fifth order ordinary differential equation (ODE) in the traveling wave variable ξ

$$-cu_\xi + \kappa u u_\xi + \alpha u_{\xi\xi\xi} - \beta u_{\xi\xi\xi\xi\xi} = 0. \tag{6}$$

By one integration this reduces to the fourth order ODE

$$-cu+\frac{\kappa}{2}u^2+\alpha u_{\xi\xi}-\beta u_{\xi\xi\xi\xi}=\mathcal{A} \tag{7}$$

with $\mathcal{A}$ an arbitrary integration constant which can be zero or not depending on the types of boundary conditions chosen. By multiplying by u_ξ and integrating once we obtain a conserved quantity for Eq. (1) in the traveling wave variable ξ

$$\mathcal{C}=-2\mathcal{A}u-cu^2+\frac{\kappa}{3}u^3+\alpha {u_\xi}^2-\beta\left[2u_\xi u_{\xi\xi\xi}-(u_{\xi\xi})^2\right]\equiv const. \tag{8}$$

In his comment to Assas' paper [4], Kudryashov developed the solutions of KE using the tanh-method [21, 22]. This method was originally used by Malfliet and Hereman [29–31] and has the advantage of reducing nonlinear ODEs into systems of algebraic equations, that might be easier to solve. Kudryashov explained that Eq. (7) does not pass the Painlevé test, but nevertheless one can find solitary waves of higher order by writing the Laurent series expansion for a function $Y(\xi)$ which must include a pole of order four [20, 23], where the function in the expansion solves the Riccati equation with constant coefficients

$$Y_\xi=\eta(1-Y^2) \tag{9}$$

with solution $Y(\xi)=\tanh(\eta\xi)$. Therefore, if we assume solutions of the form

$$u(\xi)=\sum_{i=0}^{n}C_iY^i, \tag{10}$$

once the numbers of terms n is determined using the balancing principle [39], we can write the solutions of Eq. (7) in terms of the solutions of the Riccati equation (9). Since the hyperbolic tangent solution is a particular solution of the Riccati equation, and any other solution can be found using the transformation $Y=\tanh(\eta\xi)+\frac{1}{W}$, where W satisfies a first order linear equation, then all the other solutions which are meromorphic to u can be written using a new expansion in W with the same number of terms. More than that, all forms of the general solution of the Riccati equation have the same Laurent series and they differ only by arbitrary constants [8]. Therefore, using different ODEs as generators of particular solutions, one can find a rich class of meromorphic solutions to evolution equations, which unite many approaches involving elementary functions. These methods are not restrictive to only parabolic equations ($d/dt\to -c\ d/d\xi$), as they were also successfully implemented to find solutions to hyperbolic PDEs ($d^2/dt^2\to c^2\ d^2/d\xi^2$) such as Boussinesq [52] and improved Boussinesq equation [1, 21], Klein-Gordon [59], and Allen-Cahn equation via the first integral method [40]. Note that the elliptic function method will not work if the ODE contains both even and odd derivative terms, see Lemma V.1. in the Appendix, for the explanation.

In order to determine the number of terms in the expansion of the ansatz, we compute the second order derivative $d^2/d\xi^2$ for which the leading term is $\eta^2(1-Y^2)^2\ d^2/dY^2$, while for the fourth order derivative $d^4/d\xi^4$ the leading term is $\eta^4(1-Y^2)^4\ d^4/dY^4$. Thus, when we balance the nonlinear term with the higher order derivatives, we must distinguish between two different cases. When $\beta=0$ (KdV equation), we only need to balance $u_{\xi\xi}$ with u^2 which leads to $4+(n-2)=2n\Rightarrow n=2$. For the second case (KE), we balance $u_{\xi\xi\xi\xi}$ with u^2 which leads to $8+(n-4)=2n\Rightarrow n=4$. Therefore, the solutions for the KdV or KE must take the form

$$\begin{aligned}&u(\xi)=A_0+A_2\ \tanh^2(\eta\xi), \qquad \beta=0\\ &u(\xi)=B_0+B_2\ \tanh^2(\eta\xi)+B_4\ \tanh^4(\eta\xi), \qquad \beta\neq 0\end{aligned} \tag{11}$$

with the first and third order coefficients identically zero [21].

Since KdV equation also possesses non elementary solutions in terms of Jacobi elliptic functions [19] which are not solutions of the Riccati equation (9), we extend the ansatz of the function Y and we replace the Riccati equation by an elliptic equation, i.e., the function Y in the *new* ansatz given by (10) satisfies

$${Y_\xi}^2=a_0+a_1Y+a_2Y^2+a_3Y^3, \qquad a_3\neq 0. \tag{12}$$

This new ansatz has the advantage of extending the classes of solutions to include elliptic functions [11, 24, 28], and as a special case when two of the roots of the cubic polynomial in Y collide, the solitary waves can be recovered as a limit case of cnoidal waves [33, 34]. The constants a_i which depend on the system parameters α, β, κ, the speed c, and boundary conditions $\mathcal{A}$ respectively, can be found algebraically after the ansatz passes the balancing principle [39], which will determine the number of terms n in the expansion given by (10).

Using the new ansatz, and by balancing, we obtain $n+1 = 2n \Rightarrow n = 1$ when $\beta = 0$ and $n+2 = 2n \Rightarrow n = 2$ when $\beta \neq 0$. Therefore, by replacing Riccati equation (9) with the elliptic equation (12), we reduce the numbers of terms in half in the expansion of Eq. (10) which is now only linear for KdV or quadratic for KE

$$\begin{aligned} u(\xi) &= A_0 + A_1 Y, \qquad \beta = 0 \\ u(\xi) &= B_0 + B_1 Y + B_2 Y^2, \qquad \beta \neq 0. \end{aligned} \tag{13}$$

