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Not All Traces On the Circle Come From Functions of Least Gradient in the Disk

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NOT ALL TRACES ON THE CIRCLE COME FROM FUNCTIONS OF LEAST GRADIENT IN THE DISK

GREGORY S. SPRADLIN AND ALEXANDRU TAMASAN

Abstract. We provide an example of an L^1 function on the circle, which cannot be the trace of a function of bounded variation of least gradient in the disk.

1. Introduction

Sternberg et al. in [3], and Sternberg and Ziemer in [4] considered the question of existence, uniqueness and regularity for functions of least gradient and prescribed trace. More precisely for $\Omega \subset \mathbb{R}^n$ a Lipchitz domain, and for a continuous map $g \in C(\partial\Omega)$, they formulate the problem

$$\min\{\int_\Omega |Du| : \ u \in BV(\Omega), \ u|_{\partial\Omega} = g\}, \tag{1.1}$$

where $BV(\Omega)$ denotes the space of functions of bounded varation, the integral is understood in the sense of the Radon measure $|Du|$ of Ω and the trace at the boundary is in the sense of the trace of functions of bounded varation. Solutions to the minimization problem (1.1) are called functions of least gradient. For domains Ω with boundary of non-negative curvature, and which are not locally area minimizing they prove existence, uniqueness and regularity of the solution. Moreover, if the boundary of the domain fails either of the two assumptions they provide counterexamples to existence.

It is known that traces of functions $f \in BV(\Omega)$ of bounded variation are in $L^1(\partial\Omega)$, and that conversely, any function in $L^1(\partial\Omega)$ admits an extension (in the sense of trace) in $BV(\Omega)$ (in fact in $W^{1,1}(\Omega)$), see e.g., [1]. The question we address here is whether solutions of the problem (1.1) exist in the case of traces that are merely in $L^1(\partial\Omega)$ and not continuous. We answer this question in the negative by providing a counterexample for the unit disk, which has a boundary of positive curvature and which is not locally length minimizing.

Let $\mathbb{D}$ denote the unit disk in the plane and $\mathbb{S}$ be its boundary. We prove the following:

Theorem 1.1. *There exists $f \in L^1(\mathbb{S})$ such that the minimization problem*

$$\min\{\int_\mathbb{D} |Dw| : \ w \in BV(\mathbb{D}), \ w|_\mathbb{S} = f\} \tag{1.2}$$

has no solution.

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A renewed interest in functions of least gradient with variable weights appeared recently due to its applications to current density impedance imaging, see [2] and references therein. Our counterexample sets a limit on the roughness of the boundary data one can afford to use.

2. Proof of Theorem 1.1

We will call the $L^1(\mathbb{S})$-function satisfying Theorem 1.1 "f_∞". f_∞ is the characteristic function of a fat Cantor set. Define $C_0 \supset C_1 \supset C_2 \supset \cdots$ inductively as follows:

$$C_0 = \{(\cos\theta, \sin\theta) \;\Big|\; \frac{\pi}{2} - \frac{1}{2} \le \theta \le \frac{\pi}{2} + \frac{1}{2}\},$$

and if C_n consists of 2^n disjoint closed arcs, each with arc length

$$\theta_n = \frac{1}{2^n}\prod_{i=1}^{n}(1 - \frac{1}{2^i}) \tag{2.1}$$

(if $n = 0$, the "empty product" is interpreted as 1), then C_{n+1} is obtained by removing from the center of each of those arcs an open arc of arc length $(1 - 1/2^{n+1})\theta_n$. Then C_{n+1} consists of 2^{n+1} disjoint closed arcs, each with arc length θ_{n+1}. For $n = 0, 1, 2, \ldots$, with $\mathcal{H}^1$ denoting one-dimensional Hausdorff measure,

$$\mathcal{H}^1(C_n) = 2^n\theta_n = \prod_{i=1}^{n}(1 - \frac{1}{2^i}) \equiv K_n. \tag{2.2}$$

Define

$$C_\infty = \bigcap_{n=0}^{\infty} C_n.$$

C_∞ is a compact and nowhere dense subset of $\mathbb{S}$, with

$$\mathcal{H}^1(C_\infty) = \prod_{i=1}^{\infty}(1 - \frac{1}{2^i}) = \lim_{n\to\infty} K_n \equiv K_\infty > 0.$$

Note that K_∞ is well-defined and positive, since all the terms in the infinite product are positive and $\sum_{i=1}^{\infty} 1/2^i < \infty$.

We define $f_\infty \in L^1(\mathbb{S})$ to be the characteristic function of C_∞:

$$f_\infty = \chi_{C_\infty} \in L^1(\mathbb{S}).$$

From [1, Theorem 2.16, Remark 2.17] we have that

$$\inf\{\int_{\mathbb{D}} |Du| \;\Big|\; u \in BV(\mathbb{D}),\ u|_{\mathbb{S}} = f_\infty\} \le \|f_\infty\|_{L^1(\mathbb{S})} = K_\infty.$$

We will show that for any $u \in BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$,

$$\int_{\mathbb{D}} |Du| > K_\infty,$$

proving Theorem 1.1.

The idea of the proof is as follows: we construct a compact, nowhere dense subset B_∞ of $\overline{\mathbb{D}}$ with the property that

$$\text{(i) If } u \in BV(\mathbb{D}) \text{ with } u|_{\mathbb{S}} = f_\infty \text{ and } \int_{\mathbb{D}\setminus B_\infty} |u|\, dx > 0, \text{ then } \int_{\mathbb{D}} |Du| > K_\infty,$$

and

$$\text{(ii) If } u \in BV(\mathbb{D}) \text{ with } u|_{\mathbb{S}} = f_\infty \text{ and } \int_{\mathbb{D}\setminus B_\infty} |u|\, dx = 0, \text{ then } \int_{\mathbb{D}} |Du| > K_\infty. \tag{2.3}$$

Theorem 1.1 obviously follows from this. B_∞ has the form

$$B_\infty = \bigcap_{n=1}^{\infty} B_n, \tag{2.4}$$

where $B_1 \supset B_2 \supset B_3 \supset \cdots$, and for each $n \geq 1$, B_n is a compact subset of $\overline{\mathbb{D}}$ with 2^n path components, with each path component the union of a polygon and two circular segments ("circular segment" is the standard term for the region between an arc and a chord connecting two points on a circle). That polygon will be defined precisely as the union of at least one triangle with at least one trapezoid. In Figure 2, B_1 is the union of the two shaded regions. In Figure 3, the two shaded regions constitute the upper portion of the right half of B_2. The four shaded regions in Figure 4 are indistinguishable from the top portion of B_3 (the set S_1 mentioned in the caption does not include eight tiny circular segments that hug $\mathbb{S}$ and are so small they are not visible in the figure). The entire set B_3 is formed by extending the shaded regions in Figure 4 downward to the bottom of $\mathbb{S}$, similarly to B_1 (see Figure 2). In all four figures, the arclengths and lengths of arcs and line segments are not necessarily scaled consistently with (2.1), but were chosen to try to make the figures easy to read.

