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Wave propagation and dispersion in microstructured solids

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Summary. A series of numerical simulations is carried on in order to understand the accuracy of dispersive wave models for microstructured solids. The computations are performed by means of the finite-volume numerical scheme, which belongs to the class of wave-propagation algorithms. The dispersion effects are analyzed in materials with different internal structures: microstructure described by micromorphic theory, regular laminates, laminates with substructures, etc., for a large range of material parameters and wavelengths.

Key words: microstructured solids, wave dispersion, numerical simulation

Introduction

Microstructural effects are observed in wave propagation in solids when the wavelength of a travelling signal becomes comparable with the scale of material heterogeneities. A vivid example of the influence of microstructure on wave propagation is the wave dispersion that profoundly alters both the shape and the velocity of propagating waves. The most recognizable signature of the wave dispersion is that the phase and group velocities of propagating waves differ from each other. The dispersive effects of wave propagation in microstructured solids become non-negligible for sufficiently high frequencies.

Wave propagation in heterogeneous solids has been a subject of considerable research for many years. However, micro-structural details are rarely taken into account in large-scale structural dynamics or dynamic impact simulations. The reason is the enormous complexity of wave phenomena in highly heterogeneous media. There exist two alternative approaches to the description of microstructural effects on wave propagation in solids. The first one is focused on the determining so-called effective properties of a material. It is expected that these averaged or smoothened properties reflect in some global sense the response of specimens of the material to external loads. Another approach to involve microstructural effects into the description of wave propagation is provided by higher order or generalized theories of elastic continua. These theories have been proposed in 1960s [1, 2], and later clarified, classified, and extended [3]. The well-established framework for higher grade and higher order theories is, however, accompanied by too many usually undetermined phenomenological coefficients. Nevertheless, dispersive wave equations in solid mechanics are based either on a homogenization procedure or on a generalized continuum theory.

Dispersive wave models

Wave propagation in a homogeneous medium is a well known phenomenon in mechanics. The corresponding one-dimensional wave equation is a classical example of hyperbolic partial differential equations in textbooks

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where u is the displacement, c is the elastic wave speed and subscripts denote derivatives. The wave equation (1) possesses no dispersion. Considering a harmonic wave

$$u(x,t) = \hat{u}\, exp\left[i(kx - \omega t)\right] \tag{2}$$

with wave number k and frequency ω, we obtain the dispersion relation

$$\omega^2 = c^2 k^2. \tag{3}$$

It is easy to see that here the group velocity $\partial\omega/\partial k$ is equal to the phase velocity c, which means that no dispersion is present.

To describe wave propagation in heterogeneous materials reflecting dispersion effects, several modifications of the wave equation are proposed. The simplest generalization of the wave equation is the linear version of the Boussinesq equation for elastic crystals (cf. [4])

$$u_{tt} = c^2 u_{xx} + c^2 l^2 A_{11} u_{xxxx}, \tag{4}$$

where l is an internal length parameter and A_{11} is a dimensionless coefficient. The dispersion relation is obtained by using again the harmonic wave solution (2)

$$\omega^2 = c^2 k^2 - c^2 l^2 A_{11} k^4. \tag{5}$$

This dispersion relation is nonlinear, which means that phase and group velocities are different. Another generalization of the wave equation is the Love-Rayleigh equation for rods accounting for lateral inertia (cf. [5], p.428)

$$u_{tt} = c^2 u_{xx} + l^2 A_{12} u_{xxtt}, \tag{6}$$

where A_{12} is again a dimensionless constant. The corresponding nonlinear dispersion equation has the form

$$\omega^2 = c^2 k^2 - l^2 A_{12} \omega^2 k^2. \tag{7}$$

A more general equation combining the two dispersion models gives

$$u_{tt} = c^2 u_{xx} + c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt}. \tag{8}$$

Similar model proposed by Engelbrecht and Pastrone [6] introduces additionally a contribution of microstructure on slowing down of the propagation velocity c_A^2

$$u_{tt} = \left(c^2 - c_A^2\right) u_{xx} + c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt}. \tag{9}$$

Accordingly, it has the dispersion relation in the form

$$\omega^2 = (c^2 - c_A^2)k^2 - c^2 l^2 A_{11} k^4 - l^2 A_{12} \omega^2 k^2. \tag{10}$$

Due to three additional terms combined, the last model has larger dispersion properties.