II. REDUCTION TO THE KDV EQUATION ($\beta = 0$)

When $\beta = 0$, Eq. (1) reduces to the KdV equation which describes the motion of small amplitude and large wavelength shallow waves in dispersive systems [19]

$$u_t + \kappa u u_x + \alpha u_{xxx} = 0. \tag{14}$$

By using $\beta = 0$ in Eq. (7), we obtain the second order ODE in u

$$-cu + \frac{\kappa}{2}u^2 + \alpha u_{\xi\xi} = \mathcal{A}. \tag{15}$$

Without loss of generality we may assume that $A_0 = 0$, $A_1 = 1$ so that $u = Y$. To find the solutions of Eq. (15) using our procedure, we differentiate Eq. (12) to obtain higher derivatives of Y

$$\begin{aligned} Y_{\xi\xi} &= \frac{1}{2}a_1 + a_2 Y + \frac{3}{2}a_3 Y^2 \\ Y_{\xi\xi\xi} &= (a_2 + 3a_3 Y)Y_\xi \\ Y_{\xi\xi\xi\xi} &= 3a_0 a_3 + \frac{1}{2}a_1 a_2 + \left(\frac{9}{2}a_1 a_3 + {a_2}^2\right) Y + \frac{15}{2}a_2 a_3 Y^2 + \frac{15}{2}{a_3}^2 Y^3, \end{aligned} \tag{16}$$

and by substituting them into Eq. (1) we obtain a cubic polynomial in Y

$$\sum_{i=0}^{3} s_i Y^i \equiv 0, \tag{17}$$

with coefficients s_i given by the expressions

$$\begin{aligned} s_3 &= -\tfrac{15}{2}{a_3}^2\beta \\ s_2 &= \tfrac{1}{2}\left[\kappa + 3a_3(\alpha - 5a_2\beta)\right] \\ s_1 &= -c + a_2\alpha - {a_2}^2\beta - \tfrac{9}{2}a_1 a_3\beta \\ s_0 &= -\mathcal{A} - 3a_0 a_3\beta + \tfrac{a_1}{2}(\alpha - a_2\beta). \end{aligned} \tag{18}$$

Because all these coefficients must be zero, and since $a_3 \neq 0$, from the first equation of the system (18), we immediately conclude that $\beta = 0$, so the reduced constants become

$$\begin{aligned} s_2 &= \tfrac{1}{2}(\kappa + 3a_3\alpha) \\ s_1 &= -c + a_2\alpha \\ s_0 &= -\mathcal{A} + \tfrac{a_1}{2}\alpha. \end{aligned} \tag{19}$$

By solving simultaneously $s_i = 0$ for the coefficients a_i, Eq. (12) becomes

$${Y_\xi}^2 = a_0 + \frac{2\mathcal{A}}{\alpha}Y + \frac{c}{\alpha}Y^2 - \frac{\kappa}{3\alpha}Y^3 \equiv q_3(Y), \tag{20}$$

where a_0 is an arbitrary constant. According to Eq. (8) the conserved quantity in ξ for the KdV equation is

$$\mathcal{C} = -2\mathcal{A}Y - cY^2 + \frac{\kappa}{3}Y^3 + \alpha {Y_\xi}^2, \tag{21}$$

and by comparing Eqs. (20) and (21) we identify $a_0 = \frac{\mathcal{C}}{\alpha}$, so the arbitrary constant of the elliptic equation is proportional to the conserved quantity, and inverse proportional to the cubic dispersion. The elliptic solutions of Eq. (20) depend on the type of roots of the cubic polynomial $q_3(Y)$, which automatically leads to the following sub-cases:

i) One zero root of multiplicity two, and one simple root $Y_0 = \frac{3c}{\kappa}$. This is achieved when we choose zero boundary conditions $\mathcal{A} = 0$ together with $a_0 = 0$, i.e., the fluid is undisturbed at infinity ($Y, Y_\xi, Y_{\xi\xi} \to 0$ as $|\xi| \to \infty$). By letting $\frac{\kappa}{3\alpha} = \frac{1}{\sigma}$ we factor Eq. (20) as

$${Y_\xi}^2 = \frac{1}{\sigma}Y^2(Y_0 - Y), \tag{22}$$

with solution the solitary wave [19, 45]

$$Y(\xi) = Y_0 \mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{Y_0}{\sigma}}(\xi - \xi_0)\right] \tag{23}$$

which propagates with velocity proportional to amplitude Y_0, and width inverse proportional to the square root of Y_0.

Now we must discuss the sign of σ. First possibility is that $\sigma > 0$ so κ and α are of the same sign, and because $Y_\xi \in \mathbb{R}$ we must have $Y_0 > 0$ which means that c is the same sign as α and κ. These waves travel to the right ($c > 0$) in a positive strength ($\kappa > 0$) and positive dispersive medium ($\alpha > 0$), and to the left ($c < 0$) in a negative strength ($\kappa > 0$) negative dispersive medium ($\alpha < 0$). In both cases their amplitude is always positive and they represent the *positive (compressive/ bright)* solitary waves. On the other hand, if $\sigma < 0$ so that κ and α are of opposite signs, then we must have $Y_0 < 0$. Therefore, these are waves that travel to the right ($c > 0$) in a negative strength ($\kappa < 0$) and positive dispersive medium ($\alpha > 0$) medium, and to the left ($c < 0$) in a positive strength ($\kappa > 0$) and negative dispersive medium ($\alpha < 0$). In both cases their amplitude is always negative and they represent the *negative (rarefactive/ dark)* solitary waves [19, 45]. For all other remaining combination of coefficients, the solitons become unbounded and solutions are unphysical, when the hyperbolic secant becomes periodic with poles aligned on the real ξ-axis. For the solitonic case, and regardless of the signs of the coefficients, for $\xi_0 = 0$ and $u = Y$ the solution is

$$u(x,t) = \frac{3c}{\kappa}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{c}{\alpha}}(x - ct)\right] \tag{24}$$

If the velocity is fixed, the amplitude and width can be manipulated using the strength and dispersion coefficients. When κ is big the amplitude is small, and when α is small the solitons are thin, while increasing the dispersion it increases their widths when solitons spread, see Fig. 1 for the values of $\alpha = 1$, $\kappa = 1$ and $\alpha = -5$, $\kappa = -5$ for bright solitons (top panel), and $\alpha = 1$, $\kappa = -5$ and $\alpha = -5$, $\kappa = 1$ for dark solitons (bottom panel).

ii) Now we drop the assumption that fluid must be undisturbed at infinity, so then we must have $Y_\xi = 0$ for $Y = 0$ which implies that only one root is zero. This corresponds to setting $a_0 = 0$, while the other two roots are real and distinct. Under these assumptions Eq.(20) can be factored as

$${Y_\xi}^2 = \frac{1}{\sigma}\,Y(Y_2 - Y)(Y_3 + Y), \tag{25}$$

where $Y_{2,3}$ satisfy

$$\begin{aligned} Y_2 &= \tfrac{Y_0 \pm \sqrt{\Omega}}{2} \\ Y_3 &= \tfrac{-Y_0 \pm \sqrt{\Omega}}{2}. \end{aligned} \tag{26}$$