Unfortunately, defining each B_n precisely requires a slew of definitions. For $n \geq 0$, C_n is the disjoint union of 2^n closed arcs. Call this collection of arcs $\mathcal{A}_n$. For example, $\mathcal{A}_0 = \{C_0\}$. For each $A \in \mathcal{A}_n$, we will define a set $B_A \subset \overline{\mathbb{D}}$, then define B_n as the disjoint union

$$B_n = \bigsqcup_{A \in \mathcal{A}_n} B_A. \tag{2.5}$$

Each such B_A is the connected union of a closed circular segment, n closed polygons, which are all triangles or trapezoids (including at least one triangle), and a "bottom" piece that is the union of a trapezoid and a circular segment (in Figure 2, the arc in $\mathcal{A}_1$ in the right half of the x_1-x_2 plane is called "A", and B_A is the shaded region in the right half of $\overline{\mathbb{D}}$. If we call the other arc in $\mathcal{A}_1$ "A'", then the shaded region in the left half of $\overline{\mathbb{D}}$ is $B_{A'}$. In Figure 3, the shaded region on the left is the top of B_α and the shaded region on the right is the top of B_β, where α and β are the two arcs in $\mathcal{A}_2$ in the right half of the x_1-x_2 plane. The other notations used in Figures 2 and 3 will be defined momentarily).

For an arc A, let $\operatorname{Cho}(A)$ denote the chord connecting the endpoints of A, and $W(A)$ the closed circular segment enclosed by A and $\operatorname{Cho}(A)$. For $A \in \mathcal{A}_n$ with $n \geq 1$, define $\operatorname{Par}(A)$ (the "parent" of A) to be the arc in $\mathcal{A}_{n-1}$ containing A:

$$\operatorname{Par}(A) = A' : \ A' \in \mathcal{A}_{n-1}, \ A \subset A'.$$

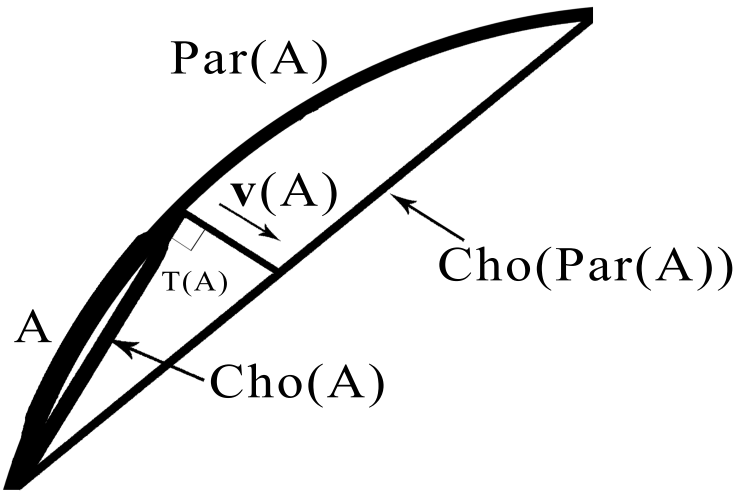

FIGURE 1

More generally, for $A \in \mathcal{A}_n$ $(n \geq 0)$, define

$$\mathrm{Par}^0(A) = A,\ \mathrm{Par}^1(A) = \mathrm{Par}(A),\ \mathrm{Par}^2(A) = \mathrm{Par}(\mathrm{Par}^1(A)),$$
$$\mathrm{Par}^3(A) = \mathrm{Par}(\mathrm{Par}^2(A)), \ldots,\ \mathrm{Par}^n(A) = C_0 \in \mathcal{A}_0.$$

For $A \in \mathcal{A}_n$ $(n \geq 0)$, define the two "children" of A, $\mathrm{Chi}_L(A)$ and $\mathrm{Chi}_R(A)$, by

$$\mathrm{Chi}_L(A), \mathrm{Chi}_R(A) \in \mathcal{A}_{n+1},\ \mathrm{Chi}_L(A), \mathrm{Chi}_R(A) \subset A,\ \mathrm{Chi}_L(A) \cap \mathrm{Chi}_R(A) = \varnothing,$$
$$\mathrm{Chi}_L(A) \text{ is "to the left" or counterclockwise from } \mathrm{Chi}_R(A).$$

For an arc A of $\mathbb{S}$ of arc length less than π, let $\mathbf{v}(A)$ denote the unit vector perpendicular to $\mathrm{Cho}(A)$ and pointing from $\mathrm{Cho}(A)$ toward $\mathbf{0} \in \mathbb{R}^2$. For $A \in \mathcal{A}_n$ with $n \geq 1$, let $T(A)$ denote the unique closed right triangle whose longer leg is $\mathrm{Cho}(A)$ and whose hypotenuse is a subset of $\mathrm{Cho}(\mathrm{Par}(A))$.

Figure 1 shows an arc A belonging to $\mathcal{A}_n$ for some $n \geq 1$, along with $\mathrm{Par}(A)$, $\mathrm{Cho}(A)$, $\mathrm{Cho}(\mathrm{Par}(A))$, $T(A)$, and $\mathbf{v}(A)$. The lengths of the segments and arcs are not necessarily to scale, and the length of $\mathbf{v}(A)$ is definitely not to scale, since $\mathbf{v}(A)$ is a unit vector and $\mathrm{Par}(A)$ is an arc of $\mathbb{S}$.

We are finally ready to define B_A (for $A \in \mathcal{A}_n$ with $n \geq 1$). We will do the $n = 1$ and $n = 2$ cases first, then the general case.

Suppose $A \in \mathcal{A}_1$ (so $A = \mathrm{Chi}_L(C_0)$ or $\mathrm{Chi}_R(C_0)$). B_A is the union of $W(A)$, $T(A)$, and a "bottom" piece that is the union of a trapezoid and a circular segment. In order to help establish the pattern for general n, we will introduce some notations that are not needed here but will be necessary later. Define the line segment $L_1(A) = \mathrm{Cho}(A)$, the triangle $T_0(A) = T(A)$, and define the line segment $L_2(A)$ to be the hypotenuse of $T_0(A)$, which can also be defined

$$L_2(A) = \{\mathbf{x} \in \partial T_0(A) \ \big| \ x_2 = \sin(\frac{\pi}{2} - \frac{1}{2})\}. \tag{2.6}$$

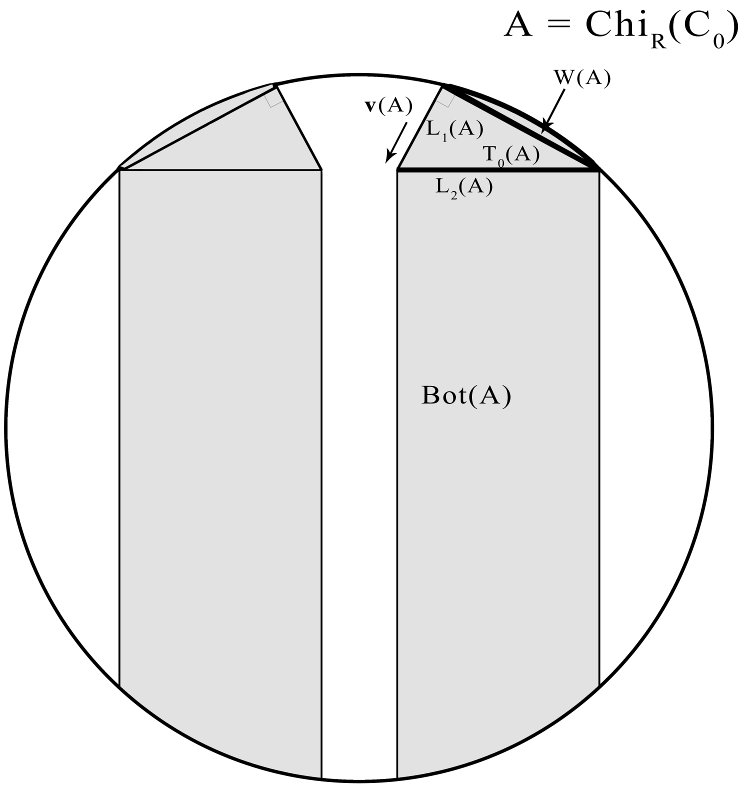

FIGURE 2. B_1

Define the "bottom" part of B_A, Bot(A), to be the set of all points in $\overline{\mathbb{D}}$ on or directly "below" $L_2(A)$, that is,

$$\mathrm{Bot}(A) = \{\mathbf{x} \in \overline{\mathbb{D}} \;\Big|\; x_2 \leq \sin(\frac{\pi}{2} - \frac{1}{2}),\ x_1 = y_1 \text{ for some } \mathbf{y} \in L_2(A)\}. \qquad (2.7)$$