In its turn, the Maxwell-Rayleigh model of anomalous dispersion [4] introduces in consideration the four-order time derivative

$$u_{tt} = c^2 u_{xx} + \frac{l^2 A_{22}}{c^2}\left(u_{tt} - c^2 u_{xx}\right)_{tt}. \tag{11}$$

Four-order time derivatives are included also in the "causal" model for the dispersive wave propagation proposed by Metrikine [7]

$$u_{tt} = c^2 u_{xx} - c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt} - \frac{l^2}{c^2} A_{22} u_{tttt}, \tag{12}$$

and in the model based on the Mindlin theory of microstructure [8], which can be represented in the form

$$u_{tt} = \left(c^2 - c_A^2\right)\ u_{xx} + l^2 P\left(u_{tt} - c^2\ u_{xx}\right)_{xx} + \frac{l^2}{c^2} Q\left(u_{tt} - c^2\ u_{xx}\right)_{tt}. \tag{13}$$

Here P and Q are dimensionless constants. It is clear hat corresponding dispersive relations are nonlinear.

As it is shown recently [9], the last two models for dispersive wave propagation can be unified as follows

$$u_{tt} = \left(c^2 - c_A^2\right)\ u_{xx} + l^2 P\left(u_{tt} - c^2\ u_{xx}\right)_{xx} + \frac{l^2}{c^2} Q\left(u_{tt} - c^2\ u_{xx}\right)_{tt} + c^2 l^2 R u_{xxxx}. \tag{14}$$

It is clear that the unified model (14) generalizes both approaches (12) and (13).

Numerical simulations

In order to understand the accuracy of the dispersive models, a series of numerical simulations is carried on. The computations are performed by means of the finite-volume numerical scheme, which belongs to the class of wave-propagation algorithms [10]. Details of the numerical scheme can be found in [11]. The dispersion effects of 1D waves are demonstrated in materials with different internal structures: microstructure described by micromorphic theory, regular laminates, laminates with substructures, etc.

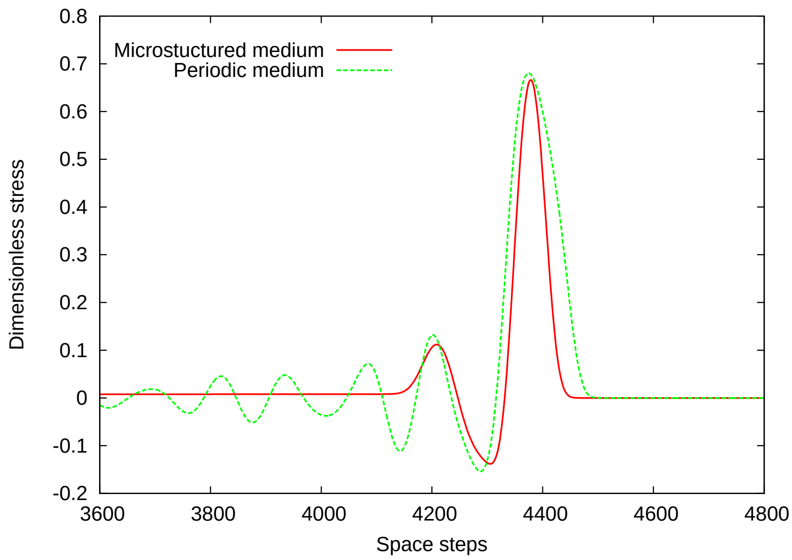

Figure 1. Deformed shape of an initially Gaussian stress pulse in periodic and microstructured solids.

One of the important problems is to compare the results obtained by means of various models. As an example, the comparison of a direct numerical simulation of a Gaussian stress pulse propagation along the elastic bar containing an inhomogeneous part constructed by periodically alternating layers and a computation based on the Mindlin-type microstructure model (14) is shown in Fig.1.

As one can see, the effect of microstructure in the model manifests itself only locally, whereas the dispersion in the periodic laminate is non-local due to consecutive reflections. In principle, the localization of the microstructure influence is expected, since the presence of the microstructure is invisible in the absence of loading.

In order to get matching results, one should critically revise the free energy function in the micromorphic theory for adequate modelling of interaction forces between macro– and microstructures. In the considered case, the pulse length is 5 times longer than the inhomogeneity size. This particular case was chosen because it clearly shows the synergy of the two microstructure models unified in [9]. The matching results are obtained by modifying the coupling between macro- and microstructures including also the dependence on gradients of the internal variables. The correlation between models is analyzed in detail for a large range of material parameters and wavelengths.

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