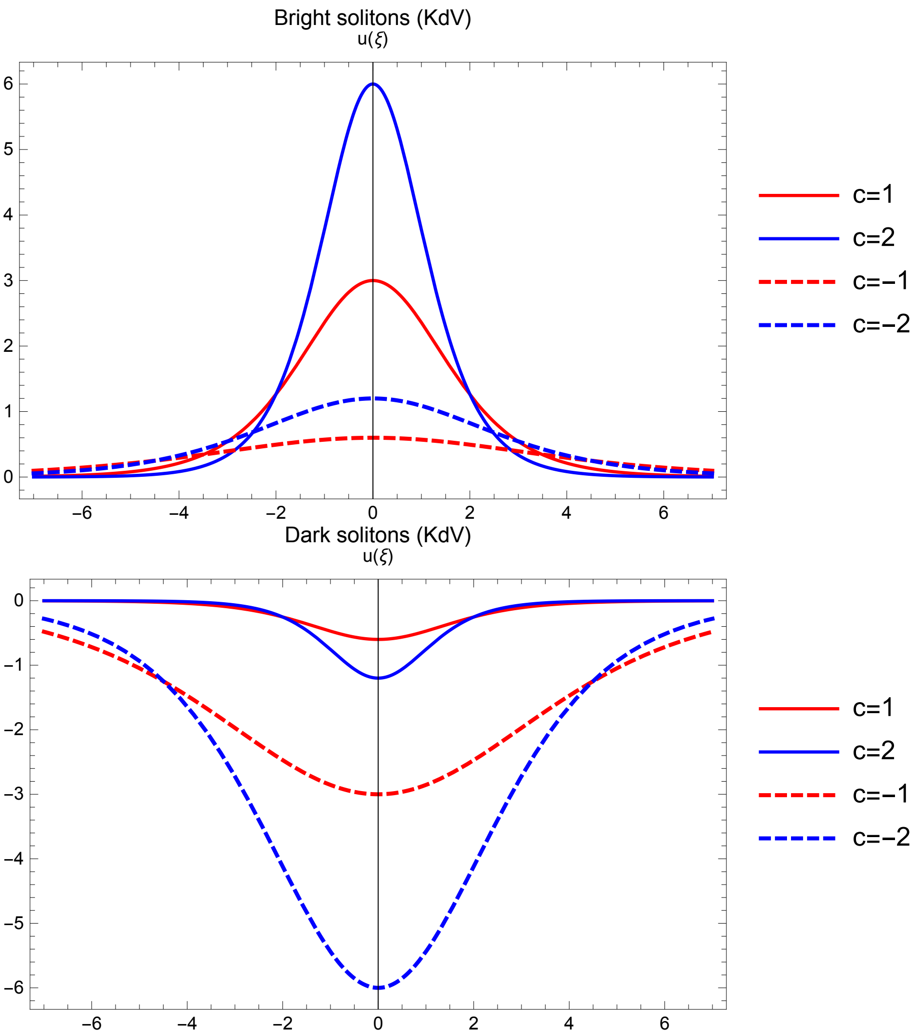

FIG. 1: Bright solitons for the KdV Eq. (14) with zero boundary conditions $\mathcal{A} = 0$ and $\alpha = 1,\ \kappa = 1$ (continuous curves) when solitons propagate to the right, and $\alpha = -5,\ \kappa = -5$ (dashed curves) when solitons propagate to the left (top panel). Dark solitons for the KdV Eq. (14) with zero boundary conditions $\mathcal{A} = 0$ and $\alpha = 1,\ \kappa = -5$ (continuous curves) when solitons propagate to the right, and $\alpha = -5,\ \kappa = 1$ (dashed curves) when solitons propagate to the left (bottom panel).

For real roots we require the discriminant $\Omega = \frac{9c^2+24\mathcal{A}\kappa}{\kappa^2} > 0$, which restricts the values for $\mathcal{A}$ and κ such that $\mathcal{A}\kappa > -\frac{3c^2}{8}$. The solution of Eq. (25) is

$$u(\xi) = Y_2 \ \mathrm{cn}^2 \left[\frac{1}{2}\sqrt{\frac{1}{\sigma}(Y_2 + Y_3)}(\xi - \xi_0); m \right], \tag{27}$$

which simplifies to

$$u(\xi) = \frac{Y_0 \pm \sqrt{\Omega}}{2} \mathrm{cn}^2 \left[\frac{1}{2}\sqrt{\pm\frac{\sqrt{\Omega}}{\sigma}}(\xi - \xi_0); \sqrt{\frac{1}{2} \pm \frac{Y_0}{2\sqrt{\Omega}}} \right], \tag{28}$$

where $\mathrm{cn}(\theta; m)$ is the Jacobian elliptic function with modulus $m = \sqrt{\frac{Y_2}{Y_2+Y_3}}$. Using the values of the roots from system (26), and $\xi_0 = 0$ the solutions of the KdV equation (14) with nonzero boundary conditions are

$$u(x,t) = \frac{3c \pm \sqrt{9c^2+24\mathcal{A}\kappa}}{2\kappa} \mathrm{cn}^2 \left[\frac{1}{2} \frac{\sqrt[4]{9c^2+24\mathcal{A}\kappa}}{\sqrt{\pm 3\alpha}}(x-ct); \sqrt{\frac{1}{2} \pm \frac{3c}{2\sqrt{9c^2+24\mathcal{A}\kappa}}} \right]. \tag{29}$$

If α, κ have same sign then $\sigma > 0$, so $Y_2 + Y_3 = \sqrt{\Omega} > 0$, and we obtain bright cnoidal waves which propagate to the right ($c > 0$) or to the left ($c < 0$), see Fig. 2 (top panel) for right waves with $\alpha = 1$, $\kappa = 2$, and left waves with $\alpha = -2$, $\kappa = -2$. Otherwise, when α, κ have opposite sign then $\sigma < 0$, so $Y_2 + Y_3 = -\sqrt{\Omega} < 0$, and we obtain dark cnoidal waves which also propagate to the right ($c > 0$) or to the left ($c < 0$), see Fig. 2 (bottom panel) for right waves with $\alpha = -1$, $\kappa = 2$, and left waves with $\alpha = 2$, $\kappa = -1$. These solutions represent trains of periodic cnoidal waves with shape and wavelength that depend on the amplitude of the waves. The wavelength is $\lambda = 4\sqrt{\frac{\pm\sigma}{\sqrt{\Omega}}}K(m)$, where $K(m)$ is the complete elliptic integral of first kind given by $K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}$. When $\mathcal{A} \to 0$, and depending