Bot(A) is the union of a closed trapezoid and a closed circular segment. Finally, define

$$B_A = W(A) \cup T_0(A) \cup \mathrm{Bot}(A). \qquad (2.8)$$

In Lemma 3.1 in the Appendix, it is proven that for any $A \in \mathcal{A}_n$ (for $n \geq 0$), $T(\mathrm{Chi}_L(A))$ and $T(\mathrm{Chi}_R(A))$ are disjoint (use $\theta = \theta_n$ and $\alpha = \theta_n/2^{n+1} \geq \theta_n^2/2$, with θ_n as in (2.1)). It follows that $B_{\mathrm{Chi}_L(A)}$ and $B_{\mathrm{Chi}_R(A)}$ are disjoint. Now B_1 is defined as in (2.5). The two shaded regions in Figure 2 comprise B_1. The arc $\mathrm{Chi}_R(C_0)$ is called A, and the parts of B_A (which is the right half of B_1) are labelled, along with the vector $\mathbf{v}(A)$, which is perpendicular to Cho(A). The lengths of the segments and arcs are not truly scaled, and the unit vector $\mathbf{v}(A)$ is drawn with shorter than unit length in order to fit in the picture.

Next, suppose $A \in \mathcal{A}_2$. B_A is the union of a chain of four sets: a closed circular segment, followed by a closed triangle, then a closed triangle or trapezoid, then finally a closed "bottom" piece which is the union of a trapezoid and a circular

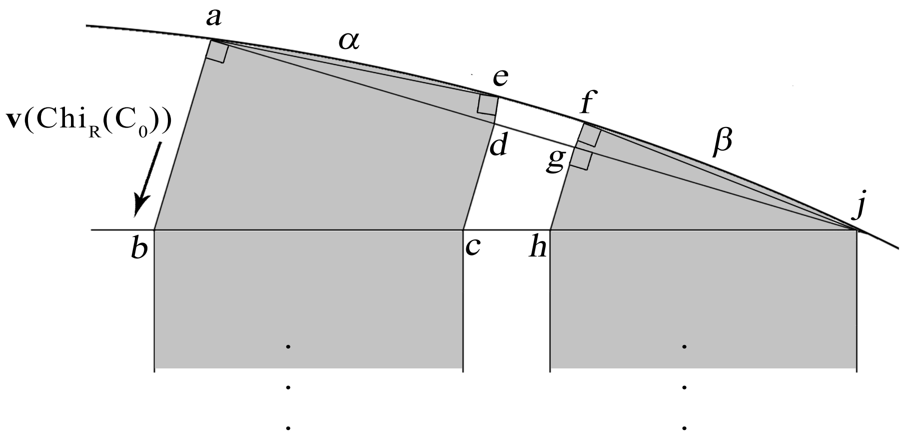

FIGURE 3. The upper part of the right half of B_2.

segment, as in the $n = 1$ case. The intersection of any two consecutive sets in the chain is a line segment.

Figure 3 shows the top of the right half of B_2, so it shows the top portions of the two rightmost of the four components of B_2. As before, the lengths of segments and arcs are not necessarily scaled truly, and $\mathbf{v}(\mathrm{Chi}_R(C_0))$ is actually a unit vector, contrary to the picture. The arc $\widehat{aj}$ is $\mathrm{Chi}_R(C_0)$. For brevity in notation we have defined $\alpha = \widehat{ae} = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$ and $\beta = \widehat{fj} = \mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$. The two connected gray regions are the upper portions of B_α and of B_β. B_α is the union of $W(\alpha)$ (a very thin circular segment in the figure), the triangle $\triangle\, ade$, the trapezoid $abcd$, and a "bottom" piece $\mathrm{Bot}(\alpha)$ consisting of all the points in $\overline{\mathbb{D}}$ on or directly below the line segment $\overline{bc}$. B_β is the union of $W(\beta)$ (a very thin circular segment in the figure), the triangle $\triangle\, fgj$, the triangle $\triangle\, ghj$, and a "bottom" piece $\mathrm{Bot}(\beta)$ consisting of all the points in $\overline{\mathbb{D}}$ on or directly below the line segment $\overline{hj}$.

Generally, for $A \in \mathcal{A}_2$, define the line segment $L_1(A) = \mathrm{Cho}(A)$ (so $L_1(\alpha) = \overline{ae}$ and $L_1(\beta) = \overline{fj}$), and the triangle $T_0(A) = T(A)$ (so $T_0(\alpha) = \triangle\, ade$ and $T_0(\beta) = \triangle\, fgj$). Define the line segment $L_2(A) = \partial T_0(A) \cap \mathrm{Cho}(\mathrm{Par}(A))$. $L_2(A)$ can also be described as the hypotenuse of $T_0(A)$. In Figure 3, $L_2(\alpha) = \overline{ad}$ and $L_2(\beta) = \overline{gj}$. Define $T_1(A) \subset T(\mathrm{Par}^1(A))$ to be the set of all points $\mathbf{x}$ in the triangle $T(\mathrm{Par}^1(A))$ with the property that for some point $\mathbf{y} \in L_2(A)$, the vector $\mathbf{x} - \mathbf{y}$ is parallel to $\mathbf{v}(\mathrm{Par}^1(A)) \equiv \mathbf{v}(\mathrm{Par}(A))$. $T_1(A)$ is either a triangle (this occurs if $A = \mathrm{Chi}_L(\mathrm{Chi}_L(C_0))$ or $\mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$) or a trapezoid (this occurs if $A = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$ or $\mathrm{Chi}_R(\mathrm{Chi}_L(C_0))$). In Figure 3, $T_1(\alpha)$ is the trapezoid $abcd$, with $\alpha = \mathrm{Chi}_L(\mathrm{Chi}_R(C_0))$, and $T_1(\beta)$ is the triangle $\triangle\, ghj$, with $\beta = \mathrm{Chi}_R(\mathrm{Chi}_R(C_0))$. $T_1(A)$ can be defined succinctly by

$$T_1(A) = \{\mathbf{x} \in T(\mathrm{Par}^1(A)) \;\Big|\; \mathbf{x} - \mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^1(A)) \text{ for some } \mathbf{y} \in L_2(A)\}.$$

Define the horizontal line segment $L_3(A)$, similarly to (2.6), to be the set of all points in $\partial T_1(A)$ with x_2-coordinate $\sin(\pi/2 - 1/2)$:

$$L_3(A) = \{\mathbf{x} \in \partial T_1(A) \;\Big|\; x_2 = \sin(\frac{\pi}{2} - \frac{1}{2})\}.$$

In other words, $L_3(A)$ is the side of the polygon $\partial T_1(A)$ that is a subset of the horizontal line $\{\mathbf{x} \mid x_2 = \sin(\pi/2 - 1/2)\}$. In Figure 3, $L_3(\alpha) = \overline{bc}$ and $L_3(\beta) = \overline{hj}$. Like in (2.7), define the "bottom" part of B_A, $\mathrm{Bot}(A)$, to be the set of all points in $\overline{\mathbb{D}}$ on or directly below $L_3(A)$, that is,

$$\mathrm{Bot}(A) = \{\mathbf{x} \in \overline{\mathbb{D}} \;\Big|\; x_2 \leq \sin(\frac{\pi}{2} - \frac{1}{2}),\ x_1 = y_1 \text{ for some } \mathbf{y} \in L_3(A)\}.$$

Like before, $\mathrm{Bot}(A)$ is the union of a trapezoid and a circular segment. Similar to (2.8), define

$$B_A = W(A) \cup T_0(A) \cup T_1(A) \cup \mathrm{Bot}(A).$$

As in the $n = 1$ case, by Lemma 3.1 in the Appendix, the sets B_A for the four elements of $\mathcal{A}_2$ are disjoint. B_2 is defined by (2.5). Clearly $B_2 \subset B_1$.

Finally we consider the $n > 2$ case. Let $A \in \mathcal{A}_n$. B_A is the union of a chain of $n + 2$ closed sets: a closed circular segment, followed by n closed polygons which are all triangles or trapezoids, and finally a bottom piece called $\mathrm{Bot}(A)$ (as before) which is the union of a closed trapezoid and a closed circular segment. Either all n of the polygons are triangles, or (more likely), the first k of them are triangles for some $1 \leq k \leq n - 1$ and the remaining $n - k$ polygons are trapezoids. The intersection of any two consecutive sets in the chain is a line segment. B_A has the form

$$B_A = W(A) \cup \bigcup_{k=0}^{n-1} T_k(A) \cup \mathrm{Bot}(A),$$

where $T_k(A)$ and $\mathrm{Bot}(A)$ are defined precisely momentarily. In order to do so, we must also name the intersections of consecutive sets in the chain, which are line segments, and which we will call $L_1(A), \ldots, L_{n+1}(A)$. We will also need to use $L_1(A), \ldots, L_{n+1}(A)$ to prove (2.3).

Define

$$
\begin{aligned}
L_1(A) &= \mathrm{Cho}(A),\\
T_0(A) &= T(A),\\
L_2(A) &= \partial T_0(A) \cap \mathrm{Cho}(\mathrm{Par}^1(A)),\\
T_1(A) &= \{\mathbf{x} \in T(\mathrm{Par}^1(A)) \;\big|\; \mathbf{x}-\mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^1(A)) \text{ for some } \mathbf{y} \in L_2(A)\},\\
L_3(A) &= \partial T_1(A) \cap \mathrm{Cho}(\mathrm{Par}^2(A)),\\
T_2(A) &= \{\mathbf{x} \in T(\mathrm{Par}^2(A)) \;\big|\; \mathbf{x}-\mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^2(A)) \text{ for some } \mathbf{y} \in L_3(A)\},\\
&\;\vdots\\
T_{n-1}(A) &= \{\mathbf{x} \in T(\mathrm{Par}^{n-1}(A)) \;\big|\; \mathbf{x}-\mathbf{y} \parallel \mathbf{v}(\mathrm{Par}^{n-1}(A)) \text{ for some } \mathbf{y} \in L_n(A)\},\\
L_{n+1}(A) &= \{\mathbf{x} \in \partial T_{n-1}(A) \;\big|\; x_2 = \sin(\frac{\pi}{2} - \frac{1}{2})\},\\
\mathrm{Bot}(A) &= \{\mathbf{x} \in \overline{\mathbb{D}} \;\big|\; x_2 \le \sin(\frac{\pi}{2} - \frac{1}{2}),\ x_1 = y_1 \text{ for some } \mathbf{y} \in L_{n+1}(A)\}.
\end{aligned}
\tag{2.9}
$$

Like before, the B_A's are disjoint for all the 2^n arcs A in $\mathcal{A}_n$, and B_n is defined by (2.5). Clearly $B_1 \supset B_2 \supset B_3 \supset \cdots$. We define B_∞ by (2.4).

Having defined B_∞, we show that it has property (2.3), from which Theorem 1.1 follows. This requires three lemmas, followed by an easy proof of (2.3)(i), then a more involved proof of (2.3)(ii).

Lemma 2.1. *Let $u \in C^\infty(\mathbb{D}) \cap BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$, $n \ge 1$, and $A \in \mathcal{A}_n$. Let $T_0(A), T_1(A), \ldots, T_{n-1}(A)$ and $\mathrm{Bot}(A)$ be as in (2.9). Then*

$$
\begin{aligned}
\int_{W(A)} |\nabla u \cdot \mathbf{v}(A)|\,dx + \sum_{k=0}^{n-1} \int_{T_k(A)} |\nabla u \cdot \mathbf{v}(\mathrm{Par}^k(A))|\,dx +&\\
+ \int_{\mathrm{Bot}(A)} |\nabla u \cdot \mathbf{j}|\,dx \ge \cos\left(\frac{K_n}{2^{n+1}}\right)\frac{K_\infty}{2^n}.&
\end{aligned}
\tag{2.10}
$$

Here, $\mathbf{j} = \langle 0, 1\rangle$, as usual. K_n is from (2.2). There is a slight abuse of notation in (2.10): the domain of u is $\mathbb{D}$, not $\overline{\mathbb{D}}$, but $W(A)$ and $\mathrm{Bot}(A)$ intersect $\mathbb{S} \equiv \partial\mathbb{D}$, and $T_k(A)$ might intersect $\mathbb{S}$. In all cases, the intersection has $\mathcal{H}^2$-measure zero. It would be better formally to replace "$W(A)$", "$T_k(A)$", and "$\mathrm{Bot}(A)$" in (2.10) with their interiors, or with their intersections with $\mathbb{D}$. However, this might make the proof of Lemma 2.1 less readable, so we will keep the notation of (2.10) in the proof of the lemma, and in the remainder of this section.