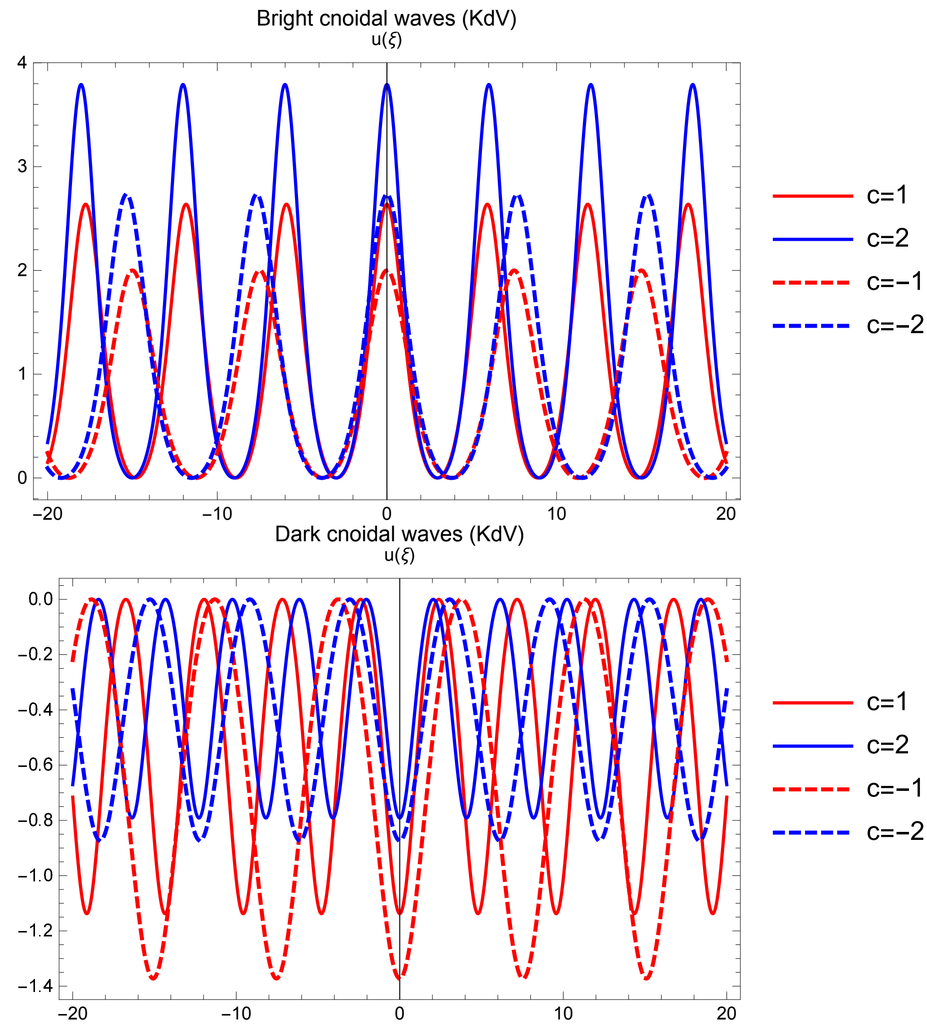

FIG. 2: Bright cnoidal waves for the KdV Eq. (14) with $\alpha = 1$, $\kappa = 2$, $\mathcal{A} = 1$ (continuous curves) when the waves propagate to the right, and $\alpha = -2$, $\kappa = -2$ $\mathcal{A} = -1$ (dashed curves) when the waves propagate to the left (top panel). Dark cnoidal waves for the KdV Eq. (14) with $\alpha = -1$, $\kappa = 2$, $\mathcal{A} = 1$ (continuous curves) when the waves propagate to the right, and $\alpha = 2$, $\kappa = -1$, $\mathcal{A} = -1$ (dashed curves) when the waves propagate to the left (bottom panel).

on the modulus m we have two extreme cases. First, by choosing the negative branch of the sqare root

$Y_2 \to 0 \Rightarrow m \to 0$, and cn $\theta \to \cos\theta$, thus cnoidal waves resemble more sinusoidal waves of unchanging shape discovered by Stokes, which in the theory of long waves constitutes a particular case of cnoidal form [19, 51]. Second, by choosing the positive branch of the square root $Y_3 \to 0 \Rightarrow m \to 1$, and cn $\theta \to$ sech θ, thus cnoidal waves loose their periodicity, and reduce to the solitary waves given by Eq. (24).

III. KAWAHARA EQUATION ($\beta \neq 0$)

By using the second equation of system (13) with $B_0 = 0$, $B_1 = 0$, $B_2 = 1$ and $u = Y^2$, the derivatives of u become

$$\begin{aligned} u_{\xi\xi} &= 2({Y_\xi}^2 + YY_{\xi\xi}) \\ u_{\xi\xi\xi} &= 2(3Y_\xi Y_{\xi\xi} + YY_{\xi\xi\xi}) \\ u_{\xi\xi\xi\xi} &= 2[3(Y_{\xi\xi})^2 + 4Y_\xi Y_{\xi\xi\xi} + YY_{\xi\xi\xi\xi}]. \end{aligned} \tag{30}$$

Using the derivatives of Y from system (16) in system (30), the second and fourth order derivatives of u as function of Y become

$$\begin{aligned} u_{\xi\xi} &= 2a_0 + 3a_1 Y + 4a_2 Y^2 + 5a_3 Y^3 \\ u_{\xi\xi\xi\xi} &= \frac{3}{2}{a_1}^2 + 8a_0 a_2 + 15(a_1 a_2 + 2a_0 a_3)Y + 2(8{a_2}^2 + 21 a_1 a_3)Y^2 \\ &\quad + 65 a_2 a_3 Y^3 + \frac{105}{2}{a_3}^2 Y^4. \end{aligned} \tag{31}$$

By substituting these derivatives in Eq. (7) we obtain the quartic polynomial in Y

$$\sum_{i=0}^{4} r_i Y^i \equiv 0, \tag{32}$$

with coefficients given by

$$\begin{aligned} r_4 &= \tfrac{1}{2}(\kappa - 105{a_3}^2\beta) \\ r_3 &= 5a_3(\alpha - 13a_2\beta) \\ r_2 &= -c + 4a_2\alpha - 16{a_2}^2\beta - 42a_1a_3\beta \\ r_1 &= -30a_0a_3\beta + 3a_1(\alpha - 5a_2\beta) \\ r_0 &= -\mathcal{A} - \tfrac{3}{2}{a_1}^2\beta + 2a_0(\alpha - 4a_2\beta). \end{aligned} \tag{33}$$