Proof of lemma: define $s_n = 2\sin(K_n/2^{n+1})$, which is the length of $\mathrm{Cho}(A)$. Let $L_1(A), \ldots, L_{n+1}(A)$ be as in (2.9). For $k = 1, 2, \ldots, n+1$, let $\phi_k : [0, s_n] \to \overline{\mathbb{D}}$ be the linear map with $\phi_k(0)$ the left endpoint of $L_k(A)$ and $\phi_k(s_n)$ the right endpoint of $L_k(A)$ ($L_k(A)$ is not vertical). Define $\phi_0 : (0, s_n) \to A$ so that $\phi_0(t)$ is the projection of $\phi_1(t)$ onto A in the direction $-\mathbf{v}(A)$ (the explicit formula for $\phi_0(t)$ is fairly complicated and we do not use it, so we omit it). Now define $g_0, g_1, \ldots, g_{n+1} \in L^1((0, s_n))$ by

$$
g_0(t) = f_\infty(\phi_0(t)),\ g_k(t) = u(\phi_k(t)) \text{ for } 1 \le k \le n+1.
$$

Now, $\mathcal{H}^1(C_\infty \cap A) = K_\infty/2^n$, so $\int_A f_\infty \, d\mathcal{H}^1 = K_\infty/2^n$. Recall θ_n from (2.5). Since the angle between A and $\mathrm{Cho}(A)$ is at most $\theta_n/2 = K_n/2^{n+1}$,

$$\|g_0\|_{L^1((0,s_n))} \geq \cos\left(\frac{K_n}{2^{n+1}}\right) \int_A f_\infty \, d\mathcal{H}^1 = \cos\left(\frac{K_n}{2^{n+1}}\right) \frac{K_\infty}{2^n}. \tag{2.11}$$

Obviously,

$$g_0 = (g_0 - g_1) + (g_1 - g_2) + (g_2 - g_3) + \cdots + (g_n - g_{n+1}) + g_{n+1},$$

so by the triangle inequality,

$$\|g_0\|_{L^1((0,s_n))} \leq \|g_1 - g_0\|_{L^1((0,s_n))} + \sum_{k=1}^{n} \|g_{k+1} - g_k\|_{L^1((0,s_n))} + \|g_{n+1}\|_{L^1((0,s_n))}. \tag{2.12}$$

Now

$$\|g_1 - g_0\|_{L^1((0,s_n))} = \int_0^{s_n} |g_1(t) - g_0(t)| \, dt \leq \int_{W(A)} |\nabla u(x) \cdot \mathbf{v}(A)| \, dx. \tag{2.13}$$

For $1 \leq k \leq n$, the Fundamental Theorem of Calculus yields

$$\|g_{k+1} - g_k\|_{L^1((0,s_n))} = \int_0^{s_n} |g_{k+1}(t) - g_k(t)| \, dt \leq \int_{T_{k-1}(A)} |\nabla u(x) \cdot \mathbf{v}(\mathrm{Par}^{k-1}(A))| \, dx. \tag{2.14}$$

Since $f_\infty = 0$ on the bottom half of $\mathbb{S}$, $u|_{\mathbb{S}} = f_\infty$, and $\mathrm{Bot}(A) \cap \mathbb{S}$ is a subset of the bottom half of $\mathbb{S}$, it follows that

$$\|g_{k+1}\|_{L^1((0,s_n))} = \int_0^{s_n} |g_{k+1}(t)| \, dt \leq \int_{\mathrm{Bot}(A)} |\nabla u \cdot \mathbf{j}| \, dx. \tag{2.15}$$

Putting (2.11) and (2.12)-(2.15) together yields (2.10).

□

Define $\mathbb{D}_- \subset \mathbb{D}$, the "lower part" of $\mathbb{D}$, by

$$\mathbb{D}_- = \{\mathbf{x} \in \mathbb{D} \;\Big|\; x_2 < \sin(\frac{\pi}{2} - \frac{1}{2})\}. \tag{2.16}$$

From Lemma 2.1, there follows:

Lemma 2.2. *Let u be as in Lemma 2.1: $u \in C^\infty(\mathbb{D}) \cap BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$. Let $n \geq 1$. Then*

$$\begin{aligned} &\sum_{A \in \mathcal{A}_n} \int_{W(A)} |\nabla u \cdot \mathbf{v}(A)| \, dx + \sum_{m=1}^{n} \sum_{A \in \mathcal{A}_m} \int_{T(A) \cap B_n} |\nabla u \cdot \mathbf{v}(A)| \, dx + \\ &\quad + \int_{B_n \cap \mathbb{D}_-} |\nabla u \cdot \mathbf{j}| \, dx \geq \cos\left(\frac{K_n}{2^{n+1}}\right) K_\infty. \end{aligned} \tag{2.17}$$

Proof: By Lemma 2.1,

$$\begin{aligned} &\sum_{A \in \mathcal{A}_n} \int_{W(A)} |\nabla u \cdot \mathbf{v}(A)| \, dx + \sum_{A \in \mathcal{A}_n} \sum_{k=0}^{n-1} \int_{T_k(A)} |\nabla u \cdot \mathbf{v}(A)| \, dx + \\ &\quad + \sum_{A \in \mathcal{A}_n} \int_{\mathrm{Bot}(A)} |\nabla u \cdot \mathbf{j}| \, dx \geq \cos\left(\frac{K_n}{2^{n+1}}\right) K_\infty. \end{aligned} \tag{2.18}$$

We must prove that the inequalities (2.17) and (2.18) are equivalent. The right-hand sides and the first terms of the left-hand sides are exactly the same. The third

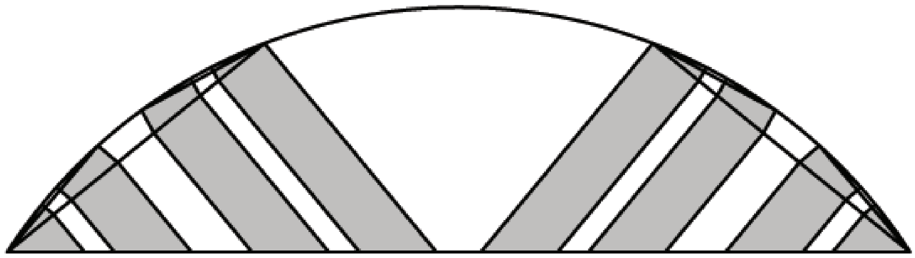

FIGURE 4. The set S_1 (which equals S_2) from the proof of Lemma 2.2

summands in the left-hand sides are equal because $B_n \cap \overline{\mathbb{D}}_-$ is the disjoint union of the sets $\mathrm{Bot}(A)$ for the 2^n arcs A in $\mathcal{A}_n$. We must show that the second summands on the left-hand sides of (2.17) and (2.18) are equal. Call the common integrand of the integrals "$g(x)$". Generally, any two distinct sets of the form $T_k(A)$, for $l \geq 1$, $A \in \mathcal{A}_l$, and $k \in \{0, \dots, l-1\}$ have intersection of $\mathcal{H}^2$-measure zero. This includes the case of $k = 0$, $T_k(A) = T_0(A) \equiv T(A)$. Therefore the second summands on the left-hand sides of (2.17) and (2.18) have the form $\int_{S_1} g\,dx$ and $\int_{S_2} g\,dx$, where

$$S_1 = \bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} (T(A) \cap B_n) = B_n \cap \left(\bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} T(A) \right),$$

$$S_2 = \bigcup_{A \in \mathcal{A}_n} \bigcup_{k=0}^{n-1} T_k(A).$$

We must show $S_1 = S_2$. This is easy to see if one uses a picture, but unfortunately it is difficult to explain in words. We will do both.