Since all coefficients $r_i = 0$, by simultaneously solving the first four equations of system (33) we find

$$\begin{aligned} a_3 &= \mp\sqrt{\tfrac{\kappa}{105\beta}} \\ a_2 &= \tfrac{\alpha}{13\beta} \\ a_1 &= \sqrt{\tfrac{5}{21\kappa\beta}}\,\tfrac{\mp 36\alpha^2 \pm 169c\beta}{2\cdot 169\beta} \\ a_0 &= \tfrac{2\alpha(36\alpha^2 - 169c\beta)}{13^3\kappa\beta^2}. \end{aligned} \tag{34}$$

For real coefficients, and since $\beta > 0$, we require that the wave steepening coefficient $\kappa > 0$. By using these coefficients together with the last equation of system (33), we find the integration constant to be

$$\mathcal{A} = \frac{(36\alpha^2 - 169\beta c)(2^2 \cdot 3^3 \cdot 17\ \alpha^2 + 5 \cdot 169\beta c)}{2^3 \cdot 7 \cdot 13^4 \kappa\beta^2}. \tag{35}$$

This shows that while for KdV equation the boundary $\mathcal{A}$ is *arbitrary*, for KE the boundary $\mathcal{A}$ is *fixed* by the system's parameters α, β, κ and the speed c. This means that traveling waves for KE can change shape depending if the speed c is chosen such a way that $\mathcal{A}$ is zero or not.

A. Transmission line equation

Now we analyze the special case of $\alpha = 0$ which from system (34) leads to $a_2 = a_0 = 0$ so that Eq. (1) includes a fifth order dispersion term only, and takes the form

$$u_t + \kappa u u_x - \beta u_{xxxxx} = 0. \tag{36}$$

This equation describes pulses over a transmission line containing a large number of LC circuits, and it occurs by making use of mutual inductance between neighboring inductors. It was first studied by Hasimoto [13] for shallow water waves near some critical value of surface tension, while Nagashima [36] performed experiments, and observed the solitary waves using an oscilloscope. Later, in a more general setting, it was also derived by Rosenau using a quasi-continuous formalism that included higher order discrete effects [46, 47].

For $\alpha = 0$ Eq. (7) becomes

$$-cu + \frac{\kappa}{2}u^2 - \beta u_{\xi\xi\xi\xi} = \mathcal{A}, \tag{37}$$

so Eq. (12) corresponds to

$${Y_\xi}^2 = Y(a_1 + a_3 Y^2), \tag{38}$$

which is in fact a special case of Eq. (25) for a real root that has multiplicity two, $Y_2 = Y_3$. The reduced coefficients obtained from system (34) are given by the expressions

$$\begin{aligned} a_3 &= \mp\sqrt{\tfrac{\kappa}{105\beta}} \\ a_1 &= \pm\tfrac{c}{2}\sqrt{\tfrac{5}{21\kappa\beta}}, \end{aligned} \tag{39}$$

and are used to factor Eq. (38) to obtain

$${Y_\xi}^2 = -a_3 Y({Y_2}^2 - Y^2), \tag{40}$$

with ${Y_2}^2 = -\frac{a_1}{a_3} = \frac{5c}{2\kappa}$. Since a_1, a_3 have opposite signs, then ${Y_2}^2 > 0$, and because $a_1, a_3 \in \mathbb{R} \Rightarrow \kappa > 0$, thus as before, the waves propagate only to the right ($c > 0$). By integrating Eq. (40), and using formula (27) we obtain the special cnoidal wave of constant modulus

$$Y(\xi) = Y_2 \operatorname{cn}^2\left[\frac{\sqrt{2}}{2}\sqrt{-a_3 Y_2}(\xi - \xi_0); \frac{\sqrt{2}}{2}\right]. \tag{41}$$

The wavelength of the waves is $\lambda = 2\sqrt{2}K\left(\frac{\sqrt{2}}{2}\right)\sqrt[4]{\frac{42\beta}{c}}$, where $K\left(\frac{\sqrt{2}}{2}\right) = 1.85407$, so the periodicity is only a function of the speed c and dispersion β. Using the values of Y_2, a_3, $u = Y^2$ and $\xi_0 = 0$ the solution to the transmission line equation (36) with nonzero boundary condition is

$$u(x,t) = \frac{5c}{2\kappa}\operatorname{cn}^4\left[\frac{\sqrt{2}}{2}\sqrt[4]{\frac{c}{42\beta}}\,(x - ct); \frac{\sqrt{2}}{2}\right]. \tag{42}$$

This solution was also obtained by Yamamoto [56], Kano [16] and Kudryashov [21], and represents a train of periodic waves with fixed modulus which tells that the shape is preserved as the pulse travels over the transmission line, see Fig. 3 for the values of $\beta = 1, \kappa = 2$.

B. KE with zero boundary conditions

Traveling waves with zero boundary condition $\mathcal{A} = 0$ yield waves which propagate with velocities $c = -\frac{2^2\cdot 3^3\cdot 17\alpha^2}{5\cdot 169\beta} < 0$ or $c = \frac{36\alpha^2}{169\beta} > 0$.

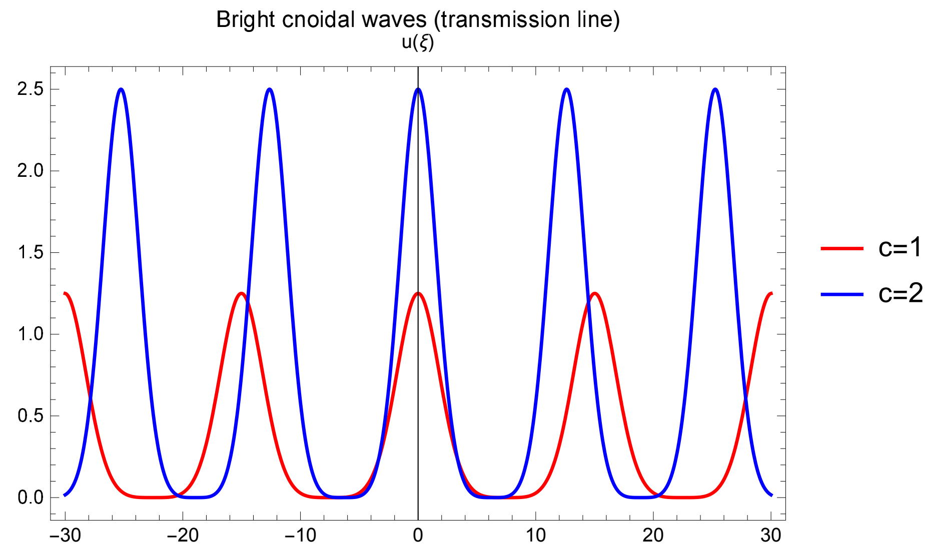

FIG. 3: Bright cnoidal waves for the transmission line equation (36) with nonzero boundary conditions for $\alpha = 0, \beta = 1, \kappa = 2$.