In Figure 4, the shaded region (comprised of eight components) is S_1 (which equals S_2) in the case $n = 3$. On one hand, the shaded region is S_2: $\mathcal{A}_3$ contains eight disjoint closed arcs A. For each such A, the union of the polygons $T_0(A), T_1(A)$, and $T_2(A)$ equals one of the eight components of the shaded region: on top, $T_0(A)$ is a tiny triangle that is barely visible, just below, $T_1(A)$ is a larger triangle or trapezoid, and on the bottom, $T_2(A)$ is a triangle or trapezoid, one of whose sides is a subset of the horizontal chord $\mathrm{Cho}(C_0)$ at the bottom of Figure 4. On the other hand, the shaded region is S_1: each component of the shaded region is the union of three polygons. The union of the polygons on the bottom of the components (there are eight such polygons) is $B_3 \cap \cup_{A \in \mathcal{A}_1} T(A)$. The union of the middle polygons of the components (there are eight such polygons) is $B_3 \cap \cup_{A \in \mathcal{A}_2} T(A)$. Finally, the union of the top polygons of each of component (there are eight such polygons, and they are all tiny triangles) is $B_3 \cap \cup_{A \in \mathcal{A}_3} T(A)$.

To formally prove $S_1 = S_2$, we show that the two sets are subsets of each other. First we show $S_1 \subset S_2$. Let $m' \in \{1, \dots, n\}$ and $A' \in \mathcal{A}_{m'}$. We will show $B_n \cap T(A') \subset S_2$. Let $A \in \mathcal{A}_n$. Since $T(A') \subset \mathbb{D} \setminus \mathbb{D}_-$ and $\mathrm{Bot}(A) \subset \overline{\mathbb{D}}_-$, $x_2 = \sin(\pi/2 - 1/2)$ along $T(A') \cap \mathrm{Bot}(A)$. Now

$$\mathrm{Bot}(A) \cap \{\mathbf{x} \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right)\} = T_{n-1}(A) \cap \{\mathbf{x} \mid x_2 = \sin\left(\frac{\pi}{2} - \frac{1}{2}\right)\}.$$

Thus $T(A') \cap \mathrm{Bot}(A) \subset T_{n-1}(A)$. Also, since $m' \leq n$, $T(A') \cap W(A) \subset T(A) \equiv T_0(A)$. Therefore,

$$B_n \cap T(A') \equiv \Big(\bigcup_{A \in \mathcal{A}_n} \big(W(A) \cup \mathrm{Bot}(A) \cup \bigcup_{k=0}^{n-1} T_k(A) \big) \Big) \cap T(A') =$$
$$= \bigcup_{A \in \mathcal{A}_n} \Big(\big(W(A) \cup \mathrm{Bot}(A) \cup \bigcup_{k=0}^{n-1} T_k(A) \big) \cap T(A') \Big) =$$
$$= \bigcup_{A \in \mathcal{A}_n} \bigcup_{k=0}^{n-1} (T_k(A) \cap T(A')) \subset \bigcup_{A \in \mathcal{A}_n} \bigcup_{k=0}^{n-1} T_k(A) = S_2,$$

and $S_1 \subset S_2$. Next we prove $S_2 \subset S_1$. Let $A' \in \mathcal{A}_n$ and $k' \in \{0, \ldots, n-1\}$. We will show $T_{k'}(A') \subset S_1$. First,

$$T_{k'}(A') \subset \bigcup_{k=0}^{n-1} T_k(A') \subset B_{A'} \subset B_n. \tag{2.19}$$

Next, let $m' = n - k' \in \{1, \ldots, n\}$. Since $T_{k'}(A') \subset T(\mathrm{Par}^{k'}(A'))$ and $\mathrm{Par}^{k'}(A') \in \mathcal{A}_{n-k'} = \mathcal{A}_{m'}$, it follows that

$$T_{k'}(A') \subset T(\mathrm{Par}^{k'}(A')) \subset \bigcup_{m=1}^{n} \bigcup_{A \in \mathcal{A}_m} T(A). \tag{2.20}$$

By (2.19), (2.20), and the definition of S_1, $T_{k'}(A') \subset S_1$. Therefore $S_2 \subset S_1$. Lemma 2.2 is proven.

□

Now as $n \to \infty$, $\mathcal{H}^2(\bigcup_{A \in \mathcal{A}_n} W(A)) \to 0$. Also $K_n \to K_\infty$ as $n \to \infty$. So taking limits of both sides of (2.17) as $n \to \infty$ yields the second inequality in the lemma below:

Lemma 2.3. *Let $u \in C^\infty(\mathbb{D}) \cap BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$. Then*

$$\int_{B_\infty \cap \mathbb{D}} |\nabla u| \, dx \geq \sum_{n=1}^{\infty} \sum_{A \in \mathcal{A}_n} \int_{T(A) \cap B_\infty} |\nabla u \cdot \mathbf{v}(A)| \, dx + \int_{\mathbb{D}_- \cap B_\infty} |\nabla u \cdot \mathbf{j}| \, dx \geq K_\infty.$$

The first inequality is obvious because any two different triangles in the collection $\{T(A) \mid A \in \mathcal{A}_l, \ l \geq 1\}$ have intersection with zero $\mathcal{H}^2$-measure, and all such triangles are disjoint with $\mathbb{D}_-$.

Now let us prove (2.3)(i). Suppose $u \in BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$ and $\int_{\mathbb{D} \setminus B_\infty} |u| \, dx > 0$. $\mathbb{D} \setminus B_\infty$ consists of countably many open components. For at least one such component Ω, $\int_\Omega |u| \, dx > 0$. $\partial\Omega$ contains an arc of $\mathbb{S}$ of positive arc length along which f_∞ equals zero. Therefore $\int_\Omega |Du| > 0$. By [1, Theorem 1.17, Remark 1.18, Remark 2.12], there exists a sequence $(u_m) \subset C^\infty(\mathbb{D}) \cap BV(\mathbb{D})$ with $u_m|_{\mathbb{S}} = f_\infty$ for all m, $u_m \to u$ in $L^1(\mathbb{D})$, and $\int_{\mathbb{D}} |\nabla u_m| \, dx \to \int_{\mathbb{D}} |Du|$ as $m \to \infty$. By [1, Theorem 1.19], $\liminf_{m \to \infty} \int_\Omega |\nabla u_m| \, dx \geq \int_\Omega |Du| > 0$. Therefore, using Lemma 2.3,

$$\int_{\mathbb{D}} |Du| = \lim_{m \to \infty} \int_{\mathbb{D}} |\nabla u_m| \, dx \geq \liminf_{m \to \infty} \Big(\int_{\mathbb{D} \cap B_\infty} |\nabla u_m| \, dx + \int_\Omega |\nabla u_m| \, dx \Big) \geq$$
$$\geq B_\infty + \int_\Omega |Du| > B_\infty.$$

Next we prove (2.3)(ii), which will complete the proof of Theorem 1.1. Suppose $u \in BV(\mathbb{D})$ with $u|_{\mathbb{S}} = f_\infty$ and $\int_{\mathbb{D}\setminus B_\infty} |u|\,dx = 0$. Recall $\mathbb{D}_-$, defined in (2.16). Since $u \neq 0$, $\int_{\mathbb{D}_-} |u|\,dx > 0$ or $\int_{\mathbb{D}\setminus\mathbb{D}_-} |u|\,dx > 0$. We examine the former case first. Assume $\int_{\mathbb{D}_-} |u|\,dx > 0$. Then there exists a closed rectangle $[a,b]\times[c,d] \subset \mathbb{D}_-$ and $\delta > 0$ with

$$\int_{[a,b]\times[c,d]} |u|\,dx > \delta.$$

B_∞ is a compact, nowhere dense subset of $\overline{\mathbb{D}}$. The restriction of χ_{B_∞} to $\mathbb{D}_-$ is constant on vertical line segments. Therefore there exists an open, dense subset U of $[a,b]$ with

$$(U \times [c,d]) \cap B_\infty = \varnothing.$$

Let $a < a_1 < b_1 < b$ with $a_1, b_1 \in U$ and

$$\int_{[a_1,b_1]\times[c,d]} |u|\,dx > \frac{\delta}{2}.$$

Let $(u_m) \subset C^\infty(\mathbb{D}) \cap BV(\mathbb{D})$ be given by the construction in [1, Theorem 1.17]: $u_m|_{\mathbb{S}} = f_\infty$ for all m, $u_m \to u$ in $L^1(\mathbb{D})$ and $\int_{\mathbb{D}} |\nabla u|\,dx \to \int_{\mathbb{D}} |Du| > 0$ as $m \to \infty$. Furthermore, the u_m's are obtained by convolving u with C^∞ mollifier functions, supported on discs, with the radii of the discs approaching 0 as $m \to \infty$ uniformly on the rectangle $[a,b]\times[c,d]$. Thus, for large enough m, $u_m = 0$ on the vertical line segments $\{a_1\}\times[c,d]$ and $\{b_1\}\times[c,d]$. By Lemma 3.2 in the Appendix,

$$\int_{[a_1,b_1]\times[c,d]} \left|\frac{\partial u_m}{\partial x_1}\right| dx \geq \frac{2}{b_1 - a_1}\int_{[a_1,b_1]\times[c,d]} |u_m|\,dx > \frac{\delta}{b_1 - a_1} \equiv \delta_2$$

for large enough m. Clearly, for large enough m,

$$\int_{[a_1,b_1]\times[c,d]} \left|\frac{\partial u_m}{\partial x_2}\right| dx \leq \int_{[a_1,b_1]\times[c,d]} |\nabla u_m|\,dx \leq \int_{\mathbb{D}} |\nabla u_m|\,dx < 2\int_{\mathbb{D}} |Du|.$$

Therefore, for large enough m, by Lemma 3.3 in the Appendix (using $g = |\partial u_m/\partial x_1|$ and $h = |\partial u_m/\partial x_2| = |\nabla u_m \cdot \mathbf{j}|$),

$$\begin{aligned}\int_{[a_1,b_1]\times[c,d]} |\nabla u_m|\,dx &\geq \int_{[a_1,b_1]\times[c,d]} |\nabla u_m \cdot \mathbf{j}|\,dx + \frac{\delta_2^2}{4\int_{\mathbb{D}}|Du| + \delta_2} \equiv \\ &\equiv \int_{[a_1,b_1]\times[c,d]} |\nabla u_m \cdot \mathbf{j}|\,dx + \delta_3.\end{aligned} \tag{2.21}$$

The collection of triangles $\{T(A) \bigm| A \in \mathcal{A}_l,\, l \geq 1\}$ is a countable family of sets, for which the intersection of any distinct pair has zero $\mathcal{H}^2$-measure. All these triangles are subsets of $\overline{\mathbb{D}} \setminus \mathbb{D}_-$. Therefore, applying (2.21) and Lemma 2.3, it follows that for large enough m,

$$\begin{aligned}\int_{\mathbb{D}} |\nabla u_m| &= \int_{\mathbb{D}_-} |\nabla u_m|\,dx + \int_{\mathbb{D}\setminus\mathbb{D}_-} |\nabla u_m|\,dx \geq \\ &\geq \int_{\mathbb{D}_-} |\nabla u_m \cdot \mathbf{j}|\,dx + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)} |\nabla u_m|\,dx \geq \\ &\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,dx + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)\cap B_\infty} |\nabla u_m \cdot \mathbf{v}(A)|\,dx \geq \\ &\geq K_\infty + \delta_3.\end{aligned}$$

Since $\int_{\mathbb{D}} |\nabla u_m|\,dx \to \int_{\mathbb{D}} |Du|$ as $m \to \infty$, it follows that $\int_{\mathbb{D}} |Du| \geq K_\infty + \delta_3 > K_\infty$.

Next, suppose $\int_{\mathbb{D}\setminus\mathbb{D}_-} |u|\,dx > 0$ (and $\int_{\mathbb{D}\setminus B_\infty} |u|\,dx = 0$). Since

$$((\mathbb{D}\setminus\mathbb{D}_-)\cap B_\infty) \subset \bigcup_{n=1}^{\infty}\bigcup_{A\in\mathcal{A}_n} T(A),$$

there exists $n' \geq 1$ and $A \in \mathcal{A}_{n'}$ with $\int_{T(A)} |u|\,dx > 0$. There then exists a closed rectangle $R \subset T(A)\cap\mathbb{D}$ with sides parallel and perpendicular to $\mathbf{v}(A)$ and $\int_R |u|\,dx > 0$.

Let (u_m) be given by the construction in [1, Theorem 1.17], as before. Arguing as before, let the line segment L be one of the two sides of R perpendicular to $\mathbf{v}(A)$. L has an open and dense (with respect to the subspace topology on L) subset X with $X \cap B_\infty = \varnothing$. From the way B_∞ is constructed, if $\mathbf{x} \in R$ and the vector $\mathbf{x}-\mathbf{y}$ is parallel to $\mathbf{v}(A)$ for some $\mathbf{y} \in X$, then $\mathbf{x} \notin B_\infty$. Arguing as in the $\int_{\mathbb{D}_-} |u|\,dx > 0$ case, there exists $\delta_3 > 0$ with

$$\int_R |\nabla u_m|\,dx \geq \int_R |\nabla u_m \cdot \mathbf{v}(A)|\,dx + \delta_3 \tag{2.22}$$

for large enough m. Using Lemma 2.3 and (2.22), for large enough m,

$$\begin{aligned}\int_{\mathbb{D}} |\nabla u_m|\,dx &= \int_{\mathbb{D}_-} |\nabla u_m|\,dx + \int_{\mathbb{D}\setminus\mathbb{D}_-} |\nabla u_m|\,dx \geq \\ &\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m|\,dx + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)} |\nabla u_m|\,dx \geq \\ &\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,dx + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)} |\nabla u_m \cdot \mathbf{v}(A)|\,dx \geq \\ &\geq \int_{\mathbb{D}_-\cap B_\infty} |\nabla u_m \cdot \mathbf{j}|\,dx + \delta_3 + \sum_{n=1}^{\infty}\sum_{A\in\mathcal{A}_n}\int_{T(A)\cap B_\infty} |\nabla u_m \cdot \mathbf{v}(A)|\,dx \geq \\ &\geq K_\infty + \delta_3.\end{aligned}$$

Like before, since $\int_{\mathbb{D}} |\nabla u_m|\,dx \to \int_{\mathbb{D}} |Du|$ as $m \to \infty$, it follows that $\int_{\mathbb{D}} |Du| \geq K_\infty + \delta_3 > K_\infty$. The proof of Theorem 1.1 is complete.

□

3. Appendix: Three Lemmas

This section contains three easy, self-contained lemmas, moved to the end of the paper in order not to interrupt the flow of the main proof.