i) For left traveling waves, the modular discriminant obtained from the last equation of system (54) becomes $\Delta = -\frac{6\cdot 1523\alpha^6}{5^4\cdot 13^6\beta^6} < 0$, so the polynomial $s_3(t)$ given by Eq. (53) has non real roots, and because is an unphysical case it will be omitted.

ii) For right traveling waves, the system (34) reduces to

$$\begin{aligned} a_3 &= \mp\sqrt{\tfrac{\kappa}{105\beta}} \\ a_2 &= \tfrac{\alpha}{13\beta} \\ a_1 &= a_0 = 0 \end{aligned} \tag{43}$$

with corresponding elliptic equation

$${Y_\xi}^2 = a_3 Y^2 \left(Y + \frac{a_2}{a_3}\right), \tag{44}$$

Letting $Y \to -\tilde{Y}$, and using solution (23) we obtain

$$Y(\xi) = -\tilde{Y}(\xi) = -\frac{a_2}{a_3}\text{sech}^2\left[\frac{1}{2}\sqrt{a_2}(\xi - \xi_0)\right]. \tag{45}$$

Therefore, right traveling wave solution of the KE with zero boundary condition and $\xi_0 = 0$ reduces to

$$u(x,t) = \frac{105\alpha^2}{169\kappa\beta}\text{sech}^4\left[\frac{1}{2}\sqrt{\frac{\alpha}{13\beta}}\left(x - \frac{36\alpha^2}{169\beta}t\right)\right]. \tag{46}$$

Since $\beta > 0 \Rightarrow \alpha > 0$, and since $a_3 \in \mathbb{R} \Rightarrow \kappa > 0$, and we obtain bright solitons which only propagate to the right. Let us rescale the solution according to the new variables

$$\gamma = 4\sqrt{\frac{\kappa}{105\beta}}, \delta = \frac{4\alpha}{13\beta} \Rightarrow c = \frac{9\beta}{4}\delta^2, \tag{47}$$

so the solitons take the simpler form

$$u(x,t) = \left(\frac{\delta}{\gamma}\right)^2 \text{sech}^4\left[\frac{1}{4}\sqrt{\delta}\left(x - \frac{9}{4}\delta^2 t\right)\right]. \tag{48}$$

This solution written in the same manner was also derived by Yamamoto [56], Yuan [58] and Rosenau [47] using different approaches, and shows once again that the speed of the wave is proportional to the height and inverse proportional to the width, as it is the case of the KdV equation, with the difference that while for the KdV the solitons propagate with arbitrary velocity, for KE they translate with velocity that is fixed by both dispersion coefficients. In Fig. 4 we plot the solitary waves that propagate to the right for the values of $\alpha = 2, \kappa = 5$ for which $\gamma = \frac{4}{\sqrt{21\beta}}$ and $\delta = \frac{8}{13\beta}$. Fixing the velocities $c = 1, 2$ we obtain the values for the dispersion coefficient to be $\beta = \frac{144}{169}, \frac{72}{169}$.

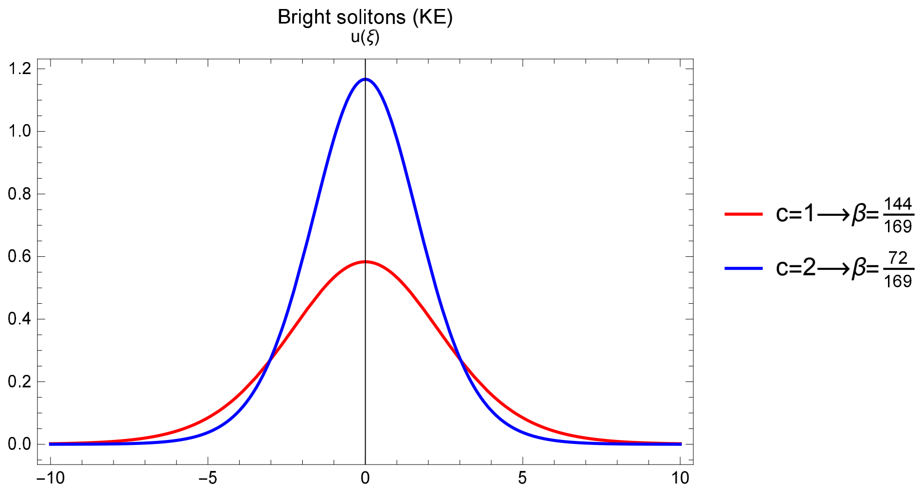

FIG. 4: Bright solitons for the KE (1) with zero boundary conditions $\mathcal{A} = 0$ and $\alpha = 2, \kappa = 5$. When $c = 1, 2$ then $\beta = \frac{144}{169}, \frac{72}{169}$.

C. KE with nonzero boundary conditions

When all the coefficients from system (34) are nonzero, we solve Eq. (12) by reduction to the Weierstrass elliptic equation [48]

$$\wp_\xi^2 = 4\wp^3 - g_2\wp - g_3, \tag{49}$$

by the scale-shift linear transformation

$$Y(\xi) = \frac{4}{a_3}\wp(\xi - \xi_0; g_2, g_3) - \frac{a_2}{3a_3}. \tag{50}$$

The invariants of the Weierstrass function are given by

$$\begin{aligned} g_2 &= \tfrac{{a_2}^2 - 3a_1a_3}{12} = 2({e_1}^2 + {e_2}^2 + {e_3}^2) \\ g_3 &= \tfrac{3^2 a_1a_2a_3 - 3^3 a_0{a_3}^2 - 2{a_2}^3}{16\cdot 3^3} = 4e_1e_2e_3, \end{aligned} \tag{51}$$

and together with the modular discriminant

$$\Delta = {g_2}^3 - 3^3{g_3}^2 = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \tag{52}$$

are used to classify the solutions of Eq. (12). The constants e_i are the roots of the cubic polynomial

$$s_3(t) = 4t^3 - g_2t - g_3 = 4(t - e_1)(t - e_2)(t - e_3) = 0, \tag{53}$$

and are related to the two periods $\omega_{1,2}$ of the $\wp$ function via the relations $e_i = \wp\left(\frac{\omega_i}{2}\right)$, and $\omega_3 = -(\omega_1+\omega_2)$. Using the constants from system (34) the invariants and discriminant are

$$\begin{aligned} g_2 &= \tfrac{169\beta c - 22\alpha^2}{2^3\cdot 3^3\cdot 13^3\beta^2} \\ g_3 &= \tfrac{\alpha(3\cdot 169\beta c - 2^7\alpha^2)}{2^5\cdot 3^3\cdot 5\cdot 13^3\beta^3} \\ \Delta &= \tfrac{(169\beta c - 36\alpha^2)(2^2\cdot 3^2\cdot 47\cdot 101\alpha^4 + 2\cdot 5^2\cdot 13^4\beta^2c^2 - 3\cdot 11\cdot 13\cdot 139\alpha^2\beta c)}{2^{10}\cdot 3^3\cdot 5^2\cdot 7^3\cdot 13^6\beta^6}. \end{aligned} \tag{54}$$

We now proceed to classify the solutions of Eq. (49) case-by-case [2, 39].

Case (1). We first consider the degenerate case of $\Delta = 0 \Rightarrow c = \frac{36\alpha^2}{169\beta}$ for which $s_3(t)$ has repeated root e_i of multiplicity two. In this case the reduced invariants are

$$\begin{aligned} g_2 &= \tfrac{\alpha^2}{2^2\cdot 3\cdot 169\beta^2} > 0 \\ g_3 &= -\tfrac{\alpha^3}{2^3\cdot 3^3\cdot 13^3\beta^3}. \end{aligned} \tag{55}$$

Depending on the sign of g_3 we have the sub-cases:

i) $\alpha > 0$. By letting $e_1 = e_2 = \hat{e} > 0$ with $\hat{e} = \frac{\alpha}{2^2\cdot 3\cdot 13\beta} > 0$ then $e_3 = -2\hat{e} = -\frac{\alpha}{78\beta} < 0$, hence

$$\begin{aligned} g_2 &= 12\hat{e}^2 > 0 \\ g_3 &= -8\hat{e}^3 < 0 \end{aligned} \tag{56}$$

the Weierstrass $\wp$ function is simplified to

$$\wp(\xi; 12\hat{e}^2, -8\hat{e}^3) = \hat{e} + 3\hat{e}\ \text{csch}^2(\sqrt{3\hat{e}}\xi) \tag{57}$$

which becomes

$$\wp(\xi; g_2, g_3) = \frac{\alpha}{2^2\cdot 3\cdot 13\beta}\left[1 + 3\ \text{csch}^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{13\beta}}\xi\right)\right]. \tag{58}$$

ii) $\alpha < 0$. By letting $e_2 = e_3 = -\tilde{e} < 0$ with $\tilde{e} = -\frac{\alpha}{2^2\cdot 3\cdot 13\beta} > 0$, then $e_1 = 2\tilde{e} = -\frac{\alpha}{78\beta} > 0$, hence

$$\begin{aligned} g_2 &= 12\tilde{e}^2 > 0 \\ g_3 &= 8\tilde{e}^3 > 0 \end{aligned} \tag{59}$$

the Weierstrass $\wp$ function is simplified to

$$\wp(\xi; 12\tilde{e}^2, 8\tilde{e}^3) = -\tilde{e} + 3\tilde{e}\ \text{csc}^2(\sqrt{3\tilde{e}}\xi) \tag{60}$$

which is

$$\wp(\xi; g_2, g_3) = \frac{\alpha}{156\beta}\left[1 - 3\ \text{csc}^2\left(\frac{1}{2}\sqrt{\frac{-\alpha}{13\beta}}\xi\right)\right]. \tag{61}$$

Using the transformation (50) together with $u = Y^2$, the solutions are

$$u(x,t) = \begin{cases} \frac{105\alpha^2}{169\kappa\beta}\text{csc}^4\left[\frac{1}{2}\sqrt{\frac{-\alpha}{13\beta}}\left(x - \frac{36\alpha^2}{169\beta}t\right)\right] & ;\alpha < 0 \\ \frac{105\alpha^2}{169\kappa\beta}\text{csch}^4\left[\frac{1}{2}\sqrt{\frac{\alpha}{13\beta}}\left(x - \frac{36\alpha^2}{169\beta}t\right)\right] & ;\alpha > 0 \end{cases} \tag{62}$$

Notice that these solutions are the second linearly independent set obtained from the reduction of the $\wp$ function corresponding exactly to the solutions of KE with zero boundary conditions of (46), since for $c = \frac{36\alpha^2}{169\beta} \Rightarrow \mathcal{A} = 0$. Because these functions are unbounded, the traveling waves are unphysical so they will also be omitted.

Case (2). If $\Delta \neq 0$ we find traveling waves with arbitrary velocity and we include two particular solutions which will fix the velocities of the traveling waves as functions of dispersion coefficients as follows: the equianharmonic ($g_2 = 0$), and lemniscatic case ($g_3 = 0$) respectively.

i) For general solution $g_2 \neq 0$, $g_3 \neq 0$ the waves travel with arbitrary velocity c and the solutions may be reduced in a manner similar to the simplification for the lemniscatic case below.

ii) For the equianharmonic case $g_2 = 0 \Rightarrow c = \frac{22\alpha^2}{169\beta}$, thus $g_3 = -\frac{31\alpha^3}{2^4 \cdot 3^3 \cdot 5 \cdot 13^3 \beta^3}$, and discriminant is $\Delta = -27{g_3}^2 < 0$ with solution to Eq. (49) given by

$$\wp(\xi - \xi_0; 0, g_3) = \wp\left(\xi - \xi_0; 0, -\frac{31\alpha^3}{2^4 \cdot 3^3 \cdot 5 \cdot 13^3 \beta^3}\right). \tag{63}$$

Since $\Delta < 0$ then the polynomial $s_3(t)$ given by Eq. (53) has non real roots, and this case will be omitted being unphysical as well. Using the transformation (50) together with $u = Y^2$, the general and equianharmonic solutions are

$$u(x,t) = \begin{cases} \frac{35}{3 \cdot 13^2 \kappa\beta}\left[\alpha - 2^2 \cdot 3 \cdot 13\beta\wp\left(x - ct; \frac{13^2 c\beta - 22\alpha^2}{2^3 \cdot 3^3 \cdot 13^3 \beta^2}, \frac{\alpha(3 \cdot 13^2 \beta c - 2^5 7\alpha^2)}{2^5 \cdot 3^3 \cdot 5 \cdot 13^3 \beta^3}\right)\right]^2 \\ \frac{35}{3 \cdot 13^2 \kappa\beta}\left[\alpha - 2^2 \cdot 3 \cdot 13\beta\wp\left(x - \frac{22\alpha^2}{13^2 \beta}t; 0, -\frac{31\alpha^3}{2^4 \cdot 3^3 \cdot 5 \cdot 13\beta^3}\right)\right]^2. \end{cases} \tag{64}$$

iii) For the lemniscatic case $g_3 = 0 \Rightarrow c = \frac{2^7\alpha^2}{3 \cdot 169\beta}$, thus $g_2 = \frac{31\alpha^2}{2^2 \cdot 3^2 \cdot 7 \cdot 169\beta^2} > 0$, and discriminant is $\Delta = {g_2}^3 > 0$, with solution to Eq. (49) given by

$$\wp(\xi - \xi_0; g_2, 0) = \wp\left(\xi - \xi_0; \frac{31\alpha^2}{2^2 \cdot 3^2 \cdot 7 \cdot 13^2 \beta^2}, 0\right). \tag{65}$$