Lemma 3.1. *Let $\theta \in (0,1]$ and $\alpha \in [\theta^2/2, \theta)$. Let P and S be points on $\mathbb{S}$ separated by arc length θ. Let Q and R lie on the arc $\overset{\frown}{PS}$, with $\overset{\frown}{QR}$ having arc length α, $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ having equal arc length, and Q between P and R. Let T and U lie on the chord $\overline{PS}$, chosen such that $\triangle PQT$ and $\triangle RSU$ are right triangles. Then $\triangle PQT$ and $\triangle RSU$ have disjoint closures.*

Proof: clearly it suffices to consider $\alpha = \theta^2/2$. By rotating the arc $\overset{\frown}{PS}$, we may assume $P = (\cos(\theta/2), \sin(\theta/2))$, $S = (\cos(\theta/2), -\sin(\theta/2))$, $Q = (\cos(\theta^2/4), \sin(\theta^2/4))$, and $R = (\cos(\theta^2/4), -\sin(\theta^2/4))$. Define $V = (\cos(\theta/2), 0)$. It suffices to show the angle $\angle PQV$ is obtuse, using a dot product. We will show $\vec{QP} \cdot \vec{QV} < 0$. Using familiar trigonometric identities,

$$\begin{aligned}
\vec{QP} &= \langle \cos(\tfrac{1}{2}\theta) - \cos(\tfrac{1}{4}\theta^2), \sin(\tfrac{1}{2}\theta) - \sin(\tfrac{1}{4}\theta^2)\rangle,\\
\vec{QV} &= \langle \cos(\tfrac{1}{2}\theta) - \cos(\tfrac{1}{4}\theta^2), -\sin(\tfrac{1}{4}\theta^2)\rangle,\\
\vec{QP}\cdot\vec{QV} &= (\cos(\tfrac{1}{4}\theta^2) - \cos(\tfrac{1}{2}\theta))^2 - (\sin(\tfrac{1}{2}\theta) - \sin(\tfrac{1}{4}\theta^2))\sin(\tfrac{1}{4}\theta^2) =\\
&= \cos^2(\tfrac{1}{4}\theta^2) + \cos^2(\tfrac{1}{2}\theta) - 2\cos(\tfrac{1}{4}\theta^2)\cos(\tfrac{1}{2}\theta) -\\
&\qquad \sin(\tfrac{1}{2}\theta)\sin(\tfrac{1}{4}\theta^2) + \sin^2(\tfrac{1}{4}\theta^2) =\\
&= 1 + (\tfrac{1}{2} + \tfrac{1}{2}\cos(\theta)) - (\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\theta^2) + \cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\theta^2)) -\\
&\qquad \tfrac{1}{2}(\cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\theta^2) - \cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\theta^2)) =\\
&= \tfrac{3}{2} + \tfrac{1}{2}\cos\theta - \tfrac{1}{2}\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\theta^2) - \tfrac{3}{2}\cos(\tfrac{1}{2}\theta - \tfrac{1}{4}\theta^2).
\end{aligned}$$

By the Maclaurin series for cos and properties of alternating series, $1 - x^2/2 < \cos x < 1 - x^2/2 + x^4/24$ for $0 < x < 1$. Both $\theta/2 + \theta^2/4$ and $\theta/2 - \theta^2/4$ are between 0 and 1. Therefore

$$\begin{aligned}
\vec{QP}\cdot\vec{QV} &< \frac{3}{2} + \frac{1}{2}(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}) - \frac{1}{2}(1 - \frac{1}{2}(\frac{\theta}{2} + \frac{\theta^2}{4})^2) - \frac{3}{2}(1 - \frac{1}{2}(\frac{\theta}{2} - \frac{\theta^2}{4})^2) =\\
&= -\frac{1}{8}\theta^3 + \frac{1}{24}\theta^4 < 0.
\end{aligned}$$

□

Lemma 3.2. *Let $a < b$, $c < d$, and $u \in C^1([a,b]\times[c,d])$ with $u = 0$ on $\{a,b\}\times[c,d]$. Then*

$$\int_{y=c}^{d}\int_{x=a}^{b} \left|\frac{\partial u}{\partial x}\right| dx\,dy \geq \frac{2}{b-a}\int_{y=c}^{d}\int_{x=a}^{b} |u(x,y)|\,dx\,dy. \tag{3.1}$$

Proof: fix $y \in [c,d]$. Let $x_0 \in (a,b)$ with $|u(x_0,y)| = \max_{[a,b]\times\{y\}} |u|$. Then

$$\begin{aligned}
\int_{x=a}^{b} |u(x,y)|\,dx &\le (b-a)|u(x_0,y)| = \\
&= (\frac{b-a}{2})(|u(x_0,y)-u(a,y)| + |u(b,y)-u(x_0,y)|) = \\
&= (\frac{b-a}{2})(\left|\int_a^{x_0} \frac{\partial u}{\partial x}\,dx\right| + \left|\int_{x^0}^{b} \frac{\partial u}{\partial x}\,dx\right|) \le \\
&\le (\frac{b-a}{2})(\int_a^{x_0} \left|\frac{\partial u}{\partial x}\right|\,dx + \int_{x_0}^{b} \left|\frac{\partial u}{\partial x}\right|\,dx) = (\frac{b-a}{2})\int_a^b \left|\frac{\partial u}{\partial x}\right|\,dx.
\end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $2/(b-a)$ and integrating from $y=c$ to $y=d$ yields (3.1).

□

Lemma 3.3. *Let $M, \delta > 0$, let Ω be an open subset of $\mathbb{R}^n$ $(n \ge 1)$, let $g, h \in L^1(\Omega)$ with $g, h \ge 0$ Lebesgue-a.e., and assume $\int_\Omega g\,dx \ge \delta$, $\int_\Omega h\,dx \le M$. Then*

$$\int_\Omega \sqrt{g^2+h^2}\,dx \ge \int_\Omega h\,dx + \frac{\delta^2}{2M+\delta}. \tag{3.3}$$

Proof (this proof is courtesy of Oleksiy Klurman of the University of Manitoba): since $x^2/(\sqrt{x^2+y^2}+|y|) \to 0$ as $(x,y) \to (0,0)$, we will interpret the expression "$g^2/(\sqrt{g^2+h^2}+h)$" as zero when $g = h = 0$ below. By the Cauchy-Schwarz Inequality,

$$\begin{aligned}
(\int_\Omega g\,dx)^2 &= \left(\int_\Omega \frac{g}{\sqrt{\sqrt{g^2+h^2}+h}} \cdot \sqrt{\sqrt{g^2+h^2}+h}\,dx\right)^2 \le \\
&\le \left(\int_\Omega \frac{g^2}{\sqrt{g^2+h^2}+h}\,dx\right)\left(\int_\Omega \sqrt{g^2+h^2}+h\,dx\right) = \\
&= \left(\int_\Omega \sqrt{g^2+h^2}-h\,dx\right)\left(\int_\Omega \sqrt{g^2+h^2}+h\,dx\right) \le \\
&\le \left(\int_\Omega \sqrt{g^2+h^2}-h\,dx\right)\left(2\int_\Omega h\,dx + \int_\Omega g\,dx\right) \le \\
&\le \left(\int_\Omega \sqrt{g^2+h^2}-h\,dx\right)\left(2M + \int_\Omega g\,dx\right).
\end{aligned}$$

Therefore,

$$\int_\Omega \sqrt{g^2+h^2}-h\,dx \ge \frac{(\int_\Omega g\,dx)^2}{2M+\int_\Omega g\,dx} \ge \frac{\delta^2}{2M+\delta}, \tag{3.4}$$

because the map $x \mapsto x^2/(2M+x)$ is an increasing function of x for $x \ge 0$. Rearranging (3.4) yields (3.3).

□

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