In this case the roots of $s_3(t)$ are real and are given by $e_3 = -\frac{\sqrt{g_2}}{2}$, $e_2 = 0$, $e_1 = \frac{\sqrt{g_2}}{2}$. Although the Weierstrass unbounded function given by (65) has poles aligned on the real axis of the $\xi - \xi_0$ complex plane, we can choose ξ_0 in such a way to shift these poles a half of period above the real axis, so that the elliptic function simplifies using the formula [16, 55]

$$\wp(\xi; g_2, 0) = e_3 + (e_2 - e_3)\mathrm{sn}^2[\sqrt{e_1 - e_3}(\xi - \xi_0'); m] \tag{66}$$

with elliptic modulus $m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$. Using the values of the roots together with $\xi_0' = 0$ we obtain

$$\wp(\xi; g_2, 0) = -\frac{\sqrt{g_2}}{2}\mathrm{cn}^2\left[\sqrt[4]{g_2}\xi; \frac{\sqrt{2}}{2}\right], \tag{67}$$

thus, the solutions for the lemniscatic case are reduced using the transformation (50) to cnoidal waves, and they become

$$u(x,t) = \frac{5\alpha^2}{3 \cdot 7 \cdot 13^2 \kappa\beta}\left\{7 + \sqrt{7 \cdot 31}\mathrm{cn}^2\left[\sqrt[4]{\frac{31}{7}}\sqrt{\frac{\alpha}{78\beta}}\left(x - \frac{2^7\alpha^2}{3 \cdot 13^2 \beta}t\right); \frac{\sqrt{2}}{2}\right]\right\}^2 \tag{68}$$

In Fig. 5 we present cnoidal wave-trains for the lemniscatic case with nonzero boundary conditions, and coefficients $\alpha = 1, \kappa = 1$. Fixing the velocities to $c = 1, 2$ the fifth order dispersion coefficient is $\beta = \frac{128}{507}, \frac{64}{507}$.

IV. CONCLUSION

In this paper we applied the generalized elliptic function method to find traveling wave solutions to Kawahara, transmission line, and Korteweg-de Vries equations, which has the advantage of finding solutions to nonlinear evolution equations as polynomial combinations of elliptic functions. Depending on the boundary conditions, these solutions can be reduced, if necessary, to the hyperbolic, periodic, or Jacobian elliptic functions.

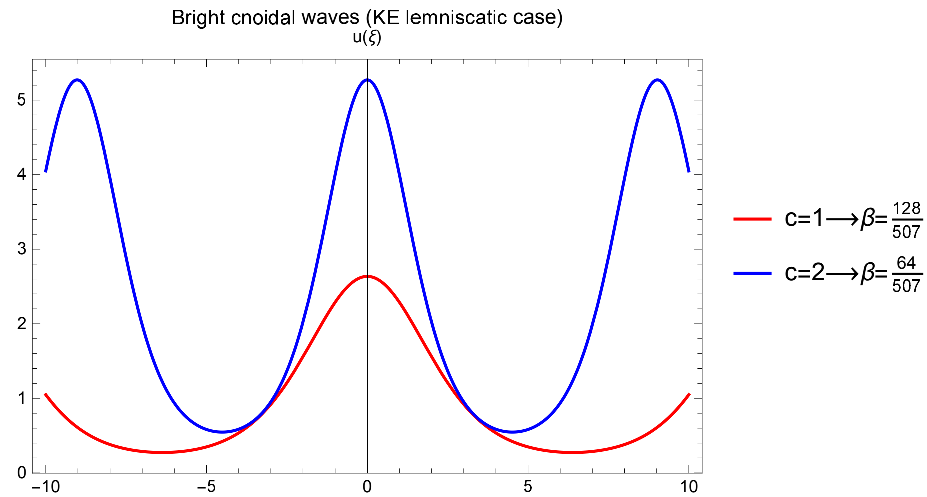

FIG. 5: Bright cnoidal waves for the KE (1) with nonzero boundary conditions for $\alpha = 1, \kappa = 1$. When $c = 1, 2$ then $\beta = \frac{128}{507}, \frac{64}{507}$.

By assuming a polynomial ansatz that satisfies an elliptic equation, in the case of KdV equation, we find the well-known solitary waves, as well as wave-trains of cnoidal waves which are either compressive or rarefactive that propagate in both directions with arbitrary velocity.

In the case of KE the traveling wave solutions are written in terms of Weierstrass elliptic functions which can be reduced to the hyperbolic (for zero boundary conditions) or Jacobi elliptic functions (for nonzero boundary conditions). For the general case the Weierstrass elliptic functions are unbounded, while for the lemniscatic case, they reduce to periodic cnoidal waves.

While for the KdV equation the solitary waves that are both compressive and rarefactive propagate with arbitrary velocity, for KE only compressive waves are found that propagate to one direction with a velocity that depends on both dispersion coefficients.

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V. APPENDIX

Lemma V.1. *Traveling wave solutions to KE (1), satisfy only the Weierstrass elliptic equation with cubic nonlinearity.*

Proof. Letting

$$Y_\xi{}^2 = \sum_{i=0}^{m} a_i Y^i, \qquad a_M \neq 0. \tag{69}$$

For KdV Eq. (14) $u = Y$ and the leading term for the second order derivative term is $u_{\xi\xi} = \frac{1}{2} m a_m Y^{m-1}$. By matching the terms u^2 with $u_{\xi\xi}$ in Eq. (15) we obtain $m = 3$. For KE (1) $u = Y^2$ and the leading term for the fourth order derivative term is $u_{\xi\xi\xi\xi} = \frac{1}{2} m(m+2)(3m-2) a_m{}^2 Y^{2(m-1)}$. By matching the terms u^2 and $u_{\xi\xi\xi\xi}$ in Eq. (7) we also obtain $m = 3$. Notice that this method will not work if an ODE contains both even and odd derivative terms.

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